

Note 6

Line integrals in three dimensions, three dimensional curl, parametrized surfaces, surface integrals, flux integrals, Stokes theorem, triple integrals, divergence theorem

1. Line integrals in three dimensions

Given a curve $\mathbf{c}(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$, and a vector field $\mathbf{F} = (P, Q, R)$, the line integral of \mathbf{F} along \mathbf{c} is

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_{\mathbf{c}} Pdx + Qdy + Rdz.$$

It is computed by putting $\mathbf{c}(t)$ into each of P, Q, R , writing $dx = x'(t)dt, dy = y'(t)dt, dz = z'(t)dt$, and integrating with respect to t from a to b .

EXAMPLE: $\mathbf{F} = (z, x, y)$, \mathbf{c} is the line segment from $(1, 1, 1)$ to $(2, 3, 4)$. Thus, $\mathbf{c}(t) = (1 + t, 1 + 2t, 1 + 3t)$, $0 \leq t \leq 1$, and

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_{\mathbf{c}} adx + xdy + ydz = \int_0^1 (1 + 3t) \cdot 1 + (1 + t) \cdot 2 + (1 + 2t) \cdot 3dt = \frac{23}{2}.$$

Exercise 1.1 Repeat the above example with the same \mathbf{F} , but using the curve $\mathbf{c}(t) = (1 + t^2, 1 + 2t^3, 1 + 3t^4)$, $0 \leq t \leq 1$. Your answer should be different from the example.

Exercise 1.2 Repeat the above example with the same curve, but using $\mathbf{F} = (yz, zx, xy)$.

Exercise 1.3 Integrate the same $\mathbf{F} = (yz, zx, xy)$, but now over the curve $\mathbf{c}(t) = (1 + t^2, 1 + 2t^3, 1 + 3t^4)$, $0 \leq t \leq 1$. You should get the same answer as in the previous exercise.

In three dimensions, there are three Fundamental Theorems of Calculus. The first one is familiar.

First FTC in \mathbb{R}^3 . If $f(x, y, z)$ is a function, then

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{c} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

Thus, the line integral of a gradient depends only on the endpoints of the curve. In particular, $\int_{\mathbf{c}} \nabla f \cdot d\mathbf{c} = 0$ for any closed curve \mathbf{c} .

Exercise 1.4 Calculate the gradient of $f(x, y, z) = xyz$. Evaluate $f(2, 3, 4) - f(1, 1, 1)$. You should get the same answer as in Exercises 1.2, 3.

2. Three dimensional curl

The curl of a vector field $\mathbf{F} = (P, Q, R)$ is a new vector field

$$\nabla \times \mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y).$$

For any unit vector \mathbf{N} , based at the point $p \in \mathbb{R}^3$, the component of $\nabla \times \mathbf{F}(p)$ in the direction of \mathbf{N} measures the counterclockwise angular velocity of \mathbf{F} around the point p in the plane normal to \mathbf{N} .

Exercise 2.1 Show that a gradient has zero curl. That is, show that

$$\nabla \times \nabla f = (0, 0, 0).$$

Compute $\nabla \times (z, x, y)$ (see previous section). Use the first part of this exercise to show that (z, x, y) is not a gradient.

Exercise 2.2 Let $\mathbf{v} = (a, b, c)$ be a vector with constant components. Compute the components of the cross-product $\mathbf{F} = (a, b, c) \times (x, y, z)$. Then compute $\nabla \times \mathbf{F}$. (Ans: $2\mathbf{v}$.)

3. Parametrized surfaces

A surface S in \mathbb{R}^3 is parametrized by a mapping

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in \mathcal{R}^*.$$

Here \mathcal{R}^* is a region in the uv -plane. Usually \mathcal{R}^* is a rectangle.

