IMAGES OF THE DISK COMPLEX

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1. Introduction

Let F be a closed and orientable surface of genus at least 2. The curve complex of F, first defined by Harvey [1], is the complex whose vertices are the isotopy classes of essential simple closed curves in F, and $k+1$ vertices determine a k-simplex if they are represented by pairwise disjoint curves. We denote the curve complex of F by $\mathcal{C}(F)$. For any two vertices in $\mathcal{C}(F)$, one can define the distance $d(x,y)$ to be the minimal number of 1–simplices in a simplicial path jointing x to y.

The definition of curve complex can be extended to torus and surfaces with boundary. If F has boundary, then the vertices are the isotopy classes of essential non-peripheral simple closed curves in F. The definition of $\mathcal{C}(F)$ for such surfaces is the same as above except for a few sporadic cases. If F is an annulus or a pair of pants, then $\mathcal{C}(F)$ is empty. If F is a torus or a once-punctured torus or a sphere with 4 punctures, then edges are placed between vertices corresponding to curves of smallest possible intersection number (the resulting $\mathcal{C}(F)$ is the Farey graph). The structure of curve complex has been extensively studied by Masur and Minsky. For example, they proved that $\mathcal{C}(F)$ is δ-hyperbolic [6].

Let S be a compact orientable surface with boundary and suppose S is not a disk or an annulus. The *arc-and-curve* complex $\mathcal{AC}(S)$ is defined as follows. Each vertex of $\mathcal{AC}(S)$ is the isotopy class of either an essential properly embedded arc or an essential non-peripheral simple closed curve in S, and a set of vertices forms a simplex of $\mathcal{AC}(S)$ if these vertices are represented by pairwise disjoint arcs or curves in S. Similar to the curve complex, the arc-and-curve complex is also very useful in the study of mapping class groups. We are mainly interested in the vertices of these complexes.

Suppose F is a closed surface and S a compact essential non-annular subsurface of F. Then, for any subcomplex $\mathcal{D}$ of $\mathcal{C}(F)$, there is a natural projection $\pi_A : \mathcal{D} \to \mathcal{AC}(S)$ as follows. For every vertex $[\gamma]$ in $\mathcal{D}$, we take a curves γ in the isotopy class so that $|\gamma \cap S|$ is minimal. If $\gamma \cap S = \emptyset$, we define $\pi_A([\gamma]) = \emptyset$. If $\gamma \cap S \neq \emptyset$, then $\pi_A([\gamma])$ is the isotopy class of a component of $\gamma \cap S$. Furthermore, there is also a natural projection π_0 from the vertices of $\mathcal{AC}(S)$ to $\mathcal{C}(S)$ which maps a properly embedded arc α to a non-peripheral boundary component of $N(\alpha \cup \partial S)$, where $N(\alpha \cup \partial S)$ is a small neighborhood of $\alpha \cup \partial S$ in S. We define the projection $\pi_C : \mathcal{D} \to \mathcal{C}(S)$ as $\pi_C = \pi_0 \circ \pi_A$. Such projections were used by Masur and Minsky [6, 7] in their study of the structure of curve complex.

Let M be a compact orientable and irreducible 3–manifold and F a component of ∂M. Let $\mathcal{D}$ be the subcomplex of $\mathcal{C}(F)$ with each vertex of $\mathcal{D}$ corresponding to a curve that bounds a compressing disk in M. We call $\mathcal{D}$ the disk complex for F. If M is a handlebody, then the disk complex is particularly interesting because it is

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related to the Heegaard splittings of a 3–manifold, for example see [2]. Masur and Minsky proved that the disk complex of a handlebody is quasiconvex in the curve complex of ∂M [8]. A set of curves Γ in F is said to be *disk-busting* in F if every compressing disk whose boundary lies in F intersects every curve in Γ.

