BOUNDARY TRAIN TRACKS OF LAMINAR BRANCHED SURFACES

TAO LI

1. Introduction

Essential laminations are codimension one objects in 3-manifolds that generalize both incompressible surfaces and taut foliations. It has been shown in [9] that if a closed orientable 3-manifold contains an essential lamination, then its universal cover is $\mathbb{R}^3$. Many 3-manifolds contain essential laminations. For example, the Property P conjecture has been proved for alternating knots [2, 13] and arborescent knots [17] by constructing essential laminations in knot manifolds. Most of the constructions of essential laminations were given by constructing a branched surface first and then showing that it carries useful essential laminations. In [10], it is shown that if a branched surface satisfies some standard conditions and it contains no sink disk, then it fully carries an essential lamination, and (except for the case of a lamination of T^3 by planes) any essential lamination is carried by such a branched surface. The conditions in [10] are easy to check in general, but in many situations we want to put some restrictions on the essential laminations. In particular, if we have a manifold with torus boundary and a properly embedded branched surface (with boundary), a natural question is whether this branched surface carries a lamination whose boundary is a union of circles with slope s. This question is interesting because if it carries such a lamination, then the lamination extends to a lamination in the manifold obtained by Dehn filling along the slope s. The next question is whether this extended lamination is essential in the closed manifold after Dehn filling. In this paper, we prove the following theorem which says that given a branched surface with certain easy-to-check conditions, for any slope realizable by its boundary train track, we always get an essential lamination in the manifold after Dehn filling (along this slope).

Theorem 2.2. *Let M be an irreducible and orientable 3-manifold whose boundary is an incompressible torus. Suppose B is a laminar branched surface and $\partial M - \partial B$ is a union of bigons. Then, for any rational slope $s \in \mathbb{Q} \cup \infty$ that can be realized by the train track ∂B, if B does not carry a torus that bounds a solid torus in $M(s)$, $M(s)$ contains an essential lamination.*

The branched surfaces (with boundary) constructed in [12, 13, 14, 15] all satisfy the hypotheses in Theorem 2.2 (after a simple splitting in a special case), and the interval of slopes computed in [12, 13, 14, 15] using affine laminations are in fact the exact intervals of realizable slopes as described in Theorem 2.2. So, Theorem 2.2 can simplify their proofs. In section 3, we also give a way to calculate realizable slopes by splitting and simplifying train tracks.

As an application, we show that:

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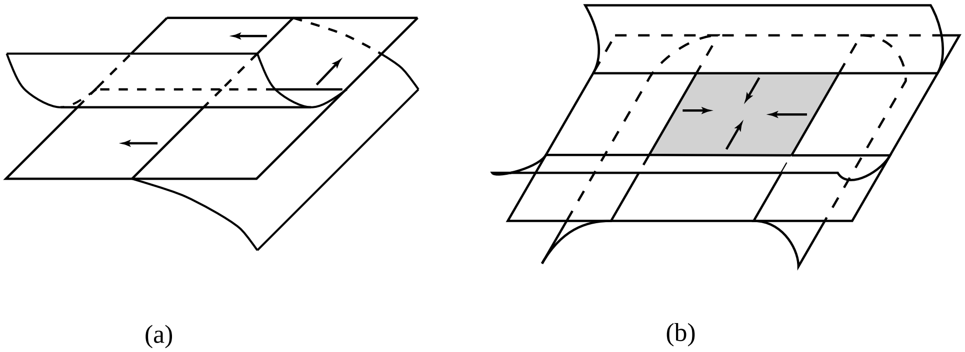


FIGURE 2.1

Theorem 4.1. *Let K be a knot in S^3 and R be a minimal genus Seifert surface. If R can be decomposed as a $2n$-Murasugi sum, then $K(s)$ has a taut foliation for any slope $s \in (1-n, n-1)$, where $K(s)$ is the closed 3-manifold after Dehn surgery along the slope s.*

The following is an easy corollary to Theorem 4.1. It says that, except for one case (i.e. the case when the Murasugi sum is a plumbing a trivial twisted band and a nonplanar surface), such a knot has property P. In section 4, we also discuss a construction of essential laminations for this exceptional case using Theorem 2.2.

Corollary 4.2. *Let K be a knot in S^3 and R be a minimal genus Seifert surface of K. Suppose R can be decomposed as a $2n$-Murasugi sum. Then, unless the Murasugi sum is a plumbing of a trivial twisted band and a nonplanar surface, K has property P. In particular, if $n \neq 2$, K has property P.*

Notation. In this paper, we denote a small regular neighborhood of X by $N(X)$, and denote the interior of X by $int(X)$.

2. LAMINAR BRANCHED SURFACES WITH BOUNDARY

In this section, we first explain the relation between essential laminations and branched surfaces without sink disks, which is the main result in [10]. Then, we consider branched surfaces in a 3-manifold with torus boundary. We show that if a branched surface does not contain any (half) sink disk and satisfies some standard conditions, then the question of whether this branched surface carries a lamination that extends to an essential lamination in the manifold after Dehn filling totally depends on its boundary train track.

Let B be a branched surface and L be the branch locus of B. We call the closure (under path metric) of each component of $B - L$ a *sector* of B. L is a collection of smooth immersed curves in B. Let Z be the union of double points of L. We associate with every component of $L - Z$ a normal vector (in B) pointing in the direction of the cusp, as shown in Figure 2.1. We call it the *branch direction* of this arc. We call a disk sector of B a *sink disk* if the branch direction of every smooth arc (or curve) in its boundary points into the disk. See Figure 2.1 for an example.

Note that if we "blow an air bubble" into the interior of a sink disk, it will destroy the sink disk by definition, but nothing really changes. So, we need to eliminate these trivial "air bubbles" first in the branched surface.

Let X be an orientable, irreducible and closed 3-manifold and B be an incompressible branched surface in X. Let $N(B)$ be a fibered neighborhood of B and K be a component of $M - int(N(B))$. If K is a 3-ball, since B is incompressible, we can give K a fiber structure $D^2 \times I$ with $D^2 \times \partial I \subset \partial_h N(B)$ and $\partial D^2 \times I \subset \partial_v N(B)$. We call K a $D^2 \times I$ *region*

Definition 2.1 ([10])**.** Let D_1 and D_2 be the two disk components of the horizontal boundary of a $D^2 \times I$ region K in $M - int(N(B))$. Hence, D_1 and D_2 are also two disk components of $\partial_h N(B)$ and $D_1 \cup D_2 = D^2 \times \partial I \subset \partial K$. Thus, $\pi(\partial D_1) = \pi(\partial D_2)$ is a circle in the branch locus L, where $\pi : N(B) \to B$ is the collapsing map. If π restricted to the interior of $D_1 \cup D_2$ is injective, i.e., the intersection of each I-fiber of $N(B)$ with $int(D_1) \cup int(D_2)$ is either empty or a single point, then we call K a *trivial* $D^2 \times I$ *region*, and we say that $\pi(D_1 \cup D_2)$ forms a *trivial bubble* in B ($\pi(D_1 \cup D_2)$ bounds a 3-ball in $M - B$).

