Math 202 extra problems for Note 6

1. Let R be the cube $0 \leq x, y, z \leq 1$, let S be the boundary of R, and let $\mathbf{F} = (2x - z, x^2y, -xz^2)$.

a) Compute $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$

b) Compute $\iiint_R \nabla \cdot \mathbf{F}\, dR$.

According to the Divergence Theorem, you should get the same answer for both, namely $\frac{11}{6}$.

2. Let R be the part of the ball $x^2 + y^2 + z^2 \leq 1$ with $x, y, z \geq 0$, and let S be the boundary of R.

a) Calculate $\iiint_R x^{n-1}y^{m-1}z^{k-1}\, dR$.

b) Let $\mathbf{F} = \frac{1}{n}(x^n y^{m-1} z^{k-1}, 0, 0)$. Calculate $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$.

According to the Divergence Theorem, you should get the same answer for both, namely

$$\frac{(\frac{n}{2} - 1)!(\frac{m}{2} - 1)!(\frac{k}{2} - 1)!}{8(\frac{n+m+l}{2})!}.$$

3. Let B^2 be the unit disk in $\mathbb{R}^2$. Compute the average radius of a point in B^2. Do the same for the unit ball B^3 in $\mathbb{R}^3$. Now using equation (4e) in Note 4, compute the average radius of a point in B^n. You should find that the average radius goes to 1 as n gets large. In other words, more and more of the ball is concentrated near the boundary.

4. Let R be the top half of the ball of radius a, given by $x^2 + y^2 + z^2 \leq a^2,\ z \geq 0$. Use the divergence theorem to compute

$$\iint_S xz^2 dydz + (x^2y - z^3)dzdx + (2xy + y^2z)dxdy.$$

(You can check your answer by computing the surface integral directly.)

5. Let $\mathbf{c}(t) = (x(t), 0, z(t)),\ a \leq t \leq b$ be a curve in the xz plane, with $x(t)$ always positive. Revolve $\mathbf{c}$ around the z axis to get a surface S. Show that the area of S is the length of $\mathbf{c}$ times the circumference of the circle travelled around the z axis by the center of mass of $\mathbf{c}$.

To get started, here is a parametrization of S:

$$\mathbf{r}(t, \theta) = (x(t)\cos\theta, x(t)\sin\theta, z(t)),\ 0 \leq \theta \leq 2\pi,\ a \leq t \leq b.$$

Check that the result is true for the sphere S_a (obtained by revolving a semicircle) and the torus T_{ab} (obtained by revolving a circle). This was discovered by an ancient Greek mathematician named Pappus, before the invention of surface integrals.

6. Let S be the triangle in space with vertices at $(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)$. Choose the upward pointing normal vector. Let C be the boundary of S. Let $\mathbf{F} = (z, x, y)$.

a) Compute $\nabla \times \mathbf{F}$.

b) Compute $\iint_S \nabla \times \mathbf{F} \cdot N\, dS$.

c) Compute $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$.
According to Stokes theorem, you should get the same answers for a) and b).

Stokes theorem implies that if $\nabla \times F = 0$, then $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ depends only on the endpoints of C. Use this fact in the next two problems.

7. Let $\mathbf{F} = (2xz^3 + 6y, 6x - 2yz, 3x^2z^2 - y^2)$.
a) Compute $\nabla \times \mathbf{F}$.
b) Use the result of a) to compute $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ where C is a squiggly path from $(1, -1, 1)$ to $(2, -1, 1)$. (Ans: 15).
c) Can you do b) with $\mathbf{F}$ replaced by $\mathbf{G} = (0, 0, y)$?

8. Let $r = (x^2 + y^2 + z^2)^{1/2}$. Let p be a point on S_a, and let q be a point on S_b, where $a < b$, and let C be any path from p to q. Compute

$$\int_C xr\ dx + xr\ dy + xr\ dz.$$

(Easiest method: Find a potential function for $\mathbf{F} = (xr, yr, zr)$. Answer: $(b^3 - a^3)/3$.)