Note 5

Green's Theorem, Two-dimensional Curl, Fundamental Theorem of Calculus, Divergence of a Vector field, Cauchy-Riemann equations, Most Interesting Vector Field, Jacobian, Linear mappings, Regions bounded by graphs.

1. Green's Theorem

Take a region R in the xy plane, and let $\mathbf{c}$ be the counterclockwise oriented boundary of R. Take also a vector field $\mathbf{F} = (P, Q)$.

Green's Theorem. *If P and Q have no bad points in R, then*

$$\iint_R Q_x - P_y \; dxdy = \oint_{\mathbf{c}} P \; dx + Q \; dy,$$

where the line integral on the right side is taken counterclockwise.

In practice, and in our course, the phrase "no bad points" means that P and Q are given by single formulas whose denominators are never zero in R. A more precise hypothesis for Green's Theorem is that P, Q are assumed to be continuous, with continuous first partial derivatives. Until further notice, all P's and Q's are assumed to have no bad points in R.

EXAMPLE 1: Use Green's theorem to compute $\oint_{\mathbf{c}} xy \; dx + xy \; dy$, over the counterclockwise rectangle with corners $(1,1)$, $(3,1)$, $(3,2)$, $(1,2)$: We first compute $Q_x - P_y = y - x$. Then the double integral is

$$\int_1^3 \int_1^2 y - x \; dydx = \int_1^3 \frac{3}{2} - x \; dx = 3 - \frac{8}{2} = -1.$$

You could also note that $\int_1^3 \int_1^2 y - x \; dydx = \text{Area} \cdot (\bar{y} - \bar{x}) = 2 \cdot (\frac{3}{2} - 2) = -1$.

We could check our answer by computing the line integral directly. We get

$$\oint_{\mathbf{c}} xy \; dx + xy \; dy = \int_1^3 t \; dt + \int_1^2 3t \; dt - \int_1^3 t \cdot 2 \; dt - \int_1^2 t \; dt = \frac{8}{2} + \frac{9}{2} - \frac{16}{2} - \frac{3}{2} = -1.$$

The double integral is a little easier than the line integral, because the region has so many sides. This is one use of Green's Theorem-to simplify line integrals.

Exercise 1.1: Use Green's Theorem to compute the following line integrals.

a) $\oint_{\mathbf{c}} (x^2 - y^2) \; dx + 2xy \; dy$, where $\mathbf{c}$ is the path in example 1. (You can check your answer by computing the line integral directly.)

b) $\oint_{\mathbf{c}} (x^2 - y^2) \; dx + 2xy \; dy$, where $\mathbf{c}$ is the ellipse $4(x-1)^2 + 9(y-2)^2 = 1$. (Use the center-of-mass trick for the double integral. You can check your answer by computing the line integral directly.)

c) $\oint_{\mathbf{c}} (x^2 - y^2) \; dx + 2xy \; dy$, where $\mathbf{c}$ is the counterclockwise circle $(x-2)^2 + y^2 = 1$. (Use the center-of-mass trick for the double integral. You can check your answer by computing the line integral directly.)

d) $\oint_{\mathbf{c}} e^x \sin y \; dx + e^x \cos y \; dy$, where $\mathbf{c}$ is the circle $x^2 + (y-1)^2 = 1$. (You will not be able to check your answer, since the line integral is impossible.)

You can also use Green to simplify a line integral over a non-closed curve.
EXAMPLE 2: Compute

$$\oint_{\mathbf{c}} (e^x - y)dx + (e^y + x)dy,$$

where $\mathbf{c}$ is the part of the circle $x^2 + (y-1)^2 = 1$ from $(0,0)$ to $(1,1)$.
Solution: Consider the straight line curve $\mathbf{c}_1(t) = (t,t)$. Let R be the region bounded by $\mathbf{c}$ and $\mathbf{c}_1$, the latter traversed backwards. We have $Q_x - P_y = 2$. By Green's Theorem, we have

$$\oint_{\mathbf{c}} (e^x - y)dx + (e^y + x)dy - \oint_{\mathbf{c}_1} (e^x - y)dx + (e^y + x)dy = \iint_R 2\ dR = 2 \cdot \text{Area of} R.$$

The area of R is $\frac{\pi - 2}{4}$. Therefore,

$$\oint_{\mathbf{c}} (e^x - y)dx + (e^y + x)dy = \int_0^1 (e^t - t)dt + (e^t + t)dt + \frac{\pi - 2}{2} = 2e - 3 + \frac{\pi}{2}.$$

Exercise 1.2: Use Green to compute

$$\oint_{\mathbf{c}} (1 - 2y + e^x \sin y)\ dx + (3 + 4x + e^x \cos y)\ dy$$

over the curve in example 2. (You could use the curve $\mathbf{c}_1$ from example 2, or it may be easier to go from (0,0) to (1,0) to (1,1) along horizontal and vertical line segments.)