If K is a trivial $D^2 \times I$ region as above, then $B \cup K$ is a fibered neighborhood of another branched surface $\bar{B}$ which can be obtained by collapsing the trivial bubble. This operation of collapsing a trivial bubble does not change properties of B, since any surface or lamination is carried by B if and only if it is carried by $\bar{B}$. Note that not all $D^2 \times I$ regions are trivial. For example, the $D^2 \times I$ region in the simplest Reeb branched surface is not trivial (its top is glued to its bottom). As we pointed out before, because of the combinatorial nature of the definition of sink disk, we would like to eliminate these trivial bubbles first.

Definition 2.2 ([10])**.** Let B be a branched surface in N and suppose we have eliminated all the trivial bubbles. We call B a *laminar branched surface* if B is incompressible and B contains no sink disk.

The following theorem is proved in [10], it gives some kind of equivalence relation between essential laminations and branched surfaces without sink disks.

Theorem 2.1 ([10])**.** *Suppose N is a closed and orientable 3-manifold. Then*

(a) *Every laminar branched surface in N fully carries an essential lamination.*
(b) *Any essential lamination in N that is not a lamination by planes is fully carried by a laminar branched surface.*

Furthermore, if $\lambda \subset N$ is a lamination by planes (hence $M = T^3$), then any branched surface carrying λ must contain a sink disk and hence is not a laminar branched surface.

In this paper, we are interested in obtaining a closed manifold through Dehn surgery, and we would like to see that, given a branched surface with boundary, whether this branched surface carries useful laminations. We first define the relative version of sink disk and trivial bubble.

Let M be an irreducible 3-manifold whose boundary is an incompressible torus. Suppose B is a properly embedded branched surface with boundary ($\partial B \subset \partial M$). Let L be the branch locus of B, and D be the closure (in the path metric) of a disk component of $B - L$. We call D a *half sink disk* if $\partial D \cap \partial M \neq \emptyset$ and the branch direction of each arc in $\partial D - \partial M$ points into D. Note that $\partial D \cap \partial M$ may not be connected.

Now, we define the relative version of trivial $D^2 \times I$ regions and trivial bubbles in the same way as before. Let B be an incompressible branched surface properly embedded in a 3-manifold M ($B \cap \partial M = \partial B$), and let $N(B)$ be a fibered neighborhood of B. Suppose K is a component of $M - int(N(B))$ and K admits a fiber structure $D^2 \times I$ such that $D^2 \times \partial I = K \cap \partial_h N(B)$, and $\partial D^2 \times I = K \cap (\partial M \cup \partial_v N(B))$ with the same fiber structure as that of $\partial_v N(B)$ at $(\partial D^2 \times I) \cap \partial_v N(B)$. Let D_1 and D_2 be the two components of $D^2 \times \partial I$. If π restricted to the interior of $D_1 \cup D_2$ is injective, i.e., the intersection of any I-fiber of $N(B)$ with $int(D_1) \cup int(D_2)$ is either empty or a single point, then we call K a *trivial* $D^2 \times I$ *region*, and we say that $\pi(D_1 \cup D_2)$ forms a *trivial bubble* in B. As before, a trivial bubble can be eliminated by pinching B.

Definition 2.3. Let B is a properly embedded branched surface in a 3-manifold M. We say B is a *laminar branched surface* if, after eliminating trivial bubbles, the followings are satisfied.

1. $\partial_h N(B)$ is essential in the following sense: $\partial_h N(B)$ is incompressible and ∂-incompressible in $M - int(N(B))$, there is no monogon in $M - int(N(B))$ and no component of $\partial_h N(B)$ is a sphere or a disk properly embedded in M.
2. $M - int(N(B))$ is irreducible and $\partial M - int(N(B))$ is incompressible in $M - int(N(B))$.
3. B contains no Reeb branched surface (see [9] for the definition of Reeb branched surface).
4. B has no sink disk or half sink disk.

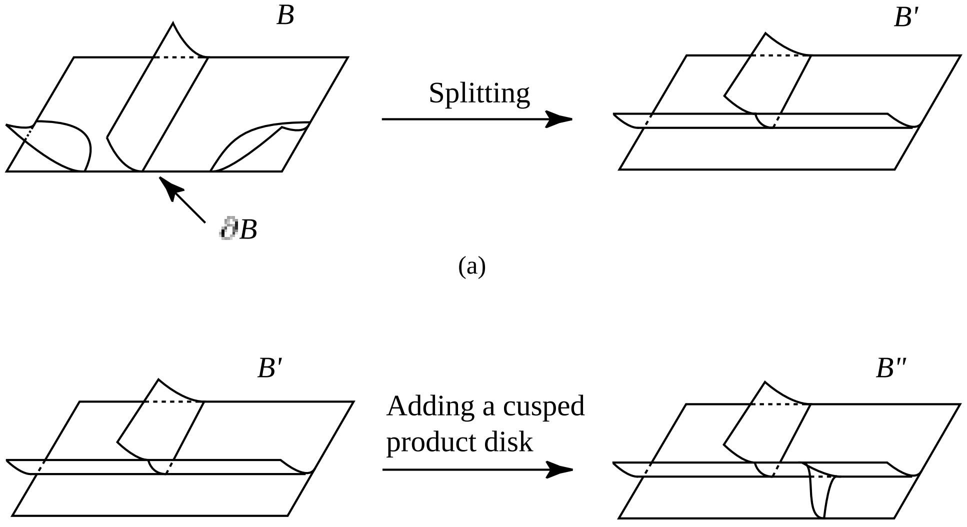
Let T be a torus, then the isotopy class of every nontrivial simple closed curve in T uniquely corresponds to a rational slope in $\mathbb{Q} \cup \infty$. Let τ be a train track in T and $s \in \mathbb{Q} \cup \infty$ be a rational slope. We say that s can be *realized* by τ if τ fully carries a union of simple closed curves with slope s, i.e., we can split τ into a union of simple closed curves with slope s. For any slope $s \in \mathbb{Q} \cup \infty$, we denote by $M(s)$ the manifold after Dehn filling along s.

Theorem 2.2. *Let M be an irreducible and orientable 3-manifold whose boundary is an incompressible torus. Suppose B is a laminar branched surface and $\partial M - \partial B$ is a union of bigons. Then, for any rational slope $s \in \mathbb{Q} \cup \infty$ that can be realized by the train track ∂B, if B does not carry a torus that bounds a solid torus in $M(s)$, $M(s)$ contains an essential lamination.*

Remark 2.1.

1. Since ∂M is a torus and $\partial_h N(B)$ is essential as in definition 2.3, by a standard index argument, $\partial M - \partial B$ must be a union of bigons and annuli. If $\partial M - \partial B$ has an annular component, ∂B can realize at most one slope which is the slope of the boundary of this annular component.
2. Note that if B carries a torus, then this torus must be incompressible in M. Thus, if $int(M)$ admits a hyperbolic structure and B does not carry a ∂-parallel torus (which is easy to check in practice), then B does not carry any torus at all and the last requirement on B in Theorem 2.2 is satisfied.

Proof. If s can be realized by ∂B, s can also be realized by the boundary train track of the new branched surface after we collapse a trivial bubble. Thus, we can assume that B has no trivial bubble. Now, we split B in a small neighborhood of ∂B, as shown in Figure 2.2 (a), such that ∂B becomes a union of simple closed curves with slope s. Note that during the splitting, no new component of $\partial_v N(B)$ is created (but two components of $\partial_v N(B)$ may merge and become one component in the new branched surface after splitting). We denote the branched surface after splitting by B'.



