Note 5 Solutions to Exercises

Green's Theorem, Two-dimensional Curl, Fundamental Theorem of Calculus, Divergence of a Vector field, Cauchy-Riemann equations, Most Interesting Vector Field, Jacobian, Linear mappings, Regions bounded by graphs.

1. Green's Theorem

Exercise 1.1: Use Green's Theorem to compute the following line integrals.

a) $\oint_{\mathbf{c}}(x^2-y^2)\,dx+2xy\,dy$, where $\mathbf{c}$ is the path in example 1 (the counterclockwise rectangle with corners $(1,1),\ (3,1),\ (3,2),\ (1,2)$).

SOLUTION:

$$\oint_{\mathbf{c}}(x^2-y^2)\,dx+2xy\,dy=\int_1^3\int_1^2 2y+2y\,dydx=12.$$

b) $\oint_{\mathbf{c}}(x^2-y^2)\,dx+2xy\,dy$, where $\mathbf{c}$ is the ellipse $4(x-1)^2+9(y-2)^2=1$.

SOLUTION:

$$\oint_{\mathbf{c}}(x^2-y^2)\,dx+2xy\,dy=\iint_R 4y\,dydx=4\bar{y}\cdot\text{Area}(R)=4\cdot 2\cdot\frac{\pi}{6}=\frac{4\pi}{3}.$$

c) $\oint_{\mathbf{c}}(x^2-y^2)\,dx+2xy\,dy$, where $\mathbf{c}$ is the counterclockwise circle $(x-2)^2+y^2=1$.

SOLUTION:

$$\oint_{\mathbf{c}}(x^2-y^2)\,dx+2xy\,dy=\iint_R 4y\,dydx=4\bar{y}\cdot\text{Area}(R)=0.$$

d) $\oint_{\mathbf{c}} e^x\sin y\,dx+e^x\cos y\,dy$, where $\mathbf{c}$ is the circle $x^2+(y-1)^2=1$.

SOLUTION:

$$\oint_{\mathbf{c}} e^x\sin y\,dx+e^x\cos y\,dy=\iint_R 0\,dR=0.$$

Exercise 1.2: Use Green to compute

$$\oint_{\mathbf{c}}(1-2y+e^x\sin y)\,dx+(3+4x+e^x\cos y)\,dy$$

over the curve in example 2 (the part of the circle $x^2+(y-1)^2=1$ from $(0,0)$ to $(1,1)$).

SOLUTION: We'll use the curve $\mathbf{c}_1=(t,0)$, followed by the curve $\mathbf{c}_2(t)=(1,t)$. Then

$$\int_{\mathbf{c}_1}+\int_{\mathbf{c}_2}-\int_{\mathbf{c}}=\iint_R 6\,dR=6\cdot(1-\frac{\pi}{4})=6-\frac{3\pi}{2},$$

so

$$\begin{aligned}\int_{\mathbf{c}}(1-2y+e^x\sin y)\,dx+(3+4x+e^x\cos y)\,dy&=\frac{3\pi}{2}-6+\int_{\mathbf{c}_1}+\int_{\mathbf{c}_2}\\&=\frac{3\pi}{2}-6+\int_0^1 1\,dt+\int_0^1 3+4+e\cos t\,dt\\&=\frac{3\pi}{2}+2+e\sin(1).\end{aligned}$$

Exercise 1.3: Show how to calculate the center of mass of R entirely in terms of line integrals over the boundary of R. Then use your formulas to find the center of mass of the triangle with vertices $(0,0),(1,0),(0,1)$.

SOLUTION: We know the area is $A = \int_{\mathbf{c}} x\, dy$. By Green's Theorem,

$$\int_{\mathbf{c}} \frac{1}{2}x^2\, dy = \iint_R x\, dR, \quad \text{and} \quad \int_{\mathbf{c}} \frac{1}{2}y^2\, dy = \iint_R y\, dR.$$

2. Two dimensional curl

No exercises in this section.

3. The Fundamental Theorem of Calculus

This problem was not assigned, but here is the solution. **Exercise 3.1** Suppose $f''(x) = 0$ for all x. Show that $f(x) = cx + d$ for some constants d, d. Then show that the average of $f(x)$ over any interval $[x_0 - a, x_0 + a]$ is equal to the value of f at the center of the interval. (This is the Mean-Value property of harmonic functions in one variable).
SOLUTION: Since $f'' = 0$, we have $f' = c$ (constant), so $f = cx + d$ where d is another constant. The graph of f is a straight line, and its value at the center of an interval is its average height over the interval. The mean-value property is trivial in one dimension!

4. The Divergence of a Vector field

NOT ASSIGNED.

Reasoning similar to that of section 2 shows that $P_x + Q_y$ measures the amount of explosion, or outward flux, of $\mathbf{F}$ from each point. That is,

$$P_x(p) + Q_y(p) = \lim_{a\to 0} \frac{2}{a} \cdot [\text{Average of } \mathbf{F}\cdot\mathbf{N}_a \text{ on } S_a(p)], \tag{4c}$$

where $\mathbf{N}_a$ is the outward unit normal vector to the tiny circle $S_a(p)$.

