

## An Empirical Comparison of Stochastic Dominance among Lognormal Prospects

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### 1. Introduction

The theory of portfolio selection and diversification developed by Markowitz [22] and Tobin [33] was based primarily on the criterion of mean-variance (MV) efficiency. The objective was to select an efficient set of portfolios from which every risk averter will choose the optimal portfolio which maximizes his expected utility. The MV criterion is the appropriate rule either for the case in which the utility function is quadratic or if the returns are normally distributed and risk aversion is assumed.<sup>1</sup>

Two approaches to the choice among risky alternatives that have been developed independently over the past two decades are the geometric mean criterion (GM) and the stochastic dominance (SD) decision model. Both can be justified by the expected utility hypothesis, with the geometric mean criterion following as a result of the assumption that the decision-maker has a logarithmic utility function, and the stochastic dominance models requiring the less restrictive assumption of signs of the first few derivatives of the decision-maker's utility function. This paper compares the concept of ordinary stochastic dominance to stochastic dominance of the lognormally distributed prospects from an empirical point of view. Further light is also shed on the issue of applying the Markowitz-Tobin mean-variance rule to the lognormal distribution.

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<sup>1</sup>Recently, Levy and Markowitz [18] have argued that this is not necessary.

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## II. Review of Literature

One criterion for portfolio selection which has received a great deal of attention in the economic literature is the maximization of the geometric mean return as evidenced by the research of Latané [14], Samuelson [30, 31], Hakansson [9], and Latané and Tuttle [15]. Latané [14] justified the geometric mean criterion on the basis of the maximization of terminal wealth in a many-period reinvestment process, the optimal growth approach, and also pointed out that the criterion maximizes expected utility if the Bernoulli [2] logarithmic utility function is assumed. This application is useful especially for comparison between prospects taken from the stock market, provided that the investment horizon is relatively long; if it is, returns, being the cumulative products of random variables, tend to distribute lognormally.

Empirical evidence that investment returns are lognormally distributed for portfolios is provided by Cootner [4], Kendall [13], Moore [25], Osborne [26], and others.<sup>2</sup> These authors have also provided evidence that investment returns for individual securities are lognormally distributed. One set of theoretical conditions under which lognormally distributed portfolio returns are consistent with lognormally distributed security returns has been set forth by Merton [24]. The most recent study on the subject of lognormality of security returns, portfolio selection, and market equilibrium will be found in Lintner's long manuscript [20]. However, Fama [5] and Mandelbrot [21] called attention to the fact that there was substantial deviation from normality in the distribution of the log of stock price changes.<sup>3</sup>

The notion of stochastic dominance has been used for many years, but modern development of the approach is usually credited to Quirk and Saposnik [29], Hadar and Russell [8], Hanoch and Levy [10], and Whitmore [34].

Let  $F$  and  $G$  denote the cumulative distribution of the returns on two prospects.  $U_1$  is the set of all nondecreasing utility functions, and  $U_2$  is the set of all nondecreasing concave utility functions. That is to say,  $\mu \in U_1$  if and only if  $\mu' \geq 0$ , and  $\mu \in U_2$  if and only if  $\mu' \geq 0$ ,  $\mu'' \leq 0$ . We say that  $F$  dominates  $G$  in  $U_i$  ( $i = 1, 2$ ) if  $E_F U(x) \geq E_G U(x)$  for all  $\mu \in U_i$ .

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<sup>2</sup>Additional studies of daily stock returns will be found in [27], [28], and, most recently, in [3].

<sup>3</sup>While the resolution of this debate is beyond the scope of this paper, there is sufficient evidence to suggest that deviations from the lognormal distribution are nonexistent or of too small an order of magnitude to make the lognormal case worth further study.

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( $i = 1, 2$ ) where at least for one  $U(x)$ , a strict inequality holds.

Theorem 1. Let  $F(x)$  and  $G(x)$  be two cumulative distributions of two distinct portfolios; then, the necessary and sufficient conditions for dominance of  $F$  over  $G$  in  $U_1, U_2$  are given by I and II, respectively.

- I.  $F(x) \leq G(x)$ , for all  $x$  with at least one strict inequality. (For proof see [8], [10], [29].)
- II.  $\int_{-\infty}^x F(t) dt \leq \int_{-\infty}^x G(t) dt$ , for all  $x$  with at least one strict inequality. (For proof see [8], [10].)

