HEEGAARD SURFACES AND THE DISTANCE OF AMALGAMATION

TAO LI

Abstract. Let M_1 and M_2 be orientable irreducible 3–manifolds with connected boundary and suppose $\partial M_1 \cong \partial M_2$. Let M be a closed 3–manifold obtained by gluing M_1 to M_2 along the boundary. We show that if the gluing homeomorphism is sufficiently complicated, then M is not homeomorphic to S^3 and all small-genus Heegaard splittings of M are standard in a certain sense. In particular, $g(M) = g(M_1) + g(M_2) - g(\partial M_i)$, where $g(M)$ denotes the Heegaard genus of M. This theorem can also be extended to manifolds with multiple boundary components.

Contents

1. Introduction

One of the most useful ways of constructing a closed 3–manifold is to glue two 3–manifolds with boundary via a homeomorphism between their boundary surfaces. This construction is called amalgamation. Dehn filling and Heegaard splitting can be viewed as examples of such construction. In this paper, we study Heegaard splittings of closed 3–manifolds obtained by amalgamation. Like Dehn filling, the closed 3–manifold obtained by amalgamation depends on the gluing homeomorphism. We will show that if the gluing homeomorphism is sufficiently complicated, then the small-genus Heegaard splittings of the resulting closed 3–manifold are standard.

The complexity of the gluing homeomorphism is defined using curve complex. Let F be a closed orientable surface of genus at least 2. The curve complex of F, introduced by Harvey [5], is the complex whose vertices are the isotopy classes of essential simple closed curves in F, and $k+1$ vertices determine a k–simplex if they are represented by pairwise disjoint curves. If F is a torus, then the curve complex of F is defined to be the Farey graph. We denote the curve complex of F by $\mathcal{C}(F)$. For any two vertices in $\mathcal{C}(F)$, the distance $d(x, y)$ is the minimal number

---

Partially supported by an NSF grant.

of 1–simplices in a simplicial path jointing x to y. To simplify notation, unless necessary, we do not distinguish a vertex in $\mathcal{C}(F)$ from a simple closed curve in F representing this vertex.

Let M_1 and M_2 be orientable irreducible 3–manifolds with connected boundary. Suppose $\partial M_1 \cong \partial M_2 \cong F$. We can glue M_1 to M_2 via a homeomorphism $\phi : \partial M_1 \to \partial M_2$ and obtain a closed orientable 3–manifold $M = M_1 \cup_\phi M_2$. We may view M_1 and M_2 as submanifolds of M and $F = \partial M_1 = \partial M_2$ as a surface properly embedded in M.

Definition 1.1. Let M_1, M_2 and M be as above. If $F = \partial M_i$ is compressible in M_i, the *disk complex* of M_i is the set of vertices in $\mathcal{C}(F)$ represented by curves bounding compressing disks in M_i. If M_i is a twisted I–bundle over a non-orientable surface, the *annulus complex* of M_i is the set of vertices in $\mathcal{C}(F)$ represented by boundary curves of vertical annuli in M_i. If M_i has incompressible boundary and M_i is not a twisted I–bundle, we fix a properly embedded essential surface Ω_i in M_i with $\partial\Omega_i \neq \emptyset$ and suppose the Euler characteristic $\chi(\Omega_i)$ is maximal among all such essential surfaces. We define $\mathcal{U}_i$ to be the set of vertices in $\mathcal{C}(F)$ as follows,

$$\mathcal{U}_i = \begin{cases} \text{the disk complex of } M_i, & \text{if } \partial M_i \text{ is compressible} \\ \text{the annulus complex of } M_i, & \text{if } M_i \text{ is a twisted } I\text{–bundle} \\ \text{vertices in } \mathcal{C}(F) \text{ represented by } \partial\Omega_i, & \text{otherwise.} \end{cases}$$

We define the distance of the amalgamation to be $d(M) = d(\mathcal{U}_1, \mathcal{U}_2)$ in the curve complex $\mathcal{C}(F)$.

Note that the surface Ω_i in Definition 1.1 is not unique, but we will show in section 3 that, if M_i has incompressible boundary and is not a twisted I–bundle, then the diameter of the set of vertices in $\mathcal{C}(F)$ represented by boundary curves of small-genus essential surfaces is bounded. Thus any different choice of Ω_i only changes $d(M)$ by an explicit small number. If both M_1 and M_2 are handlebodies or more generally if both M_1 and M_2 have compressible boundary, then $d(M)$ is the same as the Hempel distance, see [5, 23]. Schleimer informed me that, similar to the disk complex, the annulus complex of a twisted I–bundle is also quai-convex in $\mathcal{C}(F)$. So $d(M)$ is arbitrarily large if the gluing map ϕ is a sufficiently high power of a pseudo-Anosov map. Like the Hempel distance, $d(M)$ also provides a natural complexity measure for a one-sided Heegaard splitting, i.e., a decomposition of M into a handlebody and a twisted I–bundle.

If one prefers, the following is a roughly equivalent way of defining $d(M)$ which does not involve a choice of Ω_i. Let k_i be the maximal Euler characteristic of essential surfaces with boundary properly embedded in M_i (we consider compressing disks as essential surfaces). Let $\mathcal{U}_i$ be the set of vertices in $\mathcal{C}(F)$ represented by boundary curves of essential surfaces whose Euler characteristic is k_i. Then one can define $d(M) = d(\mathcal{U}_1, \mathcal{U}_2)$. If $k_i = 1$, then $\mathcal{U}_i$ is the disk complex of M_i. If M_i has incompressible boundary and is not a twisted I–bundle, then by section 3, the diameter of $\mathcal{U}_i$ is bounded by a number depending on k_i.

Theorem 1.2. *Let $M = M_1 \cup_F M_2$ be as above. Then there is a number K depending on M_1 and M_2 such that if $d(M) \geq K$ then M is not homeomorphic to S^3.*

Similar to Dehn surgery, one can perform a surgery on a graph in S^3. An immediate corollary of Theorem 1.2 is that give a graph Γ in S^3, if one perform

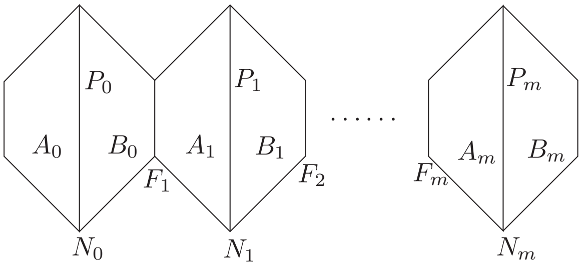


Figure 1.1

a complicated surgery on Γ, i.e., gluing back a handlebody to $S^3 - N(\Gamma)$ via a high-distance map, then the resulting closed 3-manifold cannot be S^3.

Definition 1.3. Let N be a compression body and F a closed separating surface properly embedded in N. F cuts N into two submanifolds N_1 and N_2 and we may view F as a boundary component of each N_i. Suppose F is not a 2–sphere and $\partial_+ N \subset \partial N_1$. We say F is a *middle surface* in N if both N_1 and N_2 are compression bodies, $\partial_+ N = \partial_+ N_1$, $\partial_- N_2 \subseteq \partial_- N$ and $F = \partial_+ N_2 \subseteq \partial_- N_1$. Note that if one views a compression body as a manifold obtained by adding 2–handles and 3–handles to $\partial_+ N \times I$ on the same side, then a middle surface is a middle level of this process. In particular, we can rearrange the handles such that N_1 is a compression body obtained by adding a subset of the 2– and 3–handles to $\partial_+ N \times I$, and after adding the remaining 2– and 3–handles along F, we obtain the whole of N. Therefore, although not unique, a middle surface is canonical in this handle addition process. Note that unless F is parallel to a component of $\partial_- N$, F is incompressible in N_1 but compressible in N_2.

Let S be an unstabilized Heegaard surface in M. We consider the untelescoping of the Heegaard splitting, see [24, 22]. By [24], M can be decomposed into $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$, where each F_i is incompressible in M, see Figure 1.1 for a picture. Moreover, as shown in Figure 1.1, each N_i has a strongly irreducible Heegaard surface P_i and $N_i = A_i \cup_{P_i} B_i$ where A_i and B_i are compression bodies with $\partial_- A_i = F_i = \partial_- B_{i-1}$. Let F be a closed connected surface embedded in M. We say F is a *canonical surface with respect to the untelescoping* if F is parallel to a middle surface in a compression body A_i or B_i, for some i. Note that P_i is a (trivial) middle surface of both A_i and B_i, and any component of F_i is a middle surface of both B_{i-1} and A_i.

Theorem 1.4. *Let $M = M_1 \cup_F M_2$ be as above. Then for any integer g, there is a number K depending on M_1, M_2 and g, such that for any unstabilized Heegaard surface S of M with $g(S) \leq g$, if $d(M) > K$*

1. *F is isotopic to a canonical surface with respect to any untelescoping of S,*
2. *in particular $g(M) = g(M_1) + g(M_2) - g(F)$.*

It follows from the proof that the number K in Theorem 1.4 can be chosen to be an explicit quadratic function of g and $\chi(\Omega_i)$, where Ω_i is as in Definition 1.1. The bound K depends on several distance estimates in various places in the paper. Lemma 3.7 is the only place where the bound is quadratic and all other estimates are linear functions.

If both M_1 and M_2 are simple, i.e., irreducible, ∂-irreducible, atoroidal and anannular, then Theorem 1.4 is a generalization of a theorem of Lackenby [7] and is proved in [26, 12]. Note that the complexity measure in [12] is defined using boundary curves of normal surfaces, see [9] for a relation between Heegaard surfaces and normal surfaces. The following is an immediate corollary of Theorem 1.4.

Corollary 1.5. *Let M, M_1 and M_2 be as above. If M_1 is a handlebody, then there is a number K depending on M_2 and g such that any Heegaard surface of M with genus at most g is isotopic to a Heegaard surface of M_2. In particular, $g(M) = g(M_2)$.*

Corollary 1.5 says that if the gluing map is complicated, then there is no new small-genus Heegaard surface in the resulting closed 3–manifold. In particular, if M_1 is a solid torus and M_2 is a knot manifold, Corollary 1.5 gives a weaker version of a known result on Heegaard structure and Dehn filling, see [16, 17, 18].

Although we assume M_1 and M_2 have connected boundary, the proof of Theorem 1.4 can be trivially extended to manifolds with multiple boundary components. Suppose M_i is compact irreducible and orientable and let F_i be a boundary component of M_i $(i = 1, 2)$. We can define the disk complex for F_i to be the set of vertices in $\mathcal{C}(F_i)$ represented by curves bounding compressing disks in M_i. If F_i is incompressible in M_i and M_i is not an I–bundle, we fix a properly embedded essential surface Ω_i in M_i with at least one boundary component in F_i and suppose the Euler characteristic $\chi(\Omega_i)$ is maximal among all such essential surfaces. Suppose $M_i \neq F_i \times I$. We define $\mathcal{U}_i$ to be the set of vertices in $\mathcal{C}(F_i)$ as follows,

$$\mathcal{U}_i = \begin{cases} \text{the disk complex for } F_i, & \text{if } F_i \text{ is compressible in } M_i \\ \text{the annulus complex for } F_i, & \text{if } M_i \text{ is a twisted } I\text{–bundle} \\ \text{vertices in } \mathcal{C}(F_i) \text{ represented by } F_i \cap \partial\Omega_i, & \text{otherwise.} \end{cases}$$

Suppose $F_1 \cong F_2$, then we can glue M_1 to M_2 via a homeomorphism $\phi : F_1 \to F_2$ and obtain a manifold $M = M_1 \cup_\phi M_2$. We may view $F = F_1 = F_2$ as a surface properly embedded in M. We suppose each M_i is not a product $F_i \times I$ and suppose every component of $\partial M_i - F_i$ is incompressible in M_i. We define the distance of the amalgamation to be $d(M) = d(\mathcal{U}_1, \mathcal{U}_2)$ in the curve complex $\mathcal{C}(F)$.

Theorem 1.6. *Let $M = M_1 \cup_F M_2$ be as above. Suppose M_i is not a product $F_i \times I$ and suppose every component of $\partial M_i - F_i$ is incompressible in M_i. Then for any integer g, there is a number K depending on M_1, M_2 and g, such that for any unstabilized Heegaard surface S of M with $g(S) \leq g$, if $d(M) > K$*

1. *F is isotopic to a canonical surface with respect to any untelescoping of S,*
2. *in particular $g(M) = g(M_1) + g(M_2) - g(F_i)$.*

For simplicity, we will assume M_1 and M_2 has connected boundary in this paper. The proof of Theorem 1.6 is the same as that of Theorem 1.4.

Another way of building a closed manifold is gluing two boundary components of the same manifold via an orientation-reversing map. Part (1) of Theorem 1.4 can also be extended to this case. More specifically, let M' be a compact, orientable and irreducible 3–manifold with two boundary components F_1 and F_2. Suppose $F_1 \cong F_2$. We can glue F_1 to F_2 via an orientation-reversing homeomorphism and obtain a closed manifold M. We can regard $F = F_1 = F_2$ as a surface properly embedded in M. One can define a gluing complexity similarly and show that if the gluing map is sufficiently complicated, then for any small-genus Heegaard splitting of M, F is isotopic to a canonical surface with respect to any untelescoping of the Heegaard splitting. Similar to [7], part (2) of Theorem 1.4 may not be true in this case because a minimal-genus Heegaard splitting of M' may separate F_1 and F_2.

I would like to thank Saul Schleimer for a helpful conversation on the annulus complex of a twisted I–bundle.

2. A genus calculation

Notation 2.1. Throughout this paper, we denote the interior of X by $\mathrm{int}(X)$, the closure of X (under the path-metric) by $\overline{X}$, and the number of components of X by $|X|$ for any space X.

We first show that part (2) of Theorem 1.4 follows from part (1). By Theorem 1.2, we may assume $M = M_1 \cup_F M_2$ is not S^3. Let S be an unstabilized Heegaard surface of M. Let $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$ and $N_i = A_i \cup_{P_i} B_i$ be an untelescoping for the Heegaard splitting, see Figure 1.1. As in [24, 22], one can rearrange the handle structure determined by the Heegaard splitting so that the 1– and 2–handles which occur in N_i determine the Heegaard splitting $N_i = A_i \cup_{P_i} B_i$.

Suppose S is a minimal genus Heegaard surface of M and let g be its genus. Suppose F is canonical with respect to the untelescoping of S as above. Without loss of generality, we may suppose F lies in the compression body B_j between P_j and F_{j+1} in the untelescoping, see Figure 1.1. By the definition of middle surface, F separates B_j into two compression bodies. Hence we can rearrange the 2–handles in B_j so that the 2–handles in the two compression bodies are exactly the 2–handles for B_j. Next we count the handles in M_1 and M_2.

Let a_i, b_i, c_i and d_i ($i = 1, 2$) be the numbers of 0–, 1–, 2–, and 3–handles in M_i respectively in the handle decomposition determined by the Heegaard surface S as above. The total number of 0–handles is $a_1 + a_2$ and the total number of 1–handles is $b_1 + b_2$. So the Heegaard genus $g = (b_1 + b_2) - (a_1 + a_2) + 1 = (c_1 + c_2) - (d_1 + d_2) + 1$.

Since F and M_1 are connected, as in [24], one can rearrange the 0– and 1–handles in M_1 to form a connected handlebody and obtain a Heegaard splitting of M_1 with genus $g_1 = b_1 - a_1 + 1$. Hence $g(M_1) \leq b_1 - a_1 + 1$. Similarly, one can rearrange the 2– and 3–handles in M_2 to form a handlebody and obtain a Heegaard splitting of M_2 with genus $g_2 = c_2 - d_2 + 1$. Hence $g(M_2) \leq c_2 - d_2 + 1$. Moreover, an easy calculation of the Euler characteristic of M_1 yields $g(F) = 1 - a_1 + b_1 - c_1 + d_1$. Therefore, $g(M_1) + g(M_2) - g(F) \leq (b_1 - a_1 + 1) + (c_2 - d_2 + 1) - (1 - a_1 + b_1 - c_1 + d_1) = (c_1 + c_2) - (d_1 + d_2) + 1 = g = g(M)$.

