

# Housing Markets & Top Trading Cycles

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# House Allocation Problems: A Collective Ownership Economy

A house allocation problem (Hylland & Zeckhauser, *JPE* 1979) is a triple  $\langle I, H, R \rangle$ .

$I$ : set of agents

$H$ : set of houses

$P$ : list of preferences over houses

For simplicity assume:

1.  $|H| = |I|$ , and
2. the preferences are strict.

## The Outcome: A Matching

A **(house) matching**  $\mu : I \rightarrow H$  is a one-to-one and onto function from  $I$  to  $H$ .

With everyday language it is an assignment of houses to agents such that

1. every agent is assigned one house, and
2. no house is assigned to more than one agent.

# Housing Markets: A Basic Exchange Economy

A **housing market** (Shapley & Scarf, *JPE* 1974) is a 4-tuple  $\langle I, H, R, \mu \rangle$ .

$I$ : set of agents

$H$ : set of houses with  $|H| = |I|$

$P$ : list of strict preferences over houses

$\mu$ : initial endowment matching

A matching  $\eta$  is in the **core** of the housing market  $(I, H, R, \mu)$  if there is no coalition  $T \subseteq I$  and matching  $\nu$  such that

1.  $\nu(i) \in \mu(T)$  for all  $i \in T$ ,
2.  $\nu(i) R_i \eta(i)$  for all  $i \in T$ ,
3.  $\nu(i) P_i \eta(i)$  for some  $i \in T$ .

# Gale's Top Trading Cycles Algorithm

(Described in Shapley & Scarf,  
attributed to David Gale)

**Step 1:** Each agent “points to” the owner of his favorite house. Since there are finite number of agents, there is at least one **cycle**.

Each agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment.

If there is at least one remaining agent, proceed with the next step.

**Step t:** Each remaining agent points to the owner of his favorite house among the remaining houses.

Every agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment.

If there is at least one remaining agent, proceed with the next step.

## Important Properties of the Core

**Theorem** (Roth & Postlewaite, *JME* 1977):

There is a unique matching in the core of each housing market. Moreover, this matching is the unique competitive allocation.

A **direct matching mechanism** is a systematic procedure to select a matching for each problem.

**Theorem** (Roth, *Economics Letters* 1982): Core (as a direct mechanism) is *strategy-proof*.

**Theorem** (Ma, *IJGT* 1994): Core is the only mechanism that is *Pareto efficient*, *individually rational*, and *strategy-proof*.

## Mechanisms for House Allocation

An **ordering**  $f : \{1, \dots, n\} \rightarrow A$  is a one-to-one and onto function.

**Simple serial dictatorship** induced by  $f$ :

Agent who is ordered first (by the ordering  $f$ ) gets her top choice; agent ordered second gets his top choice among those remaining; and so on.

**Core from assigned endowments**  $\mu$ : For any house allocation problem  $\langle I, H, P \rangle$ , select the core of the housing market  $\langle I, H, P, \mu \rangle$ .

# Lottery Mechanisms

A **lottery** is a probability distribution over matchings.

A **direct lottery mechanism** is a systematic procedure to select a lottery for each problem.

## Examples:

- **Random serial dictatorship (RSD):**  
Randomly select an ordering with uniform distribution and use the induced simple serial dictatorship.
- **Core from random endowments:**  
Randomly select a matching (to be interpreted as initial endowment) and select the core of the induced housing market.

## Efficient House Allocation

**Theorem** (Abdulkadiroğlu & Sönmez, *Econometrica* 1998): For any ordering  $f$ , and any matching  $\mu$ , simple serial dictatorship induced by  $f$  and core from assigned endowments  $\mu$  both yield *Pareto efficient* matchings. Moreover, for any *Pareto efficient* matching  $\eta$ , there is a *simple serial dictatorship* and a *core from assigned endowments* that yields it.

**Theorem** (Abdulkadiroğlu & Sönmez, *Econometrica* 1998): The *random serial dictatorship* is equivalent to the *the core from random endowments*. That is, for each house allocation problem they choose the same lottery.