The Jacobian of the mapping is the vector

$$\frac{\partial(x, y, z)}{\partial(u, v)} = \left(\frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right).$$

The norm of the Jacobian is the following function of u, v .

$$\left\| \frac{\partial(x, y, z)}{\partial(u, v)} \right\| = \left[\left(\frac{\partial(y, z)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(x, y)}{\partial(u, v)} \right)^2 \right]^{1/2}.$$

The Jacobian can be interpreted as a cross-product: The vectors $\mathbf{r}_u = (x_u, y_u, z_u)$ and $\mathbf{r}_v = (x_v, y_v, z_v)$ are tangent to the surface for each value of the parameters (u_0, v_0) . The vector $\mathbf{r}_u \times \mathbf{r}_v$ is therefore normal to the surface.

Exercise 3.1 Show that

$$\frac{\partial(x, y, z)}{\partial(u, v)} = \mathbf{r}_u \times \mathbf{r}_v.$$

A ZOO OF SURFACES

Parallelogram: If S is a parallelogram spanned by the vectors \mathbf{a}, \mathbf{b} , based at the point p , then

$$\mathbf{r}(u, v) = p + u\mathbf{a} + v\mathbf{b}, \quad \mathcal{R}^* : 0 \leq u, v \leq 1.$$

Moreover,

$$\mathbf{r}_u = \mathbf{a}, \quad \mathbf{r}_v = \mathbf{b}, \quad \mathbf{r}_u \times \mathbf{r}_v = \mathbf{a} \times \mathbf{b}.$$

Thus, the Jacobian is constant. If we change \mathcal{R}^* , we get a different patch of the plane spanned by \mathbf{a}, \mathbf{b} . For example, if \mathcal{R}^* is the entire uv plane, then $\mathbf{r}(u, v)$ will parametrize the entire plane spanned by \mathbf{a}, \mathbf{b} .

Sphere: Let S_a be the sphere with equation $x^2 + y^2 + z^2 = a^2$. The parametrization is

$$\mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi),$$

$$\mathcal{R}^* : 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$$

The variable ϕ is the angle down from the north pole (latitude), and the variable θ is the angle of rotation from the xz -plane, counterclockwise when viewed from the north pole (longitude). The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(\phi, \theta)} = (a^2 \sin^2 \phi \cos \theta, a^2 \sin^2 \phi \sin \theta, a^2 \cos \phi \sin \phi) = a^2 \sin \phi \mathbf{r}(\phi, \theta).$$

Since $\mathbf{r}(\phi, \theta)$ lies on S_a , it has norm a , so

$$\left\| \frac{\partial(x, y, z)}{\partial(\phi, \theta)} \right\| = a^2 \sin \phi.$$

(Note that $\sin \phi \geq 0$ for $0 \leq \phi \leq \pi$.)

Cylinder: Let S be the surface in \mathbb{R}^3 with equation $x^2 + y^2 = a^2$. This is a vertical cylinder of radius a around the z -axis. It is parametrized by

$$\mathbf{r}(\theta, z) = (a \cos \theta, a \sin \theta, z).$$

Torus: Let $T_{a,b}$ be the surface parametrized by

$$\mathbf{r}(\phi, \theta) = ((a + b \cos \phi) \cos \theta, (a + b \cos \phi) \sin \theta, b \sin \phi),$$

$$\mathcal{R}^* : 0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq 2\pi.$$

This is a Torus (it looks like a bagel), obtained by starting with the circle of radius a in the xz plane, centered at $(b, 0, 0)$, and rotating it around the z -axis.

Graph of a function. Let S be the graph of a function $z = f(x, y)$, for x, y in some region D in the xy plane. Then S can be parametrized by

$$\mathbf{r}(x, y) = (x, y, f(x, y)), \quad \mathcal{R}^* = D.$$

The Jacobian is

$$\mathbf{r}_x \times \mathbf{r}_y = (-f_x, -f_y, 1).$$

Another way to see this: Recall that S is a level surface of the function $g(x, y, z) = z - f(x, y)$, and note that $\mathbf{r}_x \times \mathbf{r}_y = \nabla(g)$, which we know is normal to the level surfaces of g .