Given two minimal-genus Heegaard splittings of M_1 and M_2, the amalgamation of the two splittings yields a Heegaard splitting of M with genus $g(M_1) + g(M_2) - g(F)$, see [7, 12, 25] for more detailed description. This means that $g(M) \leq g(M_1) + g(M_2) - g(F)$. So the equality holds and part (2) of Theorem 1.4 follows from part (1).

In Theorem 1.6, since $\partial M_i - F_i$ is incompressible in M_i, one may cap off each component of $\partial M_i - F_i$ by a handlebody and estimate Heegaard genus as above. The same calculation shows that part (2) of Theorem 1.6 follows from part (1).

3. Intersection of small surfaces

Throughout this section, we fix an orientable irreducible compact 3-manifold N with incompressible boundary. We also fix a component of ∂N and denote it by F.

Definition 3.1. We define the annulus complex $\mathcal{A}_N(F)$ to be the subcomplex of $\mathcal{C}(F)$ consisting of vertices represented by boundary curves of essential annuli in N. Note that we only consider those essential annuli with at least one boundary component in F.

The following lemma is also proved in [13].

Lemma 3.2. *Suppose N is not an I–bundle. Then the diameter of the annulus complex $\mathcal{A}_N(F)$ in $\mathcal{C}(F)$ is at most 2.*

Proof. Let J be an I–bundle in N with its horizontal boundary $\partial_h J$ in ∂N and its vertical boundary consisting of essential annuli properly embedded in N. Suppose $\partial_h J \cap F \neq \emptyset$. Note that if N contains an essential annulus A with at least one boundary component in F, then a small neighborhood of A is such an I–bundle. We may suppose J is maximal up to isotopy. This is basically from the theory of characteristic submanifolds, see [6].

As N is not an I–bundle, $J \neq N$. By our assumption, $J \cap F \subset \partial_h J$ and any component of $\partial(J \cap F)$ is a boundary component of an essential annulus in N. Let A' be a vertical boundary component of J. So A' is an essential annulus in N and $\partial A' \cap F \subset \partial(J \cap F)$. Let A be any other essential annulus in N with at least one boundary component in F. Since A and A' are essential and ∂N is incompressible, after some isotopies, either $\partial A \cap \partial A' = \emptyset$ or $A \cap A'$ consists of arcs vertical in both A and A'. If $\partial A \cap \partial A' \neq \emptyset$, then the union of a small neighborhood of $J \cup A$ and possibly some 3–balls yields a larger I–bundle contradicting the assumption that J is maximal, see [8, Section 2] for a more detailed argument. So $\partial A \cap \partial A' = \emptyset$. This means that, for any component γ of $\partial(J \cap F)$, $d(\gamma, \partial A \cap F) \leq 1$ and the lemma holds. □

Definition 3.3. A properly embedded surface is *essential* if it is incompressible and ∂-incompressible. Let P be a properly embedded separating surface in N and we allow P to be disconnected. Suppose the surface P decomposes N into two submanifolds X and Y, where X and Y are on different sides of P (note that X and Y may be disconnected). We say P is *strongly irreducible* if P has compressing disks on both sides, and each compressing disk in X meets each compressing disk in Y. We say P is *∂-strongly irreducible* if

(1) every compressing and ∂-compressing disk in X meets every compressing and ∂-compressing disk in Y, and
(2) there is at least one compressing or ∂-compressing disk on each side of P.

If P is strongly irreducible, then ∂P consists of curves essential in ∂N. To see this, suppose a component of ∂P is trivial in ∂N. Then an innermost such component bounds a disk in ∂N that is disjoint from any compressing disk on the other side of P. This contradicts that P is strongly irreducible.

Let P be a strongly irreducible and ∂-strongly irreducible surface in N and $\partial P \neq \emptyset$. Let X and Y be the closure of the two submanifolds of $N - P$ on different sides of P as in Definition 3.3. Since P is compressible on both sides, we may compress P in both X and Y. Let P^X and P^Y be the possibly disconnected surfaces obtained by maximally compressing P in X and Y respectively and removing all possible 2–sphere components. Some components of P^X and P^Y may be closed surfaces. Let P^X_∂ (resp. P^Y_∂) be the union of the components of P^X (resp. P^Y) with boundary.

Lemma 3.4. *Let P, P^X, P^Y, P^X_∂ and P^Y_∂ be as above. Then*

(a) *P^X and P^Y are incompressible in N,*
(b) *either P^X_∂ consists of non-nested ∂-parallel surfaces or there is an essential surface disjoint from P obtained by ∂-compressing P^X_∂ in X.*

Proof. Since P is strongly irreducible, part (a) of the lemma follows from [21, Lemma 5.5]. Our task is to prove part (b).

As P is separating, we call the two sides of P plus and minus sides and suppose P^X is on the plus side and P^Y is on the minus side. Moreover, any surface obtained by compression or ∂-compression on P inherits plus and minus sides. So P^X is a surface obtained by compressing P on the plus side.

Let Q be any surface obtained by some compressions and/or ∂-compressions on P on the plus side. Let Q' be the union of the components of Q with boundary. By Casson-Gordon [2] and [21, Lemma 5.5], Q' is incompressible on the minus side. We claim that Q' is also ∂-incompressible on the minus side. The main reason why the claim holds is that P is ∂-strongly irreducible and the proof is similar to [21, Lemma 5.5].

Suppose Q' is ∂-compressible on the minus side. A ∂-compressing disk for Q' on the minus side corresponds to an embedded disk D before the compression and ∂-compression on P with $\partial D = \alpha \cup \beta$, $\alpha \subset P$, $\beta \subset \partial N$, and $\partial\alpha = \partial\beta$. In particular, a neighborhood of α in D lies on the minus side of P. Since D corresponds to a ∂-compressing disk for Q' on the minus side and since P is ∂-strongly irreducible, $\text{int}(D) \cap P \neq \emptyset$. Let $\gamma_1, \dots, \gamma_n$ be the closed curves in $\text{int}(D) \cap P$ and let $\alpha_1, \dots, \alpha_k$ be the arcs in $(D - \alpha) \cap P$ with $\partial\alpha_i \subset \beta$ for each i. Moreover, we may assume the γ_i's and α_i's are essential curves and arcs in P.

Since P is strongly irreducible and by a theorem of Scharlemann [20, Theorem 2.1 and Lemma 2.2] (also see [21, Lemma 5.5]), after some isotopy (one can use the isotopy described below), we may assume the closed curves γ_i's are not nested in D. Let $\delta_1, \dots, \delta_n$ be the subdisks of D bounded by $\gamma_1, \dots, \gamma_n$ respectively. So each δ_i is a compressing disk for P in X (i.e. on the plus side). Each α_i and a subarc of β bound a subdisk D_i of D. Since P is strongly irreducible and ∂-strongly irreducible, we may assume that if $\text{int}(D_i) \cap P = \emptyset$, then D_i is a ∂-compressing disk of P on the plus side.

If some δ_i's lie inside D_j, since the δ_i's are non-nested in D, there must be a disk D_j such that $\text{int}(D_j) \cap P \neq \emptyset$ but those disks D_i's and δ_i's that lie inside D_j are pairwise disjoint (i.e. non-nested in D_j). This assumption implies that a small neighborhood of α_j in D_j must lie on the minus side of P. Thus, after replacing D by this disk D_j in our argument if necessary, we may assume all the disks D_i's and δ_i's are non-nested in D. Furthermore, we may assume $|\text{int}(D) \cap P|$, the number of components of $\text{int}(D) \cap P$, is minimal among all such disks D. So each δ_i is a compressing disk for P on the plus side and each D_i is a ∂-compressing disk for P on the plus side.

Since P is compressible on both sides, P has a compressing disk D' on the minus side. Since P is strongly irreducible and ∂-strongly irreducible, $\partial D' \cap \partial\delta_i \neq \emptyset$ and $\partial D' \cap \partial D_i \neq \emptyset$ for each i. Thus $D' \cap D \neq \emptyset$. We may assume D' is transverse to D.

After some isotopies, we may also assume $D' \cap D$ does not contain any closed curve. Let κ be an arc in $D' \cap D$ that is outermost in D', i.e., κ and a subarc of $\partial D'$ bound a subdisk Δ of D' and $\text{int}(\Delta) \cap D = \emptyset$. We have the following 5 cases to consider.

Case (1). If κ is an arc connecting two different circles γ_i and γ_j in D, then a simple isotopy that pushes a small neighborhood of Δ to the other side of D will merge γ_i and γ_j into one closed curve. This contradicts the assumption that $|\text{int}(D) \cap P|$ is minimal.

Case (2). If κ is an arc connecting a circle γ_i and an arc α_j, then the same isotopy above merges γ_i and α_j into a single arc. This again contradicts that $|\text{int}(D) \cap P|$ is minimal.

Case (3). The third case is that $\partial\kappa$ lies in the same circle $\gamma_i = \partial\delta_i$, as shown in Figure 3.1(a). Now we perform the same isotopy by pushing a small neighborhood of Δ to the other side of D. After this isotopy, the disk δ_i becomes an annulus $A \subset D$ where A can be viewed as a small neighborhood of $\delta_i \cup \kappa$ in D, see Figure 3.1(b). Moreover, A is properly embedded in X on the plus side of P.

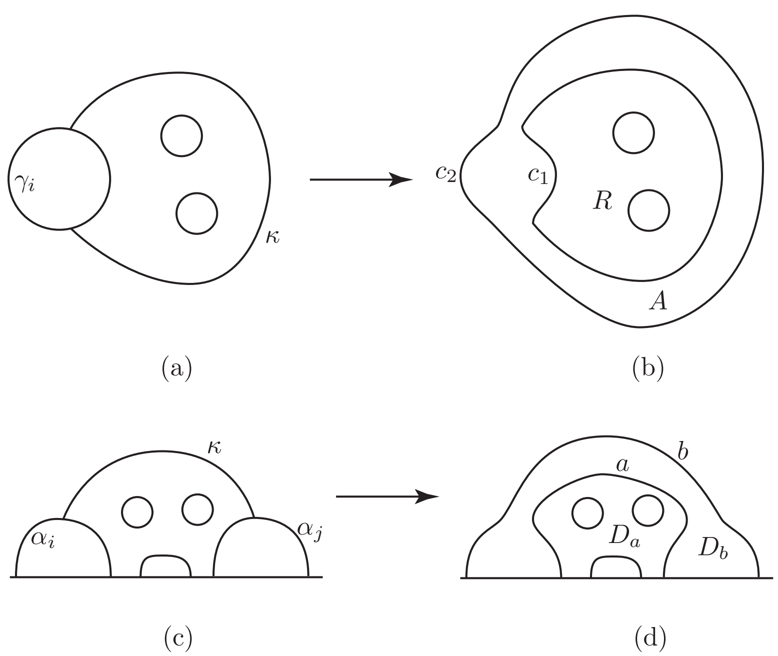
We denote the two circles of ∂A by c_1 and c_2, as shown in Figure 3.1(b). Let d_i be the disk bounded by c_i in D and suppose $d_1 \subset d_2$ and $d_2 - \text{int}(d_1) = A$. If c_i is a trivial curve in P, then a simple isotopy on P and D can reduce $|D \cap D'|$. So we may assume both c_1 and c_2 are essential curves in P. If $\text{int}(d_1) \cap P = \emptyset$, then d_1 is a compressing disk in Y (i.e., on the minus side of P) and d_1 can be isotoped disjoint from δ_i, a contradiction to the hypothesis that P is strongly irreducible. Thus we may assume $\text{int}(d_1) \cap P \neq \emptyset$.

Since the circles γ_i's are non-nested, c_1 and those γ_j's in $\text{int}(d_1)$ bound a planar surface $R \subset d_1$ and R is properly embedded in Y, see Figure 3.1(b). By a theorem of Scharlemann [20, Theorem 2.1 and Lemma 2.2] (also see [21, Lemma 5.5]), one can perform an isotopy to eliminate the nested circles in d_2. In fact, it follows from [20, Theorem 2.1 and Lemma 2.2] and [21, Lemma 5.5] that R must be ∂-parallel in Y. This contradicts the minimality assumption on $|\text{int}(D) \cap P|$.

Case (4). Now we consider the case that $\partial\kappa$ lies in the same arc α_i. After the isotopy of pushing a small neighborhood of Δ to the other side of D, α_i splits into an arc α_i' and a circle c. Let d be the subdisk of D bounded by c. By applying the same arguments for c_1 and d_1 in case (3) to c and d, we get a contradiction to either the minimality of $|\text{int}(D) \cap P|$ or the assumption that P is strongly irreducible.

Case (5). The final case is that κ is an arc connecting two different arcs α_i and α_j, see Figure 3.1(c). After the isotopy of pushing a small neighborhood of Δ to the other side of D, α_i and α_j become a pair of arcs a and b with $\partial a \cup \partial b = \partial\alpha_i \cup \partial\alpha_j$, as shown in Figure 3.1(d). Let D_a and D_b be subdisks of D cut off by a and b respectively. By our construction, D_a and D_b are nested. Suppose $D_a \subset D_b$. If a is a trivial arc in P, then a simple isotopy on D can remove the intersection arc a and lead to a contradiction to the minimality of $|\text{int}(D) \cap P|$. Suppose a is essential in P. If $\text{int}(D_a) \cap P = \emptyset$, then D_a is a ∂-compressing disk for P on the minus side and we may perturb D_a to be disjoint from D_i and D_j. This contradicts that P is ∂-strongly irreducible. Suppose $\text{int}(D_a) \cap P \neq \emptyset$. Since D_i and D_j lie on the plus side of P, a neighborhood of a in D_a is on the minus side of P. Since $|\text{int}(D_a) \cap P| < |\text{int}(D) \cap P|$, this again contradicts the minimality assumption of $|\text{int}(D) \cap P|$.

Therefore Q' must be ∂-incompressible on the minus side. Now we consider P_∂^X. The argument above implies that P_∂^X is ∂-incompressible on the minus side. Since P_∂^X is incompressible by part (a), either we can obtain an essential surface by performing ∂-compression on P_∂^X on the plus side and part (b) holds, or every component of P_∂^X is ∂-parallel. Suppose every component of P_∂^X is ∂-parallel. If



(a) (b)

(c) (d)

Figure 3.1

two components of P^X_∂ are nested (i.e. the two product regions bounded by the two ∂-parallel components are nested), then this means that after some ∂-compression on P^X_∂ on the plus side, the resulting surface becomes ∂-compressible on the minus side, a contradiction to the conclusion above. Thus part (b) holds. □

Lemma 3.5. *Let P and Q be properly embedded orientable surfaces in N. Suppose Q is essential and P is strongly irreducible and ∂-strongly irreducible. Then either*

(1) *$\partial P \cap \partial Q = \emptyset$ after isotopy, or*
(2) *after some compressions and ∂-compressions on the same side of P, one can obtain an essential surface with boundary, or*
(3) *after some isotopy, $P \cap Q$ is essential in both P and Q and $|\partial P \cap \partial Q|$ is minimal among curves isotopic to ∂P and ∂Q.*

Proof. Suppose part (1) of the lemma is not true. So we may assume that $\partial P \cap \partial Q \neq \emptyset$ and $|\partial P \cap \partial Q|$ is minimal among curves isotopic to ∂P and ∂Q.

Let X and Y be the closure of the two submanifolds of $N - P$ as in Definition 3.3. Let P^X and P^Y be the possibly disconnected surfaces obtained by maximally compressing P in X and Y respectively and removing all possible 2–sphere components. Let P^X_∂ and P^Y_∂ be the unions of the components of P^X and P^Y with boundary respectively.

By Lemma 3.4, either P^X_∂ is ∂-parallel or we can obtain an essential surface with boundary after some ∂-compressions on P^X_∂. Thus either part (2) of Lemma 3.5 holds or both P^X_∂ and P^Y_∂ are ∂-parallel and non-nested. So we may suppose P^X_∂ is ∂-parallel in X and P^Y_∂ is ∂-parallel in Y.