Let S be an essential compact non-annular subsurface of F. Then, as above, we can define projections π_A and π_C from $\mathcal{D}$ to $\mathcal{AC}(S)$ and $\mathcal{C}(S)$ respectively.

Main Theorem. *Let M be a compact orientable and irreducible 3–manifold and F a component of ∂M. Let $\mathcal{D}$ be the disk complex for F. Let S be a compact essential subsurface of F and suppose ∂S is disk-busting in F. Then either*

(1) *M is an I–bundle of which S is a horizontal boundary component, or*
(2) *the image $\pi_A(\mathcal{D})$ of the disk complex has diameter at most 10 in $\mathcal{AC}(S)$ and $\pi_C(\mathcal{D})$ has diameter at most 20 in $\mathcal{C}(S)$.*

Remark. If ∂S is not disk-busting but S is incompressible in M, then part (1) of the theorem should be changed to: (1) M contains an I–bundle J such that the horizontal boundary of J lies in ∂M, S is a component of the horizontal boundary, and the vertical boundary of J is a collection of properly embedded incompressible but ∂-compressible annuli in M.

As a corollary, let S_1 and S_2 be two essential non-annular subsurfaces of ∂M. Suppose each component of ∂S_i intersects every compressing disk and M is not an I–bundle of which S_i is a horizontal boundary component. Then if one glues S_1 to S_2 via a complicated map, the resulting manifold must have incompressible boundary.

The proof of the main theorem is essentially a technical part in the original proof of a theorem in [5] presented in the Haifa workshop in the summer of 2005. A discussion with Yoav Moriah helped me realize that this argument can be formulated into the language of the Main Theorem (in the case that ∂S is a single separating curve). Note that it is trivial to extend the theorem from the case that ∂S is a single separating curve to the general setting. I would also like to thank Yoav Moriah for helpful comments and corrections.

The main theorem, perhaps without the specific bound, was also proved by Masur and Schleimer independently but earlier. Masur and Schleimer use this theorem to study the so-called holes in the disk complex of a handlebody. I would like to thank Schleimer for several email communications which led me realize that the special case that I originally considered can be trivially extended to the full version of the Main Theorem.

2. Proof of the main theorem

To simplify notation, unless necessary, we will not distinguish between a vertex in a complex above and a well-chosen arc or curve that represents this vertex. Throughout this paper, we will use $int(X)$ to denote the interior of X and use $|X|$ to denote the number of components of X.

Proposition 2.1. *Let M, $\mathcal{D}$ and $S \subset \partial M$ be as in the Main Theorem. Let $u, v \in \mathcal{D}$ be two vertices in the disk complex, and suppose $d(\pi_A(u), \pi_A(v)) \geq 4$ in $\mathcal{AC}(S)$. Then, for any components α of $u \cap S$ and β of $v \cap S$, $\alpha \cap \beta \neq \emptyset$.*

Proof. Since $d(\pi_A(u), \pi_A(v)) \geq 4$, there is a component α' of $u \cap S$ and a component β' of $v \cap S$ such that $d(\alpha', \beta') \geq 4$ in $\mathcal{AC}(S)$. As $d(\alpha, \alpha') \leq 1$ and $d(\beta, \beta') \leq 1$, we have $d(\alpha, \beta) \geq 2$ and hence $\alpha \cap \beta \neq \emptyset$. □

Proof of the Main Theorem. Since ∂S is disk-busting in F, for any compressing disk D with $\partial D \subset F$, $\partial D \cap \partial S \neq \emptyset$. Since S is a compact subsurface of F, ∂S divides F into 2 parts, S and $F - S$, and $\partial D \cap \partial S$ contains an even number of points. Moreover, we may view D as a $2n$-gon with its vertices being the points in $\partial D \cap \partial S$. We call each component of $\partial D \cap S$ an α–edge and each component of $\partial D - int(S)$ a β–edge of ∂D. So α–edges and β–edges appear alternately on ∂D.

