School Matching

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"Matching, Auctions, and Market Design"

College Admissions, Student Placement, School Choice

- **College Admissions** (Gale & Shapley *AMM* 1962)
  - * Models (many-to-one) two-sided matching markets.
  - * Both schools and students are (potentially strategic) agents
- **Student Placement** (Balinski & Sönmez *JET* 1999)
  - * Models centralized university admissions.
  - * Students are (potentially) strategic agents
  - * School seats are goods to be consumed
  - * Priority at schools determined by exam scores
  - * Under an adequate "fairness" axiom, model isomorphic to stable college admissions.

- **School Choice** (Abdulkadiroğlu & Sönmez *AER* 2003)
  - Models centralized assignment of public school seats to K-12 students.
  - Students are (potentially) strategic agents
  - School seats are goods to be consumed
  - Priorities at schools are exogenous
  - A version of the model isomorphic to stable college admissions.

Student Placement: The Model

A **student placement problem** consists of

$S = \{s_1, \ldots, s_n\}$	a set of students
$C = \{c_1, \ldots, c_m\}$	a set of colleges
$R = (R_{s_1}, \ldots, R_{s_n})$	a list of student preferences
$q = (q_1, \ldots, q_m)$	a vector of college capacities
$T = \{t_1, \ldots, t_k\}$	a set of skill categories
$f = (f^{s_1}, \ldots, f^{s_n})$	a list of test scores
$t : C \longrightarrow T$	a function from C to T

Here

- q_i is the capacity of college c_i,
- R_{s_i} is the preference of student s_i over colleges and the no college option,
- $f^{s_i} = (f^{s_i}_{t_1}, \ldots, f^{s_i}_{t_k})$ is a vector which gives the test score of student s_i in each category, and
- t is a function which maps each college to a category.

Associated College Admissions Problem

For each student placement problem we can construct an **associated college admissions problem** by assigning each college c a preference relation R_c based on the ranking in its category $t(c)$.

Matching & Tentative Student Placement

- A **matching** is a function $\mu : S \longrightarrow C \cup \{c_0\}$ such that no college is assigned to more students than its capacity.

  $\mu(s) = c_0$: Student s is unmatched.

- A **tentative student placement** is a mapping $\mu : S \Longrightarrow C \cup \{c_0\}$ such that no college is assigned to more students than its capacity.

  Remark: *Tentative student placement* allows a student to be assigned more than one college.

- A **mechanism** is a function which assigns a *matching* for each student placement problem.

Elimination of Justified Envy

- A matching μ **eliminates justified envy** if whenever a student s prefers another student $\tilde{s}$'s assignment $\mu(\tilde{s})$ to his own, he ranks worse than $\tilde{s}$ in the category the college $\mu(\tilde{s})$ is.
  - * Closely related to *stability*: Isomorphism with stable college admissions.
  - * Critical in the context of Turkish college admissions.
- A mechanism **eliminates justified envy** if it always selects a matching that eliminates justified envy.

Simple Case: One Skill Category

- Given a priority ranking, the induced **simple serial dictatorship** assigns the first student his top choice, the next student his top choice among remaining seats, etc.

- **Proposition** (Balinski & Sönmez *JET* 1999): If there is only one category (and hence only one ranking) then there is only one mechanism that is *Pareto efficient* and *eliminates justified envy*: The *simple serial dictatorship* induced by this ranking.

Elimination of Justified Envy & Stability

- A matching is **individually rational** if no student prefers the no college option to his assignment.
- A matching is **non-wasteful** if no student prefers a college with one or more empty slots to his assignment.
- **Lemma** (Balinski & Sönmez *JET* 1999): A matching is *individually rational*, *non-wasteful* and *eliminates justified envy* if and only if it is *stable* for its associated college admissions problem.

Current Mechanism in Turkey: Multi-Category Serial Dictatorship

Step 1:

- For each category t: Consider the ranking induced by the test scores in this category and assign the college seats in this category to students with the induced simple serial dictatorship.
- Assign the no college option to all students who is not assigned a college.
- This in general leads to a tentative student placement.

- For each student s construct R_s^1 from R_s as follows:
  * If the student is not assigned more than one college then $R_s^1 = R_s$.
  * If the student is assigned more than one college then obtain R_s^1 by moving the no college option c_0 right after the best of these assignments and otherwise keeping the ranking of the colleges the same.

  Let $R^1 = (R_1^1, \ldots, R_n^1)$ be the list of adjusted preferences.

Step l: Construct R^l from R^{l-1} as it is described in Step 1.

The procedure terminates at the step in which no student is assigned more than one college. The *multi-category serial dictatorship* selects this matching.