(a)

(b)

FIGURE 2.2

Now, we can do Dehn filling along the slope s by adding a disk along each circle in $\partial B'$ and capping off the 2-spheres. We denote the branched surface in $M(s)$ obtained from B' as above by B_s. By Theorem 2.1, it suffices to show that B_s is a laminar branched surface in the closed manifold $M(s)$.

Let D be a product disk in $M - int(N(B'))$ with one vertical boundary arc in ∂M and the other vertical boundary arc in $\partial_v N(B)$, i.e., D is a compressing disk for $M - int(N(B'))$ such that $\partial D \cap \partial_v N(B')$ is a vertical arc in $\partial_v N(B')$ and $\partial D \cap \partial M$ is a vertical arc in an annular component of $\partial M - int(N(B'))$. Then, $\pi(D)$ is a triangle with vertices A, B, C such that $A, B \in \partial B'$ and $C \in L$, where L is the branch locus of B and $\pi : N(B') \to B'$ is the collapsing map. Note that the edge $\overline{AB}$ is an embedded vertical arc of (the closure of) an annular component of $\partial M - \partial B'$ and $int(D)$ is embedded (edges $\overline{AC}$ and $\overline{BC}$ may cross each other on B' from different sides). We call such a disk $\pi(D)$ a *cusped product disk*. If Δ is a cusped product disk, we can slide Δ at $\partial\Delta$ to make $B' \cup \Delta$ a new branched surface, as shown in Figure 2.2 (b). Since the effect of the splitting from B to B' on the complement of B is connecting the components of $M - B$ by a product region near ∂M, adding appropriate cusped product disks to B' (and modifying them to form a branched surface B'' as above) can be consider as the inverse of the splitting from B to B'. More precisely, we can add a set of cusped product disks to B' to form a branched surface B'' such that each component of $M - B''$ is isotopic (in M) to a corresponding component of $M - B$, as illustrated in Figure 2.2. We denote this set of cusped product disks as above by $\mathcal{D} = \{D_1, \ldots, D_k\}$.

Suppose a region P in $M - int(N(B'))$ becomes a $D^2 \times I$ region in $M(s) - int(N(B_s))$ after the Dehn filling. Since $\partial M - \partial B$ is a union of bigons, P is cut by the cusped product disks in $\mathcal{D}$ as above into a union of $D^2 \times I$ regions in $M - int(N(B''))$. In other words, there is a union of $D^2 \times I$ regions $K_1, \ldots, K_n$ in $M - int(N(B))$ that are connected and become the region P after the splitting from

B to B'. Since we have assumed that B has no trivial bubbles, by the definition of trivial $D^2 \times I$ region, there must exist an I-fiber in $N(B)$ with both endpoints in $int(D^2) \times \partial I \subset \partial K_1$ (here we identify K_1 with $D^2 \times I$). As the splitting from B to B' happens in a small neighborhood of ∂M, there must be an I-fiber in $N(B')$ with both endpoints in the horizontal boundary of P. Hence, after the Dehn filling, P cannot produce a trivial $D^2 \times I$ region. So, there is no trivial $D^2 \times I$ region in $M(s) - int(N(B_s))$.

Since the splitting from B to B' happens in a small neighborhood of ∂M, as shown in Figure 2.2(a), any sink disk in B_s must come from a half sink disk in B. As B has no sink disk or half sink disk, B_s contains no sink disk.

Next, we show that $\partial_h N(B_s)$ is incompressible in $M(s) - int(N(B_s))$. Since B is incompressible and B' is obtained by splitting B, by [9], B' is also incompressible in M (this can also be proved using the argument below). Hence, there is no compressing disk for $\partial_h N(B')$ in $M - int(N(B'))$. Suppose there is a compressing disk for $\partial_h N(B_s)$, then there is a planar surface Q properly embedded in $M - int(N(B'))$ with one boundary component (which we call the outer boundary) in $\partial_h N(B')$ and other boundary components (which we call the inner boundary components) essential simple closed curves in the annuli $\partial M - N(B')$. After the Dehn filling, Q becomes a compressing disk for $\partial_h N(B_s)$. The outer boundary of Q is an essential simple closed curve in $\partial_h N(B_s)$.

As $\partial M - \partial B$ is a union of bigons, the cusped product disks in $\mathcal{D}$ cut Q into a union of disks. Let $D \in \mathcal{D}$ be such a cusped product disk that intersects Q. We can assume D is transverse to Q and $D \cap Q$ contains no simple closed curves (by a standard cutting and pasting argument). Let A, B, C be the three vertices of D with edge $\overline{AB} \subset \partial M$ and $C \in L$. We call $\alpha \subset D$ a ∂-parallel arc if both endpoints of α lie in the same edge. Suppose there is an arc $\alpha \subset D \cap Q$ with both endpoints in the same edge $\overline{AC}$. We can assume α is outermost in D, i.e., α and a subarc of $\overline{AC}$ bound a subdisk Δ in D that contains no other intersection arcs. We cut Q along α and then glue two parallel copies of Δ to it. Note that in Q, α is an arc with both endpoints in the outer boundary of Q. So, the surgery above yields two planar surfaces. As the outer boundary of Q is essential in $\partial_h N(B_s)$, the outer boundary of at least one of the two resulting planar surfaces must be essential in $\partial_h N(B_s)$. Thus, by choosing Q to be a planar surface with the least intersections with disks in $\mathcal{D}$, we can assume that $Q \cap D$ contains no ∂-parallel arc to edges $\overline{AC}$ and $\overline{BC}$. If there is an intersection arc with both endpoints in $\overline{AB}$, we can do a surgery as above and get a planar surface with fewer inner boundary components. Therefore, we can assume that $Q \cap D$ contains no ∂-parallel arc in D for any $D \in \mathcal{D}$.

Let γ be an inner boundary component of Q. So, γ lies in an annulus H of $\partial M - int(N(B'))$. We may choose γ to be outermost in H, i.e., γ and a component of ∂H, which we denote by β, bound an annulus in H that contains no other components of ∂Q. Since $\partial M - \partial B$ is a union of bigons, there must be a cusped product disk D in $\mathcal{D}$ such that $\gamma \cap \partial D \neq \emptyset$. By our construction, $\gamma \cap \partial D$ is a single point. Suppose A, B, C are the three vertices of D as before with edge $\overline{AB} \subset \partial M$, $A \in \beta$ and $C \in L$. Let $\alpha' \subset D \cap Q$ be the intersection arc containing $\gamma \cap \partial D$, and by choosing an appropriate γ, we can assume that the other endpoint of α' is on the edge $\overline{AC}$ (note that $A \in \beta$ as above). By our choice of β, α' must be outermost in D, i.e., α' cuts off a small triangle Δ from D containing $A \in \beta$ as a vertex and Δ contains no other arcs in $Q \cap D$. Now, we can isotope Q in $M - int(N(B'))$ near the annulus H by pushing γ towards β and eventually isotoping γ off ∂M so that γ becomes a circle in $\partial_h N(B')$ parallel to β. After this isotopy, α' becomes a ∂-parallel arc in D with both endpoints in the edge $\overline{AC}$. Hence, we can do surgery on Q as before to get another planar surface with fewer boundary components. Since the outer boundary of Q is an essential curve in $\partial_h N(B_s)$ and β is trivial in $\partial_h N(B_s)$, after this surgery, the outer boundary of the new planar surface is nontrivial in $\partial_h N(B_s)$. Therefore, by choosing Q to have the least intersection with disks in $\mathcal{D}$, Q must have no inner boundary, i.e., Q is a compressing disk for $\partial_h N(B')$ in M, which contradicts that B' is an incompressible branched surface in M. Therefore, $\partial_h N(B_s)$ is incompressible in $M(s)$.