EXAMPLE 3: Take $P = 0$, $Q = x$, and let R be any region. Then $Q_x - P_y = 1$, so the double integral is the area of R. Thus, Green's Theorem says

$$\oint_{\text{boundary of } R} x\ dy = \text{Area of } R.$$

This is a formula that we have previously verified in the case that R is a polygon. Note that this particular choice of P, Q is not the only choice that would give the area. We only require that $Q_x - P_y = 1$. For example, we could take $\mathbf{F} = (P, Q) = (-\frac{1}{2}y, \frac{1}{2}x)$. We again have $Q_x - P_y = 1$, so

$$\frac{1}{2} \oint_{\text{boundary of } R} -y\ dx + x\ dy = \text{Area of } R.$$

Exercise 1.3: Show how to calculate the center of mass of R entirely in terms of line integrals over the boundary of R. Then use your formulas to find the center of mass of the triangle with vertices $(0,0), (1,0), (0,1)$.

2. Two dimensional curl

The quantity $Q_x - P_y$ appearing in Green's Theorem is called the **curl** of $\mathbf{F}$, because the value of the function $Q_x - P_y$ at the point (x,y) measures the amount of counterclockwise spinning of $\mathbf{F}$ around (x,y). This will be explained now, by deriving another version of Green's Theorem (equation (2e) below) that does not involve integrals.

Take a tiny disk $B_a(p)$ of tiny radius a, centered at $p = (x_0, y_0)$, and let $S_a(p)$ be the boundary of $B_a(p)$, so S_a is a tiny circle of radius a.

The value of $Q_x - P_y$ is approximately equal to its average over the tiny disk $B_a(p)$, and the smaller the value of a, the better the approximation. That is,

$$\text{(2a)} \qquad Q_x(p) - P_y(p) = \lim_{a \to 0} \frac{1}{\pi a^2} \iint_{B_a(p)} Q_x - P_y \, dR.$$

By Green's Theorem, term inside the limit of $(2a)$ is

$$\text{(2b)} \qquad \frac{1}{\pi a^2} \oint_{S_a(p)} P \, dx + Q \, dy.$$

We have seen that $\oint_{S_a(p)} P \, dx + Q \, dy = \oint_{S_a(p)} \mathbf{F} \cdot \mathbf{T}_a \, ds$, where $\mathbf{T}_a$ is the counterclockwise unit tangent vector along $S_a(p)$. Therefore $(2b)$ is

$$\text{(2c)} \qquad \frac{1}{\pi a^2} \oint_{S_a(p)} P \, dx + Q \, dy = \frac{1}{\pi a^2} \oint_{S_a(p)} \mathbf{F} \cdot \mathbf{T}_a \, ds.$$

Now the integral of the function $\mathbf{F} \cdot \mathbf{T}_a$ over a curve is the average of this function over the curve, times the length of the curve. Thus,

$$\text{(2d)} \qquad \frac{1}{\pi a^2} \oint_{S_a(p)} \mathbf{F} \cdot \mathbf{T}_a \, ds = \frac{2\pi a}{\pi a^2} \cdot [\text{Average of } \mathbf{F} \cdot \mathbf{T}_a \text{ on } S_a(p)].$$

Combining $(2a - d)$, we see that

$$\text{(2e)} \qquad Q_x(p) - P_y(p) = \lim_{a \to 0} \frac{2}{a} \cdot [\text{Average of } \mathbf{F} \cdot \mathbf{T}_a \text{ on } S_a(p)].$$

The function $\mathbf{F} \cdot \mathbf{T}_a$ is, at each point on $S_a(p)$, the component of $\mathbf{F}$ in the counterclockwise tangent direction to $S_a(p)$. The average of these components clearly measures the spinning of $\mathbf{F}$ around p. This explains why $Q_x(p) - P_y(p)$ is called the curl of $\mathbf{F}$ at p.