Let N_P be the submanifold of N between P^X and P^Y and we may assume P is properly embedded in N_P. By the construction of P^X and P^Y, there are graphs $G_X \subset X \cap N_P$ and $G_Y \subset Y \cap N_P$, such that $N_P - (P^X \cup G_X \cup P^Y \cup G_Y)$ is a product $P \times (0,1)$. Let $\Sigma_X = P^X \cup G_X$ and $\Sigma_Y = P^Y \cup G_Y$. We may view this as a sweepout $H : P \times (I, \partial I) \to (N_P, \Sigma_X \cup \Sigma_Y)$, where $I = [0,1]$ and $H|_{P\times(0,1)}$ is an embedding. We denote $H(P \times \{a\})$ by P_a for any $a \in I$. We may assume $P_0 = \Sigma_X$ and $P_1 = \Sigma_Y$ and each P_a $(a \neq 0,1)$ is isotopic to P.

Since P^X_∂ and P^Y_∂ are ∂-parallel and non-nested, we may assume $Q \cap P^X_\partial$ and $Q \cap P^Y_\partial$ consist of non-nested ∂-parallel arcs in Q. Moreover, after isotopy, we may assume that for each $t \in (0,1)$, Q is transverse to P_t except for at most one center or saddle tangency. We call $t \in (0,1)$ a regular level if Q is transverse to P_t, otherwise, we call t a singular level. We may assume there are only finitely many singular levels.

Let X_t and Y_t $(t \in (0,1))$ be the closure of the two submanifolds of $N - P_t$ corresponding to X and Y respectively. For each regular level t, we label it X (resp. Y) if either a closed curve in $Q \cap P_t$ bounds a compressing disk for P_t in X_t (resp. Y_t), or an arc in $Q \cap P_t$ bounds a ∂-compressing disk for P_t in X_t (resp. Y_t).

Since $\partial Q \cap \partial P \neq \emptyset$ and since P^X_∂ and P^Y_∂ are ∂-parallel in X and Y respectively, ϵ is labelled X and $1-\epsilon$ is labelled Y for any sufficiently small $\epsilon > 0$.

Since P is strongly irreducible and ∂-strongly irreducible, no regular level t is labelled both X and Y.

For any regular level t, if a closed curve (resp. an arc) in $P_t \cap Q$ is trivial in Q but essential in P_t, then we can find an innermost (resp. outermost) such curve (resp. arc) in Q that bounds a compressing disk (resp. ∂-compressing disk) for P_t. Thus if a level t has no label, then after some isotopies removing curves and arcs trivial in both P_t and Q, $P_t \cap Q$ consists of curves and arcs essential in both P_t and Q and part (3) of the lemma holds. So we may suppose every regular level t is labelled.

Since ϵ is labelled X and $1-\epsilon$ is labelled Y for small $\epsilon > 0$, the conclusions above imply that there must be a singular level $s \in (0,1)$ such that $s-\epsilon$ is labelled X but $s+\epsilon$ is labelled Y for sufficiently small $\epsilon > 0$. Moreover, $P_s \cap Q$ contains a single saddle tangency.

Let Θ be the graph component of $P_s \cap Q$ containing the saddle tangency and let $N(\Theta)$ be the closure of a small regular neighborhood of Θ in Q. Since P_s, Q and N are all orientable and P_s is separating in N, every component of $P_{s\pm\epsilon} \cap Q$ is isotopic in Q to either a component of $\partial N(\Theta)$ or a component of $P_s \cap Q - \Theta$. Since $s-\epsilon$ is labelled X and $s+\epsilon$ is labelled Y and $\partial N(\Theta) \subset Q$, this contradicts that P_s is strongly irreducible and ∂-strongly irreducible, similar to the case that a regular level is labelled both X and Y. □

For any integers g and b, let $C_{g,b}$ be the collection of orientable surfaces properly embedded in N, such that any $P \in C_{g,b}$ has at least one boundary component in F, ∂P is essential in ∂N, $g(P) \leq g$ and $|\partial P - F| \leq b$. Note that surfaces in $C_{g,b}$ need not to be essential and there is no restriction on the number of components of $\partial P \cap F$.

Lemma 3.6. *Let P be a surface in $C_{g,b}$. Let $P = P_0, P_1, \ldots, P_k$ be surfaces in N such that each P_i is obtained by performing a ∂-compression on P_{i-1}. Suppose ∂P_i is essential in ∂N for each i. Then the distance $d(\partial P \cap F, \partial P_k \cap F) \leq \max\{1,\ 4g + 2b - 2\}$ in $\mathcal{C}(F)$.*

Proof. Since ∂N is incompressible in N and ∂P_i is essential in ∂N, $\chi(P_i) \leq 0$ for each i. Let b_F be the number of components of $\partial P \cap F$. Since $P \in C_{g,b}$, the total number of boundary components of P is at most $b_F + b$. By our hypotheses, the total number of ∂-compressions is at most $-\chi(P)$, so $k \leq -\chi(P) \leq 2g-2+b+b_F$.

Let D_i be the ∂-compressing disk for P_i such that P_{i+1} is obtained by the ∂-compression along D_i. Let $\partial D_i = \alpha_i \cup \beta_i$ with $\alpha_i \subset P_i$ and $\beta_i \subset \partial N$. Since we are only concerned about how the curves change in $\mathcal{C}(F)$, we may assume $\beta_i \subset F$ for all i. Clearly, for any components γ_i and γ_{i+1} of $\partial P_i \cap F$ and $\partial P_{i+1} \cap F$ respectively, $d(\gamma_i, \gamma_{i+1}) \leq 1$. Note that if $\partial\alpha_i \cap \gamma_i = \emptyset$, then the ∂-compression does not change γ_i and γ_i can be viewed as a component of ∂P_{i+1}.

We may view each α_i above as an arc properly embedded in $P = P_0$. As above, we may assume the endpoints of these α_i's all lie in $\partial P \cap F$. We have k such arcs α_i and $|\partial P \cap F| = b_F$. Hence there is a component γ of $\partial P \cap F$ that contains at most $2k/b_F$ endpoints of these arcs α_i's. Since $k \leq 2g-2+b+b_F$, if $2g-2+b \geq 0$, then the number of those endpoints in γ is at most

$$\frac{2k}{b_F} \leq \frac{4g-4+2b+2b_F}{b_F} = \frac{4g-4+2b}{b_F} + 2 \leq (4g-4+2b)+2 = 4g+2b-2.$$

If $2g-2+b<0$, we have $\frac{2k}{b_F} \leq \frac{4g-4+2b+2b_F}{b_F} = \frac{4g-4+2b}{b_F} + 2 < 2$, which means that the number of those endpoints in γ is at most 1.

This means that at most $\max\{1,\ 4g+2b-2\}$ ∂-compressions occur at the curve γ. Since each ∂-compression changes a curve by at most distance one in $\mathcal{C}(F)$, $d(\gamma, \partial P_k \cap F) \leq \max\{1,\ 4g+2b-2\}$ and hence the lemma holds. Note that this bound is not sharp and one can easily reduce the bound by a more delicate argument. ☐

Lemma 3.7. *Suppose N is not an I–bundle. Let P and Q be surfaces in $C_{g,b}$. Suppose Q is essential and suppose P is either essential or strongly irreducible and ∂-strongly irreducible. Then there exists a number K' that depends only on g and b, such that the distance $d(\partial P \cap F, \partial Q \cap F) \leq K'$ in $\mathcal{C}(F)$. Moreover, K' can be chosen to be an explicit quadratic function of g and b.*

Proof. We first consider the case that N contains an essential annulus A with at least one boundary component in F. We claim that there is a number K_1 that depends only on g and b, such that the distance $d(\partial P \cap F, \partial A \cap F) \leq K_1$ for any essential annulus A.

Suppose $d(\partial P \cap F, \partial A \cap F) \geq 2$, then every component of $\partial P \cap F$ intersects every component of $\partial A \cap F$. Let $\omega = \min\{|\alpha \cap \beta| : \alpha$ is a component of $\partial P \cap F$ and β is a component of $\partial A \cap F\}$. By [15, Lemma 2.1] (see [5, Lemma 2.1] for a better bound), if the minimal intersection number of two curves α and β is k, then the distance $d(\alpha, \beta) \leq 2k+1$. Thus if $d(\partial P \cap F, \partial A \cap F)$ is large, so is ω. We should note that the proof of [15, Lemma 2.1] assumes the genus of F at least 2. However, it follows from the same argument that the inequality also holds if F is a torus.

If P is essential in N, we may assume $\partial P \cap \partial A$ is minimal in their isotopy classes and $P \cap A$ consists of essential arcs. If P is strongly irreducible and ∂-strongly irreducible, by Lemma 3.5, either $\partial A \cap \partial P = \emptyset$, or one can obtain an essential surface P' by compressing and ∂-compressing P on the same side, or $A \cap P$ consists of essential arcs in both A and P. If $\partial A \cap \partial P = \emptyset$, then $d(\partial A \cap F, \partial P \cap F) \leq 1$ and the claim holds. If one can obtain such an essential surface P' as in Lemma 3.5, then by Lemma 3.6, $d(\partial P' \cap F, \partial P \cap F) \leq \max\{1,\ 4g+2b-2\}$, and since P' is essential, we may assume $P' \cap A$ is essential in both P' and A and we can apply the argument below to P' instead of P. Thus to prove the claim, we only need to consider the case that $\partial P \cap \partial A$ is minimal in their isotopy classes and $P \cap A$ consists of arcs that are essential in both P and A.

Next we allow A to be an essential Möbius band. Note that if A is an essential Möbius band, then the boundary of a small neighborhood of A in N is an essential annulus disjoint from A. By Lemma 3.2 the diameter of the annulus complex is at most 2. So to prove the claim, we may choose A so that $|A \cap P|$ is minimal among all essential annuli and Möbius bands with the properties that $\partial A \cap F \neq \emptyset$ and $P \cap A$ consists of arcs essential in both P and A.

If P is an annulus, then by Lemma 3.2, $d(\partial P \cap F, \partial A \cap F) \leq 2$. So we may assume $\chi(P) < 0$.

Let g_P be the genus of P and let b_F and b_P be the numbers of components of $\partial P \cap F$ and $\partial P - F$ respectively. By the definition of $C_{g,b}$, $g_P \leq g$ and $b_P \leq b$. Since each component of $\partial P \cap F$ intersects every component of $\partial A \cap F$ in at least ω points, the number of arcs of $P \cap A$ with an endpoint in F is at least $\omega b_F/2$. Since $\chi(P) < 0$, there are at most $6g_P + 3(b_F + b_P) - 6 \leq 3b_F + 6g + 3b - 6$ pairwise nonparallel arcs in P. Thus if $\omega > \frac{2(3b_F+6g+3b-6)}{b_F}$, then at least two arcs of $P \cap A$ are parallel in P. Similar to the proof of Lemma 3.6, if $6g+3b-6 \geq 0$, $\frac{2(3b_F+6g+3b-6)}{b_F} = 6 + \frac{12g+6b-12}{b_F} \leq 12g + 6b - 6$, and if $6g + 3b - 6 < 0$, $\frac{2(3b_F+6g+3b-6)}{b_F} < 6$. Thus if $\omega > \max\{5,\ 12g+6b-6\}$ then there are 2 arcs in $P \cap A$, denoted by α and β, such that α and β are parallel in P, $\partial\alpha \cap F \neq \emptyset$ and $\partial\beta \cap F \neq \emptyset$.

Let R_P and R_A be rectangles bounded by α and β in P and A respectively. We may choose α and β so that $\mathrm{int}(R_P) \cap A = \emptyset$. Thus $A' = R_P \cup R_A$ is an embedded annulus or Möbius band. Since N is irreducible, if A' is a Möbius band, then $\partial A'$ must be essential in ∂N. Since $|\partial P \cap \partial A|$ is minimal in the isotopy classes of ∂P and ∂A and since ∂N is incompressible, if A' is an annulus, $\partial A'$ consists of essential curves. Hence A' is incompressible. Since A is essential, α and β are essential arcs in N, which implies that A' is ∂-incompressible. Therefore A' must be an essential annulus or Möbius band in N. Moreover, after a small perturbation, $A' \cap P$ has fewer components than $A \cap P$, a contradiction to the minimality assumption on $|A \cap P|$. This means that $\omega \leq \max\{5, 12g + 6b - 6\}$. Since the diameter of the annulus complex is at most 2 by Lemma 3.2, for any essential annulus A, $d(\partial P \cap F, \partial A \cap F)$ is bounded by a number K_1 which depends only on g and b. In fact, by [15, Lemma 2.1] mentioned earlier, we may choose K_1 to be an explicit linear function of g and b.

The argument above also implies that $d(\partial Q \cap F, \partial A \cap F) \leq K_1$, since Q is an essential surface. Therefore, if N contains an essential annulus with at least one boundary component in F, $d(\partial P \cap F, \partial Q \cap F) \leq 2K_1$.

Next we suppose N contains no essential annulus with a boundary component in F.

For simplicity, we assume P is strongly irreducible and ∂-strongly irreducible and the proof for the case that P is essential is the same. As in the argument above, by Lemma 3.5 and Lemma 3.6, we only need to consider the case that $|\partial P \cap \partial Q|$ is minimal in their isotopy classes and each curve or arc of $P \cap Q$ is essential in both P and Q.

Let $\omega = \min\{|\alpha \cap \beta| : \alpha$ is a component of $\partial P \cap F$ and β is a component of $\partial Q \cap F\}$. Let b_1 and b_2 be the numbers of components in $P \cap F$ and $Q \cap F$

respectively. Thus the number of arcs of $P \cap Q$ with an endpoint in F is at least $\omega b_1 b_2/2$.

Since N contains no essential annulus with a boundary component in F and since ∂N is incompressible, $\chi(P) < 0$ and $\chi(Q) < 0$. As in the argument above, the maximal numbers of pairwise nonparallel arcs in P and Q are at most $6g + 3(b + b_1) - 6$ and $6g + 3(b + b_2) - 6$ respectively. Thus if ω is sufficiently large, there are a pair of arcs α and β in $P \cap Q$ such that $\partial\alpha \cap F \neq \emptyset$, $\partial\beta \cap F \neq \emptyset$, and α and β are parallel in both P and Q. Note that the bound for ω is an explicit quadratic function of g and b. Let R_P and R_Q be the rectangles bounded by $\alpha \cup \beta$ in P and Q respectively. We can choose α and β so that $\mathrm{int}(R_P) \cap \mathrm{int}(R_Q) = \emptyset$. So $A = R_P \cup R_Q$ is an embedded annulus or Möbius band in N. As $\alpha \cap F \neq \emptyset$, at least one component of ∂A lies in F. Since N is irreducible, if A is a Möbius band, ∂A must be essential in ∂N. Since $|\partial P \cap \partial Q|$ is minimal in their isotopy classes, if A is an annulus, ∂A is essential in ∂N. Hence A is incompressible. Since Q is an essential surface, α is an essential arc in N. Hence A is ∂-incompressible in N. So A is essential in N. If A is a Möbius band, a double cover of A is an essential annulus. This contradicts our assumption that no such annulus exists. Therefore, ω and $d(\partial P \cap F, \partial Q \cap F)$ must be bounded by a number K' that depends only on g and b. As above, K' can be chosen to be an explicit quadratic function of g and b. □

Lemma 3.8. *Suppose N is a twisted I–bundle over a closed non-orientable surface and $F = \partial N$. Let P be a properly embedded orientable genus-g surface with boundary and suppose P is either essential or strongly irreducible and ∂-strongly irreducible. Then there is a number K depending only on g, such that $d(\partial P, \mathcal{A}_N(F)) \leq K$, where $\mathcal{A}_N(F)$ is the annulus complex defined in Definition 3.1. Moreover, K can be chosen to be an explicit linear function of g.*

Proof. Note that if P is an essential surface, since N is an I–bundle, P must be an annulus and $d(\partial P, \mathcal{A}_N(F)) = 0$. So we may assume that P is strongly irreducible and ∂-strongly irreducible.