By the same argument as above, it is easy to see that $M(s) - int(N(B_s))$ is irreducible and there is no monogon in $M(s) - B_s$.

Now, by Theorem 2.2, we only need to check that B_s does not contain any Reeb branched surface, i.e., B_s does not carry a sublamination of a Reeb foliation of a solid torus. Suppose B_s contains a Reeb branched surface. Then, there is a torus T embedded in $N(B_s)$ and transverse to the I-fibers of $N(B_s)$ such that T bounds a solid torus V in $M(s)$ and $N(B_s) \cap V$ fully carries a sublamination of Reeb foliation. Let U be the solid torus glued to ∂M during the Dehn filling. By the construction of B_s, $U \cap B_s$ is a union of meridional disks of U. By our hypothesis that B does not carry a torus that bounds a solid torus in $M(s)$, $U \cap T \neq \emptyset$. Since B_s has no disk of contact and since every leaf (except for the boundary torus) in a Reeb foliation is a plane, $V - int(N(B_s))$ must be a union of $D^2 \times I$ regions.

Let K be a $D^2 \times I$ region in $M(s) - int(N(B_s))$ that intersects U, and let Y be a component of $U \cap K$. We denote the core of U by C (U is considered as a small tubular neighborhood of this simple closed curve C) and let η be the component of $C \cap Y$. Since $\partial M - \partial B$ is a union of bigons, there must be a cusped product disk in $\mathcal{D}$ that intersects ∂Y and the intersection of Y with this cusped product disk must be isotopic to η in Y. Since K is a $D^2 \times I$ region, η must be a vertical arc, in other words, η is isotopic to an I-fiber of K. Therefore, $K \cap U$ is a union of vertical cylinders for each $D^2 \times I$ region K, i.e., after some isotopy, every component of $K \cap U$ is a cylinder consisting of I-fibers in the $D^2 \times I$ structure of K. If we extend the lamination in V to a Reeb foliation of the solid torus by adding plane leaves to fill out all the $D^2 \times I$ regions in V, as the intersection of C with each $D^2 \times I$ region is a union of vertical arcs, we can assume each arc in $C \cap V$ is transverse to the Reeb foliation. However, this is impossible since there is no compact arc properly embedded in a solid torus and transverse to the Reeb foliation. Thus, B_s contains no Reeb branched surface and hence B_s fully carries an essential lamination in $M(s)$ by [10]. □

Theorem 2.2 reduces the question of finding a lamination (carried by branched surface B) that extends to an essential lamination in the closed manifold after Dehn fillings to the question of calculating slopes that can be realized by the boundary train track of B. In the following section, we discuss realizable slopes for some simple train tracks and how to reduce a complicated train track to simple ones. These discussions will be used in section 4.

3. Boundary train tracks

Let T be a torus and τ be a train track in T. The nonmanifold points of τ are called *switches*. Let D be the closure (under the path metric) of a component D_0 of $T - \tau$. We say that D is an *embedded bigon* if D is a bigon and the closure

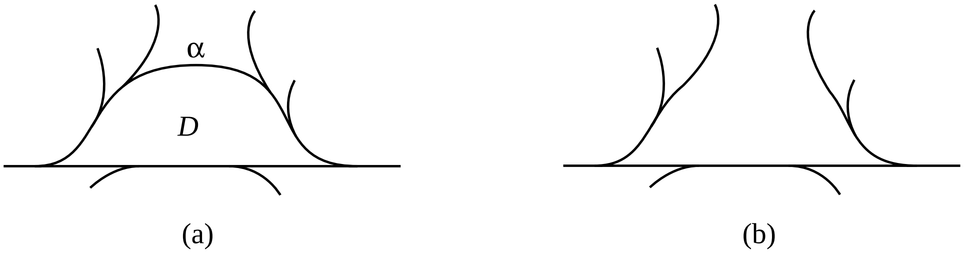


FIGURE 3.1

of D_0 in T is the same as D, in other words, there is no arc in τ which bounds D_0 on both sides. Let Z be the union of switches in τ. Then, each endpoint of (the closure of) an arc γ in $\tau - Z$ has a cusp direction pointing into or out of γ similar to the branch direction for branched surfaces. We call (the closure under path metric of) an arc γ in $\tau - Z$ a *sink arc* if the cusp directions of both endpoints of γ point into γ. We call D a *trivial bigon* if D is an embedded bigon and there is no sink arc in at least one boundary edge of D. Note that if D is a trivial bigon and the boundary edge η_1 of D contains no sink arc then there must be an arc α in $\eta_1 - Z$ with cusp directions of both endpoints pointing out of α, as shown in Figure 3.1(a). Moreover, the cusp directions at the switches in $\eta_1 - \alpha$ are coherent in each component, and $\tau - \alpha$ is a sub train track of τ as shown in Figure 3.1(b).

Lemma 3.1. *Let τ be a train track in a torus T and D be a trivial bigon as above. Let η_1 and η_2 be the two boundary edges of D and suppose η_1 contains no sink arc. Let α be the arc in $\eta_1 - Z$ whose cusp directions at both endpoints point out of α as above. Then, a slope can be realized by τ if and only if it can be realized by the train track $\tau - \alpha$.*

Proof. Suppose τ fully carries a lamination μ with slope s. Since D is an embedded bigon, we can pinch those arcs in μ that are carried by α to the edge η_2. Thus, $\tau - \alpha$ also fully carries μ.

Suppose μ is a lamination fully carried by $\tau - \alpha$. We can isotope some arcs carried by the edge η_2 across this bigon D to the edge η_1. By taking two parallel copies of μ (when μ consists of finitely many curves) if necessary, we can assume η_2 still carries some arcs in μ after this isotopy, i.e., τ fully carries a lamination that is either μ or two copies of μ. Hence, the slope of μ can also be realized by τ. ☐

If $T - \tau$ is a union of bigons, we can always split τ (in different ways) and apply Lemma 3.1 to get several simple train tracks without losing realizable slopes. Next, we calculate the sets of realizable slopes for some train tracks that we will use in section 4.