Example: $S =$ the part of the paraboloid $z = 1 - x^2 - y^2$ for $z \geq 0$. Then D is the unit disk,

$$\mathbf{r}(x, y) = (x, y, 1 - x^2 - y^2), \quad \mathbf{r}_x \times \mathbf{r}_y = (-2x, -2y, 1), \quad \|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{4x^2 + 4y^2 + 1}.$$

Exercise 3.2 Calculate the Jacobians and their norms for the cylinder of radius a , and the Torus $T_{a,b}$.

4. Surface Integrals

Take a surface S parametrized by $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in \mathcal{R}^*$, as in the previous section. To calculate integrals over S , first compute the norm of the Jacobian:

$$\left\| \frac{\partial(x, y, z)}{\partial(u, v)} \right\| = \left[\left(\frac{\partial(y, z)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(x, y)}{\partial(u, v)} \right)^2 \right]^{1/2}.$$

The integral of a function $f(x, y, z)$ over S is

$$(4.1a) \quad \iint_S f \, dS = \iint_{\mathcal{R}^*} f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial(x, y, z)}{\partial(u, v)} \right\| \, dudv.$$

The right side is an ordinary double integral over the region \mathcal{R}^* in the uv plane.

EXAMPLE: When $f = 1$ the integral is the surface area of S :

$$(4.1b) \quad \text{Area of } S = \iint_S dS = \iint_{\mathcal{R}^*} \left\| \frac{\partial(x, y, z)}{\partial(u, v)} \right\| \, dudv.$$

It is sometimes helpful to abbreviate (4.1b) as

$$(4.1c) \quad dS = \left\| \frac{\partial(x, y, z)}{\partial(u, v)} \right\| \, dudv.$$

The precise meaning of (4.1c) is that if you integrate the left side over any patch of S , and then integrate the right side over the corresponding patch in \mathcal{R}^* , you will get the same answer for both integrals, namely the area of the patch.

Integrals over Spheres Let S_a be the sphere with equation $x^2 + y^2 + z^2 = a^2$, parametrized as in the previous section. We calculated

$$\left\| \frac{\partial(x, y, z)}{\partial(\phi, \theta)} \right\| = a^2 \sin \phi.$$

Therefore, for any function $f(x, y, z)$, the integral over S_a is

$$\iint_{S_a} f \, dS = \int_0^{2\pi} \int_0^\pi f(a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi) a^2 \sin \phi \, d\phi d\theta.$$

For example, the surface area of S_a is

$$\iint_{S_a} dS = \int_0^{2\pi} \int_0^\pi a^2 \sin \phi \, d\phi d\theta = 4\pi a^2.$$

Exercise 4.1 Calculate the integral of each of the functions $f = x^{2n}$, $g = y^{2n}$, $h = z^{2n}$ over S_a . (Use the trig form of the beta integral. All three integrals have the same answer, but the calculations are different.)

Exercise 4.2 Calculate the integral of $f = x^k y^l z^m$ over the the octant of the sphere S_a given by $0 \leq \phi, \theta \leq \frac{\pi}{2}$. Your answer will be in terms of k, l, m . If any of k, l, m

is set equal to $2n$ and the other two are set at zero, you should get $1/8$ the answer to Exercise 4.1.)

Exercise 4.3 Calculate the surface areas of the torus $T_{a,b}$, and the paraboloid $z = 1 - x^2 - y^2$, $z > 0$. (Convert the last integral to polar coordinates.)