Let b be the number of boundary components of P. Suppose we can obtain an essential annulus P' after some compressions and ∂-compressions on the same side of P. By Lemma 3.6 and since $F = \partial N$, $P \in C_{g,0}$ and $d(\partial P, \mathcal{A}_N(F)) \leq d(\partial P, \partial P') \leq \max\{1,\ 4g - 2\}$. Thus by Lemma 3.5, we may assume that for any vertical annulus Q of N, we can isotope P so that $P \cap Q$ consists of arcs essential in both P and Q.

We may choose Q to be a vertical annulus or Möbius band so that $\partial P \cap \partial Q$ is minimal among all vertical annuli and Möbius bands with the property that $P \cap Q$ consists of arcs essential in both P and Q. If $\partial P \cap \partial Q = \emptyset$ then $d(\partial P, \mathcal{A}_N(F)) \leq 1$. So we may assume $\partial P \cap \partial Q \neq \emptyset$.

Let $\omega = \min\{|\alpha \cap \beta| : \alpha$ is a component of ∂P and β is a component of $\partial Q\}$. As in the proof of Lemma 3.7, if ω is large, then there are a pair of arcs α and β in $P \cap Q$ parallel in P and we can construct of new essential annulus or Möbius band with fewer intersection with P. As in the proof of Lemma 3.7, this implies that $\omega \leq \max\{5,\ 12g - 6\}$. As before, by [15, Lemma 2.1], $d(\partial P, \mathcal{A}_N(F)) \leq d(\partial P, \partial Q) \leq 2\omega + 1 \leq \max\{11,\ 24g - 11\}$. □

Lemma 3.9. *Let P be a strongly irreducible surface properly embedded in N. If P is not ∂-strongly irreducible, then there is a surface P' obtained by ∂-compressing P and deleting any resulting ∂-parallel components, such that P' is either*

1. *strongly irreducible and ∂-strongly irreducible or*
2. *essential or ∂-parallel.*

Proof. Let D be a ∂-compressing disk. We say D is disk-busting if every compressing disk on the other side of P intersects ∂D. If P has a ∂-compressing disk D that is not disk-busting, then we perform a ∂-compression along D. As D is not disk-busting, there is a compressing disk D' of P on the other side which remains a compressing disk after the ∂-compression. Moreover, since P is compressible on both sides, the surface obtained by ∂-compression along D remains compressible on both sides and strongly irreducible.

After some ∂-compressions as above, we may assume every ∂-compressing disk of P is disk-busting. If P is still not ∂-strongly irreducible, there must be a pair of ∂-compressing disks D_1 and D_2 on different sides of P with $\partial D_1 \cap \partial D_2 = \emptyset$. Now we perform ∂-compression on P along D_1 and D_2 simultaneously and obtain a surface P'. Since both D_1 and D_2 are disk-busting and D_1 and D_2 are on different sides of P, P' is incompressible in N. Therefore, after some more ∂-compressions on P', we obtain a surface of which every component is either essential or ∂-parallel. □

In the proof of Lemma 3.9, if a component of P is ∂-parallel and outermost, then we can simply eliminate this component. Next we discuss how the boundary curves of P change during the ∂-compressions in the proof of Lemma 3.9. This discussion will be used later. Since we are mainly interested in the curves in F, we suppose all the ∂-compressions occur at F. Let D be a ∂-compressing disk for P in the proof of Lemma 3.9. Then a ∂-compression along D can be viewed as an isotopy pushing D into a product neighborhood of F. In fact, we can find a product neighborhood $F\times I$ of F in N such that every level $F\times\{t\}$ is transverse to P except for a singular level $s\in I$ where $F\times\{s\}$ is transverse to P except for a single saddle tangency. The saddle tangency corresponds to the ∂-compressing disk D. Suppose $F = F\times\{0\}$ and $N' = N - (F\times[0,1))$. Then $N'\cong N$ and $P\cap N'$ can be viewed as the surface obtained by ∂-compressing P along D. Since P is separating, the component of $P\cap(F\times I)$ that contains the saddle tangency is not a once-punctured torus. Hence $P\cap(F\times I)$ consists of a pair of pants and a collection of vertical annuli. In the proof of Lemma 3.9, the surface after the ∂-compression along D remains either incompressible or strongly irreducible, so every component of $P\cap(F\times\partial I)$ is an essential curve in $F\times\partial I$. To simplify notation, we do not distinguish a curve γ in $F\times\{t\}$ from the vertex in $\mathcal{C}(F)$ represented by $\pi(\gamma)$, where $\pi: F\times I\to F$ is the projection. It is easy to see that for any curves γ_0 and γ_1 in $P\cap(F\times\{0\})$ and $P\cap(F\times\{1\})$ respectively, $d(\gamma_0,\gamma_1)\leq 1 = -\chi(P\cap(F\times I))$ in $\mathcal{C}(F)$.

The situation is slightly more complicated when we simultaneously ∂-compressing P along two disjoint ∂-compressing disks D_1 and D_2 in the last part of the proof of Lemma 3.9. Similar to the argument above, we can find a product neighborhood $F\times I$ of F in N with $F\times\{0\} = F$ such that every level $F\times\{t\}$ is transverse to P except for a singular level s where $F\times\{s\}$ is transverse to P except for two saddle tangencies. The two saddle tangencies correspond to the ∂-compressing disks D_1 and D_2. Similarly, in the proof of Lemma 3.9, every component of $P\cap(F\times\partial I)$ is an essential curve in $F\times\partial I$. Let Θ be the possibly disconnected graph of

$P \cap (F \times \{s\})$ containing the two saddle tangencies. So Θ has two vertices of valence 4. If the genus of F is at least 2, then there must be an essential simple closed curve α in $F \times \{s\}$ that is disjoint from Θ and $P \cap (F \times \{s\})$. This implies that for any components γ_0 and γ_1 of $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ respectively, $d(\gamma_0, \gamma_1) \leq d(\gamma_0, \alpha) + d(\alpha, \gamma_1) \leq 2 = -\chi(P \cap (F \times I))$ in $\mathcal{C}(F)$. If F is a torus, since $P \cap (F \times \partial I)$ consists of essential curves, each $P \cap (F \times \{i\})$ $(i = 0, 1)$ consists of parallel curves in the torus F. So if there is an essential simple closed curve in $P \cap (F \times \{s\}) - \Theta$, then any curves γ_0 and γ_1 of $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ represent the same vertex in $\mathcal{C}(F)$ and $d(\gamma_0, \gamma_1) = 0$. Suppose every component of $P \cap (F \times \{s\}) - \Theta$ is a disk. Then $P \cap (F \times I)$ contains no vertical annulus and $P \cap (F \times I)$ can be viewed as a small neighborhood of Θ. Since P is separating, $P \cap (F \times \{0\})$ contains at least two curves and the argument above implies that $P \cap (F \times \{0\})$ contains exactly two curves which cut the torus $F \times \{0\}$ into two annuli A_1 and A_2. Moreover, the two arcs $D_1 \cap (F \times \{0\})$ and $D_2 \cap (F \times \{0\})$ from the ∂-compressing disks are essential arcs in the two annuli A_1 and A_2 respectively. So it is easy to see that $P \cap (F \times \{1\})$ also consists of exactly two curves, and for any γ_0 and γ_1 of $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ respectively, the intersection number of γ_0 and γ_1 (after projecting to the torus F) is one. Since $\mathcal{C}(F)$ is the Farey graph for the torus F, $d(\gamma_0, \gamma_1) = 1 < 2 = -\chi(P \cap (F \times I))$.

Therefore, in any case, for any curves γ_0 and γ_1 of $P \cap (F \times \{0\})$ and $P \cap (F \times \{1\})$ respectively, $d(\gamma_0, \gamma_1) \leq -\chi(P \cap (F \times I))$ and $-\chi(P \cap (F \times I))$ equals to the number of saddle tangencies in $F \times I$.

4. Case I: The amalgamation surface F is incompressible

Let M_1 and M_2 be irreducible 3–manifolds with connected boundary and $\partial M_1 \cong \partial M_2 \cong F$. Let $M = M_1 \cup_\phi M_2$ be the closed manifold obtained by gluing M_1 to M_2 via a homeomorphism $\phi : \partial M_1 \to \partial M_2$. We regard M_1 and M_2 as submanifolds of M with $F = \partial M_1 = \partial M_2$. In this section, we consider the case that both M_1 and M_2 have incompressible boundary, i.e., F is incompressible in M.

Lemma 4.1. *Let M_1, M_2, M and F be as above. Then for any integer g, there is a number K_g which depends only on M_1, M_2 and g, such that, if $d(M) > K_g$, then any closed incompressible orientable surface of genus g in M can be isotoped disjoint from F.*

Proof. Let S be a closed incompressible orientable surface of genus g in M. Suppose S cannot be isotoped disjoint from F. As both S and F are incompressible, we may assume $F \cap S$ is essential in both F and S.

Let $S_i = M_i \cap S$ $(i = 1, 2)$. So S_i has no disk component, each S_i is incompressible in M_i, $S = S_1 \cup S_2$ and $\chi(S) = \chi(S_1) + \chi(S_2)$. Since S cannot be isotoped disjoint from F, we obtain an essential surface S_i' in M_i after at most $-\chi(S_i)$ ∂-compressions on S_i. Each ∂-compression changes the boundary curves of the surface by at most distance one in $\mathcal{C}(F)$. Thus for any components γ_i and γ_i' of ∂S_i and $\partial S_i'$ respectively, the distance $d(\gamma_i, \gamma_i') \leq -\chi(S_i)$.

Suppose neither M_1 nor M_2 is a twisted I–bundle. Let Ω_i be the fixed essential surface with maximal Euler characteristic used in defining $d(M)$, i.e., $d(M) = d(\partial\Omega_1, \partial\Omega_2)$. By Lemma 3.7, for any essential surface Q with genus at most g, there is a number K_i such that $d(\partial\Omega_i, \partial Q) \leq K_i$. Thus there is a component γ_i' of $\partial S_i'$, such that $d(\partial\Omega_i, \gamma_i') \leq K_i$, $i = 1, 2$. Let γ be a component of $\partial S_1 = \partial S_2$. So we have $d(\partial\Omega_1, \partial\Omega_2) \leq d(\partial\Omega_1, \gamma_1') + d(\gamma_1', \gamma) + d(\gamma, \gamma_2') + d(\gamma_2', \partial\Omega_2) \leq K_1 - \chi(S_1) - \chi(S_2) + K_2 = K_1 + K_2 - \chi(S) = K_1 + K_2 + 2g - 2$. Thus Lemma 4.1 holds in the case that neither M_1 nor M_2 is a twisted I–bundle.

If M_i is a twisted I–bundle then S_i' must be a vertical annulus and each component of $\partial S_i'$ represents a vertex in the annulus complex of M_i. By the definition of $d(M)$ in the case that M_i is a twisted I–bundle, the argument above also proves Lemma 4.1 in the case that some M_i is a twisted I–bundle. □

Let S be an unstabilized Heegaard surface of genus g. The untelescoping of the Heegaard splitting [24] gives a decomposition $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$, where each F_i is incompressible in M and $g(F_i) \leq g$. By Lemma 4.1, we may assume $d(M)$ is so large that $F_i \cap F = \emptyset$ for each i after isotopy. So we may suppose $F \subset \mathrm{int}(N_i)$ for some i. Note that if S is strongly irreducible, then $N_i = M$. By the untelescoping construction, N_i has a strongly irreducible Heegaard surface P_i and $g(P_i) \leq g$. The following Lemma of Bachman-Schleimer-Sedgwick [1, Lemma 3.3] says that we can isotope P_i so that P_i intersects F nicely. If F is parallel to some F_j above, then Theorem 1.4 holds. Thus we may assume F is not parallel to ∂N_i.

Lemma 4.2 (Bachman-Schleimer-Sedgwick [1])**.** *Let N_i be a compact, irreducible, orientable 3–manifold with ∂N_i incompressible, if non-empty. Suppose P_i is a strongly irreducible Heegaard surface of N_i. Suppose further that N_i contains an incompressible, orientable, closed, non-boundary parallel surface F. Then either*

1. *P_i may be isotoped to be transverse to F, with every component of $P_i - N(F)$ incompressible in the respective submanifold of $N_i - N(F)$, where $N(F)$ is a small neighborhood of F in N_i,*
2. *P_i may be isotoped to be transverse to F, with every component of $P_i - N(F)$ incompressible in the respective submanifold of $N_i - N(F)$ except for exactly one strongly irreducible component, or*
3. *P_i may be isotoped to be almost transverse to F (i.e., P_i is transverse to F except for one saddle point), with every component of $P_i - N(F)$ incompressible in the respective submanifold of $N_i - N(F)$.*

Let $N(F) = F \times I$ be a product neighborhood of F in N_i and let X and Y be the two components of $N_i - \mathrm{int}(N(F))$. As P_i is a Heegaard surface of N_i and F is not parallel to ∂N_i, $F \cap P_i \neq \emptyset$. Let $S_X = P_i \cap X$ and $S_Y = P_i \cap Y$. By Lemma 4.2, we may assume that each component of S_X and S_Y is either incompressible or strongly irreducible in X and Y respectively. Moreover, both S_X and S_Y are essential subsurfaces of P_i and hence $\chi(S_X) + \chi(S_Y) \geq \chi(P_i)$. By projecting $F \times I$ to F, we may view ∂S_X and ∂S_Y as curves in F. By Lemma 4.2, P_i is transverse to every level surface $F \times \{t\}$ in $F \times I$ except for at most one saddle tangency which only occurs in case (3) of Lemma 4.2. Thus, for any components γ_X and γ_Y of ∂S_X and ∂S_Y respectively, $d(\gamma_X, \gamma_Y) \leq 1$ in $\mathcal{C}(F)$.

Since $F \cap P_i \neq \emptyset$ after any isotopy, S_X or S_Y cannot be changed to a set of ∂-parallel surfaces by ∂-compressions on S_X or S_Y. Thus by Lemma 3.9, we can obtain a pair of surfaces S_X' and S_Y' by some ∂-compressions on S_X and S_Y respectively, such that S_X' and S_Y' are either essential or strongly irreducible and ∂-strongly irreducible in X and Y respectively. The numbers of ∂-compressions on S_X and S_Y are at most $-\chi(S_X)$ and $-\chi(S_Y)$ respectively. Since each ∂-compression changes a curve by distance at most one in the $\mathcal{C}(F)$, by the argument after Lemma 3.9, for any components γ_X and γ'_X of ∂S_X and $\partial S'_X$ respectively, $d(\gamma_X, \gamma'_X) \leq -\chi(S_X)$.

Since $-\chi(S_X) - \chi(S_Y) \leq -\chi(P_i) \leq 2g - 2$ and since $d(\gamma_X, \gamma_Y) \leq 1$ for any components γ_X and γ_Y of ∂S_X and ∂S_Y respectively, for any components γ'_X and γ'_Y in $\partial S'_X$ and $\partial S'_Y$, we have $d(\gamma'_X, \gamma'_Y) \leq d(\gamma'_X, \gamma_X) + d(\gamma_X, \gamma_Y) + d(\gamma_Y, \gamma'_Y) \leq -\chi(S_X) + 1 - \chi(S_Y) \leq 1 - \chi(P_i) \leq 2g - 1$.

Note that since N_i is a submanifold of $M = M_1 \cup_F M_2$ with $F \subset \text{int}(N_i)$, to simplify notation, we will regard X and Y as submanifolds of M_1 and M_2 respectively with $F = \partial M_1 \subset \partial X$ and $F = \partial M_2 \subset \partial Y$. Since F is not parallel to a component of ∂N_i, X and Y are not I–bundles unless M_1 or M_2 is a twisted I–bundle.

We first suppose neither M_1 nor M_2 is a twisted I–bundle. Let Ω_j $(j = 1, 2)$ be the fixed essential surface in M_j used in defining $d(M)$, i.e., $d(M) = d(\partial\Omega_1, \partial\Omega_2)$. Since ∂X and ∂Y are incompressible in M_1 and M_2 respectively, we may assume $\Omega_1 \cap X$ and $\Omega_2 \cap Y$ are essential surfaces in X and Y respectively. Moreover, we may assume $\Omega_1 \cap X$ and $\Omega_2 \cap Y$ are essential subsurfaces of Ω_1 and Ω_2 respectively, and in particular, $\chi(\Omega_1 \cap X) \geq \chi(\Omega_1)$ and $\chi(\Omega_2 \cap Y) \geq \chi(\Omega_2)$.