## A Slight Variation: One Existing Tenant

- A number of houses should be allocated to a group of agents through RSD.
- One of the houses is already occupied and its tenant is given two options:
  1. Keep the house, or
  2. Give it up and enter the lottery.
- Since there are no guarantees to get a better house, the existing tenant may choose the first option which in turn may result in loss of potential gains from trade.

**Example:** There are three agents  $i_1, i_2, i_3$  and three house  $h_1, h_2, h_3$ . Agent  $i_1$  is a current tenant and he occupies house  $h_1$ . Agents  $i_2, i_3$  are new applicants and house  $h_2, h_3$  are vacant houses.

- Utilities are:

	$h_1$	$h_2$	$h_3$
$i_1$	3	4	1
$i_2$	4	3	1
$i_3$	3	4	1

- Agent  $i_1$  has two options:
  1. he can keep house  $h_1$  or
  2. he can give it up and enter the lottery.
- His utility from keeping house  $h_1$  is 3.

- The following table summarizes the possible outcomes, in case he enters the lottery:

ordering	$i_1$	$i_2$	$i_3$
$i_1 - i_2 - i_3$	$h_2$	$h_1$	$h_3$
$i_1 - i_3 - i_2$	$h_2$	$h_3$	$h_1$
$i_2 - i_1 - i_3$	$h_2$	$h_1$	$h_3$
$i_2 - i_3 - i_1$	$h_3$	$h_1$	$h_2$
$i_3 - i_1 - i_2$	$h_1$	$h_3$	$h_2$
$i_3 - i_2 - i_1$	$h_3$	$h_1$	$h_2$

- Expected utility from entering the lottery:

$$\frac{1}{6}u(h_1) + \frac{3}{6}u(h_2) + \frac{2}{6}u(h_3) = \frac{4}{6} + \frac{9}{6} + \frac{2}{6} = \frac{15}{6}.$$

- Optimal strategy: Keep house  $h_1$ .

## When $i_1$ Keeps $h_1$ :

- Since both  $i_2, i_3$  prefer  $h_2$  to  $h_3$ , the eventual outcome is either

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_2 & h_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_3 & h_2 \end{pmatrix}$$

both with  $1/2$  probability.

- **Inefficiency:** The first one is Pareto dominated by

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_1 & h_3 \end{pmatrix}$$

## Avoiding Inefficiency with One Existing Tenant

The cause for the inefficiency is the lack of the mechanism to guarantee the existing tenant a house that is at least as good as the one he already holds. One natural modification that will fix this “deficiency” is the following:

1. Order the agents with a lottery.
2. Assign the first agent his top choice, the second agent his top choice among the remaining houses, and so on, *until someone demands the house the existing tenant holds.*
3. (a) If the existing tenant is already assigned a house, then do not disturb the procedure.  
(b) If the existing tenant is not assigned a house, then modify the remainder of the ordering by inserting him to the top, and proceed with the procedure.

## A General Model: House Allocation with Existing Tenants

- A set of houses should be allocated to a set of agents by a centralized clearing house.
- Some of the agents are existing tenants each of whom already occupies a house and the rest of the agents are newcomers.
- In addition to occupied houses, there are vacant houses.
- Existing tenants are not only entitled to keep their current houses but also apply for other houses.

## Formal Model

A house allocation problem with existing tenants (Abdulkadiroğlu & Sönmez *JET* 1999) is a five-tuple  $\langle A_E, A_N, H_O, H_V, P \rangle$  where

1.  $A_E$  is a finite set of existing tenants,
  2.  $A_N$  is a finite set of newcomers,
  3.  $H_O = \{h_a\}_{a \in A_E}$  is a finite set of occupied houses,
  4.  $H_V$  is a finite set of vacant houses where  $h_0 \in H_V$  denotes the null house, and
  5.  $P = (P_a)_{a \in A_E \cup A_N}$  is a list of strict preference relations.
- Assume that the null house  $h_0$  is the last choice for each agent.

- A **matching**  $\mu$  is an assignment of houses to agents such that
  1. every agent is assigned one house, and
  2. only the null house  $h_0$  can be assigned to more than one agent.
- A matching is **Pareto efficient** if there is no other matching which makes all agents weakly better off and at least one agent strictly better off.
- A matching is **individually rational** if no existing tenant strictly prefers his endowment to his assignment.