Example 1. Let α be a simple closed curve with slope ∞ and β be a simple closed curve with slope 0 in a torus T. Suppose $\alpha \cap \beta$ is a single point O. Then, we can cut α (at O) into a compact arc and move the two endpoints of this arc apart a little along β. There are two ways to move the two endpoints apart as shown in Figure 3.2 (a, b). Then, we can deform them into train tracks as shown in Figure 3.2 (c, d, e, f). Then, by assigning a positive weight to each arc in the train track, one can easily calculate the slopes that can be realized by each of the four train tracks:

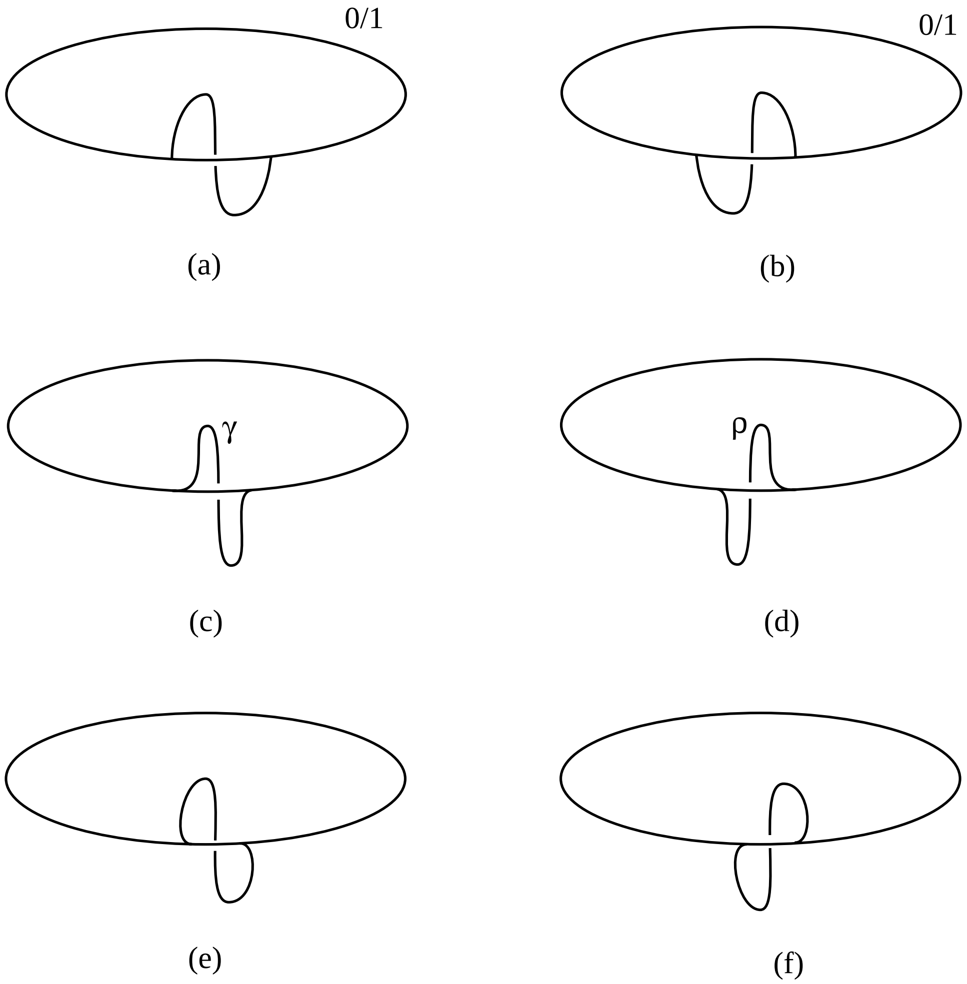


FIGURE 3.2

- Any slope in the interval $(0, 1)$ can be realized by the train track in Figure 3.2 (c).
- Any slope in the interval $(-1, 0)$ can be realized by the train track in Figure 3.2 (d).
- Any slope in the interval $(-\infty, 0)$ can be realized by the train track in Figure 3.2 (e).
- Any slope in the interval $(0, \infty)$ can be realized by the train track in Figure 3.2 (f).

Example 2. Suppose we have n copies of the arc γ in Figure 3.2 (c) forming a train track as shown in Figure 3.3 (a). Similar to example 1, by assigning positive weights to arcs in the train track, it is easy to see that the interval of realizable slopes for the train track in Figure 3.3 (a) is $(0, n)$. Note that this can also be calculated by splitting the train track and applying Lemma 3.1.

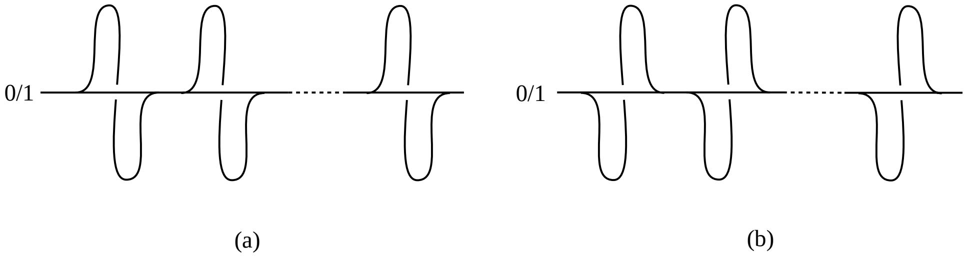


Figure 3.3

Suppose we have n copies of the arc ρ in Figure 3.2 (d) forming a train track as shown in Figure 3.3 (b). As above, the interval of realizable slopes for the train track in Figure 3.3 (b) is $(-n, 0)$.

Example 3. If we put the arcs γ and ρ in Figure 3.2 (c) and (d) together along the $0/1$ curve, we can form another train track as shown in Figure 3.4 (a). There are 3 ways to split this train track. The first splitting yields the train track in Figure 3.4 (b) and this train track only carries circles with slope 0. The second splitting yields the train track in Figure 3.4 (c), and the third splitting yields the train track in Figure 3.4 (d). Note that the train track in Figure 3.4 (d) is the same train track (on the torus T) as the one in Figure 3.4 (e) which contains a trivial bigon. By Lemma 3.1, the train track in Figure 3.4 (e) has the same interval of realizable slopes as the train track in Figure 3.4 (f), and by example 1, this interval is $(0, 1)$. Similarly, the interval of realizable slopes for the train in Figure 3.4 (c) is $(-1, 0)$. Thus, the interval of realizable slopes for the train track in Figure 3.4 (a) is $(-1, 1)$. Note that this interval can also be calculated easily by assigning a positive weight to each arc.

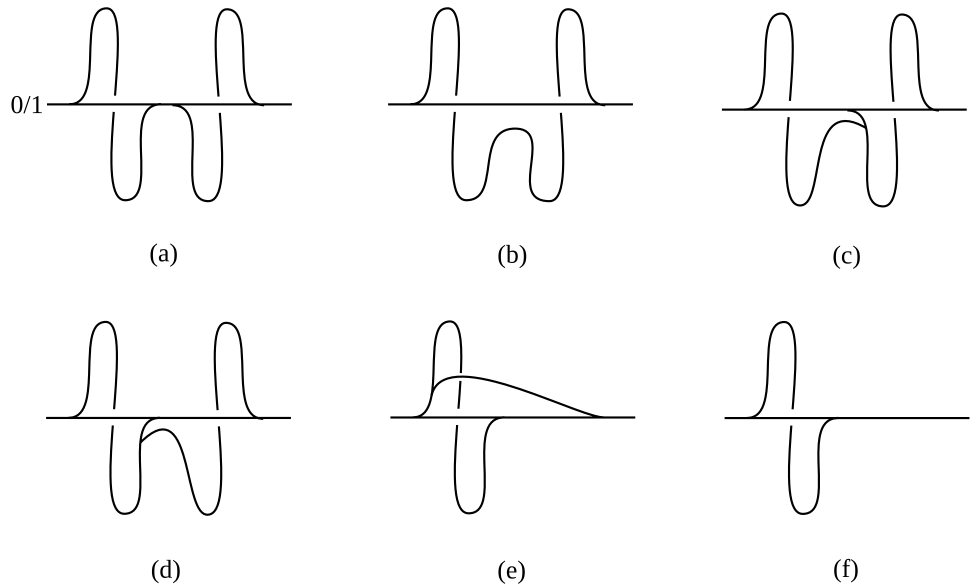
Example 4. If we put n copies of Figure 3.4 (a) along the $0/1$ curve, as shown in Figure 3.5, we can split each copy in the train track similarly. Hence, the interval of slopes realizable by the train track in Figure 3.5 is $(-n, n)$.