As $F \cap \partial(\Omega_1 \cap X) = \partial\Omega_1$ and $F \cap \partial(\Omega_2 \cap Y) = \partial\Omega_2$, by applying Lemma 3.7 to X and Y, we conclude that there is a number K depending only on g and $\max\{-\chi(\Omega_1), -\chi(\Omega_2)\}$, such that $d(\partial S'_X, \partial\Omega_1) \leq K$ and $d(\partial S'_Y, \partial\Omega_2) \leq K$. Let γ'_X and γ'_Y be any components of $\partial S'_X$ and $\partial S'_Y$ respectively. We have concluded earlier that $d(\gamma'_X, \gamma'_Y) \leq 2g - 1$. So we have $d(M) = d(\partial\Omega_1, \partial\Omega_2) \leq d(\partial\Omega_1, \gamma'_X) + d(\gamma'_X, \gamma'_Y) + d(\gamma'_Y, \partial\Omega_2) \leq K + (2g - 1) + K = 2K + 2g - 1$.

If M_j is a twisted I–bundle, then it is possible that $X = M_1$ or $Y = M_2$ is a twisted I–bundle. In this case, we can replace $\partial\Omega_j$ by the annulus complex $\mathcal{A}_F(M_j)$ in the argument above. We can apply Lemma 3.8 instead of Lemma 3.7 and get the same inequalities. Therefore, Theorem 1.4 holds in the case that both M_1 and M_2 have incompressible boundary.

5. Case II: The amalgamation surface F is compressible on both sides

The case that both M_1 and M_2 have compressible boundary in Theorem 1.4 basically follows from a theorem of Scharlemann and Tomova [23] and a theorem of Hartshorn [5], also see [11].

Let $\mathcal{D}_i$ be the disk complex of M_i $(i = 1, 2)$. Recall that in this case $d(M)$ is defined to be $d(\mathcal{D}_1, \mathcal{D}_2)$. We may assume $d(\mathcal{D}_1, \mathcal{D}_2) \geq 2$ which implies that F is strongly irreducible in M. By Casson-Gordon [2] and Haken's lemma [3], this also implies that $M = M_1 \cup_F M_2$ is irreducible.

By Haken's lemma and Casson-Gordon [2], if F is strongly irreducible in M, then M is not S^3 and Theorem 1.2 holds in this case. Next we suppose $d(M) > 2g$. Let S be an unstabilized Heegaard surface of genus g. The untelescoping of the Heegaard splitting [24] gives a decomposition $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$, where each F_i is incompressible in M and $g(F_i) \leq g$. By Hartshorn's theorem [5], see [11] for another proof, either $F_i \cap F = \emptyset$ after isotopy or $d(M) = d(\mathcal{D}_1, \mathcal{D}_2) \leq 2g(F_i) \leq 2g$ for each i. Since $d(M) > 2g$, we may assume $F \subset \text{int}(N_k)$ for some k. By the untelescoping construction, there is a strongly irreducible Heegaard surface P_k of the 3–manifold N_k and $g(P_k) \leq g$.

Let Q_j $(j = 1, 2)$ be the surface obtained by maximally compressing F in M_j and removing all resulting 2–sphere components. We may assume $Q_j \subset \text{int}(M_j)$, $Q_1 \cup Q_2$ bounds a submanifold M_F in M, and F is a strongly irreducible Heegaard surface of M_F. Since $F \subset \operatorname{int}(N_k)$ and ∂N_k is incompressible in M, we may assume the compressions occur in N_k and $M_F \subset N_k$.

Since P_k is strongly irreducible, a theorem of Scharlemann and Tomova [23] says that either $d(\mathcal{D}_1, \mathcal{D}_2) \leq 2g(P_k) \leq 2g$, or F and P_k are well-separated, or F and P_k are parallel. Note that F and P_k are well-separated means that $M_F \cap N_k = \emptyset$, which is not the case by our assumptions. Since we have assumed $d(M) > 2g$, F and P_k must be parallel. Therefore, Theorem 1.4 holds if both M_1 and M_2 have compressible boundary, and in this case we may choose the bound $K = 2g$.

6. Case III: The amalgamation surface F is compressible on one side

In the next two sections, we suppose F is compressible in M_1 but incompressible in M_2. We denote the disk complex of M_1 by $\mathcal{D}_1$.

Proposition 6.1. *Let γ be a nontrivial simple closed curve in F. Suppose γ bounds an embedded disk in $M = M_1 \cup_F M_2$. Then $d(\gamma, \mathcal{D}_1) \leq 1$.*

Proof. Let D be the embedded disk bounded by γ in M. We may assume that $|D \cap F|$ is minimal among all disks bounded by γ and transverse to F. Since F is incompressible in M_2, if $\operatorname{int}(D) \cap F = \emptyset$ then D must be a compressing disk of M_1 and $d(\gamma, \mathcal{D}_1) = 0$.

Let γ' be a component of $D \cap F$ that is innermost in D and let δ be the subdisk of D bounded by γ'. If γ' is a trivial curve in F, then a standard cutting and pasting yields a new disk bounded by γ with fewer intersection curves with F. So δ must be a compressing disk in M_1. Since D is embedded, γ and γ' are disjoint in F. Therefore, $d(\gamma, \mathcal{D}_1) \leq d(\gamma, \gamma') \leq 1$. □

Lemma 6.2. *Let M' be a compact submanifold of $M = M_1 \cup_F M_2$ with $F \subset \operatorname{int}(M')$ and suppose $\partial M'$ is incompressible in M'. Let P be an orientable connected surface properly embedded in M'. Suppose P is either incompressible or strongly irreducible in M', $P \cap F \neq \emptyset$, and each component of $P \cap F$ is essential in F. Let $M_2' = M_2 \cap M'$ and $P_2 = P \cap M_2'$. Suppose P_2 is either incompressible or strongly irreducible in M_2' and P_2 does not lie in a product neighborhood of F in M_2'. Then there is a surface Q obtained by performing some ∂-compressions on P_2 in M_2' and removing all resulting ∂-parallel components, such that $d(Q \cap F, (P \cap F) \cup \mathcal{D}_1) \leq \max\{3 - \chi(P), 2\}$ and Q is either an essential or a strongly irreducible and ∂-strongly irreducible surface properly embedded in M_2'.*

Proof. First note that in the lemma if $M' = M$ then P is a closed surface. If P_2 is incompressible in M_2', then after performing some ∂–compressions on P_2 in M_2', we get a surface Q such that each component of Q is either essential or ∂–parallel in M_2'. Similarly, if P_2 is strongly irreducible but not ∂-strongly irreducible, as in Lemma 3.9, we can obtain a surface Q after some ∂-compressions on P_2 in M_2' such that Q is either essential, or ∂-parallel, or strongly irreducible and ∂-strongly irreducible in M_2. Since P_2 does not lie in a product neighborhood of F, after discarding all the ∂–parallel components, we get a surface Q which is either essential or strongly irreducible and ∂-strongly irreducible in M_2'. Since $\partial P_2 \cap F = P \cap F$, to prove the lemma, we need to study the distance between ∂P_2 and ∂Q in the curve complex $\mathcal{C}(F)$.

The surface F is a boundary component of M_2'. Since we are only interested in how the curves in $\partial P_2 \cap F$ change during ∂-compressions, to simplify notation, we will assume that all the ∂-compressions on P_2 in the construction above occur at F, i.e., for any ∂-compressing disk D for P_2, we assume $D \cap \partial M_2' \subset F$.

A ∂-compression on P_2 is basically the same as an isotopy that pushes the ∂-compressing disk into a product neighborhood of F. Thus we can find a product neighborhood $F \times I$ of F in M_2' and assume $Q = P_2 \cap \overline{M_2' - (F \times I)}$. We denote $F \times \{t\}$ by F_t and suppose $F_0 = F \subset \partial M_2'$. By the discussion after the proof of Lemma 3.9, we may describe each ∂-compression using a saddle tangency in $F_t \cap P_2$. In the proof of Lemma 3.9, we have to simultaneously perform two ∂-compressions, so we allow two saddle tangencies at the same level surface F_t. Since Q is obtained by a sequence of ∂-compressions and pushing away the ∂-parallel components, we may assume that there are finitely many numbers $0 = s_0 < s_1 < \cdots < s_k = 1$, such that

1. P_2 is transverse to each F_{s_i}, and each component of $P_2 \cap F_{s_i}$ is essential in F_{s_i},
2. there is one special component of $P_2 \cap (F \times [s_i, s_{i+1}])$ that is transverse to every F_t except for a singular level $t_i \in (s_i, s_{i+1})$ where it is transverse to F_{t_i} except for one or two saddle tangencies, and
3. every other component of $P_2 \cap (F \times [s_i, s_{i+1}])$ is either a vertical annulus or a ∂-parallel surface in $F \times [s_i, s_{i+1}]$ with boundary in F_{s_i}.

Each saddle tangency in the special component in (2) above corresponds to a ∂-compression on P_2 and the ∂-parallel components in (3) are the possible ∂-parallel components after a ∂-compression. Note that it is possible to have two saddle tangencies at the same level F_{t_i} because in the proof of Lemma 3.9, we have to simultaneously ∂-compressing the surface on both sides in order to obtain an incompressible surface, see the discussion after the proof of Lemma 3.9. We regard $Q = P_2 \cap \overline{M_2' - (F \times I)}$, so $\partial Q \subset F_1 \cup (\partial M_2' - F_0)$.

To simplify notation, we do not distinguish a nontrivial curve γ in F_t and a vertex in $\mathcal{C}(F)$ representing $\pi(\gamma)$, where $\pi : F \times I \to F$ is the projection. Next we will show that $d(Q \cap F_1, (P \cap F_0) \cup \mathcal{D}_1) \le \max\{3 - \chi(P), 2\}$. The argument is similar to [11, Claims 1 and 3 of Lemma 2.2].

Let γ_i be any component of $P_2 \cap F_{s_i}$ and let Q_i be the component of $P_2 \cap (F \times [s_{i-1}, s_i])$ that contains γ_i. By our assumption on $P_2 \cap (F \times [s_{i-1}, s_i])$ above, Q_i is either a vertical annulus or a special component in (2) above. Since the saddle tangencies in a special component correspond to ∂-compressions, $\partial Q_i \cap F_{s_{i-1}} \neq \emptyset$. Let γ_{i-1} be a component of $\partial Q_i \cap F_{s_{i-1}}$. By the discussion after the proof of Lemma 3.9, $d(\gamma_{i-1}, \gamma_i) \le n_i$, where n_i is the number of saddle tangencies in the special component of $P_2 \cap (F \times [s_{i-1}, s_i])$ and n_i is either 1 or 2. Thus we can find a curve γ_i in each $P_2 \cap F_{s_i}$ such that $d(\gamma_{i-1}, \gamma_i) \le n_i$ and in particular, $d(P_2 \cap F_{s_{i-1}}, P_2 \cap F_{s_i}) \le n_i$ for each i, where $n_i = 1$ or 2 is the number of saddle tangencies in the special component of $P_2 \cap (F \times [s_{i-1}, s_i])$.

We say a component γ of $P_2 \cap F_{s_i}$ is *good* if the component of $P_2 \cap (F \times [s_i, 1])$, denoted by Q_γ, that contains γ has a boundary component in F_1, i.e. $Q_\gamma \cap F_1 \neq \emptyset$. Moreover, every component of $P_2 \cap F_1 = Q \cap F_1$ is good. Let C_i be the set of good components of $P_2 \cap F_{s_i}$. As $s_k = 1$, $C_k = Q \cap F_1$. Since P_2 does not lie in a product neighborhood of F in M_2', $C_i \neq \emptyset$ for all i.

Suppose the lemma is not true and $d(Q \cap F_1, (P \cap F_0) \cup \mathcal{D}_1) > 2$. Since $s_k = 1$ and $C_k = Q \cap F_1$, we have $d(C_k, (P \cap F_0) \cup \mathcal{D}_1) > 2$. As $s_0 = 0$ and $P_2 \cap F_{s_0} = P \cap F_0$, $d(P_2 \cap F_{s_0}, (P \cap F_0) \cup \mathcal{D}_1) = 0$ and $d(C_0, P \cap F_0) \cup \mathcal{D}_1) = 0$. Let m be the smallest number ($1 \leq m \leq k$) such that $d(C_m, (P \cap F_0) \cup \mathcal{D}_1) \geq 2$. Since m is the smallest such number and $m \geq 1$, $d(C_{m-1}, (P \cap F_0) \cup \mathcal{D}_1) \leq 1$. By the discussion after the proof of Lemma 3.9 and as above, for any curves α and β in C_{m-1} and C_m respectively, $d(\alpha, \beta)$ is smaller than or equal to the number of saddle tangencies in the special component of $P_2 \cap (F \times [s_{m-1}, s_m])$ and $d(\alpha, \beta) \leq 2$. This implies that for any curve β in C_m, $d(\beta, (P \cap F_0) \cup \mathcal{D}_1) \leq d(\beta, \alpha) + d(C_{m-1}, (P \cap F_0) \cup \mathcal{D}_1) \leq 3$.

Let Q' be a component of $P_2 \cap (F \times [s_m, 1])$ connecting F_{s_m} and F_1, i.e. $\partial Q'$ contains curves in both F_{s_m} and F_1. By the definition of C_i, $\partial Q' \cap F_{s_m} \subset C_m$ and $\partial Q' \cap F_1 \subset C_k$. Since $d(C_m, (P \cap F_0) \cup \mathcal{D}_1) \geq 2$, we have $d(C_m, \mathcal{D}_1) \geq 2$. Similarly, $d(C_k, \mathcal{D}_1) \geq 2$ by our assumption. Hence $d(\partial Q', \mathcal{D}_1) \geq 2$. If a curve γ in $\partial Q'$ is trivial in P, then γ bounds a disk in P and by Proposition 6.1, $d(\partial Q', \mathcal{D}_1) \leq d(\gamma, \mathcal{D}_1) \leq 1$, a contradiction. Thus Q' must be an essential subsurface of P, and P cannot be a 2–sphere or disk. In particular, $\chi(P) \leq \chi(Q') \leq 0$. Let Γ be the total number of saddle tangencies in those special components of $Q' \cap (F \times [s_i, s_{i+1}])$, $i = m, \dots, k-1$. Note that we are only counting the saddle tangencies in Q' not all saddle tangencies. As $\partial Q' \subset F_{s_m} \cup F_{s_k}$ ($s_k = 1$), by our construction, $-\chi(Q') \geq \Gamma$ (note that this is an inequality because a component of $Q' \cap (F \times [s_i, s_{i+1}])$ may be ∂-parallel as in part (3) of our assumption above). Hence $-\chi(P) \geq -\chi(Q') \geq \Gamma$.

Let γ_k be a component of $\partial Q' \cap F_{s_k} \subset C_k = Q \cap F_1$. Recall that we can successively find a curve γ_i in each $Q' \cap F_{s_i}$ ($i = 1, \dots, k$) such that $d(\gamma_{i-1}, \gamma_i) \leq n_i$ for all i, where n_i is the number of saddle tangencies in the special component of $Q' \cap (F \times [s_{i-1}, s_i])$. Thus $d(\gamma_m, \gamma_k) \leq \Gamma \leq -\chi(Q') \leq -\chi(P)$. By our conclusion earlier, we have $d(\gamma_m, (P \cap F_0) \cup \mathcal{D}_1) \leq 3$. Since $s_k = 1$ and γ_k is a component of $P_2 \cap F_{s_k} = Q \cap F_1$,

$$d(Q \cap F_1, (P \cap F_0) \cup \mathcal{D}_1) \leq d(\gamma_k, \gamma_m) + d(\gamma_m, (P \cap F_0) \cup \mathcal{D}_1) \leq \Gamma + 3 \leq 3 - \chi(P).$$

□

Corollary 6.3. *There is a number K depending on M_2 such that if $d(M) \geq K$ then $M = M_1 \cup_F M_2$ is irreducible.*

Proof. Suppose M is reducible and let P be an essential 2–sphere. Since both M_1 and M_2 are irreducible, $P \cap F \neq \emptyset$. If $P \cap M_2$ is compressible in M_2, then we can compress $P \cap M_2$ and obtain a new essential 2–sphere in M. After finitely many such operations, we may assume $P \cap M_2$ is incompressible in M_2. Moreover, as in Lemma 6.2, after pushing parts of $P \cap M_2$ into M_1 via ∂-compressions, we may assume that $Q = P \cap M_2$ is an essential planar surface in M_2. However, since each component of ∂Q bounds a disk in P, by Proposition 6.1, $d(\gamma, \mathcal{D}_1) \leq 1$ for each component γ of ∂Q.

