THE RANK EXTENDED TRANSLOG

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1. Rank Three, Quadratic Budget Share, Exactly Aggregable, IB Demand Systems

Let total expenditure be M, and prices be p. Having budget shares be linear in $\ln M$ (corresponding to PIGLOG demands or Working Leser Engel curves) requires indirect utility functions of the form

$$\ln \widetilde{U} = \theta_0(p) + \theta_1(p) \ln M$$

for some functions $\theta_0(p)$ and $\theta_1(p)$. This class of demand systems, while fitting many data sets reasonably well, is rank two. Many authors have found that empirical demands are actually rank three. More specifically, Banks, Blundell, and Lewbel (1997) found that while budget shares linear in $\ln M$ fits demands for some goods quite well, other goods appear to have budget shares quadratic in $\ln M$. They also show (based closely on Gorman 1981) that utility derived rank three demand systems in which some goods have linear budget shares and others quadratic must have indirect utility functions of the form

$$\begin{aligned}(\ln U)^{-1} &= (\ln \widetilde{U})^{-1} + \theta_2(p) \\ &= [\theta_0(p) + \theta_1(p) \ln M]^{-1} + \theta_2(p)\end{aligned}$$

for some functions $\theta_0(p)$, $\theta_1(p)$, and $\theta_2(p)$.

Next consider household attributes A that affect preferences. For welfare calculations, it is desirable to impose the IB, (Independence of Base) property (see Lewbel 1989, 1991). By Lemma 1 of Lewbel (1989), IB requires $\ln M = g_1(p, U) + g_2(p, A)$ for some functions g_1 and g_2. Solving the above indirect utility function for $\ln M$, yields the log cost function

$$\ln M = \frac{1}{\left[(\ln U)^{-1} - \theta_2(p)\right]\theta_1(p)} - \frac{\theta_0(p)}{\theta_1(p)}$$

which clearly has the IB form only if $\theta_0(p)$ is allowed to depend on A, but not $\theta_1(p)$ or $\theta_2(p)$.

Finally, consider exact aggregation. Exact aggregation requires that Marshallian budget shares be linear in functions of A and M. By construction, the budget shares are linear in $\ln M$ and $(\ln M)^2$. By the IB property, Only θ_0 is permitted to depend on A, and applying Roy's identity to the indirect utility function shows that the dependence of budget shares on θ_0 is that they are linear in $\partial\theta_0(p)/\partial \ln p$, $\theta_0(p)$, and $\theta_0(p)^2$. The demands will therefore be exactly aggregable if θ_0 is linear in functions of A, that is, if we replace $\theta_0(p)$ with $\theta_0(p) + \phi(p)'\varphi(A)$, for some vector valued functions ϕ and φ.

We therefore have that demands are rank three, exactly aggregable in income and attributes, have the Independence of Base property for welfare comparisons, and have budget shares quadratic in log income, if and only if the indirect utility function takes the form

$$(\ln U)^{-1} = [\theta_0(p) + \phi(p)'\varphi(A) + \theta_1(p)\ln M]^{-1} + \theta_2(p)$$

which in turn implies a log cost function of the form

$$\ln M = \frac{1}{\left[(\ln U)^{-1} - \theta_2(p)\right]\theta_1(p)} - \frac{\theta_0(p) + \phi(p)'\varphi(A)}{\theta_1(p)}$$

and budget shares

$$w = \frac{-\frac{\partial\phi(p)'}{\partial \ln p}\varphi(A) - \frac{\partial\theta_1(p)}{\partial \ln p}\ln M + \frac{\partial\theta_2(p)}{\partial \ln p}[\theta_0(p) + \phi(p)'\varphi(A) + \theta_1(p)\ln M]^2}{\theta_1(p)}$$

2. The Rank Extended Translog

The indirect translog demand system of Jorgenson, Lau, and Stoker (1983) has all of the properties of the above class of models, except that it is rank two, having budget shares linear rather than quadratic in $\ln M$. We will now consider extending the translog model to give it rank three quadratic budget shares.

