MT830 Groups of order p^3

Let G be a non-abelian group of order p^3 and let Z be the center of G. Then $|Z| = p$ since $Z \neq 1$ and G/Z is not cyclic. Since G/Z has order p^2, it is abelian, so $Z = [G, G]$. Since $G/Z \simeq C_p \times C_p$, there are exactly $p + 1$ subgroups A of G with $|A| = p^2$ and $Z < A < G$. There are two possible isomorphism classes of this group A of order p^2. Can we find an A of a given type?

Lemma 1: Let G be a group, let $x, y \in G$, and let $[y, x] = yxy^{-1}x^{-1}$ be the commutator. Suppose $[y, x]$ commutes with x and y. Then for all $n \in \mathbb{N}$ we have

$$(xy)^n = x^n y^n [y, x]^{n(n-1)/2}.$$

Proof: We start with

$$(xy)^n = (xy)(xy)\cdots(xy). \tag{1}$$

We want to put all the y's to the right of all the x's. The commutator is the price to pay for replacing yx by xy:

$$yx = [y, x]xy,$$

which we can write as

$$yx = xy[y, x],$$

since x, y commute with $[y, x]$. The left-most y in (1) moves past $n - 1$ x's, so contributes $[y, x]^{n-1}$. The new left-most y then contributes $n - 2$ x's, and so on. Thus, we get

$$(xy)^n = x^n y^n [y, x]^{(n-1)+(n-2)+\cdots+1} = x^n y^n [y, x]^{n(n-1)/2},$$

as claimed. ■

Corollary Let G be a non-abelian group of order p^3, where p is an odd prime. Then G has normal subgroups

$$Z < A < G,$$

where Z, the center of G, has order p and $A \simeq C_p \times C_p$.

Proof: Since G is a p-group its center Z is non-trivial. Since G is non-abelian, the quotient G/Z cannot be cyclic, so $|Z| = p$ and $G/Z \simeq C_p \times C_p$ is abelian. Hence $[G, G] \subset Z$. But G is non-abelian, so $[G, G] = Z$. Hence Lemma 1 applies to any $x, y \in G$.

We apply Lemma 1 with $n = p$. Since p is odd, it divides $p(p-1)/2$, so $[y,x]^{p(p-1)/2} \in Z^p = \{1\}$. Therefore, we have

$$(xy)^p = x^p y^p \qquad \text{for all} \quad x, y \in G.$$

This means that the p-power map is a homomorphism $p : G \to G$. Note that $Z \subset \ker p$. But Z cannot be the whole of $\ker p$. For otherwise,

$$\operatorname{im} p \simeq G/Z \simeq C_p \times C_p$$

would be a subgroup of $\ker p = Z$, contradicting $|Z| = p$. It follows that there is an element $x \in G - Z$ of order p. The subgroup $A = \langle Z, x \rangle$ is then isomorphic to $C_p \times C_p$.

In a p-group, any subgroup of index p is normal, so A is normal in G. This can also be seen directly: for all $y \in G$, we have $yxy^{-1} = [y,x]x \in Zx \subset A$. ■

Remark: The corollary is false for $p = 2$: the quaternion group Q_8 has no subgroup isomorphic to $C_2 \times C_2$. The proof breaks down because 2 does not divide $2(2-1)/2$. Nevertheless, a similar result holds: Any non-abelian group G of order 8 has normal subgroups $Z < A < G$, where Z, the center of G, has order 2 and $A \simeq C_4$. It is a nice exercise to prove this without using the classification of groups of order 8.

We turn now to representation theory. Choose any subgroup A of G with $Z < A < G$ and $|A| = p^2$. Since $G/Z \simeq C_p \times C_p$, the group G has exactly $p^2 - 1$ linear characters of order p.

For each nontrivial character λ of Z, let

$$\operatorname{Irr}(A, \lambda) = \{\chi \in \operatorname{Irr}(A) : \ \chi|_Z = \lambda\}.$$

I claim that the group G/A acts simply-transitively on $\operatorname{Irr}(A, \lambda)$. First, if $\chi, \chi' \in \operatorname{Irr}(A, \lambda)$, then $\chi'\chi^{-1} \in \operatorname{Irr}(A/Z)$, so $|\operatorname{Irr}(A, \lambda)| = p$. Next, if $g \in G$ fixes $\chi \in \operatorname{Irr}(A, \lambda)$, then $\chi([a, g]) = 1$ for all $a \in A$. If $g \notin A$ then $[A, g] \neq 1$ since G is nonabelian, so $[A, g] = Z$. But $\chi|_Z = \lambda$ is a nontrivial character of Z. Hence $g \in A$. This proves that G/A acts simply-transitively on $\operatorname{Irr}(A, \lambda)$.

For each $\chi \in \operatorname{Irr}(A, \lambda)$ we form the induced representation $\operatorname{Ind}_A^G \chi$. Given two such characters $\chi, \chi' \in \operatorname{Irr}(A, \lambda)$, we have, by Frobenius reciprocity and Mackey's formula:

$$\langle \rho_{\chi'}, \rho_\chi \rangle_G = \sum_{g \in G/A} \langle \chi', \chi^g \rangle_A = |\{g \in G/A : \ \chi^g = \chi'\},$$

since A is normal in G.

Since G/A is simply-transitive on $\mathrm{Irr}(A, \lambda)$, it follows that $\mathrm{Ind}_A^G \chi$ is irreducible and that its equivalence class depends only λ; we write

$$\pi_\lambda := \mathrm{Ind}_A^G \chi, \qquad \text{for any} \quad \chi \in \mathrm{Irr}(A, \lambda).$$

Thus we get $p - 1$ distinct irreducible characters π_λ of G, each of dimension p. Since

$$p^2 + (p-1)p^2 = p^3,$$

we have found all irreducible representations of G.

The character of π_λ is given by

$$\mathrm{tr}(g, \pi_\lambda) = \sum_{\substack{y \in G/A \\ g \in A^y}} \chi(ygy^{-1}).$$

Since A is normal in G, this character vanishes off A, where we have

$$\pi_\lambda|_A = \sum_{\chi \in \mathrm{Irr}(A, \lambda)} \chi.$$

Since we can take $A \simeq C_4$ or $C_p \times C_p$, the group $\mathrm{Aut}(A)$ has a unique conjugacy-class of elements of order p. It follows that the character of π_λ depends only on A and not on G, so that the character tables of any two non-abelian groups of order p^3 are identical.