If M_2 is not a twisted I–bundle, let Ω_2 be the fixed essential surface in M_2 used in defining $d(M) = d(\partial\Omega_2, \mathcal{D}_1)$. Since Q is planar, by Lemma 3.7, there is a number K' depending on $g(\Omega_2)$ such that $d(\partial\Omega_2, \gamma) \leq K'$, where γ is a component of ∂Q. Hence $d(M) = d(\partial\Omega_2, \mathcal{D}_1) \leq d(\partial\Omega_2, \gamma) + d(\gamma, \mathcal{D}_1) \leq K' + 1$.

If M_2 is a twisted I–bundle, then Q must be an essential annulus and hence $d(M) = d(\mathcal{A}_{M_2}, \mathcal{D}_1) \leq 1$, where $\mathcal{A}_{M_2}$ is the annulus complex of the twisted I–bundle. □

Corollary 6.4. *Let F' be the surface obtained by maximally compressing F in M_1 and removing all resulting 2–sphere components. Suppose $F' \neq \emptyset$. Then there is a number K depending on M_2 such that if $d(M) \geq K$, F' is incompressible in M.*

Proof. We may assume that $F' \subset \text{int}(M_1)$. By our construction, F' is incompressible in M_1. Suppose F' is compressible in M and let D be a compressing disk. So $D \cap F \neq \emptyset$. As in Corollary 6.3, we may assume a component Q of $D \cap M_2$ is essential in M_2. By Proposition 6.1, $d(\gamma, \mathcal{D}_1) \leq 1$ for each component γ of ∂Q. Now the proof is the same as the proof of Corollary 6.3. $\square$

Theorem 1.2 basically follows from the arguments above. To prove Theorem 1.2, we also need the following theorem from [14] which says that a graph complement in S^3 always contains a nice planar surface.

Lemma 6.5 ([14])**.** *Let Γ be any graph in S^3. Then there is a planar surface P properly embedded in $S^3 - N(\Gamma)$ such that all but at most one of the components of ∂P bound compressing disks in the handlebody $\overline{N(\Gamma)}$ and P is either*

1. *strongly irreducible and ∂-strongly irreducible, or*
2. *essential, or*
3. *nonseparating and incompressible in $S^3 - N(\Gamma)$.*

Proof of Theorem 1.2. Suppose $M \cong S^3$. Since $M = M_1 \cup_F M_2$ and $M \cong S^3$, F is compressible. As in Section 5, Casson-Gordon [2] implies that if F is compressible on both sides, then F cannot be strongly irreducible and hence $d(M) < 2$. Therefore we only need to consider the case that F is compressible on one side. Suppose F is compressible in M_1 but incompressible in M_2.

Since $M = S^3$ does not contain an incompressible surface, by Corollary 6.4, $F' = \emptyset$ and M_1 must be a handlebody. We may view M_1 as a neighborhood of a graph in $M = S^3$. So there is a planar surface P properly embedded in M_2 as in Lemma 6.5. As F is incompressible in M_2, P is not a compressible disk and hence a component of ∂P bounds a compressing disk in M_1 and $d(\mathcal{D}_1, \partial P) = 0$.

If P is nonseparating and incompressible as in part (3) of Lemma 6.5, then one can perform some ∂-compressions and obtains an essential planar surface Q. Moreover, by Lemma 3.6, $d(\partial P, \partial Q) \leq \max\{1,\ 4g + 2b - 2\} = 1$ since $g = 0$ and $b = 0$ in this case. Since $d(\mathcal{D}_1, \partial P) = 0$, this means that $d(\mathcal{D}_1, \partial Q) \leq 2$. Thus in any possibility of Lemma 6.5, we have a planar surface Q in M_2 that is either essential or strongly irreducible and ∂-strongly irreducible such that $d(\mathcal{D}_1, \partial Q) \leq 2$.

Since $M = S^3$, M_2 cannot be a twisted I–bundle. Let Ω_2 be the fixed essential surface in M_2 used in defining $d(M)$. Since Q is planar, by Lemma 3.7, there is a number K' depending on $g(\Omega_2)$ such that $d(\partial\Omega_2, \gamma) \leq K'$, where γ is a component of ∂Q. Hence $d(M) = d(\partial\Omega_2, \mathcal{D}_1) \leq d(\partial\Omega_2, \gamma) + d(\gamma, \mathcal{D}_1) \leq K' + 2$. $\square$

In the remainder of the paper, we assume M is not S^3 and hence our Heegaard surfaces are not S^2.

Lemma 6.6. *For any $g \geq 1$, there is a number K depending only on M_2 and g, such that if $d(M) \geq K$ then any incompressible surface in M of genus g can be isotoped disjoint from F.*

Proof. By Corollary 6.3, we may assume $d(M)$ is so large that M is irreducible. Let P be an incompressible surface in M of genus g and suppose $F \cap P \neq \emptyset$ after any isotopy.

Let D be a compressing disk for F in M_1. If $P \cap D$ contains a closed curve, since P is incompressible, a standard isotopy on P can remove this intersection curve. Moreover, by shrinking D to be sufficiently small while fixing P, we can also isotope F to eliminate all the arcs in $P \cap D$. Thus, after isotopy, we may assume $P \cap D = \emptyset$. Since P is incompressible and M is irreducible, we may also assume every curve in $P \cap F$ is essential in F. Since $D \cap P = \emptyset$, for any component γ of $P \cap F$, $d(\gamma, \mathcal{D}_1) \leq d(\gamma, \partial D) \leq 1$.

Since P is incompressible in M and M is irreducible, after some isotopy, we may assume that $P \cap M_2$ is incompressible in M_2. Now we apply Lemma 6.2, setting M', P and P_2 in Lemma 6.2 to be M, P and $P \cap M_2$ above respectively. By Lemma 6.2 and since $F \cap P \neq \emptyset$, there is an essential surface Q in M_2 obtained by ∂-compressing $P \cap M_2$ such that $d(\partial Q, (P \cap F) \cup \mathcal{D}_1) \leq 3 - \chi(P) = 2g+1$. For any component γ of $P \cap F$, by our earlier assumption, $d(\gamma, \mathcal{D}_1) \leq 1$. This implies that $d(\partial Q, \mathcal{D}_1) \leq 2g+2$. Moreover, by our construction, the genus of Q is at most g.

If M_2 is a twisted I–bundle, then Q must be an essential annulus and hence $d(M) = d(\mathcal{A}_{M_2}, \mathcal{D}_1) \leq d(\partial Q, \mathcal{D}_1) \leq 2g+2$, where $\mathcal{A}_{M_2}$ is the annulus complex of the twisted I–bundle.

If M_2 is not a twisted I–bundle, let Ω_2 be the fixed essential surface in M_2 used in defining $d(M) = d(\partial\Omega_2, \mathcal{D}_1)$. Since $g(Q) \leq g$, by Lemma 3.7, there is a number K' depending on Ω_2 and g, such that $d(\partial\Omega_2, \partial Q) \leq K'$. Hence we can find a component γ_Q of ∂Q such that $d(M) = d(\partial\Omega_2, \mathcal{D}_1) \leq d(\partial\Omega_2, \gamma_Q) + d(\gamma_Q, \mathcal{D}_1) \leq K' + 2g + 3$. □

Let S be an unstabilized Heegaard surface of genus g. The untelescoping of the Heegaard splitting [24] gives a decomposition $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$, where each F_i is incompressible in M and $g(F_i) \leq g$. By Lemma 6.6, we may assume $d(M)$ is so large that each F_i is disjoint from F after some isotopy. Thus we may assume $F \subset N_j$ for some j. Now we consider the strongly irreducible Heegaard surface P_j of N_j in the untelescoping construction. Let X and Y be the two compression bodies in the splitting of N_j. We have $P_j = \partial_+ X = \partial_+ Y$ and $g(P_j) \leq g$.

Let F' be the surface obtained by maximally compressing F in M_1 and removing all resulting 2–sphere components. If $F' \neq \emptyset$, then by Corollary 6.4 F' is incompressible in M. We may assume $F' \subset \text{int}(M_1)$. Let M_F be the compression body bounded by F and F' in M_1. If $F' = \emptyset$ then $M_F = M_1$ is a handlebody.

Lemma 6.7. *Let E be an orientable incompressible surface in M and $E \times I$ a product neighborhood of E. Suppose $M_2 \subset \text{int}(E \times I)$, then $d(M) < K$ for some K depending only on M_2.*

Proof. Since E is incompressible in M and $F \subset \text{int}(E \times I)$, every compressing disk of F can be isotoped into $E \times I$. Thus, after isotopy, we may assume the compression body M_F described above lies in $E \times I$. As $M_2 \subset E \times I$, $F' \neq \emptyset$.

By Corollary 6.4, if $d(M)$ is sufficiently large, then F' is incompressible in M. So we may assume F' is incompressible in M. Since $F' \subset E \times I$, each component of F' must be parallel to E. Since $M_2 \subset \text{int}(E \times I)$, F' bounds a connected submanifold of $\text{int}(E \times I)$ which is obtained by adding 2–handles and 3–handles to M_2 in $\text{int}(E \times I)$. This implies that F' consists of 2 parallel copies of E in $\text{int}(E \times I)$ and M_2 lies in the product region bounded by F'. Moreover, there is a graph G properly embedded in $E \times I$ connecting the two boundary components of $E \times I$ such that $M_2 = (E \times I) - N(G \cup (E \times \partial I))$, where $N(G \cup (E \times \partial I))$ is a regular neighborhood of $G \cup (E \times \partial I)$ in $E \times I$. Furthermore, we may view $F' = E \times \partial I$ and $M_F = \overline{N(G \cup (E \times \partial I))}$.

Next we show that $M_F = \overline{N(G \cup (E \times \partial I))}$ does not contain a properly embedded incompressible annulus A whose two boundary circles lie in different components of $F' = E \times \partial I$. As $M_F = \overline{N(G \cup (E \times \partial I))}$, there is a compressing disk D for F in M_F that separates the two components of F' in M_F. If there is a properly embedded annulus A described above, since $\partial D \subset F$ and $\partial A \subset F'$, $A \cap \partial D = \emptyset$. Hence $A \cap D = \emptyset$ after isotopy. However, this is impossible since D separates the two components of F' in M_F but A connects the two components of F'.

Let A be a vertical annulus in $E \times I$. We may assume either $A \cap G = \emptyset$ or $A \cap G$ consists of a finite number of points in $\operatorname{int}(A)$. Hence $P = \overline{A - M_F}$ is a planar surface properly embedded in M_2. After some standard cutting and pasting, we may assume P is incompressible in M_2.

The conclusion earlier says that A cannot be isotoped totally into M_F. This means that, after ∂-compressions on P, we obtain an essential planar surface Q ($Q \neq \emptyset$) properly embedded in M_2. Since we can view a ∂-compression on P as part of an isotopy on A pushing the ∂-compressing disk into M_F, we may view Q as a possibly disconnected subsurface of A and $Q = A \cap M_2$.

Next we show that there is a curve $\gamma_Q \subset \partial Q$ such that $d(\gamma_Q, \mathcal{D}_1) \leq 1$. Since F is incompressible in M_2, no component of Q is a disk. If a component of Q is not an essential subannulus of A, then there is a component γ_Q of ∂Q that bounds a disk in A. By Proposition 6.1, $d(\gamma_Q, \mathcal{D}_1) \leq 1$. If every component of Q is an essential subannulus of A, then there is a component A' of $A - \operatorname{int}(Q)$ such that A' is an annulus in M_F with one component of $\partial A'$ in $F' = E \times \partial I$ and the other component of $\partial A'$, denoted by γ_Q, in $\partial Q \subset F$. Since ∂A is essential, A' is incompressible in M_F. Since the two components of $\partial A'$ lie in different components of ∂M_F, A' is also ∂-incompressible. After some standard cutting and pasting, one can always find a compressing disk of M_F disjoint from any essential annulus in M_F. Thus $d(\gamma_Q, \mathcal{D}_1) \leq 1$. Hence, in any case, there is a curve $\gamma_Q \subset \partial Q$ such that $d(\gamma_Q, \mathcal{D}_1) \leq 1$.

Note that M_2 cannot be a twisted I–bundle, since $E \times I$ does not contain any closed embedded non-orientable surface. Let Ω_2 be the essential surface used in defining $d(M)$. Since Q is a planar surface, by Lemma 3.7, there is a number K' depending on $g(\Omega_2)$ such that $d(\partial\Omega_2, \gamma_Q) \leq K'$. Therefore $d(M) = d(\partial\Omega_2, \mathcal{D}_1) \leq d(\partial\Omega_2, \gamma_Q) + d(\gamma_Q, \mathcal{D}_1) \leq K' + 1$. □

Lemma 6.8. *Let F_i be an incompressible surface in the untelescoping construction described earlier. Then there is a number K depending only on g and M_2 such that if F lies in a product neighborhood of F_i in M, then $d(M) < K$.*

Proof. Let $F_i \times I$ be a product neighborhood of F_i in M and suppose $F \subset \operatorname{int}(F_i \times I)$. By Lemma 6.7, we may assume $M_2 \not\subset F_i \times I$. So at least one component of $F_i \times \partial I$ lies in M_2. Let $N = M_2 \cap (F_i \times I)$. By Corollary 6.3, we may assume M is irreducible. This means that F does not lie in a 3–ball in $F_i \times I$ and hence we can find a vertical annulus A of $F_i \times I$ that cannot be isotoped disjoint from F. Let $P_2 = A \cap N$. Since at least one component of $F_i \times \partial I$ lies in M_2, one or two components of ∂P_2 lie in $F_i \times \partial I$.

Let D be a compressing disk for F in M_1. Since F_i is incompressible in M and $F \subset \operatorname{int}(F_i \times I)$, D can be isotoped into $F_i \times I$. By shrinking D to be sufficiently small, we may assume $D \cap A = \emptyset$ and hence $\partial P_2 \cap \partial D = \emptyset$. This means that $d(\gamma, \mathcal{D}_1) \leq 1$ for any component γ of $\partial P_2 \cap F = A \cap F$. Moreover, since M is irreducible, after some standard cutting and pasting, we may assume that P_2 is incompressible in N.

Now we apply Lemma 6.2, setting M', P and P_2 in Lemma 6.2 to be $F_i \times I$, A and P_2 above respectively. After performing some ∂-compressions on P_2 in N, we obtain an essential surface Q such that $Q \cap F \neq \emptyset$ and $d(Q \cap F, (A \cap F) \cup \mathcal{D}_1) \leq 3 - \chi(A) = 3$. So there is a component δ of $Q \cap F$ such that $d(\delta, (A \cap F) \cup \mathcal{D}_1) \leq 3$. Since $d(\gamma, \mathcal{D}_1) \leq 1$ for any component γ of $\partial P_2 \cap F = A \cap F$, there is a component δ of $Q \cap F$ such that $d(\delta, \mathcal{D}_1) \leq 4$.

If M_2 is a twisted I–bundle, since F_i can be isotoped into M_2 and F_i is incompressible, F_i must be parallel to $\partial M_2 = F$. However, this contradicts that F is compressible in M_1 but F_i is incompressible in M. Thus M_2 cannot be a twisted I–bundle.