Jean [12] has shown that any degree of stochastic dominance, including first-degree stochastic dominance (FSD), implies a geometric mean ranking, and, therefore, the geometric mean ranking is a necessary condition for a stochastic dominance ranking. It is not true for general distributions that the geometric mean ranking is a sufficient condition for a stochastic dominance ranking. If a portfolio can be found that has a higher geometric mean of terminal values than any other portfolio, then Latané's [14] subgoal<sup>4</sup> may be met by that portfolio, but it cannot be concluded that the subgoal is met by that portfolio without further information.

Levy [17] studied the lognormal distribution and concluded that two conditions--a ranking of the geometric mean and equality of the variance of the logarithms of the variables--were sufficient for stochastic dominance.

Theorem 2. Let  $F$  and  $G$  be the cumulative distributions of two alternative lognormal prospects. A necessary and sufficient condition for  $E_F U(x) > E_G U(x)$  within  $U_1$  where  $U_1$  is the class of all nondecreasing functions and  $U_2$  where  $U_2$  is the class of all nondecreasing concave utility functions, are given by I and II, respectively.

- I. a)  $E_F \text{Log}(x_i) > E_G \text{Log}(x_i)$ ,  
b)  $\text{Var}_F \text{Log}(x_i) = \text{Var}_G \text{Log}(x_i)$ .  
(For proof see Levy [17].)
- II. a)  $E_F \text{Log}(x_i) \geq E_G \text{Log}(x_i)$ ,  
b)  $\text{Var}_F \text{Log}(x_i) \leq \text{Var}_G \text{Log}(x_i)$ ,  
c)  $E_F \text{Log}(x_i) - E_G \text{Log}(x_i) \geq \frac{1}{2}[\text{Var}_G \text{Log}(x_i) - \text{Var}_F \text{Log}(x_i)]$ .

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<sup>4</sup>Latané's principal justification for the geometric mean criterion in [14, p. 146] was based on his choice of subgoal:

The subgoal proposed here is the choice of the portfolio that has a greater probability ( $P'$ ) of being as valuable or more valuable than any other significantly different portfolio at the end of  $n$  years,  $n$  being large.

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where at least one of the above inequalities (a) and (b) is strict.  
(For proof see Levy [17].)<sup>5</sup>

The mean-variance analysis is applied mainly to one-period investments. When the number of investment periods is very large, Latané [14] and others propose to replace the mean-variance rule by the geometric mean criterion. Samuelson [30], on the other hand, has shown that even for very long investment sequences the geometric mean rule does not necessarily dominate all other policies. Samuelson [31] suggests, however, that looking at the pair  $E \text{ Log } (x_i)$ ,  $\text{Var Log } (x_i)$  rather than  $E \text{ Log } (x_i)$  alone, (as the geometric mean rule does), improves the decision-making process. Thus, Samuelson's procedure could be summarized as follows:

Portfolio F dominates portfolio G if, and only if,

$$\begin{aligned} E_F \text{ Log } (x_i) &\geq E_G \text{ Log } (x_i), \\ \text{Var}_F \text{ Log } (x_i) &\leq \text{Var}_G \text{ Log } (x_i) \end{aligned}$$

where at least one of the strong inequalities holds. Levy [17] has shown that, in terms of expected utility defined on terminal wealth, that even if the Samuelson conditions are met, distribution F does not necessarily truly dominate distribution G for all risk averters.

Feldstein [6] has shown that when the returns are lognormally distributed, the mean-variance rule is a sufficient, but not a necessary investment criterion. He concluded that the application of the mean-variance rule to the lognormal distribution might lead to paradoxical results. Levy [17] has shown that the only possible damage might be the generation of an extremely large efficient set. He pointed out that the mean-variance rule is an optimal rule for lognormal distributions provided that:

$$\begin{aligned} E_F (x_i) &\geq E_G (x_i), \\ \text{Var}_F \text{ Log } (x_i) &\leq \text{Var}_G \text{ Log } (x_i). \end{aligned}$$

(For proof see Levy [17].)

### III. Data Description

For our empirical comparisons of ordinary stochastic dominance to stochastic dominance of the lognormally distributed prospects, one set of 500 portfolios was analyzed. Each portfolio contained 20 securities with an equal investment in each security. These portfolios were generated randomly

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<sup>5</sup> $x_i$  represents holding period rate of return; that is, one plus the rate of return in the investment in period  $i$ .

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with replacement from the set of 685 stocks with complete data for the 1961-1976 period.