Example: $S = \{s_1, s_2, s_3, s_4, s_5\}$,
$C = \{c_1, c_2, c_3\}$, $q = (q_{c_1}, q_{c_2}, q_{c_3}) = (2, 1, 1)$,
$T = \{t_1, t_2\}$, $t(c_1) = t_1$, $t(c_2) = t(c_3) = t_2$,

$$
\begin{array}{lcr}
s_1: & c_2 - c_1 - c_0 & f^{s_1} = (9,9) \\
s_2: & c_1 - c_2 - c_3 - c_0 & f^{s_2} = (8,6) \\
s_3: & c_1 - c_3 - c_2 - c_0 & f^{s_3} = (7,7) \\
s_4: & c_1 - c_2 - c_0 & f^{s_4} = (6,8) \\
s_5: & c_2 - c_3 - c_1 - c_0 & f^{s_5} = (5,5)
\end{array}
$$

Note that these scores induce the following rankings in categories t_1 and t_2:

$$t_1 : s_1 \ s_2 \ s_3 \ s_4 \ s_5$$

$$t_2 : s_1 \ s_4 \ s_3 \ s_2 \ s_5$$

***Step 1*:**

$$t_1 : \begin{matrix} s_1 & s_2 \\ c_1 & c_1 \end{matrix} \qquad t_2 : \begin{matrix} s_1 & s_4 & s_3 \\ c_2 & - & c_3 \end{matrix}$$

Step 1 yields the following tentative student placement:

$$\nu^1 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ c_1, c_2 & c_1 & c_3 & c_0 & c_0 \end{pmatrix}$$

Since student s_1 is assigned two colleges his preferences are truncated:

$$s_1 : c_2 - c_0$$

For other students: $R^1_{s_2} = R_{s_2}$, $R^1_{s_3} = R_{s_3}$, $R^1_{s_4} = R_{s_4}$, and $R^1_{s_5} = R_{s_5}$.

***Step 2*:** In Step 2 we first find the serial dictatorship outcomes for R^1.

$$t_1 : \begin{array}{ccc} s_1 & s_2 & s_3 \\ - & c_1 & c_1 \end{array} \qquad t_2 : \begin{array}{ccc} s_1 & s_4 & s_3 \\ c_2 & - & c_3 \end{array}$$

Step 2 yields the following tentative student placement:

$$\nu^2 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ c_2 & c_1 & c_1, c_3 & c_0 & c_0 \end{pmatrix}$$

Since student s_3 is assigned two colleges his preferences are truncated:

$$s_3 : c_1 - c_0$$

For other students: $R^2_{s_1} = R^1_{s_1}$, $R^2_{s_2} = R^1_{s_2}$, $R^2_{s_4} = R^1_{s_4}$, and $R^2_{s_5} = R^1_{s_5}$.

***Step 3*:** In Step 3 we first find the serial dictatorship outcomes for R^2.

$$t_1 : \begin{array}{ccc} s_1 & s_2 & s_3 \\ - & c_1 & c_1 \end{array} \qquad t_2 : \begin{array}{cccc} s_1 & s_4 & s_3 & s_2 \\ c_2 & - & - & c_3 \end{array}$$

Step 3 yields the following tentative student placement:

$$\nu^3 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ c_2 & c_1, c_3 & c_1 & c_0 & c_0 \end{pmatrix}$$

Since student s_2 is assigned two colleges his preferences are truncated:

$$s_2 : c_1 - c_0$$

For other students: $R^3_{s_1} = R^2_{s_1}$, $R^3_{s_3} = R^2_{s_3}$, $R^3_{s_4} = R^2_{s_4}$, and $R^3_{s_5} = R^2_{s_5}$.

***Step 4*:** In Step 4 we first find the serial dictatorship outcomes for R^3.

$$t_1 : \begin{matrix} s_1 & s_2 & s_3 \\ - & c_1 & c_1 \end{matrix} \qquad t_2 : \begin{matrix} s_1 & s_4 & s_3 & s_2 & s_5 \\ c_2 & - & - & - & c_3 \end{matrix}$$

Step 4 yields the following tentative student placement (which is also a matching):

$$\nu^4 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ c_2 & c_1 & c_1 & c_0 & c_3 \end{pmatrix}$$

Since no student is assigned more than one college in ν^4 the algorithm terminates and $\varphi^{msd}(R_S, f, q) = \nu^4$.

Mechanisms via the Associated College Admissions Problem

- **Gale-Shapley college optimal stable mechanism**: The mechanism which selects the college optimal stable matching of the associated college admissions problem for each student placement problem.

- **Gale-Shapley student optimal stable mechanism**: The mechanism which selects the student optimal stable matching of the associated college admissions problem for each student placement problem.