Example 5. The interval of realizable slopes for the train track in Figure 3.6 (a) is $(-1, \infty)$, and the interval of realizable slopes for the train track in Figure 3.6 (b) is also $(-1, \infty)$. This can be calculated either by splitting as in example 3 or by assigning a positive weight to each arc of the train track.

Example 6. We consider the train track in Figure 3.5 as the union of a simple closed curve C with slope 0 and $2n$ arcs $\alpha_1, \dots, \alpha_{2n}$. We say that an α_i is of positive type if $C \cup \alpha_i$ forms the train track in Figure 3.2 (c) (it can realize any slope in $(0, 1)$), and we say α_i is of negative type if $C \cup \alpha_i$ forms the train track in Figure 3.2 (d) (it can realize any slope in $(-1, 0)$). Let τ be a train track obtained by erasing k_1 arcs of positive type and k_2 arcs of negative type from the train track in Figure 3.5. Similar to the argument before, the interval of realizable slopes for τ is $(-n + k_2, n - k_1)$.

4. Murasugi sum

Let K be a knot in S^3 and R be a minimal genus Seifert surface. Suppose R can be decomposed into a $2n$-Murasugi sum of compact oriented surfaces R_1 and R_2,



(a) (b) (c)

(d) (e) (f)

FIGURE 3.4

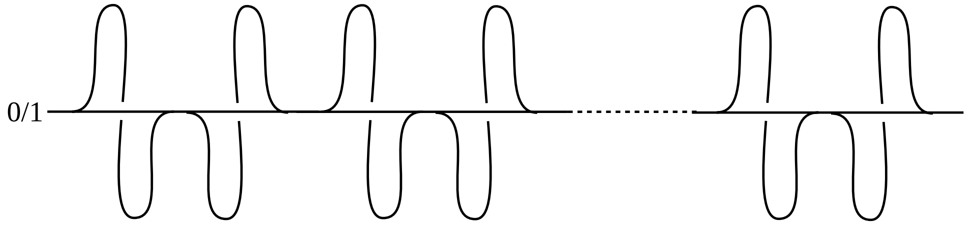
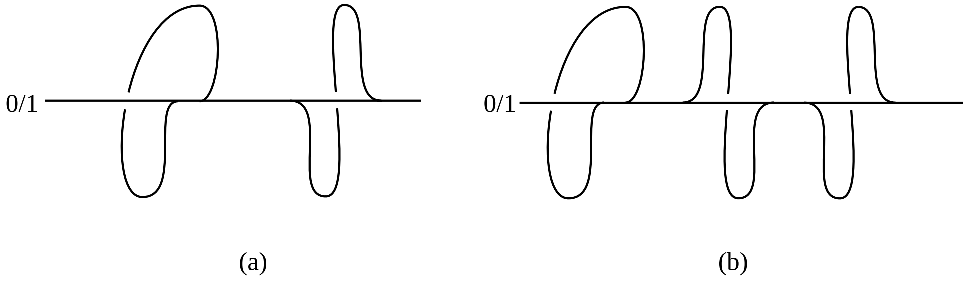


FIGURE 3.5



(a) (b)

FIGURE 3.6

i.e., there is a 2-sphere H in S^3 decomposing S^3 into two 3-balls B_1 and B_2 such that: $R = R_1 \cup R_2$, $R_i \subset B_i$ $(i = 1, 2)$, and $R_1 \cap H = R_2 \cap H = D$, where D is a $2n$-gon. See [3] for a picture of a Murasugi sum.

When $n = 1$, the Murasugi sum is known as connected sum, and when $n = 2$, the Murasugi sum is known as plumbing. It was proved in [3, 4] that R is a minimal genus Seifert surface for K if and only if R_i is a minimal genus Seifert surface for the link ∂R_i ($i = 1, 2$).

Theorem 4.1. *Let K be a knot in S^3 and R be a minimal genus Seifert surface for K. If R can be decomposed as a $2n$-Murasugi sum, then $K(s)$ has a taut foliation for any slope $s \in (1-n, n-1)$, where $K(s)$ is the closed 3-manifold obtained from Dehn surgery along the slope s.*

Proof. Let H be the 2-sphere and $D = R \cap H$ be the $2n$-gon as in the definition of Murasugi sum. Let $N(K)$ be a small tubular neighborhood of K such that $R \cap N(K)$ is an annular neighborhood of $K = \partial R$ and $H \cap N(K)$ consists of $2n$ meridional disks of the solid torus $N(K)$. To simplify notation, we still use R, D and H to denote $R - int(N(K))$, $D - int(N(K))$ and $H - int(N(K))$ respectively. Now, R is a surface properly embedded in $M = S^3 - int(N(K))$ with ∂S a circle of slope $0/1$.

By [3, 4, 5], $H - D$ gives rise to a taut sutured manifold decomposition. Hence, we can deform $H \cup R$ to a branched surface B such that $M - int(N(B))$ is a taut sutured manifold whose suture is the union of $\partial_v N(B)$ and $\partial M - int(N(B))$. B can be constructed by sliding the disk $H - int(D)$ away from D along R. The branched locus of B consists of $2n$ disjoint arcs with endpoints in ∂M (they correspond to the $2n$ edges of D). B has 4 sectors corresponding to D, $H - D$, $R_1 - D$ and $R_2 - D$. To simplify notation, we still use D, $H - D$, $R_i - D$ to denote the 4 sectors. Note that the sectors $R_1 - D$ and $R_2 - D$ have disjoint neighborhoods in M. The branch direction of L points into $R_i - D$ ($i = 1, 2$) and out of D and $H - D$. See Figure 3.2 in [5] for a picture.

Since $M - int(N(B))$ is a taut sutured manifold, B is an incompressible branched surface. Moreover, by the construction, B does not carry any closed surface and in particular B does not carry any torus. The boundary train track ∂B in the torus ∂M consists of ∂S and $2n$ arcs as shown in Figure 3.5.

Case 1. Neither $R_1 - D$ or $R_2 - D$ is a disk.

In this case, B contains no sink disk or half sink disk, and hence B is a laminar branched surface. As B does not carry a torus, by Theorem 2.2, for any slope s that is realizable by the train track ∂B, B fully carries a lamination λ whose boundary is a union of circles with slope s, and λ extends to an essential lamination in $M(s)$. Since the train track ∂B is as shown in Figure 3.5, by example 4, the slope s can be any rational slope in $(-n, n)$. Furthermore, this essential lamination extends to a taut foliation in $M(s)$ as follows. This basically follows from Gabai's sutured manifold decomposition.