5. Flux integrals

The Flux of the vector field $\mathbf{F} = (P, Q, R)$ through a surface S is given by the surface integral

$$\text{(flux integral)} \quad \iint_S \mathbf{F} \cdot \mathbf{N} \, dS$$

Explanation: \mathbf{N} is a unit normal vector field on S , in the direction of the flux we are considering. At each point $p \in S$, we have two vectors: $\mathbf{F}(p)$ and $\mathbf{N}(p)$, and their dot product $\mathbf{F}(p) \cdot \mathbf{N}(p)$ is a number, which is the component of $\mathbf{F}(p)$ in the direction of $\mathbf{N}(p)$. Thus we have a function $\mathbf{F} \cdot \mathbf{N}$ on S , whose value at p is $\mathbf{F}(p) \cdot \mathbf{N}(p)$, and this function is being integrated over S . If $\mathbf{F}(p)$ is tangent to S for every $p \in S$ then $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = 0$. If $\mathbf{F}(p)$ is normal to S then the integral is also fairly easy:

EXAMPLE 1 Take $\mathbf{F} = (x, y, z)$, $S = S_a$, and let \mathbf{N} be the outward normal to S_a . Then \mathbf{F} is normal to the sphere, and $\mathbf{F} \cdot \mathbf{N} = \|\mathbf{F}\| = a$ at all points on S_a . Thus,

$$\iint_{S_a} \mathbf{F} \cdot \mathbf{N} \, dS = \iint_{S_a} a \, dS = a \cdot \text{Area of } S_a = 4\pi a^3.$$

To calculate a general flux integral, recall that

$$\mathbf{r}_u \times \mathbf{r}_v = \left(\frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right)$$

is normal to S , so

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

is a unit normal to S . (The other normal vector field is the negative of this one.) Next recall (4.1c): $dS = \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv$. We get

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_{\mathcal{R}^*} P(\mathbf{r}(u, v)) \frac{\partial(y, z)}{\partial(u, v)} + Q(\mathbf{r}(u, v)) \frac{\partial(z, x)}{\partial(u, v)} + R(\mathbf{r}(u, v)) \frac{\partial(x, y)}{\partial(u, v)} \, du \, dv$$

To remember this, it is helpful to abbreviate

$$dydz = \frac{\partial(y, z)}{\partial(u, v)} \, du \, dv, \quad dzdx = \frac{\partial(z, x)}{\partial(u, v)} \, du \, dv, \quad dx dy = \frac{\partial(x, y)}{\partial(u, v)} \, du \, dv.$$

Then the flux integral formula can be written

$$(5a) \quad \iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_S P \, dydz + Q \, dzdx + R \, dx dy.$$

This is the analogue of the line integral formula

$$(5b) \quad \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint Pdx + Qdy + Rdz.$$

EXAMPLE 2 Let $\mathbf{F} = (x, y, z)$ and let S be the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Then $\mathbf{r}(x, y) = (x, y, 1 - x^2 - y^2)$ and \mathcal{R}^* is the unit disk. We compute $\mathbf{r}_x \times \mathbf{r}_y = (2x, 2y, 1)$, so

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} \, dS &= \iint_{\mathcal{R}^*} x(2x) + y(2y) + (1 - x^2 - y^2)(1) \, dxdy \\ &= \iint_{\mathcal{R}^*} x^2 + y^2 + 1 \, dxdy \\ &= \int_0^{2\pi} \int_0^1 (r^2 + 1)r \, dr \, d\theta \\ &= \frac{3\pi}{2}. \end{aligned}$$

EXAMPLE 3 Take $\mathbf{F} = (0, 0, 1) = \mathbf{k}$, and suppose S is the graph of a function $f(x, y)$, above the region \mathcal{R} in the xy plane. Take the parametrization $\mathbf{r}(x, y) = (x, y, f(x, y))$. The integrand on the right side of the flux integral formula is 1, so

$$\iint_S \mathbf{k} \cdot \mathbf{N} \, dS = \text{Area of } \mathcal{R}.$$

In other words, $\iint_S \mathbf{k} \cdot \mathbf{N} \, dS$ is the area of the shadow cast by S in the xy plane.

Exercise 5.1 Calculate the integrals $\iint_S \mathbf{k} \cdot \mathbf{N} \, dS$ where S is the top half of S_a , the top half of the torus $T_{a,b}$, and the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$.