An Equivalence

Theorem (Balinski & Sönmez *JET* 1999): The *multi-category serial dictatorship* is equivalent to *Gale-Shapley college optimal stable mechanism.*

Pareto Efficiency

The *multi-category serial dictatorship* is not *Pareto efficient*.

Example: $S = \{s_1, s_2\}$ $C = \{c_1, c_2\}$ $q = (1, 1)$
$T = \{t_1, t_2\}$ $t(c_1) = t_1, t(c_2) = t_2$

$$s_1 : c_1 - c_2 - c_0 \qquad f^{s_1} = (6, 8)$$
$$s_2 : c_2 - c_1 - c_0 \qquad f^{s_2} = (8, 6)$$

The algorithm terminates in one step:

$$\varphi^{msd}(R_S, f, q) = \begin{pmatrix} s_1 & s_2 \\ c_2 & c_1 \end{pmatrix}$$

Clearly the matching which assigns both students their top choices Pareto dominates this matching.

Theorem (Gale & Shapley *AMM* 1962, Balinski & Sönmez *JET* 1999): *Gale-Shapley student optimal stable mechanism* Pareto dominates any other mechanism that eliminates justified envy.

Strategy-Proofness

Example continued: Recall that

$$\varphi^{msd}(R_S, f, q) = \begin{pmatrix} s_1 & s_2 \\ c_2 & c_1 \end{pmatrix}$$

Now suppose s_1 announces a fake preference relation $\tilde{R}_{s_1}$ where only c_1 is acceptable.

In this case

$$\varphi^{msd}(\tilde{R}_{s_1}, R_{s_2}, f, q) = \begin{pmatrix} s_1 & s_2 \\ c_1 & c_2 \end{pmatrix}$$

and hence student s_1 succesfully manipulates the *multi category serial dictatorship*.

- A mechanism is **strategy-proof** if truthtelling is a dominant strategy in its associated preference revelation game.
- **Theorem**: (Dubins & Freedman *AMM* 1981, Roth *MOR* 1982): *Gale-Shapley student optimal stable mechanism* is *strategy-proof*.
- **Theorem** (Alcalde & Barberà *ET* 1994): *Gale-Shapley student optimal stable mechanism* is the only mechanism that *eliminates justified envy*, and is *individually rational, non-wasteful,* and *strategy-proof*.

Respecting Improvements

Example continued: Recall that

$$\varphi^{msd}(R_S, f, q) = \begin{pmatrix} s_1 & s_2 \\ c_2 & c_1 \end{pmatrix}$$

Now suppose students_1 scores worse in both tests and his new test scores are $\tilde{f}^{s_1} = (5, 5)$.

In this case

$$\varphi^{msd}(R_S, \tilde{f}^{s_1}, f^{s_2}, q) = \begin{pmatrix} s_1 & s_2 \\ c_1 & c_2 \end{pmatrix}$$

and student s_1 is revarded by getting his top choice as a result of a worse performance!

- A mechanism **respects improvements** if a student never receives a worse assignment as a result of an increase in one or more of his test scores.
- **Theorem** (Balinski & Sönmez *JET* 1999): *Gale-Shapley student optimal stable mechanism respects improvements.*
- **Theorem** (Balinski & Sönmez *JET* 1999): *Gale-Shapley student optimal stable mechanism* is the only mechanism that is *individually rational, non-wasteful* and that *eliminates justified envy* and *respects improvements.*

School Choice

- *School choice problem* (Abdulkadiroğlu & Sönmez *AER* 2003):
  * There are a number of students, each of whom should be assigned a seat at one of a number of schools.
  * Each school has a maximum capacity but there is no shortage of the total seats.
  * Each student has preferences over all schools and each school has a priority ordering of all students.
- Priorities: Exogenous

Differences with College Admissions & Student Placement

- **Differences with College Admissions:**
  * Students are (possibly strategic) agents; school seats are objects to be consumed.
  * Elimination of justified envy *plausable* but not a *must*. If imposed then *isomorphic* to stable college admissions.
- **Differences with Student Placement:**
  * Priorities are exogenous.
  * Elimination of justified envy *plausable* but *not a must*.

Boston Mechanism

The most common mechanisms is the mechanism used by the Boston Public Schools and it works as follows:

1. For each school a priority ordering is determined according to the following hierarchy:
   - First priority: sibling and walk zone
   - Second priority: sibling
   - Third priority: walk zone
   - Fourth priority: other students

   Students in the same priority group are ordered based on an even lottery.

2. Each student submits a preference ranking of the schools.

3. The final phase is the student assignment based on preferences and priorities:

Round 1: In Round 1 only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her first choice.

Round k: Consider the remaining students. In Round k only the k^{th} choices of these students are considered. For each school with still available seats, consider the students who have listed it as their k^{th} choice and assign the remaining seats to these students one at a time following their priority order who has listed it as her k^{th} choice. until either there are no seats left or there is no student left ho has listed it as her k^{th} choice.