Suppose on the contrary that the diameter of $\pi_A(\mathcal{D})$ is greater than 10.

Claim. Let N be the minimal value of $|\partial D \cap \partial S|$ among all the compressing disks D of M with boundary in F. If the diameter of $\pi_A(\mathcal{D})$ is at least 7, then N must be either 2 or 4. In other words, there must be a compressing disk that is either a bigon or a quadrilateral.

Proof of the Claim. By our hypotheses, N must be an even number. Suppose on the contrary that $N > 4$. Let D be an N-gon. Since the diameter of $\pi_A(\mathcal{D})$ is at least 7, there is another compressing disk E so that $\pi_A(\partial D)$ and $\pi_A(\partial E)$ have distance at least 4 in $\mathcal{AC}(S)$. Thus, by Proposition 2.1, each α–edge in ∂D intersects every α–edge of ∂E. We may assume $|D \cap \partial S|$, $|E \cap \partial S|$ and $|D \cap E|$ are minimal in the isotopy classes of D and E.

We consider $D \cap E$. After a standard cutting and pasting, we may assume $D \cap E$ does not contain any closed curve. Let δ be a component of $D \cap E$ that is outermost in E. If $\partial\delta$ lies in the same edge of ∂E, then δ together with a subarc of this edge bounds a subdisk Δ in E. As δ is outermost, $int(\Delta) \cap D = \emptyset$. We may also regard Δ as a ∂-compressing disk for D. Now we view δ as an arc in D. We have the following two cases. (1) If the endpoints of δ lie in different edges of ∂D, then we can perform a ∂–compression on D along the disk Δ. After this ∂–compression, D becomes two disks and at least one of them is not ∂–parallel in M. Moreover, both disks have fewer number of edges than D, which contradicts that D is an N-gon and N is minimal. (2) If $\partial\delta$ lies in the same edge of ∂D, then δ together with a subarc of this edge bounds a subdisk Δ' of D. So $\Delta \cup \Delta'$ forms a properly embedded disk in M with $\partial(\Delta \cup \Delta') \subset F - \partial S$. Since ∂S is disk-busting in F, $F - \partial S$ is incompressible in M. Thus, $\Delta \cup \Delta'$ must be a ∂-parallel disk. By isotoping Δ' across Δ, we can eliminate δ and obtain a disk isotopic to D but having fewer intersection arcs with E, which contradicts our assumption that $|D \cap E|$ is minimal in their isotopy classes. Therefore, the endpoints of δ must lie in different edges of ∂E. In fact, the argument implies that the two endpoints of any arc of $D \cap E$ lie in different edges of ∂E.

Let δ' be a subarc of ∂E with $\partial\delta = \partial\delta'$, and Δ the subdisk of E bounded by $\delta \cup \delta'$. Since δ is outermost, we may choose δ' so that $int(\Delta) \cap D = \emptyset$. By the conclusion above, δ' contains at least one point of $\partial E \cap \partial S$. Since each α–edge of ∂E intersects every α–edge of ∂D, δ' does not contain a whole α–edge. Hence, $\delta' \cap \partial S$ either is a single point, in which case Δ is a triangle, or contains exactly two points, in which case δ' contains the whole of a β–edge and Δ is a quadrilateral.