As in the proof of Theorem 2.2, we split B along ∂B in a small neighborhood of ∂M and get a branched surface B' such that $\partial B'$ is a union of circles with slope s and λ is fully carried by B'. For each component Z of $M - int(N(B'))$, there is a union of product disks $\mathcal{D}$ that decompose Z into some components isotopic to corresponding components of $M - int(N(B))$. Since each component of $M - int(N(B))$ is a taut sutured manifold, by Lemma 3.12 in [6], $M - int(N(B'))$ is a taut sutured manifold with suture consisting of annuli $\partial_v N(B')$ and $\partial M - int(N(B'))$. Since K is a knot in S^3, by the thin position argument in [7], there is a taut sutured manifold hierarchy for $M - int(N(B'))$ which extends λ to a foliation whose boundary is a union of circles. Note that although the slope of the boundary suture in our case is not 0, since the slope cannot be ∞, the combinatorial argument in [7] also works

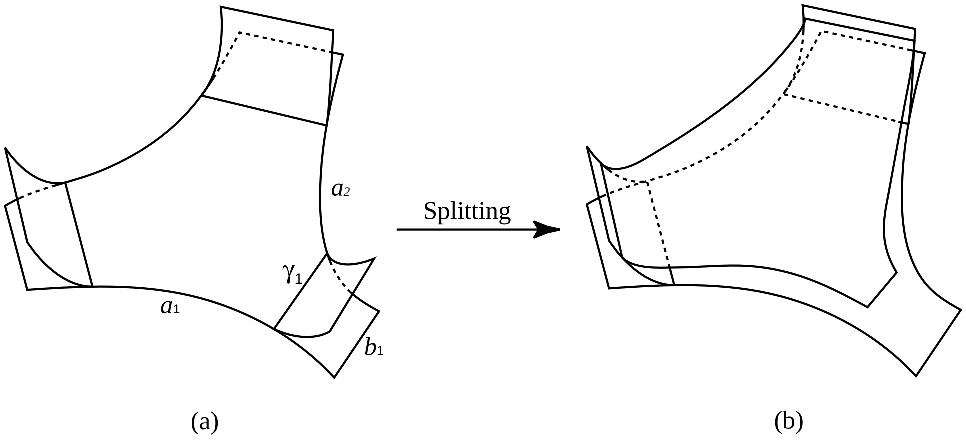


FIGURE 4.1

and we also get such a taut sutured manifold hierarchy [11], which leads to a taut foliation. A similar argument for this can also be found in Corollary 1.10 in [13].

Case 2. $R_1 - D$ is a disk but $R_2 - D$ is not a disk.

In this case, $R_1 - int(D)$ is a half sink disk. What we will do next is to split B to eliminate this half sink disk. Let N_1 be a small neighborhood of $R_1 - D$ in M, $D_1 = N_1 \cap R$ and $\hat{D}_1 = N_1 \cap B$. Thus, D_1 is a $2n$-gon with n edges in ∂M, and $\hat{D}_1$ is a branched surface consisting of the $2n$-gon D_1 and n "tails", as shown in Figure 4.1 (a). Let $a_1, a_2, \dots, a_n$ be n edges of D_1 that lie in ∂M and $b_1, b_2, \dots, b_n$ be the other n edges. The branched locus of $\hat{D}_1$ consists of n arcs, which we denote by $\gamma_1, \gamma_2, \dots, \gamma_n$. Each arc γ_j is an arc properly embedded in D_1 and parallel to b_j with endpoints in α_j and α_{j+1} ($\alpha_{n+1} = \alpha_1$). Then, we split $\hat{D}_1$ starting from γ_1, as shown in Figure 4.1, and we call this splitting a *standard splitting with respect to* γ_1. Let $\hat{B}$ be the branched surface after this splitting. Since $R_2 - D$ is not a disk, after this splitting, there is no half sink disk and hence $\hat{B}$ is a laminar branched surface.

Before this splitting, ∂B consists of a simple closed curve ∂R with slope 0 and $2n$ arcs $\alpha_1, \alpha_2, \dots, \alpha_{2n}$, as shown in Figure 3.5. As in example 6, we say that an α_i is of positive type if $\partial R \cup \alpha_i$ forms the train track in Figure 3.2 (c) (it can realize any slope in $(0, 1)$), and we say α_i is of negative type if $\partial R \cup \alpha_i$ forms the train track in Figure 3.2 (d) (it can realize any slope in $(-1, 0)$). If we split the train track ∂B by sliding an arc α_i of negative type over an arc of positive type as shown in Figure 3.4 (d), the resulting train track (as we pointed out in the Example 3) realize the same interval of slopes as the train track obtained by simply erasing this arc α_i from ∂B.

Now, we calculate the slopes that are realizable by the train track $\partial\hat{B}$. We need to see how the train track changes near $a_1, a_2, \dots, a_n$ during the splitting. Let α_i and α_j be the two arcs containing the two endpoints of γ_1 respectively in ∂B. Since R is orientable, one arc, say α_i, is of positive type and the other arc α_j is of negative type. As the two endpoints of γ_1 lie in a_1 and a_2 respectively, the effect of this spitting near a_1 and a_2 is sliding α_i and α_j over corresponding adjacent arcs as shown in Figure 3.4 (c, d). Thus, the train track after the splitting near a_1

and a_2 realize the same interval of slopes as the train track obtained by erasing α_i and α_j from ∂B. Moreover, near a_k for $k \neq 1, 2$, the effect of splitting B on the boundary train track is creating a trivial bigon, as shown in Figure 4.1. Thus, by Lemma 3.1, the interval of realizable slope does not change after this splitting near a_k for $k \neq 1, 2$. Therefore, the train track $\partial \hat{B}$ realizes the same interval of slopes as the train track obtained by erasing an arc α_i of positive type and an arc α_j of negative type, and hence by example 6, the interval of realizable slopes for $\partial \hat{B}$ is $(-(n-1), n-1)$.

Case 3. Both $R_1 - D$ and $R_2 - D$ are disks.

In this case, both $R_1 - int(D)$ and $R_2 - int(D)$ are half sink disks. Let N_i be a small neighborhood of $R_i - D$ in M, $D_i = N_i \cap R$ and $\hat{D}_i = N_i \cap B$ ($i = 1$ or 2). By choosing N_i small enough, we can assume that $\hat{D}_1$ and $\hat{D}_2$ are disjoint and as shown in Figure 4.1 (a). Now, we split $\hat{D}_1$ and $\hat{D}_2$ as follows.

We keep the same notation as in case 2, and describe ∂B as the union of ∂R and $2n$ arcs $\alpha_1, \alpha_2, \dots, \alpha_{2n}$. Suppose α_1 is of positive type and we denote the two endpoints of α_1 by x and y (x and y are two switches in the train track ∂B). Let γ_x and γ_y be the two components of the branch locus of B that contain x and y respectively. We can assume that $\gamma_x \subset \hat{D}_1$ and $\gamma_y \subset \hat{D}_2$. Since α_1 is of positive type, the arc α_j (resp. α_k) containing the other endpoint $\partial\gamma_x - x$ (resp. $\partial\gamma_y - y$) is of negative type. Now, similar to case 2, we perform standard splitting on $\hat{D}_1$ with respect to γ_x and standard splitting on $\hat{D}_2$ with respect to γ_y. The effect of the two splittings on ∂B near α_1, α_j and α_k is sliding the three arcs over adjacent arcs. Similar to case 2, the train track after the two splittings above realize the same interval of slopes as the train track obtained by erasing α_1, α_j and α_k from ∂B. Since α_1 is of positive type and α_j and α_k are of negative type, the interval of realizable slope is $(-(n-2), n-1)$ by example 6.