The data used were taken from the University of Chicago Center for Research in Security Price Relative File which contains monthly price, dividend, and adjusted price and dividend information for all securities listed on the New York Stock Exchange in the period January 1961-December 1976. The time series of holding-period returns were computed by adding the end-of-the-month market value to the distributions (dividends) paid during the month and dividing the sum by the first-of-the-month market value. No distinction was made between capital gain and ordinary income distribution, and no adjustment was made for sales or redemption fees.

#### IV. Empirical Results

The sample distribution of 192 months' returns for each portfolio was examined for consistency with a lognormal distribution.<sup>6</sup> Distributions were tested for lognormality using the Kolmogorov-Smirnov test (K-S) [16] for goodness-of-fit. The K-S test is the only goodness-of-fit test that is appropriate since the variables are measured on a continuous scale and are ungrouped [7, p. 86-87]. For the K-S test, the null hypothesis is that the distribution of each portfolio is lognormal.

The K-S test is an exact test when all parameters of the distribution are known; however, if the mean and variance of the distribution must be estimated using sample data, the K-S test is conservative in that the probability of a type I error will be smaller than that given by the usual table [23, p. 73]. Indeed when the K-S values were compared to critical values from the usual K-S table, the hypothesis that each portfolio is lognormal was not rejected once in the 500 cases. This certainly suggests that the K-S test is quite conservative. A table developed by Lilliefors [19] using Monte Carlo simulation gives estimated critical values for a sample size of 40 or less when the mean and variance of the distribution must be estimated. When the sample size is much greater than 40, the procedure for estimating the critical value given by Lilliefors produces inaccurate critical values. A procedure for developing critical values of the K-S test for any sample size was developed by Helms, Mansfield, and Strong [11]. Using this procedure, a critical value of sample size of 192 was calculated. At the 5 percent significance level the null hypothesis was rejected for 24 of the 500 portfolios.

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<sup>6</sup>Using the log of the return is equivalent to using the rate of return assuming continuous compounding as in [1], [5], [32].

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Therefore, the distribution of returns of the portfolios seems to be approximately lognormal.

Our analysis indicates that of the 500 portfolios examined (without transformation to lognormal return), 11 portfolios are efficient by the Markowitz-Tobin (MV) rule and 72 portfolios by the second-degree stochastic dominance (using Theorem 1, Part II). The first-degree stochastic dominance efficient set includes virtually all of the 500 portfolios investigated. Since the efficiency of our algorithms is a function of the frequency with which portfolios are eliminated, a complete test of the FSD rule on all 500 portfolios could not be run with monthly return data within a reasonable time period and were hence omitted.

Of more importance, however, since the distribution of 500 portfolio returns followed a lognormal distribution, Theorem 2 was applied to the 500 portfolios and yielded the results in Table 1.

Clearly, since the returns are lognormally distributed, there is no need to calculate higher moments of  $\text{Log}(x_i)$  for dominance of F over G. Thus, from the viewpoint of minimizing computer time and the corresponding costs, applying lognormal stochastic dominance is more efficient than using the regular stochastic dominance criteria.

From an examination of Table 1, it can be seen that Levy's necessary conditions for FSD under lognormal distribution is very ineffective and that the efficient set is as large as the feasible set. Levy's necessary conditions for SSD under lognormal distribution contain 20 portfolios; that is, 4 percent of the total number of portfolios examined.

One should note two major results. First, the computer time necessary for running the test is minimized. While under FSD the test on all 500 portfolios could not be run within a reasonable time, Levy's necessary conditions for FSD under lognormal distribution took less than two minutes of computer time. Second, Levy's necessary conditions for second-degree stochastic dominance with lognormal distribution applied to the data set eliminates 52 more portfolios from the efficient set than SSD with an unknown distribution. Or more clearly, the second-degree stochastic dominance set contains 100 percent of Levy's necessary conditions for SSD with lognormal distribution set. Since these same 20 are almost 28 percent of the total membership of the second-degree stochastic dominance efficient set, we conclude that the major difference in Levy's necessary conditions for lognormal distribution and SSD efficiency is that Levy's necessary conditions for SSD lognormal distribution set excludes a number of portfolios that are included in the SSD efficient set.

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Evaluation of other differences in the efficient sets obtained by the Markowitz-Tobin mean-variance (in terms of  $x_i$  rather than  $\text{Log}(x_i)$ ) rule needs further attention in the case of lognormal distribution of returns and risk aversion.

Our empirical analysis (Table 2) indicates that of the 500 portfolios examined, 9 were efficient by applying Samuelson's criterion [ $E \text{Log}(x_i)$ ,  $\text{Var Log}(x_i)$ ], and 10 were efficient by applying Levy's criterion [ $E(x_i)$ ,  $\text{Var Log}(x_i)$ ]. As previously mentioned, the [ $E(x_i)$ ,  $\text{Var}(x_i)$ ] rule contained 11 portfolios in the efficient set.