Let Ω_2 be the surface in M_2 used in defining $d(M) = d(\partial\Omega_2, \mathcal{D}_1)$. Since ∂N is incompressible in M_2, we may assume $\Omega_2 \cap N$ is an essential subsurface of Ω_2 and $-\chi(\Omega_2 \cap N) \leq -\chi(\Omega_2)$. Since F is compressible but F_i is incompressible in M, N cannot be an I–bundle. By our construction of P_2 and Q, Q is a planar surface in N with all but one or two boundary components in F. Thus by Lemma 3.7, $d(\partial\Omega_2, Q \cap F) \leq K'$ for some K' depending only on $\chi(\Omega_2)$. Since $d(\delta, \mathcal{D}_1) \leq 4$ for some component δ of $Q \cap F$, $d(M) = d(\partial\Omega_2, \mathcal{D}_1) \leq d(\partial\Omega_2, \delta) + d(\delta, \mathcal{D}_1) \leq K' + 5$. □

Lemma 6.9. *Let N_j be the submanifold of M between F_j and F_{j+1} in the untelescoping construction as shown in Figure 1.1. Let X and Y be the two compression bodies in the splitting of N_j in which P_j is the strongly irreducible Heegaard surface. Suppose $F \subset \text{int}(N_j)$ and $F \cap P_j = \emptyset$. Then there is a number K such that $P_j \subset \text{int}(M_2)$ if $d(M) > K$.*

Proof. Since $P_j \cap F = \emptyset$, P_j lies in either $\text{int}(M_1)$ or $\text{int}(M_2)$. Suppose the lemma is not true and $P_j \subset \text{int}(M_1)$. We may suppose $F \subset \text{int}(X)$ and let $Z = X \cap M_1$. Since $F \subset \text{int}(X)$ and $P_j \subset \text{int}(M_1)$, F and P_j are boundary components of Z. So we may view $Z \cup M_2$ as a submanifold of M and $X \subset Z \cup M_2$. Since P_j is compressible on both sides, there is a compressing disk D for P_j in $Z \cup M_2$. We claim that D can be isotoped into Z if $d(M)$ is large.

Suppose $D \cap F \neq \emptyset$ and we may assume $|D \cap F|$ is minimal in the isotopy class of D. Let Q be a component of $D \cap M_2$. Since $D \cap F \neq \emptyset$ and $|D \cap F|$ is minimal, we may assume Q cannot be pushed into M_1 and Q is incompressible in M_2. As in the proofs of Corollary 6.3 and Corollary 6.4, we can perform some ∂–compressions on Q in M_2 and obtain an essential planar surface Q' in M_2. We may regard Q' as a subsurface of D. Since every component of $\partial Q'$ bounds a disk in D, by Proposition 6.1, for any component γ of $\partial Q'$, $d(\gamma, \mathcal{D}_1) \leq 1$. Now similar to the proofs of Corollary 6.3 and Corollary 6.4, this implies that $d(M) \leq K$ for some K depending only on M_2. Thus if $d(M)$ is sufficiently large, every compressing disk of P_j in $Z \cup M_2$ can be isotoped into Z.

Let W be the surface obtained by maximally compressing P_j in Z and removing all resulting 2–sphere components. Since a maximal compression on P_j in X yields $\partial_- X$, the conclusion above implies that W is parallel to $\partial_- X$. This means that F must lie in a product region bounded by W and $\partial_- X$. Now Lemma 6.9 follows from Lemma 6.8. □

Lemma 6.10. *Let N_j, X and Y be as in Lemma 6.9. Suppose $F \subset \mathrm{int}(X)$ and $P_j \subset \mathrm{int}(M_2)$. Let $Z = X \cap M_2$. Suppose P_j is compressible in Z. Then there is a number K such that, if $d(M) > K$, F is isotopic to a middle surface of X.*

Proof. Similar to the proof of Lemma 6.9, since $F \subset \mathrm{int}(X)$ and $P_j \subset \mathrm{int}(M_2)$, P_j and F are boundary components of Z. Let P' be the surface obtained by maximally compressing P_j in Z and removing all resulting 2–sphere components. So P' is incompressible in Z. Since P_j is strongly irreducible, Casson-Gordon [2] implies that P' is also incompressible on the other side. Hence P' is incompressible in M_2.

Let N be the submanifold of Z between F and P'. If N is an I–bundle (i.e., if F is parallel to a component of P'), then by our construction, F is a middle surface of X. Next we suppose N is not an I–bundle and F is not parallel to $\partial_- X$.

By Lemma 6.8, we may assume P' is not parallel to $\partial_- X$. This means that P' must be compressible in X. Let D be a compressing disk for P' in X. By the construction of P', $D \cap F \neq \emptyset$. Let Q be the component of $D \cap N$ that contains ∂D. After isotopy, we may assume Q is an essential surface in N. Note that one component of ∂Q (i.e., ∂D) lies in P' and all other components of ∂Q lie in F. Moreover, $Q \cap F = \partial Q - \partial D$ and every curve of $Q \cap F$ bounds a subdisk of D. By Proposition 6.1, $d(\mathcal{D}_1, \gamma) \leq 1$ for every component γ of $Q \cap F$.

Since P' is incompressible in M_2 and F is not parallel to P', M_2 cannot be a twisted I–bundle. Let Ω_2 be the fixed essential surface in M_2 used in defining $d(M) = d(\mathcal{D}_1, \partial\Omega_2)$. Since P' is incompressible in M_2, we may assume $\Omega' = \Omega_2 \cap N$ is an essential subsurface of Ω_2 and Ω' is essential in N. Thus $-\chi(\Omega') \leq -\chi(\Omega_2)$.

By Lemma 3.7, there is a number K' depending on $\chi(\Omega_2)$ such that $d(Q \cap F, \partial\Omega' \cap F) \leq K'$. Since $\partial\Omega' \cap F = \partial\Omega_2$ and $d(\mathcal{D}_1, \gamma) \leq 1$ for every γ in $Q \cap F$, $d(M) = d(\mathcal{D}_1, \partial\Omega_2) \leq d(\mathcal{D}_1, \gamma) + d(\gamma, \partial\Omega_2) \leq 1 + K'$. □

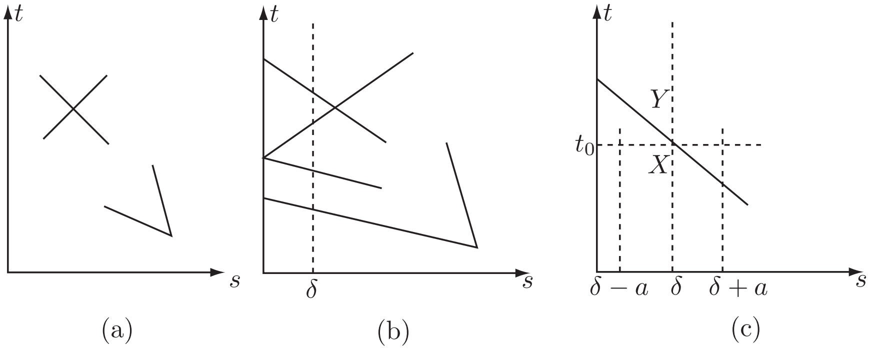
Lemma 6.9 and Lemma 6.10 say that if $F \subset N_j$ and $F \cap P_j = \emptyset$, then either Theorem 1.4 holds, or (1) P_j must lies in M_2 and (2) P_j cannot be compressible on both sides in M_2.

7. Intersection of F with sweepout surfaces

Let $M = M_1 \cup_F M_2$ be as in section 6 and S an unstabilized genus g Heegaard surface of M. Let $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$ be the untelescoping of S, where each F_i is incompressible in M, and let P_i be the strongly irreducible Heegaard surface of N_i. By Lemma 6.6, we may assume that $d(M)$ is so large that each F_i is disjoint from F. Suppose $F \subset \mathrm{int}(N_j)$.

Let F' be the surface obtained by maximally compressing F in M_1 and capping off the 2–sphere components by 3–balls. We consider the compression body M_F bounded by F' and F. So $\partial_+ M_F = F$ and $\partial_- M_F = F'$. Since $F \subset \mathrm{int}(N_j)$ and ∂N_j is incompressible in M, every compressing disk of F in M_1 can be isotoped into N_j. Thus we may assume $M_F \subset \mathrm{int}(N_j)$.

Let X and Y be the compression bodies in the splitting of N_j where $P = P_j$ is the strongly irreducible Heegaard surface of N_j. Let graphs G_X and G_Y be the cores of the compression bodies X and Y respectively, $\Sigma_X = G_X \cup \partial_- X$ and $\Sigma_Y = G_Y \cup \partial_- Y$, such that $N_j - (\Sigma_X \cup \Sigma_Y) \cong P \times (0,1)$. We consider the sweepout $f : P \times I \to N_j$ such that $f|_{P \times (0,1)}$ is an embedding, $f(P \times \{0\}) = \Sigma_X$



(a) (b) (c)

FIGURE 7.1

and $f(P \times \{1\}) = \Sigma_Y$. We denote $f(P \times \{t\})$ by P^t. Each P^t is isotopic to $P_j = P$ if $t \in (0,1)$.

Similarly, let graph G_F be the core of the compression body M_F and $\Sigma_F = G_F \cup F'$ such that $M_F - \Sigma_F \cong F \times (0,1]$. We consider the (half) sweepout $g : F \times I \to M_F$ where $g|_{F\times(0,1]}$ is an embedding, $g(F \times \{0\}) = \Sigma_F$ and $g(F \times \{1\}) = F$. We denote $g(F \times \{t\})$ by F^t. Each F^t is isotopic to F if $t \in (0,1]$.

After isotopy, we may assume that the graphs G_X, G_Y and G_F are pairwise disjoint in N_j.

The graphic Λ of the sweepouts, defined in [19], is the set of points $(s,t) \in (0,1) \times (0,1)$ such that F^s is not transverse to P^t. We briefly describe the graphic below and refer to [19] for more details. As in [19], Cerf theory implies that after some isotopy, we may assume that Λ is a graph in $(0,1) \times (0,1)$ whose edges are the set of points (s,t) for which F^s is transverse to P^t except for a single saddle or center tangency. There are two types of vertices in Λ, birth-and-death vertices and crossing vertices, as shown in Figure 7.1(a). The complement of Λ, $(0,1)\times(0,1)-\Lambda$, is a finite collection of regions. For every (s,t) in $(0,1)\times(0,1)-\Lambda$, F^s is transverse to P^t, and for any two points (s,t) and (s',t') in the same region, $F^s \cap P^t$ and $F^{s'} \cap P^{t'}$ have the same intersection pattern. As in [19], Λ can be naturally extended to a graph in $[0,1] \times [0,1]$ whose boundary vertices have valences one or two.

For any P^t, $t \in (0,1)$, we use X_t (resp. Y_t) to denote the closure of the component of $N_j - P^t$ that contains Σ_X (resp. Σ_Y). Recall that $P = P_j$ is a Heegaard surface of N_j and P^t is parallel to P. So if $t \in (0,1)$, X_t and Y_t are the two compression bodies corresponding to X and Y respectively.

By the properties of the graphic Λ [19], we can fix a number $\delta > 0$ so small that $(0,\delta] \times I$ does not contain any vertex of Λ and each edge in $\Lambda \cap ([0,\delta] \times I)$ is an arc with one endpoint in $\{0\} \times I$ and the other endpoint in $\{\delta\} \times I$, see Figure 7.1(b) for a picture.

Labelling. Since F^δ is isotopic to F, to simplify notation, we still use M_1 and M_2 to denote the closure of the two components of $M - F^\delta$. A number $t \in \{\delta\} \times I - \Lambda$ is labelled X (resp. Y) if, there is an essential curve γ in P^t such that

(1) γ bounds a compressing disk in X_t (resp. Y_t) and
(2) $\gamma \subset M_2$ and γ bounds an embedded disk D in M_2 that is transverse to P^t.

This is a labelling for all $t \in \{\delta\} \times I - \Lambda$ and we do not assign any label for t if F^δ is tangent to P^t.

Remark 7.1. Since δ is small, F^δ can be viewed as a boundary component of a small neighborhood of Σ_F. The graph G_F can be viewed as the core of the 1–handles in the compression body M_F. So by our assumption on δ, for any point $(\delta, t) \notin \Lambda$, one can find a compressing disk (which can be chosen to be a cocore of the 1–handles) for F^δ in M_F that is disjoint from P^t. Thus $d(\gamma, \mathcal{D}_1) \leq 1$ for any component γ of $F^\delta \cap P^t$ that is essential in F^δ.

Theorem 1.4 follows from the following 4 claims.

Claim 1. Either Theorem 1.4 holds, or we can fix a small δ as above so that, if $\epsilon > 0$ is sufficiently small, then ϵ is labelled X and $1 - \epsilon$ is labelled Y.

Proof. Since the compression body M_F lies in N_j, F^δ is disjoint from $\partial N_j = \partial_- X \cup \partial_- Y$. The graph G_X cannot totally lie in M_1, because otherwise P^ϵ lies in M_1 for a sufficiently small ϵ, contradicting Lemma 6.9. Thus $G_X \cap \text{int}(M_2) \neq \emptyset$ and X_ϵ must have a compressing disk D lying in M_2. Hence ϵ is labelled X.

We can apply the same argument to G_Y and conclude that $1 - \epsilon$ is labelled Y for sufficiently small ϵ. □

Claim 2. If some $t \in \{\delta\} \times I - \Lambda$ has no label, then Theorem 1.4 holds.

Proof. Let $P_2 = P^t \cap M_2$. By Lemma 6.9, $P_2 \neq \emptyset$. We first suppose P_2 is compressible in M_2 and let D be a compressing disk for P_2 in M_2. Since t is not labelled, D cannot be a compressing disk for P^t and ∂D must be trivial in P^t but essential in P_2. We compress P^t along D and delete the resulting 2–sphere component. Let P' be the remaining surface after this operation. Since M is irreducible, P' is isotopic to P^t. Suppose $P' \cap M_2$ is still compressible in M_2 and let D' be a compressing disk of $P' \cap M_2$ in M_2. Suppose $\partial D'$ is essential in P'. Since $D' \cap P' = \partial D'$, by the operation above and after a slight perturbation on D' if necessary, we may view $\partial D'$ as an essential curve in P^t bounding an embedded disk $D' \subset M_2$. Since D' may intersect the 2–sphere component that we eliminated in the operation above, $\text{int}(D') \cap P^t$ may not be empty. Since $\partial D'$ is essential in P^t, by Scharlemann's no-nesting lemma [20, Lemma 2.2], $\partial D'$ bounds a compressing disk for P^t in X_t or Y_t. This means that t is labelled X or Y, contradicting our hypothesis. Thus $\partial D'$ must also be trivial in P' and we can perform the same operation on P', i.e. compress P' along D' and remove the resulting 2–sphere component.

After finitely many such operations, we may assume that $P_2 = P^t \cap M_2$ is incompressible in M_2. If $P^t \cap F^\delta = \emptyset$, then by Lemma 6.9, $P^t \subset \text{int}(M_2)$. However, since P^t is separating in N_j and compressible on both sides, P^t must be compressible in M_2, a contradiction. Thus $P^t \cap F^\delta \neq \emptyset$. Moreover, $\partial P_2 = P^t \cap F^\delta$ consists of curves essential in F^δ.

By Lemma 6.9, P_2 does not lie in a product neighborhood of F^δ. So after some ∂–compressions on P_2, we get an essential surface Q properly embedded in M_2. By Lemma 6.2, $d(\partial Q, (P^t \cap F^\delta) \cup \mathcal{D}_1) \leq 3 - \chi(P^t) \leq 2g + 1$. By our assumption on δ, see Remark 7.1, $d(\alpha, \mathcal{D}_1) \leq 1$ for every component α of $P^t \cap F^\delta$. Hence, $d(\partial Q, \mathcal{D}_1) \leq 2g + 2$.

If M_2 is a twisted I–bundle, then Q must be a vertical annulus. Hence $d(M) \leq d(\partial Q, \mathcal{D}_1) \leq 2g + 2$.