We have used Green's Theorem to interpret the quantity $Q_x - P_y$ in terms of the curl of $\mathbf{F}$ at each point. Conversely, if you accept the curl interpretation of $Q_x - P_y$, then Green's Theorem can be made plausible, as follows.

At each point p in R, draw a tiny counterclockwise circle, indicating the curl of F at p in the center of the circle. The double integral $\iint_R Q_x - P_y dR$ is the "sum" of all the curls. Two nearby interior points will have nearly the same curl, but their circles will pass one another in different directions. So it is plausible that the interior curls cancel out. Only the boundary curls survive, since we do not count the cancelling curls from circles exterior to R. This surviving boundary curl is the counterclockwise flow of $\mathbf{F}$ along the boundary of R, and this flow is the line integral $\oint_{\mathbf{C}} P dx + Q dy$.

3. The Fundamental Theorem of Calculus

Green's Theorem is the two-dimensional analogue of the Fundamental Theorem of Calculus (FTC). To explain this, we first revisit the one-dimensional version.

The one-dimensional FTC is the formula

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{3a}$$

On the left side we have a one-dimensional integral, over the interval $[a,b]$. The right side is a zero-dimensional integral over the boundary of $[a,b]$. The interval is a path, so it has a direction: from a to b. This direction appears on the right side of (3a) as well, telling us in what order to evaluate f at the endpoints a, b.

Let us imitate the previous section, by using (3a) to interpret $f'(x)$ as the average of something. Take a number x_0. Let $B_a(x_0) = [x_0 - a, x_0 + a]$ be the one-dimensional ball centered at x_0, with tiny radius a. Then $f'(x_0)$ is approximately equal to the average of $f'(x)$ over the tiny interval $B_a(x_0)$, and the measure (length) of $B_a(x_0)$ is $2a$, so

$$f'(x_0) = \lim_{a\to 0} \frac{1}{2a} \int_{x_0-a}^{x_0+a} f'(x)\,dx. \tag{3b}$$

By the FTC, the term inside the limit of (3b) is

$$\frac{1}{2a}[f(x_0+a) - f(x_0-a)]. \tag{3c}$$

Combining (3b,c), we have

$$f'(x_0) = \lim_{a\to 0} \frac{1}{a}\left[\frac{f(x_0+a) - f(x_0-a)}{2}\right]. \tag{3d}$$

Compare this with equation (2e). The integrands f' and $Q_x - P_y$ in FTC and Green, respectively, are determined by the average of f and $\mathbf{F}\cdot\mathbf{T}_a$, respectively, over the boundary of the region of integration. It is no surprise that tangent vectors do not occur in the one-dimensional FTC, since points do not have tangents. The only real difference between (2e) and (3d) is the in the factors: $\frac{2}{a}$ in (2e) versus $\frac{1}{a}$ in (3d). You might guess that the difference has to do with the dimension, and this is correct.

Exercise 3.1 Suppose $f''(x) = 0$ for all x. Show that $f(x) = cx + d$ for some constants d, d. Then show that the average of $f(x)$ over any interval $[x_0 - a, x_0 + a]$ is equal to the value of f at the center of the interval. (This is the Mean-Value property of harmonic functions in one variable).

4. The Divergence of a Vector field

The form of Green's Theorem that we have used so far involves the *flow* of $\mathbf{F}$ along the boundary. Consider now the *flux* of $\mathbf{F}$ through the boundary. Recall that the boundary curve $\mathbf{c}$ is supposed to be counterclockwise. If $\mathbf{c}(t) = (x(t), y(t))$ is a parametrization of this curve, then the outward unit normal vector at $\mathbf{c}(t)$ is

$$\mathbf{N}(t) = \frac{1}{||\mathbf{c}'(t)||}(y'(t), -x'(t)),$$

so the outward flux of $\mathbf{F}$ through $\mathbf{c}$ is

$$\oint_{\mathbf{c}} \mathbf{F}\cdot\mathbf{N}\ ds = \oint_{\mathbf{c}} -Q\ dx + P\ dy. \tag{4a}$$

Apply Green's Theorem to the right side of (4a). The curl of $(-Q, P)$ is $P_x - (-Q)_y = P_x + Q_y$, so we have the formula

$$\oint_{\mathbf{c}} \mathbf{F}\cdot\mathbf{N}\ ds = \iint_R P_x + Q_y\ dR. \quad \text{(Flux version of Green's Thm)} \tag{4b}$$

The quantity $P_x + Q_y$ is called the **divergence** of $\mathbf{F}$.