## A Popular Real-Life Mechanism: RSD with squatting rights

- Each existing tenant decides whether she will enter the housing lottery or keep her current house. Those who prefer keeping their houses are assigned their houses. All other houses become available for allocation.
- An ordering of agents in the lottery is randomly chosen from a given distribution of orderings. This distribution may be uniform or it may favor some groups.
- Once the agents are ordered, available houses are allocated using the induced *simple serial dictatorship*: The first agent receives her top choice, the next agent receives her top choice among the remaining houses and so on so forth.

**Major deficiency:** Neither *individually rational* nor *Pareto efficient*.

## Solution: Top Trading Cycles Mechanism

Fix an ordering  $f$  of agents. Interpret this as a **priority ordering**.

**Step 1:** Define the set of **available houses** for this step to be the set of vacant houses.

- \* Each agent points to his favorite house,
  - \* each occupied house points to its occupant,
  - \* each available house points to the agent with highest priority.
- 
- There is at least one cycle. Every agent in a cycle is assigned the house that he points to and removed from the market with his assignment.
  - If there is at least one remaining agent and one remaining house then we go to the next step.

## **TTC: Adjustment of Available Houses**

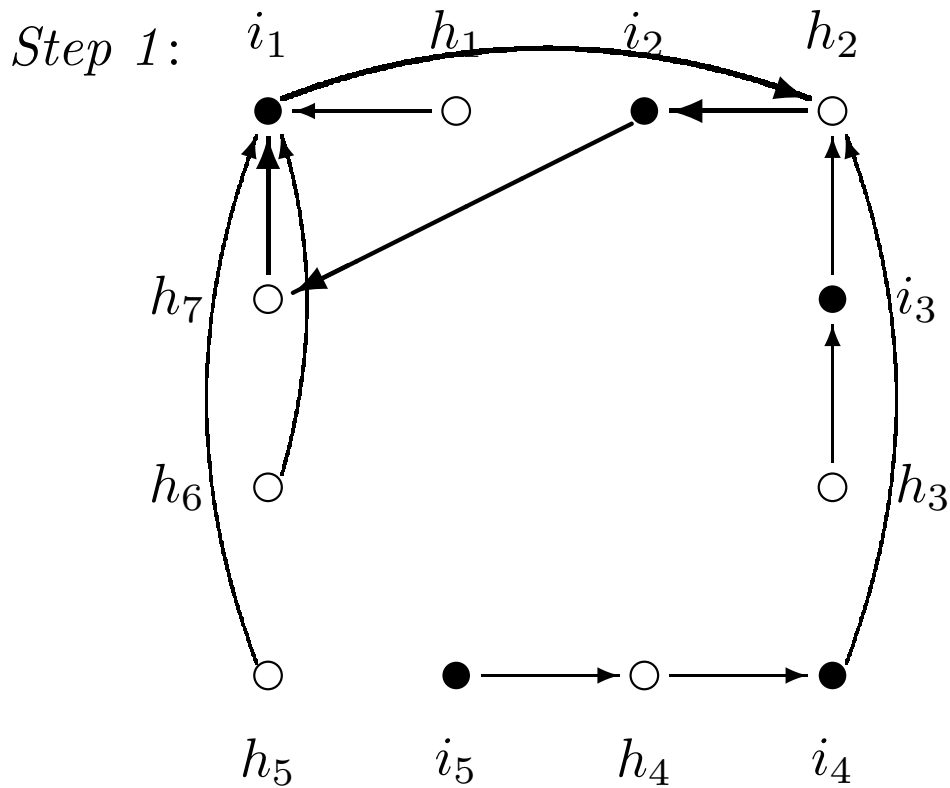
- Whenever there is an available house in a cycle, the agent with the highest priority, i.e. agent  $f(1)$ , is also in the same cycle.
- If this agent is an existing tenant, then his house  $h_{f(1)}$  cannot be in any cycle and it becomes available for the next step.
- All available houses that are not removed remain available.

*Step t*: The set of **available houses** for Step  $t$  is defined at the end of Step  $(t-1)$ .

