Shape Invariant Demand Functions

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Abstract

Shape invariance is a property of demand functions that is widely used for parametric and semiparametric modeling, and is associated with a commonly employed class of equivalence scale models used for welfare calculations. This paper derives the set of all shape invariant demand functions and associated preferences. All previously known shape invariant demands were derived from utility functions that, up to monotonic transformation, are called IB/ESE (independent of base - equivalence scale exact) utility functions, because they yield IB/ESE equivalence scales. This paper shows that there exist exceptional shape invariant demands that are not derived from a transform of IB/ESE utility, and provides some simple tests for these exceptions. In particular, all the exceptions have rank two, so any rank three or higher demand system is shape invariant if and only if it is derived from a transform of IB/ESE utility.

Keywords: Shape Invariance, Equivalence Scales, Engel curves, Consumer demand, Demand Systems, Utility, Cost Functions.

JEL Codes: D11, D12, C31, C51
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1 Introduction

One common way to write an Engel curve is to express the budget share w_i (defined as the fraction of income a consumer spends on the good of type i) as a function of the consumer's log income y. To account for heterogeneity of preferences, this function can also depend on a vector A of observable attributes of the consumer, such as age, gender, etc.,. An Engel curve is said to be shape invariant if, when graphed in the w_i and y plane, the taste shifters A affect the curve only by translating its location vertically and horizontally. Equivalently, under shape invariance the Engel curves of a good i, when evaluated at different values of A, all look identical except for the location of the w_i and y axes. Shape invariance is associated with equivalence scale welfare comparisons (see below for details), provides a parsimonious way to model the dependence of demand functions on taste shifters A, is consistent with utility maximization, and has been empirically found to hold at least approximately in some data sets.

Without reference to demand systems, nonparametric estimators of collections of shape invariant regressions were proposed by Hardle and Marron (1990) and Pinkse and Robinson (1995). Tests and estimators of parametric and nonparametric shape invariant Engel curves include Gozalo (1997), Blundell, Duncan and Pendakur (1998), Pendakur (1998), Blundell, Chen, and Kristensen (2003), Yatchew, Sun, and Deri (2003), and Stengos, Sun, and Wang (2006). Many empirically popular specifications for demand systems possess shape invariant Engel curves, including Jorgenson, Lau, and Stoker's (1982) Translog demand systems and the verision of the Quadratic Almost Ideal Demand System implemented by Banks, Blundell, and Lewbel (1997).

Up until now, the only class of utility functions that were known to generate shape invariant demand functions were utility functions that, up to monotonic transformation, are called IB/ESE (independence of base - equivalence scale exactness) utility functions. These are utility functions that yield IB/ESE equivalence scales, which have many important implications for social welfare calculations and are widely used in practice. Any set of consumers that have IB/ESE preferences must also have shape invariant Engel curves. Estimates, tests, and theory regarding IB/ESE scales include Lewbel (1989), Blundell and Lewbel (1991), Blackorby and Donaldson (1993), Jorgenson and Slesnick (1993), Dickens, Fry, and Pashardes (1993), Pashardes (1995), Gozalo (1997), Pendakur (1998), Blundell, Duncan and Pendakur (1998), Lyssiotou, (2003), Yatchew, Sun, and Deri (2003), and Stengos, Sun, and Wang (2006).

Despite this extensive literature on IB/ESE utility and shape invariant demands, the set of utility functions that give rise to *all* shape invariant demand functions was not known, until now. In particular, prior to this paper it was an open question whether demands have shape invariance if and only if preferences are transforms of IB/ESE utility. Theorem 1 below provides the answer. This theorem shows that all shape invariant demand systems are derived either from transforms of IB/ESE utility or from a restrictive class of alternative models. This paper also provides an example member of this hitherto unknown class of shape invariant demands that are not derived from a transform of IB/ESE utility.

Some simple sufficient conditions on Engel curves are then provided that can be applied empirically to test for these exceptional non IB/ESE cases. For example, it is shown that all the exceptions have demand rank two. Therefore, any utility derived demand system that has rank higher than two also has shape invariant Engel curves if and only if it is derived from transforms of IB/ESE utility.

2 Definitions

Let y denote the logarithm of the total dollars a consumer spends on goods (call it log income for short), and for goods $i = 1, ..., n$, let w_i denote the consumer's budget share for good i, that is, the fraction of total dollar expenditures e^y that the consumer devotes to purchases of good i. Let A denote a vector of attributes (fixed or random utility parameters) of the consumer that affect his tastes. Let p denote the n vector of logged prices $(p_1, ..., p_n)$ the consumer faces for goods. Assume that the consumer determines $w = (w_1, ..., w_n)$ to maximize a regular utility function given a linear budget constraint. Specifically, Assumption A1 is assumed to hold.