Exercise 5.2 Let $\mathbf{F} = (x, 0, 0)$. Calculate the integrals $\iint_S \mathbf{k} \cdot \mathbf{N} \, dS$ where $S = S_a$, $T_{a,b}$, the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$, and the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$. (Answers: In each case, the flux integral equals the volume of the space enclosed by S . This will be explained by the Divergence Theorem, later on.)

Exercise 5.3 Let $\mathbf{F} = (yz, zx, xy)$ and let S be the part of the paraboloid $z = 1 - x^2 - y^2$ with $x, y, z \geq 0$. Calculate $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$. (Answer: 7/24)

Exercise 5.4 Let $S = S_a$, and

$$\mathbf{G} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Calculate $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$ using the method of Example 1. (The answer is independent of a .) This \mathbf{G} is the three dimensional "world's most interesting vector field". Up to a constant, it is the gravitational or electrical field due to a point mass or charge, respectively.

6. Stokes Theorem

Stokes theorem is the second FTC in three dimensions. It is a direct analogue of Green's theorem. Take a surface S , with a normal vector \mathbf{N} on S . Let C be the boundary of S , oriented counterclockwise when N is pointing at you. In other words, if your right thumb points along \mathbf{N} , then your fingers contract in the direction of C .

Stokes Theorem. If \mathbf{F} is a vector field with no bad points on S , then the flux of the curl of \mathbf{F} in the direction of \mathbf{N} is equal to the flow of \mathbf{F} along the boundary C . In other words,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{N} \, dS = \oint_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

If S has no boundary, then

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{N} \, dS = 0.$$

Using the definition of curl, and formulas (5a,b), Stokes theorem can be written

$$\iint_S (R_y - Q_z) dydz + (P_z - R_x) dzdx + (Q_x - P_y) dxdy = \int_C P dx + Q dy + R dz.$$

You can use Stokes theorem to calculate difficult line integrals. **EXAMPLE:** Let C be the curve obtained by slicing the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 1$. Thus C is a slanted ellipse around the z axis. Orient C to be counter clockwise when viewed from high up the z axis. To integrate over C , we take S to be the part of the plane $y + z = 1$ which is interior to the cylinder. So S is the graph of the function $z = 1 - y$. It is parametrized by $\mathbf{r}(x, y) = (x, y, 1 - y)$, and \mathcal{R}^* is the unit disk $x^2 + y^2 \leq 1$.

Exercise 6.1 For the curve C above, use Stokes theorem to compute the line integrals

$$\int_C z(x^2 - 1) dx + y(x + 1) dz, \quad \int_C y(z - 1) dx + x(z + 1) dy.$$

(Answers: 0, 2π .)

Exercise 6.2 Let C be the intersection of the saddle $z = x^2 - y^2$ with the cylinder $x^2 + y^2 = 1$. Use Stokes theorem to calculate $\int_C xy \, dz$. (You can check your answer by computing the line integral directly, using the parametrization $\mathbf{c}(t) = (\cos t, \sin t, \cos 2t)$.)

Exercise 6.3 Let $\mathbf{F} = (\sin x, \sin y, \sin z)$. Calculate $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$, where C is the curve in exercise 6.2.

Stokes theorem says the integral of a curl over any closed surface is zero. (A closed surface is one without a boundary, like a sphere or an ellipsoid or a cube.)

Exercise 6.4 Let $\mathbf{v} = (a, b, c)$ be a constant vector field, and let S be a sphere. Compute $\iint_{S_a} \mathbf{v} \cdot \mathbf{N} \, dS$. (Use exercise 2.2.)

7. Triple integrals

The third and final FTC in three dimensions involves triple integrals. We have a region R in three dimensional space, and we have a function $f(x, y, z)$, and we want to integrate f over R .

EXAMPLE 1: R is a rectangle.

