Note 4
Gaussian Integral, Factorial Function,
Beta Integral, Volumes of Spheres

1. Gaussian Integral

The **Gaussian Integral** is

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx. \tag{1a}$$

This integral cannot be computed in an elementary way, because the antiderivative of e^{-x^2} is not one of the basic functions we ordinarily study. One can compute the Gaussian integral using double integrals, using a trick, which turns out to be useful for other integrals as well. The essence of the trick is that the entire plane is both a disk and a rectangle, so we can compute double integrals over the plane in two ways.

The starting point is to compute the square of the integral, and use the product formula (see Note 3) to get an integral over the plane $-\infty \leq x \leq \infty,\ -\infty \leq x \leq \infty$. Then view the plane as the disk $0 \leq \theta \leq 2\pi,\ 0 \leq r \leq \infty$, which gives an extra r, that enables us to compute the disk integral directly. Then take the square root to get (1). Here we go:

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy\\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\, rdrd\theta\\
&= 2\pi\cdot\frac{1}{2}\int_{0}^{\infty} e^{-t}\, dt\\
&= 2\pi\cdot\frac{1}{2}\cdot 1 = \pi.
\end{aligned}$$

In the next to last line, we made the substitution $t = r^2$, and computed

$$\int_{0}^{\infty} e^{-t}\, dt = 1. \tag{1b}$$

We therefore have

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \pi,$$

and so we have computed

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \qquad \text{(GAUSSIAN INTEGRAL)}$$

2. Factorial Function

The Gaussian Integral is related to factorials. The usual definition of factorials works only for positive integers:

$$n! = n(n-1)(n-2)\cdots 2\cdot 1.$$

There is also the Royal Decree that

$$0! = 1,$$

with the explanation that "it makes the formulas work out nicely", which is true, but is not really an explanation.

The *true* Factorial function is a formula for $n!$ that works for any number $n > -1$, whether or not n is an integer. The formula was found by Euler (what follows is slightly different from his approach, but becomes the same after a change of variables).

Consider the integral $\int_0^\infty x^3 e^{-x}\,dx$. Use integration by parts ($u = x^3$, $dv = e^{-x}\,dx$) and the fact that $e^{-\infty} = 0$, and you get

$$\int_0^\infty x^3 e^{-x}\,dx = 3\int_0^\infty x^2 e^{-x}\,dx.$$

Do integration by parts on this new integral (now $u = x^2$) and get

$$\int_0^\infty x^3 e^{-x}\,dx = 3\int_0^\infty x^2 e^{-x}\,dx = 3\cdot 2\int_0^\infty x e^{-x}\,dx.$$

One more integration by parts gives

$$\int_0^\infty x^3 e^{-x}\,dx = 3\cdot 2\cdot 1\int_0^\infty e^{-x}\,dx = 3\cdot 2\cdot 1\cdot 1 = 3!\ .$$

The last integral is (1b) from section 1.

For any number n, using integration by parts with $u = x^n$, you get

(2a) $$\int_0^\infty x^n e^{-x}\,dx = n\int_0^\infty x^{n-1} e^{-x}\,dx.$$

If n is an integer, you can repeat this over and over, you get

(2b) $$\int_0^\infty x^n e^{-x}\,dx = n!\ .$$

The advantage of the formula (2b) is that the integral makes sense for any number $n > -1$, regardless of whether it is an integer. (You need $n > -1$ to make the integral converge.)

The true definition of factorial is, therefore,

$$n! = \int_0^\infty x^n e^{-x}\,dx. \qquad \text{(FACTORIAL INTEGRAL)}$$

First of all, this explains why $0! = 1$. It is because

$$0! = \int_0^\infty x^0 e^{-x}\, dx = 1.$$

Equation (2a) says

$$n! = n(n-1)!, \tag{2c}$$

for any number $n > -1$ (not necessarily an integer).

We can now compute factorials of fractions. For example,

$$(-\frac{1}{2})! = \int_0^\infty x^{-1/2} e^{-x}\, dx.$$

In this integral, let $u = x^{1/2}$, so $2du = x^{-1/2}dx$, and you get

$$(-\frac{1}{2})! = 2\int_0^\infty e^{-x^2}\, dx.$$

Now e^{-x^2} is an even function, so twice the integral from 0 to ∞ is the same as the integral from $-\infty$ to ∞. Therefore we get

$$(-\frac{1}{2})! = \int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}. \tag{2d}$$

To compute other factorials, use equation (2c,d). For example,

$$(\frac{1}{2})! = (\frac{1}{2})(\frac{1}{2}-1)! = (\frac{1}{2})(-\frac{1}{2})! = (\frac{1}{2})\sqrt{\pi}.$$

Thus, using (2c,d), you can compute any factorial $(\frac{n}{2})!$, where n is an integer ≥ -1. Another example:

$$(5/2)! = (5/2)(3/2)! = (5/2)(3/2)(\sqrt{\pi}/2) = \frac{15\sqrt{\pi}}{8}.$$

Exercise 2.1 Show that

$$\int_0^\infty x^a e^{-x^2}\, dx = \frac{1}{2}(\frac{a-1}{2})!$$

(Let $u = x^2$, to turn the integral into the factorial integral.)