Reasoning similar to that of section 2 shows that $P_x + Q_y$ measures the amount of explosion, or outward flux, of $\mathbf{F}$ from each point. That is,

$$P_x(p) + Q_y(p) = \lim_{a\to 0} \frac{2}{a}\cdot[\text{Average of } \mathbf{F}\cdot\mathbf{N}_a \text{ on } S_a(p)], \tag{4c}$$

where $\mathbf{N}_a$ is the outward unit normal vector to the tiny circle $S_a(p)$.

Exercise 4.1 Derive equation (4c), by imitating the method of section 2 using $\mathbf{N}_a$ instead of $\mathbf{T}_a$.

5. The Cauchy-Riemann equations

A while ago, we considered vector fields with "no flow and no flux". These are vector fields $\mathbf{F}$ such that

$$\oint_{\mathbf{c}} \mathbf{F}\cdot\mathbf{T}\ ds \ = \ 0 \ = \ \oint_{\mathbf{c}} \mathbf{F}\cdot\mathbf{N}\ ds \tag{5a}$$

for every closed curve $\mathbf{c}$. We noticed that such vector fields could be obtained by finding solutions to the Cauchy Riemann equations:

$$Q_y = P_x, \qquad -P_y = Q_x.$$

These equations can be rewritten as

$$Q_x - P_y = 0, \qquad P_x + Q_y = 0. \tag{5b}$$

The question arises as to whether this is the only way to find vector fields with no flow and no flux. That is, if all we know is that (5a) holds for $\mathbf{F} = (P, Q)$, does it follow that (5b) holds as well?

Exercise 5.1 Using equations (2e) and (4b), show that that answer to this question is Yes. (In other words, take equations (5a) as given, and use (2e), (4b) to derive (5b).)

6. The Most Interesting Vector Field

Take the vector field

$$\mathbf{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}).$$

Let us check Green's Theorem for $\mathbf{F}$, for the disk B_a of radius a centered at $(0,0)$. The boundary of B_a is the circle parametrized by $\mathbf{c}_a(t) = (a\cos t, a\sin t)$.

Exercise 6.1 Compute

$$\int_{\mathbf{c}_a} \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy.$$

Your answer will be nonzero, and independent of a.

Exercise 6.2 Show that $Q_x - P_y = 0$.

These two calculations show that Green's theorem is not valid for this vector field and this region:

$$\int_{\mathbf{c}_a} \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2} \neq 0 = \iint_{B_a} Q_x - P_y \; dR.$$

The problem is that $\mathbf{F}$ has a bad point in B_a. The point $(0,0)$ makes the denominators of P and Q zero, so the hypothesis of Green's Theorem does not hold, and the conclusion fails as well. The curious thing is that neither integral in Green's Theorem involves integrating over a bad point. In the line integral, the curve does not go through the bad point, and in double integral the integrand is zero, hence has no bad points. Nevertheless, the line integral of $\mathbf{F}$ can detect that the path is going around (0,0).

On the other hand, Green is valid in any region that does not contain $\mathbf{F}$'s bad point (0,0). Since $Q_x - P_y = 0$, Green predicts correctly that

$$\int_{\mathbf{c}} \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy = 0$$

as long as $(0,0)$ is not on, or inside $\mathbf{c}$.

Exercise 6.3 Compute

$$\int_{\mathbf{c}} \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy,$$

where $\mathbf{c}$ is the square with corners $(1,0)$, $(1,2)$ $(2,1)$ $(1,1)$. Do the line integral directly, by parametrizing each side of the square. The four integrals should add up to zero, as predicted by Green's Theorem.

Exercise 6.4 Compute

$$\int_{\mathbf{c}} \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$$

where $\mathbf{c}$ is the top half of the circle $(x-2)^2 + y^2 = 1$, from $(3,0)$ to $(1,0)$. (Hint: Let $\mathbf{c}_1$ be the line segment from $(1,0)$ to $(3,0)$. Note that $\int_{\mathbf{c}} + \int_{\mathbf{c}_1}$ is the integral over a closed curve. Apply Green to this closed curve, then compute the integral over $\mathbf{c}_1$. It is the same method as example 2 of section 1.)