- \* Each remaining agent points to his favorite house among the remaining houses,
- \* each remaining occupied house points to its occupant,
- \* each available house points to the agent with highest priority among the remaining agents.
- Every agent in a cycle is assigned the house that he points to and removed from the market with his assignment.
- If the most senior (remaining) agent's house is vacated, then it is added to the set of available houses for the next step. All available houses that are not removed remain available.
- If there is at least one remaining agent and one remaining house then we go to the next step.

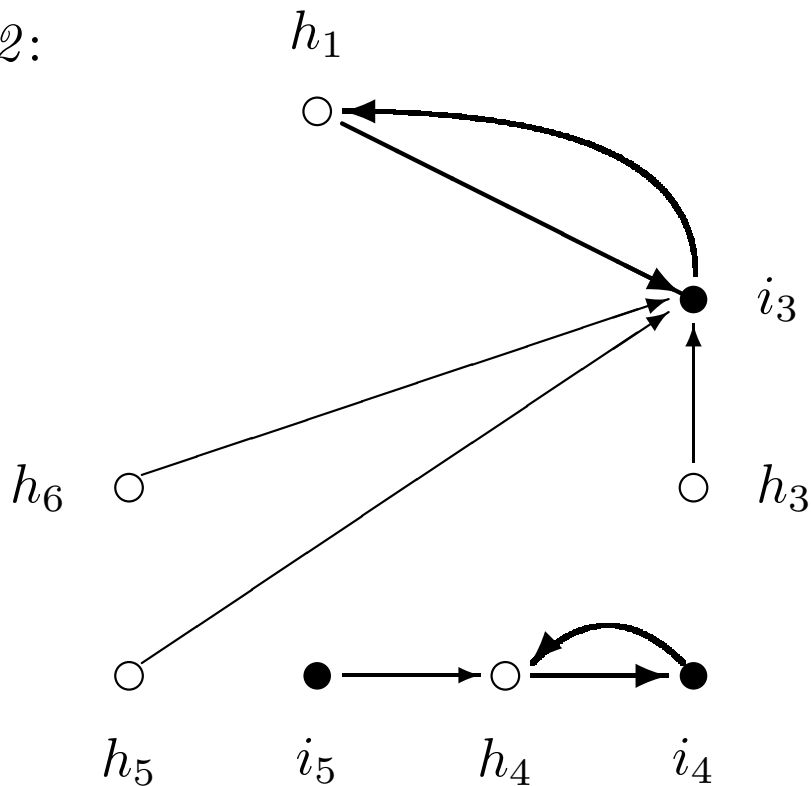
**Example:** Let  $I_E = \{i_1, i_2, i_3, i_4\}$ ,  $I_N = \{i_5\}$ ,  
 $H_O = \{h_1, h_2, h_3, h_4\}$ , and  $H_V = \{h_5, h_6, h_7\}$ . Let  
the ordering  $f$  order the agents as  
 $i_1 - i_2 - i_3 - i_4 - i_5$  and the preferences (from  
best to worst) be as follows:

$P_{i_1}$	$P_{i_2}$	$P_{i_3}$	$P_{i_4}$	$P_{i_5}$
$h_2$	$h_7$	$h_2$	$h_2$	$h_4$
$h_6$	$h_1$	$h_1$	$h_4$	$h_3$
$h_5$	$h_6$	$h_4$	$h_3$	$h_7$
$h_1$	$h_5$	$h_7$	$h_6$	$h_1$
$h_4$	$h_4$	$h_3$	$h_1$	$h_2$
$h_3$	$h_3$	$h_6$	$h_7$	$h_5$
$h_7$	$h_2$	$h_5$	$h_5$	$h_6$
$h_0$	$h_0$	$h_0$	$h_0$	$h_0$