We first consider the case that Δ is a triangle, i.e., $\partial\Delta$ consists of δ, a subarc of an α–edge and a subarc of a β–edge. Now we view δ as an arc in D. In this case, the two endpoints of δ lie in an α–edge and a β–edge of ∂D. Hence, both components of $\partial D - \partial\delta$ intersect ∂S. The arc δ divides D into two subdisks which we denote by D_1 and D_2. If D_1 is a triangle (i.e., $\partial D_1 \cap \partial S$ is a single point), then $D_1 \cup \Delta$ is a bigon disk properly embedded in M. Since we have assumed $N > 4$, $D_1 \cup \Delta$ must be a ∂-parallel disk in M. So $D_2 \cup \Delta$ is isotopic to D. After perturbing $D_2 \cup \Delta$ a little, we get a disk isotopic to D but having fewer intersection arcs with E than $D \cap E$, a contradiction to our assumption that $|D \cap E|$ is minimal. Thus, we may assume that $\partial D_i \cap \partial S$ contains more than one point for both $i = 1, 2$. Therefore, both $D_1 \cup \Delta$ and $D_2 \cup \Delta$ have fewer intersection points with ∂S than $\partial D \cap \partial S$. Moreover, at least one of $D_1 \cup \Delta$ and $D_2 \cup \Delta$ is not ∂-parallel. This contradicts the assumption that $N = |\partial D \cap \partial S|$ is minimal.

The case that Δ is a quadrilateral (i.e., δ' contains a whole β–edge of ∂E and $|\delta' \cap \partial S| = 2$) is similar. The arc δ divides D into two subdisks D_1 and D_2. In this case, both endpoints of δ lie in α–edges, so both $|\partial D_1 \cap \partial S|$ and $|\partial D_2 \cap \partial S|$ are even numbers. If $\partial D_1 \cap \partial S = \emptyset$, then $D_1 \cup \Delta$ is a bigon disk properly embedded in M. Since we have assumed $N > 4$, $D_1 \cup \Delta$ must be ∂-parallel in M. This implies that the β–edge of ∂E that lies in δ' is isotopic (relative to the boundary) to a subarc of ∂S, which contradicts our assumption that $|E \cap \partial S|$ is minimal in the isotopy class of E. So we may assume that $\partial D_i \cap \partial S \neq \emptyset$ for both $i = 1, 2$. If $\partial D_1 \cap \partial S$ contains exactly two points, then ∂D_1 contains the whole of a β–edge of ∂D, and $D_1 \cup \Delta$ is a quadrilateral disk properly embedded in M. Again, since $N > 4$, $D_1 \cup \Delta$ must be ∂-parallel in M. Thus, $D_2 \cup \Delta$ is isotopic to D. After perturbing $D_2 \cup \Delta$ a little we get a disk isotopic to D but has fewer intersection arcs with E, a contradiction. So we may assume $D_i \cap \partial S$ contains more than two points for both $i = 1, 2$. Therefore, $D_i \cup \Delta$ is a disk whose intersection with ∂S contains fewer points than $|\partial D \cap \partial S| = N$. Since at least one of $D_1 \cup \Delta$ and $D_2 \cup \Delta$ is not ∂–parallel, we get a contradiction to the assumption that $N = |\partial D \cap \partial S|$ is minimal. □

The claim reduces the argument to two cases:

<u>Case 1</u>: $N = 2$. Any bigon D in M can be viewed as a product disk as follows. Let A be a small neighborhood of ∂S in $F - int(S)$, and we denote the closure of $F - (S \cup A)$ by S'. Then we can view D as a product $I \times I$ with $\partial I \times I$ a pair of vertical arcs in A and $I \times \partial I$ a pair of arcs in S and S'.

Since $F - \partial S$ is incompressible in M, as in [3, 4], M has a characteristic submanifold J such that any bigon disk D can be isotoped into J. Since ∂S is separating and $\partial J \cap A \neq \emptyset$, J is a product of a compact surface and an interval I.

If $J = M$, then M is a product $S \times I$ and the first claim of the Main Theorem holds. Otherwise, $J \cap S$ is a nontrivial subsurface of S whose frontier consists of essential arcs and closed curves in S. Let α be any arc or closed curve in the frontier of $J \cap S$. Since ∂S is disk-busting and $J \neq M$, α is non-peripheral in S. For any bigon (or product disk) D, since D can be isotoped into J, the distance between α and $D \cap S$ in $\mathcal{AC}(S)$ is at most one. Suppose there is another compressing disk E with $d(\pi_A(E), \alpha) > 4$ in $\mathcal{AC}(S)$. Let D be a bigon such that $|D \cap E|$ is minimal among all the nontrivial bigon disks. So $d(\pi_A(E), D \cap S) \geq d(\pi_A(E), \alpha) - d(\alpha, D \cap S) \geq 4$ and by Proposition 2.1, $D \cap S$ intersects every component of $E \cap S$.