Further consideration shows that the 10 efficient portfolios by Levy's criterion and 9 efficient portfolios obtained by Samuelson's criterion are also efficient by the Markowitz mean-variance rule. Thus, the Markowitz mean-variance efficient set contains 100 percent of Levy's and Samuelson's mean-variance efficient set. Moreover, the 9 efficient portfolios by Samuelson's criterion (i.e.,  $E \text{Log}(x_i)$ ,  $\text{Var Log}(x_i)$ ) are also efficient by Levy's criterion [ $E(x_i)$ ,  $\text{Var Log}(x_i)$ ]. However, both Levy's and Samuelson's efficient portfolios are subsets of the Markowitz efficient portfolios. In addition, Levy's efficient set is also a subset of Samuelson's efficient set.

#### V. Conclusion

In this paper, the distribution of 500 randomly selected portfolio returns were tested for lognormal distributions. In most portfolios, the lognormality assumption was confirmed. Secondly, Levy's Theorems 4 and 5 [17] were applied to the 500 portfolios and it was found that the efficient set in one case is the same and, in another case, is smaller than the efficient set under FSD and SSD. The fact that the portfolios are distributed lognormally allows the application of Levy's Theorems, which reduces the size of the SSD efficient set and reduces the computer time needed to find the efficient set. Finally, the empirical results confirm that the application of the Markowitz mean-variance rule (i.e.,  $E(x)$ ,  $\text{Var}(x)$ )--which is a sufficient but not a necessary criterion--to the lognormal distribution will result in a larger efficient set than the optimal rule [ $E(x)$ ,  $\text{Var Log}(x)$ ] as pointed out by Levy [17].

TABLE 1

SIZE OF THE STOCHASTIC DOMINANCE EFFICIENT SETS  
FOR 500 PORTFOLIOS LOGNORMALLY DISTRIBUTED

	Theorem 2.1	Theorem 2.11	Theorem 1.11
Efficiency Criteria	$E_F \log(x_i) > E_G \log(x_i)$ $\text{Var}_F \log(x_i) = \text{Var}_G \log(x_i)$	$E_F \log(x_i) \geq E_G \log(x_i)$ $\text{Var}_F \log(x_i) \leq \text{Var}_G \log(x_i)$ $E_F \log(x_i) - E_G \log(x_i) \geq \frac{1}{2} [\text{Var}_G \log(x_i) - \text{Var}_F \log(x_i)]$	$\int_{-\infty}^x F(t) dt \leq \int_{-\infty}^x G(t) dt$
Number of Portfolios in Efficient Set	500	20	72

TABLE 2

SIZE OF THE MEAN VARIANCE EFFICIENT SETS FOR 500 PORTFOLIOS LOGNORMALLY DISTRIBUTED

	Samuelson's Criterion	Levy's Criterion	Markowitz Criterion
Efficiency Criteria	$E_F \log(x_i) \geq E_G \log(x_i)$ $\text{Var}_F \log(x_i) \leq \text{Var}_G \log(x_i)$	$E_F(x_i) \geq E_G(x_i)$ $\text{Var}_F \log(x_i) \leq \text{Var}_G \log(x_i)$	$E_F(x_i) \geq E_G(x_i)$ $\text{Var}_F(x_i) \leq \text{Var}_G(x_i)$
Number of Portfolios in Efficient Set	9	10	11