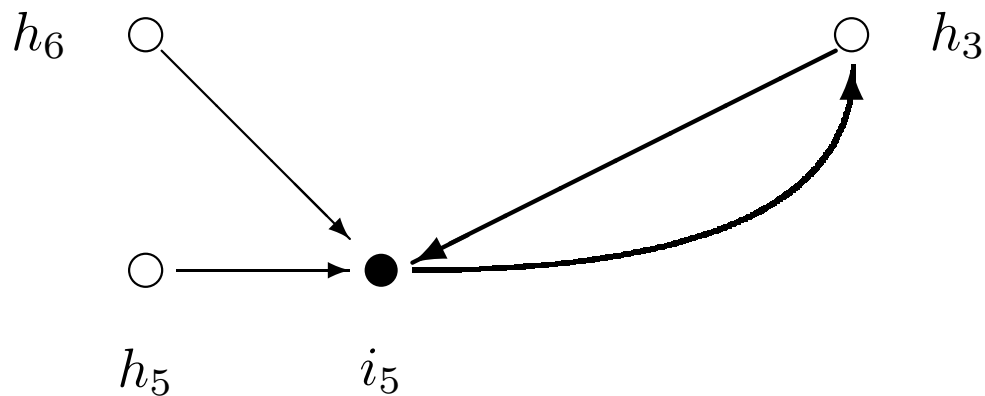
- The set of available houses in Step 1 is  $H_V = \{h_5, h_6, h_7\}$ .
- The only cycle that is formed in this step is  $(i_1, h_2, i_2, h_7)$ . Therefore  $i_1$  is assigned  $h_2$  and  $i_2$  is assigned  $h_7$ .
- Since  $i_1$  leaves the market, his house  $h_1$  becomes available for the next step. Therefore the set of available houses for Step 2 is  $\{h_1, h_5, h_6\}$ .

Step 2:



- There are two cycles  $(i_3, h_1)$  and  $(i_4, h_4)$  in Step 2.
- Therefore  $i_3$  is assigned  $h_1$  and  $i_4$  is assigned  $h_4$ .
- Since  $i_3$  leaves the market his house  $h_3$  becomes available for the next step. Therefore the set of available houses for Step 3 is  $\{h_3, h_5, h_6\}$ .

Step 3:



- There is one cycle  $(i_5, h_3)$  in Step 3.
- Therefore  $i_5$  is assigned  $h_3$ .
- There are no remaining agents so the algorithm terminates and the matching it induces is:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ h_2 & h_7 & h_1 & h_4 & h_3 \end{pmatrix}$$

## Efficiency, Individual Rationality, and Strategy-Proofness

- TTC reduces to Gale's TTC for housing markets and RSD for house allocation problems.
- Moreover:

**Theorem** (Abdulkadiroğlu & Sönmez *JET* 1999): For any ordering  $f$ , the induced *top trading cycles mechanism* is

- \* individually rational,
- \* Pareto efficient, and
- \* strategy-proof.

## Simpler Efficient, Individual Rational, and Strategy-Proof Mechanism

- Consider the case where:  $|H_V| = |A_N|$ .
- Simpler PE, IR and S-P mechanism:
  1. Construct an initial allocation by
    - (a) assigning each existing tenant her own house and
    - (b) randomly assigning the vacant houses to newcomers with uniform distribution, and
  2. choose the core of the induced housing market to determine the final outcome.
- **Theorem** (Sönmez & Ünver *GEB* 2005):

The above mechanism is equivalent to an extreme case of TTC where newcomers are randomly ordered first and existing tenants are randomly ordered next.

## A More Intuitive Algorithm

We can find the outcome of the TTC mechanism using the following **you ask my house-I get your turn (YRMH-IGYT)** algorithm:

1. For any given ordering  $f$ , assign the first agent his top choice, the second agent his top choice among the remaining houses, and so on, until someone demands the house of an existing tenant.
2. If at that point the existing tenant whose house is demanded is already assigned a house, then do not disturb the procedure. Otherwise modify the remainder of the ordering by inserting him to the top and proceed with the procedure.

3. Similarly, insert any existing tenant who is not already served at the top of the line once his house is demanded.
  4. If at any point a loop forms, it is formed by exclusively existing tenants and each of them demands the house of the tenant next in the loop. (A **loop** is an ordered list of agents  $(i_1, i_2, \dots, i_k)$  where agent  $i_1$  demands the house of agent  $i_2$ , agent  $i_2$  demands the house of agent  $i_3, \dots$ , agent  $i_k$  demands the house of agent  $i_1$ .) In such cases remove all agents in the loop by assigning them the houses they demand and proceed with the procedure.
- **Theorem** (Abdulkadiroğlu & Sönmez *JET* 1999): For a given ordering  $f$ , YRMH-IGYT algorithm yields the same outcome as the TTC algorithm.