Note: In advanced mathematics, the Factorial function is replaced by the Gamma function: $\Gamma(x) = (x+1)!$. There are good reasons for the shift from x to $x+1$, but they won't be apparent in our course. We will stick with the terminology "Factorial".

3. The Beta Integral

We can compute many integrals in terms of factorials. A famous example is the Beta Integral:

$$\int_0^1 x^p(1-x)^q\,dx$$

where p and q can be any numbers > -1.

The Beta integral is actually a family of integrals. For each choice of p and q you get a different integral, and there is a single formula, involving factorials of p and q, that works for all of them. For example, if we take $p = q = \frac{1}{2}$, the Beta integral formula will tell us that

$$\int_0^1 \sqrt{x - x^2}\,dx = \frac{\pi}{8}.$$

If we take $p = \frac{1}{2}$, $q = -\frac{1}{2}$, the Beta integral formula will tell us that

$$\int_0^1 \sqrt{\frac{x}{1-x}}\,dx = \frac{\pi}{2}.$$

In this section you will find the formula for the Beta Integral, using the steps outlined in the next three exercises.

Exercise 3.1 Show that

$$\int_0^1 x^p(1-x)^q\,dx = 2\int_0^{\pi/2} \cos^{2p+1}\theta\,\sin^{2q+1}\theta\,d\theta.$$

(Start with the integral on the left side and set $x = \cos^2\theta$. Remember that you are not trying to compute either integral, you just want to show that one integral is equal to the other.)

Exercise 3.2 Show that

$$\Big(\int_0^\infty x^{2p+1}e^{-x^2}\,dx\Big)\cdot\Big(\int_0^\infty y^{2q+1}e^{-y^2}\,dy\Big)$$
$$= \Big(\int_0^\infty r^{2p+2q+3}e^{-r^2}\,dr\Big)\Big(\int_0^{\pi/2} \cos^{2p+1}\theta\,\sin^{2q+1}\theta\,d\theta\Big).$$

(Start with the left side, change it into a double integral over the first quadrant, using the product formula. Change to polar coordinates, and then use the product formula again.)

Exercise 3.3 Apply Exercise 2.1 to the first three integrals in Exercise 3.2, to show that

$$2\int_0^{\pi/2} \cos^{2p+1}\theta\,\sin^{2q+1}\theta\,d\theta = \frac{p!q!}{(p+q+1)!}\,. \qquad \text{(TRIG FORM OF BETA INTEGRAL)}$$

Combining Exercises 3.1,3.3, you have found the formula (for $p, q > -1$)

$$\int_0^1 x^p(1-x)^q \, dx = \frac{p!q!}{(p+q+1)!} \, . \qquad \text{(BETA INTEGRAL)}$$

The Beta integral occurs frequently in very simple problems. For example, suppose we want to integrate a polynomial, like

$$x^7y^3 - 3x^6y^8 + 2x^4y^5$$

over the unit disk. Naturally, we try to integrate each term separately. The first and last terms contain odd exponents, so their integral will be zero (the values in one quadrant will be the negative of the values in another quadrant, so will cancel each other). So we only have to integrate the middle term x^6y^8. This term takes the same values in all four quadrants, so its integral over the disk is four times the integral over a quadrant. Let R be the first quadrant. We get

$$\begin{aligned}\iint_{\text{Disk}} x^7y^3 - 3x^6y^8 + 2x^4y^5 dR &= -3 \iint_{\text{Disk}} x^6y^8 dR \\ &= -12 \iint_R x^6y^8 \, dR \\ &= -12 \int_0^{\pi/2} \int_0^1 (r\cos\theta)^6 (r\sin\theta)^8 \, r dr d\theta.\end{aligned}$$

Exercise 3.4 Use the trigonometric form of the Beta integral (Exercise (3.3)) to finish computing $\iint_{\text{Disk}} x^7y^3 - 3x^6y^8 + 2x^4y^5 dR$.