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REFERENCES

- [1] Barnes, Amir, and David H. Downes. "A Reexamination of the Empirical Distribution of Stock Price Changes." *Journal of the American Statistical Association* (June 1973), pp. 348-350.
- [2] Bernoulli, Daniel. "Exposition of a New Theory on the Measurement of Risk." *Econometrica* (January 1954), pp. 23-36.
- [3] Blattberg, Robert C., and Nicholas J. Gonedes. "A Comparison of the Stable and Student Distributions as Statistical Models of Stock Prices." *Journal of Business* (April 1974), pp. 244-280.
- [4] Cootner, P. H., ed. *The Random Character of Stock Market Prices*. Cambridge, MA: MIT Press (1964).
- [5] Fama, Eugene F. "The Behavior of Stock Market Prices." *Journal of Business* (January 1965), pp. 34-105.
- [6] Feldstein, M. S. "Mean-Variance Analysis in the Theory of Liquidity Preference and Portfolio Selection." *Review of Economic Studies* (January 1969), pp. 5-12.
- [7] Gibbons, J. D. *Nonparametric Statistical Inference*. NY: McGraw-Hill (1971).
- [8] Hadar, J., and W. R. Russel. "Rules for Ordering Uncertain Prospects." *American Economic Review* (March 1969), pp. 25-34.
- [9] Hakansson, Nils H. "Capital Growth and the Mean-Variance Approach to Portfolio Selection." *Journal of Financial and Quantitative Analysis* (January 1971), pp. 517-557.
- [10] Hanoch, G., and H. Levy. "The Efficiency Analysis of Choices Involving Risk." *Review of Economic Studies* (July 1969), pp. 335-346.
- [11] Helms, B. P.; E. R. Mansfield; and G. Q. Strong. "Distribution of Returns on Common Stock." Working Paper presented at the American Statistical Association meeting in Detroit (August 1981).
- [12] Jean, W. H. "The Geometric Mean and Stochastic Dominance." *The Journal of Finance* (March 1980), pp. 151-158.
- [13] Kendall, M. G. "The Analysis of Economic Time Series." *Journal of the Royal Statistical Society, Ser. A*, XCVI (1953), pp. 11-25.
- [14] Latané, Henry. "Criteria for Choice among Risky Ventures." *Journal of Political Economy* (April 1959), pp. 144-155.
- [15] Latané, Henry, and Donald L. Tuttle. "Criteria for Portfolio Building." *Journal of Finance* (September 1967), pp. 359-373.
- [16] Leitch, R. A., and A. S. Paulson. "Estimation of Stable Law Parameters: Stock Price Behavior Application." *Journal of the American Statistical Association* (September 1975), pp. 690-697.
- [17] Levy, H. "Stochastic Dominance among Log-normal Prospects." *International Economic Review* (October 1973), pp. 601-614.

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- [18] Levy, H., and H. M. Markowitz. "Approximating Expected Utility by a Function of Mean and Variance." *American Economic Review* (June 1979), pp. 308-317.
- [19] Lilliefors, H. W. "On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown." *Journal of the American Statistical Association*, Vol. 62, No. 318 (June 1967), pp. 399-402.
- [20] Lintner, John. "The Lognormality of Security Returns, Portfolio Selection and Market Equilibrium." Working Paper, Graduate School of Business Administration, Harvard University (1980).
- [21] Mandelbrot, Benoit. "The Variation of Speculative Prices." *The Journal of Business* (October 1963), pp. 394-410.
- [22] Markowitz, H. M. *Portfolio Selection*, Cowles Foundation Monograph 16. NY: John Wiley and Sons, Inc. (1959).
- [23] Massey, F. J., Jr. "The Kolmogorov-Smirnov Test for Goodness of Fit." *Journal of the American Statistical Association*, Vol. 45, No. 253 (March 1951), pp. 68-78.
- [24] Merton, Robert. "Optimal Consumption and Portfolio Rules in a Continuous Time Model." *Journal of Economic Theory* (December 1971), pp. 373-413.
- [25] Moore, Arnold. "A Statistical Analysis of Common Stock Prices." Unpublished Doctoral Dissertation, Graduate School of Business, University of Chicago (1962).
- [26] Osborne, M. F. M. "Brownian Motion in the Stock Market." *Operations Research* (March 1959), pp. 145-173.
- [27] Praetz, P. D. "The Distribution of Share Price Changes." *Journal of Business* (January 1972), pp. 49-55.
- [28] Press, S. J. "A Compound Events Model for Security Prices." *Journal of Business* (July 1967), pp. 317-335.
- [29] Quirk, J. P., and R. Saposnik. "Admissibility and Measurable Utility Function." *Review of Economic Studies* (February 1962), pp. 104-144.
- [30] Samuelson, Paul. "Lifetime Portfolio Selection by Dynamic Stochastic Programming." *Review of Economics and Statistics* (August 1969), pp. 139-146.
- [31] \_\_\_\_\_. "The Fallacy of Maximizing the Geometric Mean in Long Sequences in Investing or Gambling." *Proceeding-National Academy of Science* (October 1971), pp. 2493-2496.
- [32] Teichmoeller, John. "A Note on the Distribution of Stock Price Changes." *Journal of the American Statistical Association* (June 1971), pp. 282-284.
- [33] Tobin, J. "Liquidity Preference as Behavior towards Risk." *Review of Economics Studies* (February 1958), pp. 65-85.
- [34] Whitmore, G. A. "Third Degree Stochastic Dominance." *American Economic Review* (June 1970), pp. 457-459.