7. The Jacobian

The Jacobian is the higher dimensional analogue of a "u-substitution". However, the main use of the Jacobian is to simplify the region of a double integral, not the integrand.

Suppose x and y are functions of other variables u, v. So we have functions $x(u, v),\ y(u, v)$. We put these together and get a "mapping"

$$R(u, v) = (x(u, v), y(u, v))$$

that sends points in the uv plane over to points in the xy-plane.

The Jacobian of the mapping is defined to be

$$\frac{\partial(x, y)}{\partial(u, v)} = x_u y_v - x_v y_u.$$

The Jacobian is another function of u and v.

EXAMPLE 1: (Polar Coordinates) Here we use r, θ instead of u, v. The mapping is

$$R(r, \theta) = (r\cos\theta, r\sin\theta).$$

The Jacobian is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = x_r y_\theta - x_\theta y_r = (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\cos\theta) = r.$$

EXAMPLE 2: (Scaling)
We have two constants a, b, and

$$R(u, v) = (au, bv).$$

The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = ab.$$

The Jacobian is used to change coordinates in double integrals. Say we want to integrate over a complicated region R in the xy-plane. We look for a simple region R^* in the uv plane (usually R^* is a rectangle), and a mapping $R(u, v)$ sending R^* into R. More precisely, each point (x, y) in the interior of R must be equal to $R(u, v)$ for exactly one point (u, v) in the interior of R^*.

Think of this as a parametrization the region R, just as we parametrize a curve $\mathbf{c}$ using a one dimensional mapping $\mathbf{c}(t) = (x(t), y(t)),\ t \in [a, b]$.

If we have a function $f(x, y)$ on R, then we have a transformed function

$$f^*(u, v) = f \circ R(u, v) = f(x(u, v), y(u, v))$$

on R^*. You change coordinates in double integrals using the following:
CHANGE OF VARIABLES FORMULA:

$$\text{(7a)} \qquad \iint_R f(x, y)\, dR = \iint_{R^*} f^*(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dudv.$$

Thus, the price we pay for integrating over the simpler region R^* is the absolute value of the Jacobian. In practice, the Jacobian is usually positive on R^*, and you can ignore the absolute value signs.

In the example of Polar Coordinates, formula (7a) says the integral over the disk B_a is

$$\iint_{B_a} f(x,y)\ dR = \iint_{R^*} f^*(r,\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\ drd\theta$$

$$= \int_{0,2\pi}\int_0^a f(r\cos\theta, r\sin\theta)\ rdrd\theta,$$

which is a formula we have been using without explanation.

EXAMPLE 3: (Ellipse)

Our region R is the interior of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. We use a modified polar coordinates mapping:

$$R(r,\theta) = (ar\cos\theta, br\sin\theta), \qquad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = abr.$$

Formula (7a) then gives the following integration formula over the ellipse R:

$$\iint_R f(x,y)\ dR = \int_0^{2\pi}\int_0^1 f(ar\cos\theta, br\sin\theta)abr\ drd\theta.$$

Note that a and b are constants, so they can be pulled out of the integral sign.

Exercise 7.1 Compute the average of the function $f(x,y)=x^2$ over the interior of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

8. Linear Mappings

Besides polar coordinates and its elliptical variations, the most basic mappings are Linear. These mappings look like

$$\text{(8a)} \qquad R(u,v) = (x_0 + au + bv, y_0 + cu + dv).$$

Here (x_0, y_0) is a given point, and a, b, c, d are constants. These are called "Linear" because they take straight lines to other straight lines. Note that $(x_0, y_0) = R(0,0)$.

Exercise 8.1 Show that the Jacobian of the linear map (8a) is

$$\frac{\partial(x,y)}{\partial(u,v)} = ad - bc.$$

There are different kinds of linear maps.