Easy to manipulate

- *Major difficulty:* The Boston mechanism is not *strategy-proof*.
- Even if a student has very high priority at school s, unless she lists it as her top choice she loses her priority to students who have top ranked school s.
- Hence the Boston mechanism gives parents strong incentives to overrank schools where they have high priority.

- **Evidence From Media:**

  Consider the following quotation from St.Petersburg Times:

  > Make a realistic, informed selection on the school you list as your first choice. It's the cleanest shot you will get at a school, but if you aim too high you might miss.
  > Here's why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That's because you then fall in line behind everyone who wanted your second choice school as their first choice. You can fall even farther back in line as you get bumped down to your third, fourth and fifth choices.

- **Evidence from Education Literature**:
  Glenn [PI 1991] states

  > As an example of how school selections change, analysis of first-place preferences in Boston for sixth-grade enrollment in 1989 (the first year of controlled choice in Boston) and 1990 shows that the number of relatively popular schools doubled in only the second year of controlled choice. The strong lead of few schools was reduced as others "tried harder."

  Overly optimistic scenario!

  More plausible scenario: Learning

- **Evidence from BPS School Guide**:

  For a better choice of your "first choice" school . . . consider choosing less popular schools.

Elimination of Justified Envy

- **Elimination of Justified Envy:** There should be no unmatched student-school pair i, s where student i prefers school s to her assignment and she has higher priority than some other student who is assigned a seat at school s.

- Boston mechanism does not eliminate justified envy: Priorities are lost unless school ranked as top choice.

- Balinski & Sönmez *JET* (1999): If *elimination of justified envy* is plausable, then Gale-Shapley Student Optimal Stable Mechanism is the big winner!

Gale-Shapley Student Optimal Stable Mechanism

It's outcome can be found with the *student-proposing deferred acceptance algorithm* (Gale & Shapley *AMM* 1962):

Step 1: Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.

In general, at

Step k: Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected.

Equilibria of the Boston Mechanism

- **Theorem** (Ergin & Sönmez *JPubE* in press): The set of Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism is equal to the set of stable matchings of the associated college admissions game under the true preferences.
- **Corollary**: The dominant-strategy equilibrium outcome of the Gale-Shapley student optimal stable mechanism either Pareto dominates or equal to the Nash equilibrium outcomes of the Boston mechanism.
- The preference revelation game induced by the Boston mechanism is a "coordination game" among large numbers of parents in which there is incomplete information. So it is unrealistic to expect to reach a Nash equilibrium in practice.

Efficiency Cost of Elimination of Justified Envy

Example (Roth *MOR* 1982): There are three students i_1, i_2, i_3 and three schools s_1, s_2, s_3, each of which has only one seat.

The priorities of schools and the preferences of students are as follows:

$$s_1 : i_1 - i_3 - i_2 \qquad i_1 : s_2 - s_1 - s_3$$

$$s_2 : i_2 - i_1 - i_3 \qquad i_2 : s_1 - s_2 - s_3$$

$$s_3 : i_2 - i_1 - i_3 \qquad i_3 : s_1 - s_2 - s_3$$

Only one matching that eliminates justified envy:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

Pareto dominated by:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$$

Back to TTC

Step 1:

- Assign a *counter* for each school which keeps track of how many seats are still available at the school. Initially set the counters equal to the capacities of the schools.
- Each student "points to" her favorite school. Each school points to the student who has the highest priority.
- There is at least one cycle. Every student in a cycle is assigned a seat at the school she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed. Counters of all other schools stay put.

Step k: Repeat Step 1 for the remaining "economy."

Efficiency & Strategy-Proofness

- TTC simply trades priorities of students among themselves starting with the students with highest priorities.
- TTC inherits the plausible properties of Gale's TTC:
- **Theorem** (Abdulkadiroğlu & Sönmez *AER* 2003): The TTC mechanism is Pareto efficient.
- **Theorem** (Abdulkadiroğlu & Sönmez *AER* 2003): The TTC mechanism is strategy-proof.

Recent Developments

- NYC (Abdulkadiroğlu, Pathak & Roth *AER P&P* 2005): Gale-Shapley Student Optimal Stable Mechanism adopted in Fall 2003 with Al Roth's leadership. Hybrid between college admissions and school choice with strategic schools.

- Boston (Abdulkadiroğlu, Pathak, Roth & Sönmez *AER P&P* 2005): TTC had a head start but Superintendent Payzant recently recommended Gale-Shapley Student Optimal Stable Mechanism. Parag Pathak will tell more this afternoon.

- Chicago: Experimenting with the idea of adopting a centralized strategy-proof mechanism.