We consider $D \cap E$. Let δ be a component of $D \cap E$ that is outermost in E. As before, we view E as a polygon with $E \cap \partial S$ being its vertices, and we may assume

$|E \cap \partial S|$ and $|E \cap D|$ are minimal in their isotopy classes. If $\partial\delta$ lies in the same edge of ∂E, since D is a bigon, $\partial\delta$ lies in the same edge of ∂D. As in the proof of the claim, we may perform an isotopy to eliminate the intersection arc δ. So we may assume that the endpoints of δ lie in different edges of E.

Let δ' be a subarc of ∂E with $\partial\delta = \partial\delta'$ and Δ the subdisk of E bounded by $\delta \cup \delta'$. Since δ is outermost, we may choose δ' so that $int(\Delta) \cap D = \emptyset$. As the endpoints of δ lie in different edges of E, $\delta' \cap \partial S \neq \emptyset$. Since $D \cap S$ intersects every α–edge of ∂E, as in the proof of the claim, Δ is either a triangle ($\delta' \cap \partial S$ is a single point) or a quadrilateral ($\delta' \cap \partial S$ consists of two point and δ' contains the whole of a β–edge). We use D_1 and D_2 to denote the two subdisks of D divided by δ. If Δ is a triangle, then δ must be an arc in D which separates the vertices i.e., a vertical arc, and both $D_1 \cup \Delta$ and $D_2 \cup \Delta$ are bigon disks properly embedded in M. Since D is a compressing disk, at least one of $D_1 \cup \Delta$ and $D_2 \cup \Delta$ is not ∂-parallel. After some small perturbation, both $D_1 \cup \Delta$ and $D_2 \cup \Delta$ have fewer intersection arcs with E than $D \cap E$, which contradicts the assumption that $|D \cap E|$ is minimal among all the nontrivial bigon disks D. Therefore, Δ must be a quadrilateral.

Since Δ is a quadrilateral, both endpoints of δ must lie in the α–edges of ∂E. Hence $\partial\delta$ lies in the α–edge of ∂D and δ is a ∂-parallel arc in the bigon D. Let D_1 be the subdisk of D bounded by δ and a subarc of $D \cap S$. Then $D_1 \cup \Delta$ is a bigon disk properly embedded in M. If $D_1 \cup \Delta$ is a ∂-parallel disk in M, then the β–edge of ∂E that lies in δ' must be a ∂-parallel arc in $F - int(S)$, which contradicts our assumption that $|\partial E \cap \partial S|$ is minimal in the isotopy class of E. Thus the bigon $D_1 \cup \Delta$ is a compressing disk. However, after perturbing $D_1 \cup \Delta$ a little, $(D_1 \cup \Delta) \cap E$ has fewer components than $D \cap E$, which again contradicts the assumption that $|D \cap E|$ is minimal among all the nontrivial bigons.

The arguments above implies that if $N = 2$, then for any compressing disk E, there is a bigon D such that the distance between $\pi_A(D)$ and $\pi_A(E)$ is less than 4 in $\mathcal{AC}(S)$. Since $\pi_A(D)$ has distance at most one from a fixed frontier curve α, we have that $d(\pi_A(E), \alpha) \leq 4$ for all compressing disks E and the diameter of $\pi_A(\mathcal{D})$ is at most 8.

<u>Case 2</u>: $N = 4$. The proof is similar. For any quadrilateral compressing disk D of M, we call a properly embedded arc δ in D a *vertical arc* if the two endpoints of δ lie in different α–edges of ∂D.