- Translations:

$$R(u,v) = (x_0 + u, y_0 + v). \tag{8b}$$

This takes a uv-region R^* and translates every point by adding the point $p = (x_0, y_0)$. For example, if R^* is the disk B_a of radius a centered at $(0,0)$, then the target region will be the disk $B_p(a)$ of radius a centered at $p = (x_0, y_0)$. If we combine (8b) with the polar coordinate mapping, we get a new mapping

$$(r, \theta) \longrightarrow (r \cos\theta, r \sin\theta) \longrightarrow (x_0 + r\cos\theta, y_0 + r\sin\theta) \tag{8c}$$

The Jacobian of (8c) is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r,$$

so the formula for integrating over the disk $B_a(p)$ is

$$\iint_{B_a(p)} f(x,y)\ dR = \int_0^{2\pi} \int_0^a f(x_0 + r\cos\theta, y_0 + r\sin\theta)\ r dr d\theta.$$

Exercise 8.2 Compute the averages of the following functions $f(x,y)$ over $B_a(p)$. Your answer will always involve $p = (x_0, y_0)$, but only sometimes will it involve a.

a) $f(x,y) = xy$
b) $f(x,y) = x^2$
c) $f(x,y) = x^2 - y^2$
d) $f(x,y) = x^3 - 3xy^2$.

After you've computed these averages, compute $f_{xx} + f_{yy}$ for these four functions. What is the connection between $f_{xx} + f_{yy}$ and the appearance of a in the average of f?

- Rotations:

Take a fixed angle α, and let

$$R(u,v) = (\cos\alpha u - \sin\alpha v, \sin\alpha v + \cos\alpha v).$$

This takes a region R^* and rotates it by the counterclockwise angle α, about the origin. The Jacobian is 1.

Exercise 8.3 Compute the integral of $f(x,y) = x^2$ over the square R with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$. (This is a bad square, because its sides are not parallel to the axes. It is the rotation by $\alpha = \pi/4$ of the good square R^* whose vertices are $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$.)

- Parallelograms:

This is not the name of the mapping, but the kind of region we map onto. Suppose R is the parallelogram with vertices $(2,1)$, $(4,2)$, $(3,3)$, $(5,4)$. We will take R^* to be the square $0 \le u, v \le 1$. Our linear mapping will be of the form

$$R(u,v) = (x_0 + au + bv, y_0 + cu + dv),$$

and we have to determine the constants x_0, y_0, a, b, c, d. The idea is that the vertices of R^* must go to the vertices of R, and this requirement determines the constants.

First, pick a vertex of R, say $(2, 1)$, and take $(x_0, y_0) = (2, 1)$, so

$$R(u, v) = (2 + au + bv, 1 + cu + dv). \tag{8d}$$

Even though we haven't finished determining $R(u, v)$, this already means that $R(0, 0) = (2, 1)$. Next, the vertices connected to $(0, 0)$ in R^* must go to the vertices connected to $(2, 1)$ in R. There are two ways to do this, and one way is slightly better. We take

$$R(1, 0) = (4, 2), \qquad R(0, 1) = (3, 3). \tag{8e}$$

But (8d) says

$$R(1, 0) = (2 + a, 1 + c), \qquad R(0, 1) = (2 + b, 1 + d) \tag{8f}$$

Comparing (8e) and (8f), we see that

$$a = 2, \quad c = 1, \quad b = 1, \quad d = 2,$$

So

$$R(u, v) = (2 + 2u + v, 1 + u + 2v).$$

Note that $R(1, 1) = (5, 4)$: The fourth vertex takes care of itself, since $R(u, v)$ is linear.

The Jacobian of R is, by Exercise 8.1,

$$\frac{\partial(x, y)}{\partial(u, v)} = 2 \cdot 2 - 1 \cdot 1 = 3. \tag{8g}$$

Therefore the formula for integrating over the parallelogram R is

$$\iint_R f(x, y)\, dR = 3 \int_0^1 \int_0^1 f(2 + 3u + v, 1 + u + 2v)\, dudv.$$

If we had made the opposite choice in (8e), we would have been turning the square "upside down" and (8g) would have been -3, and we would have had to take the absolute value of -3 in the integral formula.

Exercise 8.4 Find the formula for integrating over the parallelogram R with vertices $(-1, -1)$, $(3, 1)$, $(2, 4)$, $(6, 6)$. Then use your formula to find the center of mass of R.