If M_2 is not a twisted I–bundle, let Ω_2 be the fixed essential surface used in defining $d(M)$. As the genus of Q is at most g, by Lemma 3.7, there is a K' depending on Ω_2 and g, such that $d(\partial\Omega_2, \partial Q) \leq K'$. Since $d(\partial Q, \mathcal{D}_1) \leq 2g+2$, this means that $d(M) = d(\Omega_2, \mathcal{D}_1) \leq K' + 2g + 3$. □

Claim 3. If some $t \in \{\delta\} \times I - \Lambda$ is labelled both X and Y, then Theorem 1.4 holds.

Proof. Let D be an embedded disk in M_2 transverse to P^t and $\partial D \subset P^t \cap M_2$. We call D an *almost compressing disk* for X_t (resp. Y_t) if ∂D bounds a compressing disk in X_t (resp. Y_t).

Suppose $t \in \{\delta\} \times I - \Lambda$ is labelled both X and Y. Then by definition, M_2 contains almost compressing disks D_X and D_Y for X_t and Y_t respectively. Since P^t is strongly irreducible, $\partial D_X \cap \partial D_Y \neq \emptyset$.

Let $P_2 = P^t \cap M_2$ and let Δ be a compressing disk for P_2 in M_2. We say Δ is a *trivial compressing disk* if $\partial\Delta$ is essential in P_2 but trivial in P^t. Suppose a trivial compressing disk Δ lies in $X_t \cap M_2$ and there is an almost compressing disk D_Y for Y_t such that $\partial D_Y \cap \partial\Delta = \emptyset$. Then we can compress P^t along Δ and delete the resulting 2–sphere component. As in Claim 2, the remaining surface P' is isotopic to P^t. Since $\partial D_Y \cap \partial\Delta = \emptyset$, $\partial D_Y \subset P'$ and D_Y remains an almost compressing disk for P'.

For any almost compressing disk D_X for X_t, if a component γ of $\text{int}(D_X) \cap P^t$ is essential in P^t, then by Scharlemann's no-nesting lemma [20, Lemma 2.2], γ must bound a compressing disk for P^t. Since P^t is strongly irreducible and ∂D_X bounds a compressing disk in X_t, the subdisk of D_X bounded by γ must also be an almost compressing disk for X_t. Thus we may choose an almost compressing disk D_X for X_t so that every component of $\text{int}(D_X) \cap P^t$ is trivial in P^t. Since ∂D_X bounds a compressing disk in X_t, this implies that a small neighborhood of ∂D_X in D_X lies in X_t. Now we consider $D_X \cap \Delta$, where Δ is the trivial compressing disk in $X_t \cap M_2$ above. If $D_X \cap \Delta \neq \emptyset$, similar to the proof of Lemma 3.4, we can push the arcs in $D_X \cap \Delta$ across Δ. More specifically, we may suppose $D_X \cap \Delta$ does not contain any closed curve and let α be an arc in $D_X \cap \Delta$ that is outermost in Δ. Then α and a subarc of $\partial\Delta$ bound a subdisk E of Δ and $\text{int}(E) \cap D_X = \emptyset$. Since a small neighborhood of ∂D_X in D_X lies in X_t, $E \cap D_X = \alpha$. So, similar to the proof of Lemma 3.4, we can perform an isotopy by pushing α and D_X across E to eliminate α. Since each component of $\text{int}(D_X) \cap P^t$ is trivial in P^t, after the isotopy, we obtain a new almost compressing disk for X_t with fewer intersection arcs with Δ. After finitely many these operations, we can construct an almost compressing disk D'_X (for X_t) that is disjoint from Δ.

The arguments above say that if there is a trivial compressing disk Δ in $X_t \cap M_2$ such that $\partial\Delta \cap \partial D_Y = \emptyset$ for some almost compressing disk D_Y for Y_t, then after compressing P^t along Δ and deleting the 2–sphere component, the resulting surface still has two almost compressing disks for X_t and Y_t respectively. Therefore, after finitely many such operations on trivial compressing disks as above, we may assume that for each trivial compressing disk Δ, if $\Delta \subset X_t$ then $\partial\Delta \cap \partial D_Y \neq \emptyset$ for every almost compressing disk D_Y for Y_t, and if $\Delta \subset Y_t$ then $\partial\Delta \cap \partial D_X \neq \emptyset$ for every almost compressing disk D_X for X_t. Note that this implies that every curve of $P^t \cap F^\delta$ must be essential in F^δ, because otherwise the subdisk of F^δ bounded by an innermost such curve is either a trivial compressing disk disjoint from all almost compressing disks, or a compressing disk of X_t (resp. Y_t) disjoint from an almost compressing disk D_Y (resp. D_X), which contradicts that P^t is strongly irreducible.

Next we show that $P_2 = P^t \cap M_2$ has compressing disks in both $X_t \cap M_2$ and $Y_t \cap M_2$. Suppose P_2 does not have any compressing disk lying in X_t. Let D_X be an almost compressing disk for X_t and we may assume $|\mathrm{int}(D_X) \cap P^t|$ is minimal among all almost compressing disks for X_t. If $\mathrm{int}(D_X) \cap P^t = \emptyset$, then D_X is a compressing disk for P_2 lying in X_t, contradicting our assumption. So we may suppose $\mathrm{int}(D_X) \cap P^t \neq \emptyset$. Let γ be an innermost component of $\mathrm{int}(D_X) \cap P^t$ and let d_γ be the subdisk of D_X bounded by γ. If γ is trivial in P_2, then we can perform a simple isotopy on D_X to remove γ and get a contradiction to the minimality assumption of $|\mathrm{int}(D_X) \cap P^t|$. Thus γ is essential in P_2 and d_γ is a compressing disk for P_2. Since we have assumed that P_2 does not have any compressing disk lying in X_t, $d_\gamma \subset Y_t \cap M_2$. If γ is also essential in P^t, then d_γ is a compressing disk for P^t in Y_t. However, since ∂D_X bounds a compressing disk for P^t in X_t and $\gamma \cap \partial D_X = \emptyset$, this contradicts that P^t is strongly irreducible. Hence γ must be trivial in P^t and d_γ is a trivial compressing disk for P_2 in Y_t, but this contradicts our earlier assumption that every trivial compressing disk in Y_t intersects every almost compressing disk for X_t because $\gamma \cap \partial D_X = \emptyset$. Therefore, $P_2 = P^t \cap M_2$ must have compressing disks in both $X_t \cap M_2$ and $Y_t \cap M_2$.

Suppose P_2 is not strongly irreducible in M_2. Then there are compressing disks Δ_X and Δ_Y for P_2 in M_2 such that $\Delta_X \subset X_t$ and $\Delta_Y \subset Y_t$ and $\partial\Delta_X \cap \partial\Delta_Y = \emptyset$. By our assumption above, both Δ_X and Δ_Y must be trivial compressing disks. Now we compress P^t along Δ_X and Δ_Y simultaneously and delete the two resulting 2–sphere components. The remaining surface P' is isotopic to P^t. Suppose $P' \cap M_2$ has an almost compressing disk D'. As in Claim 2, after some perturbation, we may view $\partial D'$ as an essential curve in P^t and view D' as an almost compressing disk of P_2. However, since $D' \cap \Delta_X = \emptyset$ and $D' \cap \Delta_Y = \emptyset$ after isotopy, this contradicts our earlier assumption that every trivial compressing disk must intersect every almost compressing disk on the other side. So P' does not contain any almost compressing disk in M_2, and this implies that every compressing disk of $P' \cap M_2$ in M_2 is a trivial compressing disk for P'. We can compress P' along each trivial compressing disk of $P' \cap M_2$ in M_2 and delete the resulting 2–sphere component. By the argument above, the resulting surface does not contain any almost compressing disk in M_2 neither. Therefore, after finitely many such operations, we obtain a surface P'' isotopic to P^t and $P'' \cap M_2$ is incompressible in M_2.

The arguments above imply that, after some isotopies/operations on P^t described above, we may assume that $P_2 = P^t \cap M_2$ is either strongly irreducible or incompressible in M_2. If $P^t \cap F^\delta = \emptyset$ after the operations above, then $P_2 = P^t$. Since P^t is separating in N_j and compressible on both sides, $P^t \cap F^\delta = \emptyset$ implies that $P_2 = P^t$ cannot be incompressible in M_2. Hence $P_2 = P^t$ is strongly irreducible and in particular P_2 is compressible on both sides in M_2. In this case, Theorem 1.4 follows from Lemma 6.10. Therefore we may assume $P^t \cap F^\delta \neq \emptyset$.

By Lemma 6.9, P_2 does not lie in a product neighborhood of F^δ. As P_2 is either strongly irreducible or incompressible in M_2, Claim 3 basically follows from Lemma 6.2. By Lemma 6.2, we can perform some ∂-compressions on P_2 and obtain a surface Q which is either essential or strongly irreducible and ∂-strongly irreducible, such that $d(\partial Q, (P^t \cap F^\delta) \cup \mathcal{D}_1) \leq 3 - \chi(P^t) \leq 2g + 1$. By our assumption on δ, see Remark 7.1, $d(\alpha, \mathcal{D}_1) \leq 1$ for every component α of $P^t \cap F^\delta$. Hence, $d(\partial Q, \mathcal{D}_1) \leq 2g + 2$.

Suppose M_2 is not a twisted I–bundle and let Ω_2 be the essential surface used in defining $d(M)$. As the genus $g(Q) \leq g$, by Lemma 3.7, there is a number K' depending on Ω_2 and g such that $d(\partial\Omega_2, \partial Q) \leq K'$. Thus $d(M) = d(\partial\Omega_2, \mathcal{D}_1) \leq K' + 2g + 3$.

If M_2 is a twisted I–bundle, then we can apply Lemma 3.8 instead of Lemma 3.7 in the argument above and get the same inequality. □

Claim 4. Suppose every $t \in \{\delta\} \times I - \Lambda$ is labelled, then Theorem 1.4 holds.

Proof. By Claim 3, we may assume that no t is labelled both X and Y. By Claim 1, as t increases from ϵ to $1-\epsilon$, its label changes from X to Y. If every $t \in \{\delta\} \times I - \Lambda$ is labelled, then as shown in Figure 7.1(c), there is a number $t_0 \in (\{\delta\} \times I) \cap \Lambda$ such that $t_0 - \epsilon$ is labelled X and $t_0 + \epsilon$ is labelled Y for sufficiently small $\epsilon > 0$. Since $(\delta, t_0) \in \Lambda$, $F^\delta \cap P^{t_0}$ contains a single tangency. Since $t_0 - \epsilon$ and $t_0 + \epsilon$ have different labels, the tangency in $F^\delta \cap P^{t_0}$ must be a saddle tangency.

After a small perturbation if necessary, we may assume that $(\delta + a, t_0)$ and $(\delta - a, t_0)$ lie in the same regions as $(\delta, t_0 + \epsilon)$ and $(\delta, t_0 - \epsilon)$ respectively for a small $a > 0$, see Figure 7.1(c). So the intersection patterns of $F^{\delta+a} \cap P^{t_0}$ and $F^{\delta-a} \cap P^{t_0}$ are the same as $F^\delta \cap P^{t_0+\epsilon}$ and $F^\delta \cap P^{t_0-\epsilon}$ respectively.

Let $M_1^\pm$ and $M_2^\pm$ be the closure of $M - F^{\delta\pm a}$ corresponding to M_1 and M_2 respectively. Clearly $M_1^- \subset \mathrm{int}(M_1^+)$ and $M_2^+ \subset \mathrm{int}(M_2^-)$. Since $t_0 - \epsilon$ is labelled X and since $(\delta, t_0 - \epsilon)$ and $(\delta - a, t_0)$ lies in the same region, there is a curve γ_X in $P^{t_0} \cap M_2^-$ such that (1) γ_X bounds a compressing disk in X_{t_0}, and (2) γ_X bounds an almost compressing disk D_X in M_2^-. Similarly, since $t_0 + \epsilon$ is labelled Y, there is a curve γ_Y in $P^{t_0} \cap M_2^+$ such that (1) γ_Y bounds a compressing disk in Y_{t_0}, and (2) γ_Y bounds an almost compressing disk D_Y in M_2^+. Since $M_2^+ \subset \mathrm{int}(M_2^-)$, $D_Y \subset M_2^+ \subset M_2^-$ and this means that $t_0 - \epsilon$ is also labelled Y. Now Theorem 1.4 follows from Claim 3. □

References

[1] David Bachman and Saul Schleimer and Eric Sedgwick, *Sweepouts of amalgamated 3-manifolds.* arXiv:math.GT/0507490

[2] Andrew Casson and Cameron Gordon, *Reducing Heegaard splittings.* Topology and its Applications, **27** 275–283 (1987).

[3] W. Haken, *Some results on surfaces in 3–manifolds.* Studies in Modern Topology (Math. Assoc. Amer., distributed by Prentice-Hall, 1968) 34–98.

[4] Kevin Hartshorn, *Heegaard splittings of Haken manifolds have bounded distance*, Pacific J. Math. **204** (2002), 61–75.

[5] John Hempel, *3–manifolds as viewed from the curve complex.* Topology, **40** (2001) 631–657.

[6] William Jaco, *Lectures on Three-Manifold Topology.* CBMS Regional Conference Series in Mathematics, **43** (1977).

[7] Marc Lackenby, *The Heegaard genus of amalgamated 3–manifolds.* Geom. Dedicata, **109** (2004), 139–145.

[8] Tao Li, *Immersed essential surfaces in hyperbolic 3-manifolds.* Comm. Anal. Geom. **10** (2002), 275–290.

[9] Tao Li, *Heegaard surfaces and measured laminations I: the Waldhausen conjecture.* Invent. Math., **167** (2007) 135–177.

[10] Tao Li, *Heegaard surfaces and measured laminations II: non-Haken 3-manifolds.* J. Amer. Math. Soc., **19** (2006) 625–657.

[11] Tao Li, *Saddle tangencies and the distance of Heegaard splittings* Algebraic & Geometric Topology, **7** (2007) 1119–1134.

[12] Tao Li, *On the Heegaard splittings of amalgamated 3–manifolds* Geometry & Topology Monographs, **12** (2007) 157–190

[13] Tao Li, *Images of the disk complex.* Preprint: www2.bc.edu/˜taoli/publications.html

[14] Tao Li, *Thin position and planar surfaces for graphs in the 3-sphere.* Preprint available at: www2.bc.edu/˜taoli/publications.html

[15] Howard A. Masur, Yair N. Minsky, *Geometry of the Complex of Curves I: Hyperbolicity.* Invent. Math. **138** (1999), 103–149.

[16] Yoav Moriah and Hyam Rubinstein, *Heegaard structures of negatively curved 3–manifolds*, Comm. Anal. Geom., **5** (1997)375–412

[17] Yoav Moriah and Eric Sedgwick, *The Heegaard structure of Dehn filled manifolds*, arXiv:0706.1927

[18] Yoav Rieck and Eric Sedgwick, *Persistence of Heegaard structures under Dehn filling*, Topology Appl., **109** (2001) 41–53

[19] Hyam Rubinstein and Martin Scharlemann, *Comparing Heegaard splittings of non-Haken 3–manifolds.* Topology **35** (1996) 1005–1026

[20] Martin Scharlemann, *Local detection of strongly irreducible Heegaard splittings.* Topology and its Applications, **90** (1998) 135–147.

[21] Martin Scharlemann, *Proximity in the curve complex: boundary reduction and bicompressible surfaces.* Pacific J. Math. **228** (2006), 325–348. arXiv:math.GT/0410278

[22] Martin Scharlemann, *Heegaard splittings of compact 3–manifolds*, Handbook of Geometric Topology, Elsevier (2002), 921–953

[23] Martin Scharlemann and Maggy Tomova, *Alternate Heegaard genus bounds distance.* Geometry and Topology, **10** (2006) 593–617.

[24] Martin Scharlemann and A. Thompson, *Thin position for 3–manifold*, Comptemp. Math., **164** (1992), 231–238

[25] Jenifer Schultens, *The classification of Heegaard splittings for (compact orientable surface)*$\times S^1$, Proc. London Math. Soc. **67** (1993), 425–448.

[26] Juan Souto, *The Heegaard genus and distances in the curve complex.* Preprint.

Department of Mathematics, Boston College, Chestnut Hill, MA 02467
E-mail address: `taoli@bc.edu`