ASSUMPTION A1: A consumer has an indirect utility function $u = U(p, y, A)$ that possesses the following standard regularity properties: Nonincreasing, twice differentiable, and quasiconvex in p, strictly increasing and differentiable in y, and homogeneous of degree zero in (e^p, e^y). The consumer's budget share Marshallian demand functions are given by Roy's identity:

$$w_i = -\frac{\partial U(p, y, A)/\partial p_i}{\partial U(p, y, A)/\partial y}.$$

Engel curves are defined as the functions ω_i such that, in a given price regime, $w_i = \omega_i(y, A)$. Engel curves are often highly nonlinear in y, but not necessarily in A, and the vector of attributes A is often long with many discrete elements. In a regression context, for dimension reduction one would therefore typically specify a partly linear form $w_i = A'b_i + G_i(y)$, or more generally an additive model $w_i = H_i(A) + G_i(y)$ where H_i may be relatively tightly parameterized while G_i is semiparametric or nonparametric. The difficulty with these specifications is that, except for very specific choices of the functions G_i, these models are inconsistent with utility maximization.

A general class of solutions to this problem, proposed by Blundell, Duncan and Pendakur (1998), are shape invariant Engel curves. The set of Engel curves $w_i = \omega_i(y, A)$ for goods $i = 1, ...n$ are defined to satisfy shape invariance if there exist functions H_i, G_i, and B such that

$$w_i = H_i(A) + G_i\left(y - B(A)\right). \tag{1}$$

See also Pendakur (1998), who provides a general nonparametric test for shape invariance, Blundell, Chen, and Kristensen (2003) who implement semiparametric estimators of shape invariant Engel curves, and the many references given in the introduction for other tests, estimators, and models that possess shape invariance. An implication of equation (1) is that the shape of the Engel curve (in terms of the effect of y on w_i), which is given by the function G_i, does not depend on attributes A and so look the same for all consumers up to possible translations in both axis. These translations, or location shifts, are given by the functions $H_i(A)$ and $B(A)$, which shift w_i and y, respectively.

Define shape invariant (Marshallian) demand functions as

$$w_i = H_i(p, A) + G_i\left(y - B(p, A), p\right) \tag{2}$$

which is just equation (1), where the log price regime vector p is now explicitly included in the functions H_i, G_i, and B where it was formerly implicit.

Up until the present, the only way shape invariant Engel curves were known to be consistent with utility maximization is as follows. Suppose consumer's have indirect utility functions of the form

$$u = V(y - B(p, A), p) \tag{3}$$

for some functions V and B. This is equivalent to a log cost (expenditure) function of the form $y = C(p, u, A) = \Psi(p, u) + B(p, A)$ for some functions Ψ and B, obtained by solving equation (3) for y. Equation (3) is the indirect utility function associated with IB/ESE equivalence scales. Consider the log change in the cost of living that results from changing an attribute, e.g., changing from a young man A_0 to an older man A_1. This type of cost of living index, which in log form equals $C(p, u, A_1) - C(p, u, A_0)$, or equivalently $C(p, U(p, y, A_0), A_1) - y$, is known as an adult equivalence scale (see, e.g, Lewbel 1997 for a survey on equivalence scales).

IB/ESE preferences are exactly the class of preferences having the property that this equivalence scale is independent of the level of utility u or consumption y at which it is evaluated. For IB/ESE preferences, $C(p, u, A_1) - C(p, u, A_0) = B(p, A_1) - B(p, A_0)$ which only depends on prices and attributes, and does not vary with u or y. As seen below, if a set of consumers have IB/ESE preferences, then those consumers also have shape invariant Engel curves.

Demand functions are unaffected by monotonic transformations of utility, so consider what could be called Transformed IB (TIB) indirect utility functions, defined as indirect utility functions of the form

$$u = T\left[V(y - B(p, A), p), A\right], \tag{4}$$

for any function T that is strictly monotonically increasing in its first argument. Solving equation (4) for y gives the TIB log cost function

$$y = C(p, u, A) = \Psi\left[p, \tau(u, A)\right] + B(p, A) \tag{5}$$

where τ is the inverse of the function T. Unlike IB preferences, TIB preferenes need not have equivalence scales that are independent of utility or income. However, the difference between IB and TIB preferences is unobservable by revealed preference, since both yield the same demand functions. This can be seen by applying Roy's identity to equation (3) or to equation (4). Either one gives the same Marshallian budget share demand functions

$$w_i = \frac{\partial B(p, A)}{\partial p_i} - \frac{\partial V(y - B(p, A), p)/\partial p_i}{\partial V(y - B(p, A), p)/\partial y} \tag{6}$$

which is in the form of equation (2) and so is shape invariant. This shows that if a set of consumers have TIB indirect utility functions (or IB which is a special case of TIB), then those consumers also have shape invariant Engel curves. This paper examines whether the converse holds.