Then R is defined by inequalities $a \leq x \leq b$, $c \leq y \leq d$, $h \leq z \leq k$. then

$$\iiint_R f \, dR = \int_a^b \int_c^d \int_h^k f(x, y, z) \, dz dy dx.$$

EXAMPLE 2: R is the ball B_a of radius a defined by $x^2 + y^2 + z^2 \leq a^2$. Then

$$\begin{aligned} \iiint_{B_a} f \, dR &= \int_0^a \left[\iint_{S_r} f \, dS \right] dr \\ &= \int_0^a \int_0^{2\pi} \int_0^\pi f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) r^2 \sin \phi \, dr d\phi d\theta. \end{aligned}$$

8. The Divergence Theorem

Let $\mathbf{F} = (P, Q, R)$ be a vector field in three dimensions. The divergence of \mathbf{F} is the function

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z.$$

It measures the amount of explosion of \mathbf{F} at each point. For example,

$$\nabla \cdot (x, y, z) = 1 + 1 + 1 = 3.$$

Exercise 8.1 Show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any vector field $\mathbf{F} = (P, Q, R)$. Use this to show that $(\sin y, \sin z, \sin x)$ is not the curl of any vector field.

The Divergence Theorem is the third FTC in three dimensions. It says that the outward flux of \mathbf{F} through a closed surface equals the triple integral of $\nabla \cdot \mathbf{F}$ over the region inside S . That is,

Divergence Theorem. *If R is a three dimensional region, and S is the boundary of R , then*

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_R \nabla \cdot \mathbf{F} \, dR.$$

In other words,

$$\iint_S P \, dydz + Q \, dzdx + R \, dxdy = \iiint_R P_x + Q_y + R_z \, dR.$$

Many vector fields in physics, such as the velocity of a liquid, like water, have zero divergence, at least in regions containing no sources or sinks. If R is such a region, then the Divergence Theorem simply says $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = 0$. In other words, what goes in must come out.

The next level of complexity is if the divergence is not zero, but constant. Then the Divergence Theorem says the flux is the volume, times this constant.

For example, taking $P = x$, $Q = R = 0$, we get the formula

$$\text{Volume of } R = \iint_S x \, dydz,$$

which is the three-dimensional analogue of our old formula

$$\text{Area of } R = \int_C x dy$$

for planar regions R .

Exercise 8.2 Calculate $\iint_S x dy dz$ for the following surfaces S .

- $S = S_a$
- $S = T_{a,b}$
- $S =$ the box $a \leq x \leq b$, $c \leq y \leq d$, $h \leq z \leq k$.

The next level of complexity is when the divergence is not constant, but linear.

Exercise 8.3 The center of mass of a region R is the point $(\bar{x}, \bar{y}, \bar{z})$, where \bar{x} is the average of the function x over R , similarly for \bar{y} and \bar{z} . Show how to compute the center of mass of a region using flux integrals over the boundary. Use your formula to find the center of mass when

- R is the inside of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$, and
- R is the box $0 \leq x \leq 2$, $0 \leq y, z \leq 1$.

Exercise 8.4 Calculate the surface integral

$$\iint_S x dy dz + y dz dx + z dx dy,$$

where S is the surface of the cube $0 \leq x, y, z \leq 1$. Do the calculation in two ways: First directly, using 6 different surface integrals, and second using the Divergence Theorem.

Exercise 8.5 The (negative) Gravity vector field is

$$\mathbf{G} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

It has a bad point at $(0, 0, 0)$.

- Show that $\nabla \cdot \mathbf{G} = 0$.
- Show that if S is a surface not containing $(0, 0, 0)$ in its interior or boundary, then

$$\iint_S \mathbf{G} \cdot \mathbf{N} dS = 0.$$

- Show that $\iint_{S_a} \mathbf{G} \cdot \mathbf{N} dS = 4\pi$.
- Show that if S is a surface containing $(0, 0, 0)$ in its interior, then

$$\iint_S \mathbf{G} \cdot \mathbf{N} dS = 4\pi.$$

(Surround $(0, 0, 0)$ by a small S_a , then apply the divergence theorem to the region R between S and S_a .)

The vector field G is a constant times the electric field due to a point charge at the origin. Part d) of Exercise 8.4 says there is zero total flux of the electric field out of any region not containing the charge (even though the field is nonzero in the region).