Note 4
SOLUTIONS TO EXERCISES

1. Gaussian Integral

No exercises in this section

2. Factorial Function

Exercise 2.1 Show that

$$\int_0^\infty x^a e^{-x^2}\ dx = \frac{1}{2}\left(\frac{a-1}{2}\right)!$$

SOLUTION: $u = x^2,\ x = u^{1/2},\ dx = \frac{1}{2}u^{-1/2}\ du$, and the limits do not change. The integral becomes

$$\int_0^\infty u^{a/2}e^{-u}\frac{1}{2}u^{-1/2}\ du = \frac{1}{2}\int_0^\infty u^{(a-1)/2}e^{-u}\ du = \frac{1}{2}\left(\frac{a-1}{2}\right)!.$$

3. The Beta Integral

Exercise 3.1 Show that

$$\int_0^1 x^p(1-x)^q\ dx = 2\int_0^{\pi/2}\cos^{2p+1}\theta\sin^{2q+1}\theta\ d\theta.$$

SOLUTION: $x = \cos^2\theta,\ dx = -2\cos\theta\sin\theta\ d\theta$, and the limits change from $x = 0, 1$ to $\theta = \pi/2, 0$. We get

$$\int_0^1 x^p(1-x)^q\ dx = -2\int_{\pi/2}^0 \cos^{2p}\theta(1-\cos^2\theta)^q(\cos\theta\sin\theta)\ d\theta = 2\int_0^{\pi/2}\cos^{2p+1}\theta\sin^{2q+1}\theta\ d\theta.$$

Exercise 3.2 Show that

$$\Big(\int_0^\infty x^{2p+1}e^{-x^2}\ dx\Big)\cdot\Big(\int_0^\infty y^{2q+1}e^{-y^2}\ dy\Big)$$
$$= \Big(\int_0^\infty r^{2p+2q+3}e^{-r^2}\ dr\Big)\Big(\int_0^{\pi/2}\cos^{2p+1}\theta\sin^{2q+1}\theta\ d\theta\Big).$$

SOLUTION:

$$\begin{aligned}\Big(\int_0^\infty x^{2p+1}e^{-x^2}\ dx\Big)\cdot\Big(\int_0^\infty y^{2q+1}e^{-y^2}\ dy\Big) &= \int_0^\infty\int_0^\infty x^{2p+1}y^{2q+1}e^{-(x^2+y^2)}\ dxdy\\ &= \int_0^{2\pi}\int_0^\infty (r\cos\theta)^{2p+1}(r\sin\theta)^{2q+1}e^{-r^2}\ rdrd\theta\\ &= \int_0^{2\pi}\int_0^\infty r^{2p+2q+3}e^{-r^2}\cos^{2p+1}\theta\sin^{2q+1}\theta\ drd\theta\\ &= \Big(\int_0^\infty r^{2p+2q+3}e^{-r^2}\ dr\Big)\Big(\int_0^{\pi/2}\cos^{2p+1}\theta\sin^{2q+1}\theta\ d\theta\Big).\end{aligned}$$

Exercise 3.3 Apply Exercise 2.1 to the first three integrals in Exercise 3.2, to show that

$$2\int_0^{\pi/2} \cos^{2p+1}\theta \sin^{2q+1}\theta \; d\theta = \frac{p!q!}{(p+q+1)!}\ . \qquad \text{(TRIG FORM OF BETA INTEGRAL)}$$

SOLUTION: Applying 2.1 to 3.2 we get

$$\frac{1}{2}\left(\frac{2p+1-1}{2}\right)! \cdot \frac{1}{2}\left(\frac{2q+1-1}{2}\right)! = \frac{1}{2}\left(\frac{2p+2q+3-1}{2}\right)! \cdot \left(\int_0^{\pi/2} \cos^{2p+1}\theta \sin^{2q+1}\theta \; d\theta\right).$$

Simplifying, and isolating the last integral gives the desired formula.