Exercise 3.5 Using the same method, compute

$$\iint_{\text{Disk}} x^{2n}y^{2m} \, dR$$

for $n, m \geq 0$. (Answer: $\frac{(n-\frac{1}{2})!(m-\frac{1}{2})!}{(n+m+1)!}$)

Exercise 3.6 Use the Beta integral to compute

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

Let $u = x^4$ and put the integrand in the form $u^p(1-u)^q$.
(Answer: $\sqrt{\pi}(1/4)!/(-1/4)!$)

4. The measure of Spheres

- The circle has equation $x^2 + y^2 = 1$, and its length is 2π.
- The sphere has equation $x^2 + y^2 + z^2 = 1$, and its surface area is 4π.
- The hypersphere has equation $x^2+y^2+z^2+w^2 = 1$, and its volume is... what?

The Gaussian integral can be used to compute the volume of the hypersphere. We will in fact work with the sphere in n-dimensions:

$$S^{n-1} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \ x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

We call this sphere S^{n-1} because a close up view of it looks like a slightly curved $\mathbb{R}^{n-1}$ (for example, the surface of the earth is S^2, and it looks like the plane $\mathbb{R}^2$). We use the single word "measure" to mean either the length of S^1, or the area of S^2, or the volume of S^3, or.... Thus, our goal is to compute the measure of S^{n-1}, for any n.

To see the idea, we review the calculation of the Gaussian Integral from section1. There we had

$$\text{(4a)} \qquad \int_{\mathbb{R}^2} e^{x^2+y^2}\ dR = \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\ r dr d\theta.$$

We used the product formula to write the r, θ integral as

$$\Big(\int_0^{\infty} e^{-r^2}\ r dr\Big) \cdot \Big(\int_0^{2\pi} d\theta\Big).$$

Look at the last integral: $\int_0^{2\pi} d\theta$. It is the integral of the function $f = 1$ over the circle S^1:

$$\text{(4b)} \qquad \int_0^{2\pi} d\theta = \int_{S^1} ds = \text{measure of } S^1.$$

So (4a) becomes

$$\text{(4c)} \qquad \Big(\int_{-\infty}^{\infty} e^{-x^2}\ dx\Big)\Big(\int_{-\infty}^{\infty} e^{-y^2}\ dx\Big) = \Big(\int_0^{\infty} e^{-r^2}\ r dr\Big) \cdot (\text{measure of } S^1).$$

If we didn't know the length of the circle, we could use (4c) to find it: The first two integrals in (4b) are Gaussian Integrals and the third is easy. Substituting their values into (4c), we get

$$\sqrt{\pi} \cdot \sqrt{\pi} = \frac{1}{2} \cdot (\text{measure of } S^1).$$

Solving for measure of S^1, we get

$$\text{measure of } S^1 = 2\pi.$$

Therefore the Gaussian Integral tells us the measure of S^1. This may not seem interesting, because we already know the length of a circle, but the same idea will tell us the measure of S^{n-1} for any n. All we have to do is take more integrals on the left side of (4c).

First, a few words about integration in higher dimensions. Our integrals will be over the simplest region: $\mathbb{R}^n$ itself. On the one hand,

$$\int \cdots \int_{\mathbb{R}^n} f(x_1, x_2, \ldots, x_n)\ dR = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\ dx_1 dx_2 \cdots dx_n.$$

You do the dx_1 integral first, holding the other variables constant. Then do the dx_2 integral, and so on, just like double integrals, except that now you do n integrals.

There is also a product formula: If $f(x_1, x_2, \ldots, x_n) = g_1(x_1) g_2(x_2) \cdots g_n(x_n)$, then

$$\begin{aligned} &\int_{\mathbb{R}^n} g_1(x_1) g_2(x_2) \cdots g_n(x_n)\ dR \\ \text{(4d)} \qquad &= \Big(\int_{-\infty}^{\infty} g_1(x_1)\ dx_1 \Big) \cdot \Big(\int_{-\infty}^{\infty} g_2(x_2)\ dx_2 \Big) \cdots \Big(\int_{-\infty}^{\infty} g_n(x_n)\ dx_n \Big). \end{aligned}$$

On the other hand, you can use polar coordinates in $\mathbb{R}^n$. Let

$$r^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

The value of r at a point in $\mathbb{R}^n$ is the distance from the point to the origin $(0, \ldots, 0)$. To simplify things, suppose the function f that we want to integrate is a function of r only. That is, suppose the value of f at a point depends only on the distance from the point to the origin. Then the polar coordinates formula for integration is

$$\text{(4e)} \qquad \int \cdots \int_{\mathbb{R}^n} f(x_1, x_2, \ldots, x_n)\ dR = \Big(\int_0^{\infty} f(r) r^{n-1} dr \Big) \cdot \Big(\int \cdots \int_{S^{n-1}} ds \Big).$$

The $r^{n-1}dr$ is the higher dimensional analogue of $r dr$. The integral $\int \cdots \int_{S^{n-1}} ds$ (with $n-1$ integral signs) is the higher dimensional analogue of the integral over a circle (of the constant function 1). Compare with equation (4b). At the moment, all we need to know about this integral is that

$$\int \cdots \int_{S^{n-1}} ds = \text{measure of } S^{n-1}.$$

For example,

$$\int_{S^1} ds = 2\pi, \qquad \iint_{S^2} ds = 4\pi,$$

and $\iiint_{S^3} ds$ is what we want to compute, along with all higher dimensional measures.