Let D_1 be a quadrilateral compressing disk and let E be another compressing disk such that $d(\pi_A(E), \pi_A(D_1)) \geq 4$. By Proposition 2.1, each α–edge of ∂D_1 intersects every α–edge of ∂E. We may suppose $|E \cap D_1|$ is minimal in the isotopy classes of E and D_1. Let δ be an arc in $E \cap D_1$ that is outermost in E. Let δ' be the subarc of ∂E such that $\partial\delta = \partial\delta'$ and $\delta \cup \delta'$ bounds a subdisk Δ of E with $int(\Delta) \cap D_1 = \emptyset$. Since each α–edge of ∂D_1 intersects every α–edge of ∂E, as before, Δ is either a triangle ($\delta' \cap \partial S$ is a single point) or a quadrilateral ($\delta' \cap \partial S$ has two points and δ' contains the whole of a β–edge of ∂E). If Δ is a triangle, then one endpoint of δ lies in an α–edge and the other endpoint lies in a β–edge. Since D_1 is a quadrilateral disk, δ cuts off a corner (i.e., a triangle) Δ_1 from D_1 and $\Delta \cup \Delta_1$ is a bigon disk properly embedded in M. Since $N = 4$, $\Delta \cup \Delta_1$ must be ∂-parallel in M. Hence $(D_1 - \Delta_1) \cup \Delta$ is isotopic to D_1. After a slight perturbation, $(D_1 - \Delta_1) \cup \Delta$ has fewer intersection arcs with E, which contradicts that $|E \cap D_1|$ is minimal. Thus, Δ must be a quadrilateral, i.e., δ' contains the whole of a β–edge and $\partial\delta$ lies in α–edges.

As $\partial\delta$ lies in α–edges, δ is either a vertical arc in D_1 or an arc with both endpoints in the same α–edge of ∂D_1. In the later case, we have a subarc δ_1 of an α–edge of ∂D_1 such that $\delta \cup \delta_1$ bound a subdisk Δ_1 of D_1. Since Δ is a quadrilateral, $\Delta \cup \Delta_1$ is a bigon disk properly embedded in M. Since $N = 4$, $\Delta \cup \Delta_1$ must be ∂-parallel in M. As before, this implies that the β–edge in δ' must be ∂-parallel in $F - int(S)$, which contradicts that $|E \cap \partial S|$ is minimal. So δ must be a vertical arc in D_1.

The arc δ cuts D_1 into two subdisks, which we denote by X_1 and X_2. Since δ is vertical, both $X_1 \cup \Delta$ and $X_2 \cup \Delta$ are quadrilateral disks, at least one of which is not ∂-parallel in M. Suppose $X_1 \cup \Delta$ is not ∂-parallel. We can perturb $X_1 \cup \Delta$ into a quadrilateral disk D_2 so that $D_2 \cap D_1 = \emptyset$ and $|D_2 \cap E| < |D_1 \cap E|$.

If the distance between $\pi_A(D_2)$ and $\pi_A(E)$ is also at least 4, then we can repeat the argument above, replacing D_1 by D_2, and construct a new essential quadrilateral disk D_3. By the construction, $D_3 \cap D_2 = \emptyset$ and more importantly $D_1 \cap D_3$ consists of vertical arcs in both D_1 and D_3. These vertical arcs $D_1 \cap D_3$ are all parallel and close to some arcs of $D_1 \cap E$.

Therefore, we can inductively construct a finite set of quadrilateral compressing disks, $D_1, D_2, \dots, D_n$, such that

(1) $d(\pi_A(D_n), \pi_A(E))$ is at most 3,
(2) For any i, j, $D_i \cap D_j$ consists of vertical arcs in both D_i and D_j, and every arc in $D_i \cap D_j$ is parallel and close to an arc of $D_1 \cap E$.