Exercise 3.4 Use the trigonometric form of the Beta integral (Exercise (3.3)) to finish computing $\iint_{\text{Disk}} x^7y^3 - 3x^6y^8 + 2x^4y^5 dR$.
SOLUTION:

$$\begin{aligned}
\iint_{\text{Disk}} x^7y^3 - 3x^6y^8 + 2x^4y^5 dR &= -12\int_0^{\pi/2}\int_0^1 (r\cos\theta)^6(r\sin\theta)^8 r\,dr\,d\theta \\
&= -12\left(\int_0^1 r^{15}\;dr\right)\cdot\int_0^{\pi/2}\cos^6\theta\sin^8\theta\;d\theta \\
&= -12\cdot\frac{1}{16}\cdot\frac{(5/2)!(7/2)!}{2(5/2+7/2+1)!} \\
&= -\frac{3}{8}\frac{(5/2)(3/2)(\sqrt{\pi}/2)!(7/2)(5/2)(3/2)(\sqrt{\pi}/2)!}{7!} \\
&= -\frac{15\pi}{2^{14}}
\end{aligned}$$

Exercise 3.5 Using the same method, compute

$$\iint_{\text{Disk}} x^{2n}y^{2m}\; dR$$

for $n, m \geq 0$.
SOLUTION:

$$\begin{aligned}
\iint_{\text{Disk}} x^{2n}y^{2m}\; dR &= 4\int_0^{\pi/2}\int_0^1 r^{2n+2m+1}\cos^{2n}\theta\sin^{2m}\theta\;dr\,d\theta \\
&= \frac{4}{2n+2m+2}\,\frac{(n-\frac{1}{2})!(m-\frac{1}{2})!}{2(n+m)!} \\
&= \frac{(n-\frac{1}{2})!(m-\frac{1}{2})!}{(n+m+1)!}.
\end{aligned}$$

Exercise 3.6 Use the Beta integral to compute

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

SOLUTION: Let $u = x^4$, so $x = u^{1/4}$, $dx = (1/4)u^{-3/4}du$, and the limits don't change. We get

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 (1/4)u^{-3/4}(1-u)^{-1/2}\;du = \frac{1}{4}\cdot\frac{(-3/4)!(-1/2)!}{(-3/4-1/2+1)!} = \sqrt{\pi}\frac{(1/4)!}{(-1/4)!}.$$

4. The measure of Spheres

The formula we found in this section is

$$\text{measure of } S^{n-1} = \frac{2\pi^{n/2}}{(\frac{n}{2}-1)!}. \tag{4f}$$

Exercise 4.1 Check that (4f) gives the right answer for the length of a circle and the area of sphere. Then use (4f) to find the volume (measure) of the hypersphere S^3, and the measure of S^4. Do you see a pattern in the powers of π that appear?
SOLUTION:

$$\frac{2\pi^{2/2}}{(\frac{2}{2}-1)!} = 2\pi. \quad \text{(circle)}$$

$$\frac{2\pi^{3/2}}{(\frac{3}{2}-1)!} = \frac{2\pi^{3/2}}{(\sqrt{\pi}/2)} = 4\pi. \quad \text{(usual sphere)}$$

$$\frac{2\pi^{4/2}}{(\frac{4}{2}-1)!} = 2\pi^2. \quad \text{(hypersphere)}$$

$$\frac{2\pi^{5/2}}{(\frac{5}{2}-1)!} = \frac{8}{3}\pi^2. \quad (S^4)$$

The power of π in the measure of S^k is the smallest integer $\geq k/2$.

For the next two problems, formula (4e), applied to the function

$$f(r) = \begin{cases} 1, & \text{for } 0 \leq r \leq a \\ 0, & \text{for } a < r < \infty, \end{cases}$$

along with (4f), gives

$$\text{measure of } B^n(a) = \frac{a^n}{n} \cdot \frac{2\pi^{n/2}}{(\frac{n}{2}-1)!}.$$

Exercise 4.2 Compute the right side of (4e) with the above function $f(r)$, and thus determine the hypervolume of the hyperball $B^4(a)$.
ANSWER: $\pi^2 a^4/2$.

Exercise 4.3 Let $n = 2k$, and find the measure of $B^{2k}(a)$ using the same method as in the previous exercise. Let $b_{2k}(a)$ denote the measure of $B^{2k}(a)$. Now show that

$$1 + b_2(a) + b_4(a) + b_6(a) + \cdots = e^{\pi a^2}.$$

SOLUTION: Using the above formula, the measure of $B^{2k}(a)$ is found to be $\pi^k a^{2k}/k!$. Therefore

$$1 + b_2(a) + b_4(a) + b_6(a) + \cdots = 1 + \pi a^2 + \frac{(\pi a^2)^2}{2!} + \frac{(\pi a^3)^2}{3!} + \cdots + \frac{(\pi a^2)^k}{k!} + \cdots = e^{\pi a^2}.$$