9. Regions bounded by graphs

Suppose R is the triangle with vertices $(0, 2)$, $(0, -3)$, $(1, 0)$. We can describe R by the inequalities

$$x - 3 \leq y \leq 2 - x, \qquad 0 \leq x \leq 1. \tag{9a}$$

To parametrize R, we let R^* be the usual square $0 \leq u, v \leq 1$, and take

$$R(u,v) = (u, (1-v)(u-3) + v(2-u)).$$

The line $v = 0$ in R^* goes to the lower boundary of R, and the line $v = 1$ goes to the upper boundary of R. Vertical lines in R^* go to vertical lines in R, except that the line $u = 1$ gets squashed into the point $(1,0)$ in R. Since $x_u = 1$ and $y_u = 0$, the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = (2-u) - (u-3).$$

We don't do the subtraction, in order to make it clear that the Jacobian is the difference between the upper and lower boundaries. In particular, it is positive in R^*. The integral formula for R is (writing the v integral first)

$$\text{(9b)} \quad \iint_R f(x,y)\, dR = \int_0^1 \int_0^1 f(u, (1-v)(u-3) + v(2-u))[(2-u) - (u-3)]\, dvdu.$$

This can be simplified, using a substitution in the v integral. In this integral, u is constant. Let $w = (1-v)(u-3) + v(2-u)$. Then $dw = [(2-u) - (u-3)]\, dv$. When $v = 0, 1$, then w is $u-3$, $2-u$ respectively, so (9b) becomes

$$\iint_R f(x,y)\, dR = \int_0^1 \int_{u-3}^{2-u} f(u,w)\, dwdu.$$

In the last integral, the names of the letters doesn't matter, so let us replace u by x, and w by y. We get, finally,

$$\text{(9c)} \qquad \iint_R f(x,y)\, dR = \int_0^1 \int_{x-3}^{2-x} f(x,y)\, dydx.$$

For example to integrate the function $f(x,y) = x + y$, we compute

$$\begin{aligned}
\iint_R x + y\, dR &= \int_0^1 \int_{x-3}^{2-x} x + y\, dydx \\
&= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=x-3}^{y=2-x} dx \\
&= \int_0^1 x[(2-x) - (x-3)] + \frac{1}{2}[(2-x)^2 - (x-3)^2]\, dx = -\frac{1}{6}.
\end{aligned}$$

Now the order of integration is important. It would make no sense to write

$$\text{(WRONG)} \qquad \int_{x-3}^{2-x} \int_0^1 f(x,y)\, dxdy$$

because the final answer would contain x's, when it should be a number. The order of integration in (9c) was dictated by the inequalities (9a) defining R.

Exercise 9.1 Find the center of mass of the triangle R considered above, using the integration formula (9c).

In deriving formula (9c), we made no use of the fact that R was a triangle. We need only to have R described by inequalities

$$\phi(x) \leq y \leq \psi(x), \quad a \leq x \leq b.$$

The good region R^* is still the square $0 \leq u, v \leq 1$. Now our mapping is

$$R(u,v) = (u, (1-v)\phi(u) + v\psi(u)). \tag{9d}$$

Just as above, we arrive at the integration formula

$$\iint_R f(x,y)\ dxdy = \int_a^b \int_{\phi(x)}^{\psi(x)} f(x,y)\ dydx. \tag{9e}$$

When using (9e) it is not necessary to remember the mapping $R(u,v)$. However, in exercise (9.4) below, it will be necessary to use the formula (9d).

Likewise, if R is described by inequalities

$$\xi(y) \leq x \leq \eta(y), \quad c \leq y \leq d,$$

then we have the integration formula

$$\iint_R f(x,y) = \int_a^b \int_{\xi(y)}^{\eta(y)} f(x,y)\ dxdy. \tag{9f}$$

Exercise 9.3 Use (9e) to find the integral of $f(x,y) = x^2 + y^2$ over the region R defined by the inequalities $0 \leq y \leq 1 - x^2, \quad 0 \leq x \leq 1$.

Exercise 9.4 In this exercise, you will combine the methods of the two previous sections to find the integration formula for a general triangle in the plane. Let R be the triangle with vertices $(a,b),\ (c,d),\ (e,f)$. Let R^* be the triangle with vertices $(0,0),\ (1,0),\ (0,1)$.

a) Using the method of section 8, find a linear map $R(u,v)$ mapping R^* to R, with $R(0,0) = (a,b)$. Calculate the Jacobian of $R(u,v)$, and use the Change of Variables Formula (7a) to express the integral of a general function $f(x,y)$ over R as the integral of the transformed function over R^*.

b) Using the method of this section, express the integral that you found in a) as a double integral with limits $0, 1$ and $0, 1-x$ (as in (9c)).

c) Use the formula in b) to find the area and center of mass of the triangle R.