Exercise 4.1 Derive equation (4c), by imitating the method of section 2 using $\mathbf{N}_a$ instead of $\mathbf{T}_a$.
SOLUTION:

$$\begin{aligned} P_x(p) + Q_y(p) &= \lim_{a\to 0} \frac{1}{\pi a^2} \iint_{B_a(p)} P_x + Q_y\, dR \\ &= \lim_{a\to 0} \frac{1}{\pi a^2} \oint_{S_a(p)} \mathbf{F}\cdot\mathbf{N}_a\, ds \\ &= \lim_{a\to 0} \frac{2\pi a}{\pi a^2} [\text{Average of } \mathbf{F}\cdot\mathbf{N}_a \text{ on } S_a(p)]. \end{aligned}$$

5. The Cauchy-Riemann equations

(NOT ASSIGNED.)

Exercise 5.1 Using equations (2e) and (4b), show that that answer to this question is Yes. (In other words, take equations (5a) as given, and use (2e), (4b) to derive (5b).)
SOLUTION: Take a point p. Then from equation (2e) we have

$$Q_x(p) - P_y(p) = \lim_{a\to 0} \frac{2}{a} \cdot [\text{Average of } \mathbf{F}\cdot\mathbf{T}_a \text{ on } S_a(p)].$$

This average is zero, since $\oint_{S_a(p)} \mathbf{F} \cdot \mathbf{T}_a \; ds = 0$, by assumption.

6. The Most Interesting Vector Field

Exercise 6.1 Compute

$$\int_{\mathbf{c}_a} \frac{-y}{x^2+y^2}\; dx + \frac{x}{x^2+y^2}\; dy.$$

Your answer will be nonzero, and independent of a.
ANSWER: 2π.

Exercise 6.2 Show that $Q_x - P_y = 0$.

SOLUTION: Calculate Q_x and P_y separately. Omit this next problem

OMIT THIS NEXT PROBLEM **Exercise 6.3** Compute

$$\int_{\mathbf{c}} \frac{-y}{x^2+y^2}\; dx + \frac{x}{x^2+y^2}\; dy,$$

where $\mathbf{c}$ is the square with corners $(1,0)$, $(1,2)$ $(2,1)$ $(1,1)$. Do the line integral directly, by parametrizing each side of the square. The four integrals should add up to zero, as predicted by Green's Theorem.

Exercise 6.4 Compute

$$\int_{\mathbf{c}} \frac{-y}{x^2+y^2}\; dx + \frac{x}{x^2+y^2}\; dy$$

where $\mathbf{c}$ is the top half of the circle $(x-2)^2+y^2=1$, from $(3,0)$ to $(1,0)$.
SOLUTION: By Green, we have $\int_{\mathbf{c}} + \int_{\mathbf{c}_1} = 0$. But

$$\frac{-y}{x^2+y^2}\; dx + \frac{x}{x^2+y^2}\; dy = 0$$

on $\mathbf{c}_1$. So

$$\int_{\mathbf{c}} \frac{-y}{x^2+y^2}\; dx + \frac{x}{x^2+y^2}\; dy = 0.$$

7. The Jacobian

Exercise 7.1 Compute the average of the function $f(x,y) = x^2$ over the interior of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
SOLUTION:

$$\frac{1}{\pi ab} \iint_R x^2 \; dR = \frac{1}{6\pi} \int_0^{2\pi} \int_0^1 (3r\cos\theta)^2 6r \; dr d\theta = \frac{9}{4}.$$

8. Linear Mappings

Exercise 8.1 Show that the Jacobian of the linear map (8a) is

$$\frac{\partial(x,y)}{\partial(u,v)} = ad - bc.$$

SOLUTION: Just do it!

Exercise 8.2 Compute the averages of the following functions $f(x,y)$ over $B_a(p)$. Your answer will always involve $p = (x_0, y_0)$, but only sometimes will it involve a.

a) $f(x,y) = xy$ **ANSWER:** $x_0 y_0$
b) $f(x,y) = x^2$ **ANSWER:** $x_0^2 + \frac{a^2}{4}$
c) $f(x,y) = x^2 - y^2$ **ANSWER:** $x_0^2 - y_0^2$
d) $f(x,y) = x^3 - 3xy^2$. **ANSWER:** $x_0^3 - 3x_0 y_0^2$

When $f_{xx} + f_{yy} = 0$, the average over the disk equals the value of f at the center of the disk.

Exercise 8.3 Compute the integral of $f(x,y) = x^2$ over the square R with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$.
SOLUTION:

$$\iint_R x^2 \; dR = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2 \; dxdy = \frac{1}{3}.$$

Exercise 8.4 Find the formula for integrating over the parallelogram R with vertices $(-1,-1)$, $(3,1)$, $(2,4)$, $(6,6)$. Then use your formula to find the center of mass of R.
SOLUTION:

$$R(u,v) = (-1 + 4u + 3v, -1 + 2u + 5v).$$

The Jacobian is 14. The integral formula is

$$\iint_R f(x,y) \; dR = 14 \int_0^1 \int_0^1 f(-1 + 4u + 3v, -1 + 2u + 5v) \; dudv.$$

The center of mass is $(5/2, 5/2)$.

9. Regions bounded by graphs

(OMIT THIS SECTION FOR NOW)