Note that by our construction, property (2) above implies that the intersections of $D_1, D_2, \dots, D_n$ have no triple point. Therefore, a regular neighborhood of $\bigcup_{k=1}^{n} D_k$ is an I–bundle J, where each I–fiber of J is parallel and close to a vertical arc of some D_i.

We call $J \cap S$ the horizontal boundary of J and denote it by $\partial_h J$. Clearly, $\partial_h J \subset S$ is a neighborhood of $\bigcup_{k=1}^{n}(D_k \cap S)$ in S. We define the vertical boundary of J to be $\partial J - int(\partial_h J)$ and denote it by $\partial_v J$. Note that $J \cap (F - int(S))$ consists of neighborhoods of the β–edges of the D_i's. The closure of $\partial_v J - \partial M$, which we denote by $\overline{\partial_v J - \partial M}$, consists of quadrilateral disks and annuli properly embedded in M.

Now we consider $\partial_h J$ as a subsurface of S. Let η be a component of the frontier of $\partial_h J$. So η is either a closed curve or a properly embedded arc in S. We first consider the case that η is a trivial closed curve in S. In this case, η bounds a disk Δ_η in S and $\Delta_\eta \cap J = \eta$. Let A_η be the component of $\partial_v J$ that contains η. Clearly A_η is an annulus. Let $\eta' = \partial A_\eta - \eta$ be the other boundary component of A_η. Since η' bounds the disk $\Delta_\eta \cup A_\eta$ and S is incompressible, η' must also bound a disk, say Δ'_η, in S. Since M is irreducible, $\Delta_\eta \cup A_\eta \cup \Delta'_\eta$ is a 2–sphere bounding a 3–ball in M. The union of J and this 3-ball is also an I–bundle.

Next we consider the case that η is a ∂-parallel arc in S. In this case, there is a subarc of ∂S, say ζ, such that $\eta \cup \zeta$ bounds a disk Δ_η in S and $\Delta_\eta \cap J = \eta$. Since J is an I–bundle, the component of $\overline{\partial_v J - \partial M}$ that contains η is a quadrilateral disk properly embedded in M. We denote this quadrilateral disk by Q and clearly η is an α–edge of ∂Q. Let $\eta' = (Q \cap S) - \eta$ be the other α–edge of ∂Q. If Q is not ∂-parallel in M, since η is ∂-parallel in S, we can perform an isotopy on Q by pushing η across Δ_η into $F - S$. After this isotopy, Q becomes an essential bigon disk, which contradicts that $N = 4$. Therefore, Q is ∂-parallel in M, and Q and a subdisk of $F \subset \partial M$ bound a 3–ball in M. Similar to the previous case, the union of J and this 3-ball is an I–bundle.

If every component η of the frontier of $\partial_h J$ is trivial in S, then by adding 3-balls to J as above, we conclude that the whole M is an I–bundle of which S is its horizontal boundary. In this case, $\partial M - S$ consists of annuli and M is a twisted I–bundle over a non-orientable surface.

Thus we may assume that at least one component, say $\hat{\eta}$, of the frontier of $\partial_h J$ in S is an essential arc or closed curve. Since ∂S is disk-busting and $J \neq M$, $\hat{\eta}$ is non-peripheral in S. Since each D_k $(k = 1, \dots, n)$ lies in J, $d(\pi_A(D_k), \hat{\eta}) \leq 1$. Since $d(\pi_A(E), \pi_A(D_n)) \leq 3$, we have

$$d(\pi_A(E), \pi_A(D_1)) \leq d(\pi_A(E), \pi_A(D_n)) + d(\pi_A(D_n), \hat{\eta}) + d(\hat{\eta}, \pi_A(D_1)) \leq 5.$$

Since D_1 is a fixed essential quadrilateral disk and E is an arbitrary compressing disk, the diameter of $\pi_A(\mathcal{D})$ in $\mathcal{AC}(S)$ is at most 10.