Let ι be a vector of ones. The Translog indirect utility function $\widetilde{U}$ is defined as

$$\ln \widetilde{U} = \alpha_0 + \ln\left(\frac{p}{M}\right)'\alpha_p + 0.5\ln\left(\frac{p}{M}\right)'\beta_{pp}\ln\left(\frac{p}{M}\right) + \ln\left(\frac{p}{M}\right)'\beta_{pA}A$$

A free normalization from scaling $\ln \widetilde{U}$ is $\iota'\alpha_p = -1$. Let $\widetilde{w}$ be the translog budget shares. By applying the log form of Roy's identity,

$$\widetilde{w} = \frac{\partial \ln \widetilde{U}/\partial \ln(p/M)}{\iota'\left[\partial \ln \widetilde{U}/\partial \ln(p/M)\right]}$$

we obtain the translog budget shares

$$\widetilde{w} = \frac{\alpha_p + \beta_{pp} \ln\left(\frac{p}{M}\right) + \beta_{pA} A}{-1 + \iota' \beta_{pp} \ln\left(\frac{p}{M}\right) + \iota' \beta_{pA} A}$$

Define the Retranslog (Rank Extended Translog) indirect utility function U to be

$$(\ln U)^{-1} = (\ln \widetilde{U})^{-1} - \ln\left(\frac{p}{M}\right)' \gamma_p$$

By Roy's identity, the Retranslog has budget shares given by

$$w = \frac{\alpha_p + \beta_{pp} \ln\left(\frac{p}{M}\right) + \beta_{pA} A + \gamma_p (\ln \widetilde{U})^2}{-1 + \iota' \beta_{pp} \ln\left(\frac{p}{M}\right) + \iota' \beta_{pA} A + \iota' \gamma_p (\ln \widetilde{U})^2}$$

Unlike the Translog, the Retranslog is a rank three, meaning that the dimension of the space spanned by its Engel curves is three. To make the Retranslog exactly aggregable, impose the constraints

$$\begin{aligned} \iota' \beta_{pp} \iota &= 0 \\ \iota' \beta_{pA} &= 0 \\ \iota' \gamma_p &= 0 \end{aligned}$$

The resulting budget shares are then given by

$$w = \frac{\alpha_p + \beta_{pp} \ln\left(\frac{p}{M}\right) + \beta_{pA} A + \gamma_p (\ln \widetilde{U})^2}{D}$$

where

$$D = -1 + \iota' \beta_{pp} \ln p$$

Relative to the Translog, the Retranslog shares have the added term $(\ln \widetilde{U})^2 \gamma_p$. Note that γ_p is a vector of constants and $(\ln \widetilde{U})^2$ is a scalar. The scalar $\ln \widetilde{U}$ is linear in $\ln M$ and in A, so the added term makes the Retranslog is quadratic in $\ln M$ and A.

This exactly aggregable Retranslog is a special case of the general class of rank three, quadratic budget share, exactly aggregable, IB demand systems, in particular, the Retranslog has the required form $(\ln U)^{-1} = [\theta_0(p) + \phi(p)' \varphi(A) +$

$\theta_1(p)\ln M]^{-1}+\theta_2(p)$ with

$$\begin{aligned}\theta_0(p) &= \alpha_0+\ln(p)'\alpha_p+0.5\ln(p)'\beta_{pp}\ln(p)\\ \theta_1(p) &= -1+\imath'\beta_{pp}\ln p\\ \theta_2(p) &= \ln(p)'\gamma_p\\ \phi(p) &= \ln(p)\\ \varphi(A) &= \beta_{pA}A\end{aligned}$$

Notes:

1. Unlike the ordinary translog, in the Retranslog the term α_0 is identified, and affects the budget shares, because α_0 appears inside $\widetilde{U}$ and γ_p multiplies a function of $\widetilde{U}$ in the budget shares.

2. The ordinary translog can be made equivalent to a Barten scaled model of attributes by replacing α_0 with a relatively complicated function $\delta(A)$, defined by equation (4.3) of Lewbel (1989). The Barten scales s would then be given by $\ln s=B_{pp}^{-1}B_{pA}A$.

The Retranslog can also be made Barton scaled in the same way, but unlike the ordinary Translog, doing so will change the estimated budget shares, because replacing α_0 with $\delta(A)$ changes the budget shares. Also, the additional, otherwise unnecessary constraint $\gamma'B_{pp}^{-1}B_{pA}=0$ needs to be imposed to make the model Barten scaled. These extra complications are probably not worth the trouble.