Now, take the function

$$f(x_1, x_2, \ldots, x_n) = e^{-x_1^2} \cdot e^{-x_2^2} \cdots e^{-x_n^2} = e^{-(x_1^2 + x_2^2 + \cdots + x_n^2)} = e^{-r^2}.$$

Since f is a product function and also a function of r alone, we can use both equations (4d) and (4e) to compute $\int_{\mathbb{R}^n} f\, dR$. We get

$$\begin{aligned}\Big(\int_{-\infty}^{\infty} e^{-x_1^2}\, dx_1\Big)\cdot\Big(\int_{-\infty}^{\infty} e^{-x_2^2}\, dx_2\Big)\cdots\Big(\int_{-\infty}^{\infty} e^{-x_n^2}\, dx_n\Big)\\ = \int\cdots\int_{\mathbb{R}^n} f\, dR\\ = \Big(\int_0^{\infty} e^{-r^2} r^{n-1} dr\Big)\cdot\Big(\int\cdots\int_{S^{n-1}} ds\Big).\end{aligned}$$

The first bunch of integrals are the same Gaussian Integral; each one is equal to $\sqrt{\pi}$. You showed in Exercise 2.1 that

$$\int_0^{\infty} e^{-r^2} r^{n-1} dr = \frac{1}{2}\Big(\frac{n-2}{2}\Big)!\ .$$

Finally, the last integral is the measure of S^{n-1}. Therefore,

$$(\sqrt{\pi})^n = \frac{1}{2}\Big(\frac{n-2}{2}\Big)! \cdot (\text{measure of } S^{n-1}).$$

We get the formula

$$\text{(4f)} \qquad \text{measure of } S^{n-1} = \frac{2\pi^{n/2}}{(\frac{n}{2}-1)!}.$$

Exercise 4.1 Check that (4f) gives the right answer for the length of a circle and the area of sphere. Then use (4f) to find the volume (measure) of the hypersphere S^3, and the measure of S^4. Do you see a pattern in the powers of π that appear?

Aside from the intrinsic interest of spheres, we need to know the measure of S^{n-1} in order to use formula (4e) to compute integrals of other functions $f(r)$. Here is one example.

- The disk of radius a is the set of points (x, y) in $\mathbb{R}^2$ satisfying $x^2 + y^2 \leq a^2$. Let's call this disk $B^2(a)$. Its area is πa^2.
- The ball of radius a is the set of points (x, y, z) in $\mathbb{R}^3$ satisfying $x^2+y^2+z^2 \leq a^2$. Let's call this ball $B^3(a)$. Its volume is $\frac{4}{3}\pi a^3$.
- The hyperball radius a is the set of points (x, y, z, w) in $\mathbb{R}^4$ satisfying $x^2+y^2+z^2+w^2 \leq a^2$. Let's call this hyperball $B^4(a)$. Its hypervolume is ... what?

To answer this, use (4e), with the function

$$f(r) = \begin{cases} 1, & \text{for } 0 \leq r \leq a \\ 0, & \text{for } a < r < \infty. \end{cases}$$

The left side of (4e) is the integral of the function 1 over the hyperball, so the left side of (4e) is the hypervolume of the hyperball.

Exercise 4.2 Compute the right side of (4e) with the above function $f(r)$, and thus determine the hypervolume of the hyperball $B^4(a)$. Remember, you now know the measure of S^3, from Exercise 4.1.

Once again, we use the word "measure" to mean the area of a disk, the volume of a ball, the hypervolume of a hyperball, etc. In the next exercise, you will show that the measures of the even dimensonal balls have a simple formula, and that the sum of all of these measures is very simple indeed.

Exercise 4.3 Let $n = 2k$, and find the measure of $B^{2k}(a)$ using the same method as in the previous exercise. Let $b_{2k}(a)$ denote the measure of $B^{2k}(a)$. Now show that

$$1 + b_2(a) + b_4(a) + b_6(a) + \cdots = e^{\pi a^2}.$$