Next we show that the image $\pi_C(\mathcal{D})$ has diameter at most 20 in the curve complex $\mathcal{C}(S)$. We may assume that S is not an annulus or a pair of pants, since the curve complex is empty in the two cases.

Given two arcs or closed curves α and β in S with distance one in $\mathcal{AC}(S)$, i.e., $\alpha \cap \beta = \emptyset$, the map π_0 (see section 1 for the definition of π_0) sends them to a pair of closed curves in S. Recall that $\pi_0(\alpha)$ is chosen to be a non-peripheral boundary component of $N(\partial S \cup \alpha)$. So $\pi_0(\alpha) \cap \pi_0(\beta)$ has at most two intersection points.

If $\pi_0(\alpha) \cap \pi_0(\beta)$ is a single point, a regular neighborhood $N(\pi_0(\alpha) \cup \pi_0(\beta))$ is a once-punctured torus. If S is not a once-punctured torus, then $\partial N(\pi_0(\alpha) \cup \pi_0(\beta))$ is an essential non-peripheral curve which is disjoint from both $\pi_0(\alpha)$ and $\pi_0(\beta)$. Hence $d(\pi_0(\alpha), \pi_0(\beta)) \leq 2$ in $\mathcal{C}(S)$. If S is a once-punctured torus, by definition $\mathcal{C}(S)$ is the Farey graph and $d(\pi_0(\alpha), \pi_0(\beta)) = 1$.

If $\pi_0(\alpha) \cap \pi_0(\beta)$ has two points, by the construction of $\pi_0(\alpha)$ and $\pi_0(\beta)$, a regular neighborhood $N(\pi_0(\alpha) \cup \pi_0(\beta))$ must be a 4-hole sphere. If S is not a 4-hole sphere, then a component of $\partial N(\pi_0(\alpha) \cup \pi_0(\beta))$ is an essential non-peripheral curve which is disjoint from both $\pi_0(\alpha)$ and $\pi_0(\beta)$. Hence $d(\pi_0(\alpha), \pi_0(\beta)) \leq 2$ in $\mathcal{C}(S)$. If S is a 4-hole sphere, then by definition $\mathcal{C}(S)$ is the Farey graph and $d(\pi_0(\alpha), \pi_0(\beta)) = 1$.

Therefore in any case, if two vertices u and v have distance k in $\mathcal{AC}(S)$, then $d(\pi_0(u), \pi_0(v)) \leq 2k$ in $\mathcal{C}(S)$. Thus the diameter of $\pi_C(\mathcal{D})$ is at most 20 in $\mathcal{C}(S)$. □

References

[1] W. J. Harvey, *Boundary structure of the modular group*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (I. Kra, B. Maskit eds.), Ann. of Math. Stud. **97**, Princeton (1981) 245-251.

[2] John Hempel, *3–manifolds as viewed from the curve complex.* Topology, **40** (2001) 631–657.

[3] William Jaco, *Lectures on Three-Manifold Topology*, CBMS Regional Conference Series in Mathematics, **43** (1977).

[4] Tao Li, *Immersed essential surfaces in hyperbolic 3-manifolds*, Communications in Analysis and Geometry **10** (2002), 275 - 290.

[5] Tao Li, *On the Heegaard splittings of amalgamated 3–manifolds.* Preprint available at http://www2.bc.edu/˜taoli/amalg.pdf

[6] Howard A. Masur, Yair N. Minsky, *Geometry of the Complex of Curves I: Hyperbolicity.* Invent. Math. **138** (1999), 103–149.

[7] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves II: Hierarchical structure.* Geom. Funct. Anal. **10** (2000), 902–974.

[8] Howard A. Masur and Yair N. Minsky. *Quasiconvexity in the curve complex.* In the tradition of Ahlfors and Bers, III, 309–320, Contemp. Math., 355, Amer. Math. Soc., Providence, RI, 2004.

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