Then, we replace α_1 by an arc of negative type and repeat the splitting above to eliminate the two half sink disks. By the same argument, the boundary train track of the resulting branched surface realizes the same interval of slopes as the train track obtained by erasing one arc of negative type and two arcs of positive type from ∂B. Hence, the interval we get is $(-(n-1), n-2)$. By combining the two intervals together, we get that any slope in $(-(n-1), n-1)$ can be realized by an essential lamination carried by B. As in case 1, this essential lamination extends to a taut foliation. □

Corollary 4.2 is a quick application of Theorem 4.1 and some well-known results.

Corollary 4.2. *Let K be a knot in S^3 and R be a minimal genus Seifert surface of K. Suppose R can be decomposed as a $2n$-Murasugi sum. Then, unless the Murasugi sum is a plumbing of a trivial twisted band and a nonplanar surface, K has property P. In particular, if $n \neq 2$, K has property P.*

Proof. If $n = 1$, the Murasugi sum is a connected sum and the Property P for connected sum is well-known. So, we assume $n \neq 1$ and by Theorem 4.1 $K(s)$ has infinite fundamental group for any $s \in (1-n, n-1)$. If $n \neq 2$ (i.e., $n \geq 3$), $[-1, 1] \subset (1-n, n-1)$. As $K(s)$ is a homology sphere only if $s = \pm 1/m$ for some integer m, K has property P if $n > 2$.

Now we consider the case that $n = 2$, i.e., the minimal genus Seifert surface is a plumbing of R_1 and R_2. If R_i is not an annulus for both $i = 1, 2$, $R_i - D$ is not a disk and hence the branched surface B as above contains no sink disk or half sink disk. By case 1 in the proof of Theorem 4.1, $K(s)$ contains a taut foliation for any $s \in (-2, 2)$ and the property P for K follows.

If R_1 is an annulus, then a small neighborhood of R_1 can be considered as a tubular neighborhood of a knot k'. If k' is a nontrivial knot, then K is a satellite knot. Property P is known for satellite knots [8]. If K is not a satellite knot and R_1 is an annulus, then R_1 must be a trivial twisted band. Furthermore, if R_2 is also an annulus, then R is a plumbing of two trivial twisted bands and hence K is a 2-bridge knot. Property P for 2-bridge knots is also known [16, 12, 1]. Hence, unless R is a plumbing of a trivial twisted band and a nonplanar surface, K has property P. □

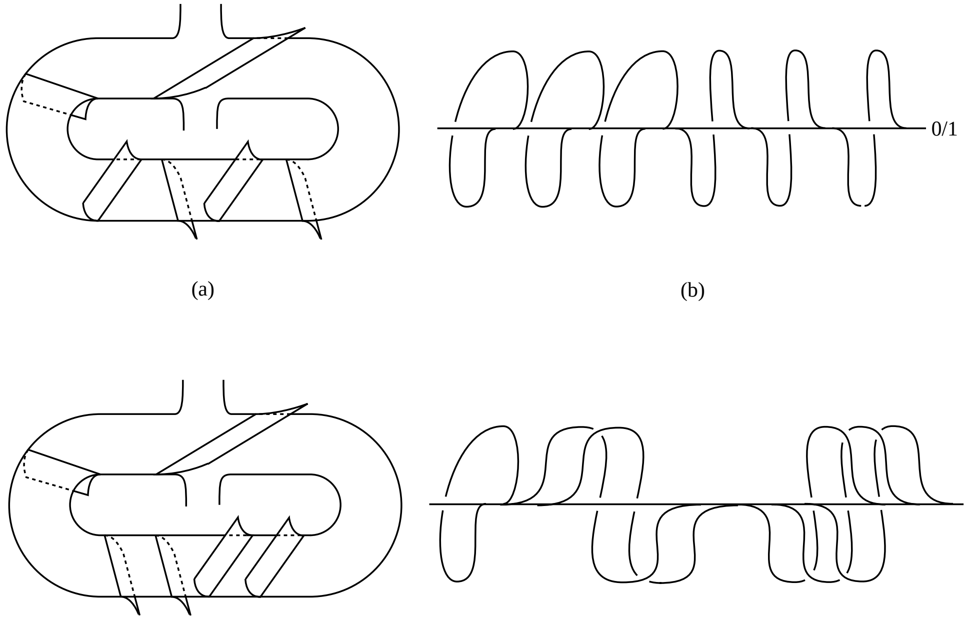
In the next theorem, we consider the case that R is a plumbing of a trivial twisted band and another surface. This theorem was first proved by Roberts (in a different situation) [13] using techniques of affine lamination. We give another proof here by splitting the branched surface and using Theorem 2.2.

Theorem 4.3. *Let K be a knot in S^3 and R be a minimal genus Seifert surface of K. Suppose R is a plumbing of two surfaces R_1 and R_2, and R_1 is a trivial twisted band. Then, depending on the direction of the twist for R_1, $K(s)$ has a taut foliation for any slope s in $(-\infty, 1)$ or $(-1, \infty)$.*

Proof. Let $L = \partial R_1$. We can regard R_1 as a surface properly embedded in $M_L = S^3 - int(N(L))$. Then, we cut M_L open along R_1 and get a manifold $T = M_L - int(N(R_1))$. Since R_1 is a trivial twisted band, T is a solid torus. The two sides of R_1 are two annuli in ∂T. Moreover, we can consider B_2 ($R_2 \subset B_2$) as a 3-ball in T whose intersection with ∂T is a square corresponding to the plumbing square. As T is a solid torus, there is a compressing disk E for T that does not intersect $int(B_2)$. So, we can view E as a properly embedded disk in $M - int(N(R))$, where $M = S^3 - int(N(K))$, and by [6], E gives rise to a taut sutured manifold decomposition. Hence, we can add E to R forming a branched surface B such that $M - int(N(B))$ is a taut sutured manifold. In fact, we can view E directly from the picture of R_1 (see Figure 11 in [13] for a picture). Suppose R_1 is a band with m full twists. Figure 4.2 (a) gives a schematic picture (in the case $m = 3$) of the branched surface $B = R \cup E$ near the annulus R_1 (see Figure 14 in [13] for a picture of B near ∂M). In particular, the branch locus of B is a union of disjoint arcs on R_1. The boundary train track of ∂B is as shown in Figure 4.2 (b).

If the number of twists $m > 1$, then B contains half sink disks on R_1 as shown in Figure 4.2(a) (with $m = 3$). Now, we can split B by moving E along R_1 changing B from Figure 4.2(a) to a branched surface as shown in Figure 4.2(c) which does not contain any half sink disk. This splitting changes the boundary train track from Figure 4.2(b) to Figure 4.2(d), and after this splitting we get a laminar branched surface. The boundary train track of Figure 4.2(d) contains many trivial bigons. After eliminating these trivial bigons as in Lemma 3.1, we get a train track in Figure 3.6 (b). Therefore, by Theorem 2.2, Lemma 3.1 and example 5, we get that, depending on the direction of the twist for R_1, $K(s)$ contains an essential lamination for any slope $s \in (-1, \infty)$ or $(-\infty, 1)$. As in the proof of Theorem 4.1, this lamination extends to a taut foliation.

If R_1 is a Hopf band, i.e., $m = 1$, then B contains no half sink disk and ∂B is the train track as in Figure 3.6 (a). In this case, we do not need to do any splitting and the theorem follows from the same argument as above. □



(a) (b)

(c) (d)

FIGURE 4.2

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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078
E-mail address: `tli@math.okstate.edu`