As an example, consider indirect utility functions of the form

$$u = \sum_{i=1}^{n-1}(p_n - p_i)\Gamma_i\left[y - B(p, A)\right] \tag{7}$$

for some functions Γ_i, $i = 1, ..., n - 1$. Applying Roy's identity to equation (7) yields shape invariant demand functions (2) where $G_i\,(x, p)$ is proportional to $\left[\partial\Gamma_i(x)/\partial x\right]^{-1}$ for all goods i except $i = n$, which has budget share demands given by one minus the sum of the other goods shares. The only constraint on the Γ_i functions and hence on the associated Engel curve components G_i are the inequalities implied by Assumption A1. Based on these minimal restrictions on G_i implied by equation (4) utility maximization, Blundell, Chen, and Kristensen (2003) among others use unconstrained nonparametric regressions to estimate these functions. In addition to the connection with IB/ESE equivalence scales, this feature of shape invariant demands makes them desirable for empirical work.

3 Characterizing Shape Invariance

Theorem 1 and Corollary 1 below provide necessary and sufficient conditions on demands and on preferences (in terms of cost functions) for a consumer to have shape invariant demand functions.

THEOREM 1: Let Assumption A1 hold. A consumer has shape invariant demand functions if and only if the consumer either has a log cost function of the form

$$y = \Psi\left[p, \tau(u, A)\right] + B(p, A) \tag{8}$$

for some functions Ψ, τ and B, or has utility given by the log cost function

$$y = c\left[S(p), u, A\right] + B(p, A) \tag{9}$$

for some functions c, B, and S, where the function c is a solution to the differential equation

$$\frac{\partial c(s, u, A)}{\partial s} = h(s, A) + g\left[c(s, u, A), s\right] \tag{10}$$

for some functions h and g.

Equation (8) is the exactly the log cost function (5), which corresponds to the TIB indirect utility function (4), while equations (9), and (10) are a previously unknown alternative class of preferences that also yield shape invariant demands.

Roughly speaking, the additive structure of shape invariant demands requires a corresponding additive form in cost functions, since demand functions are obtained by differentiating cost functions via Shephard's lemma. The TIB log cost function (5) directly provides the necessary additivity. The alternative cost function (9) obtains the necessary additivity via the differential equation (10).

Theorem 1 is proved by first plugging shape invariant demands into Shephard's lemma, which provides a differential equation that any cost function yielding shape invariant demands must satisfy. Differentiating the result yields a representation of Slutsky symmetry for this class of demands. The proof then applies a long series of simplifying substitutions to reduce the resulting differential equation into the forms given in Theorem 1.

COROLLARY 1: Let Assumption A1 hold. A consumer has shape invariant demand functions if and only if, for every good i, the consumer either has budget share demand functions of the form

$$w_i = \frac{\partial B(p, A)}{\partial p_i} - \frac{\partial V(y - B(p, A), p)/\partial p_i}{\partial V(y - B(p, A), p)/\partial y} \tag{11}$$

for some functions B and V, or the consumer has budget share demand functions of the form

$$w_i = \frac{\partial B(p, A)}{\partial p_i} + \frac{\partial S(p)}{\partial p_i}\left[h(S(p), A) + g\left(y - B(p, A), S(p)\right)\right] \tag{12}$$

for some functions B, S, h, and g, where $g\left(y - B, S\right)$ is nonlinear in y and there exists some p, A for which $h\left(S\left(p\right), A\right) \neq 0$.

Equation (11) is the same as equation (6), obtained from the TIB indirect utility function (4) which in turn is comes from inverting the log cost function (8). Equation (12) is derived from equations (9) and (10) Cost functions (8) and (9), or equivalently demand functions (11) and (12), are not mutually exclusive, that is, there can exist preferences that can be written in both forms. An example is when h is zero. The restrictions in Corollary 1 that g be nonlinear and h be nonzero are included without loss of generality because when these restrictions do not hold, equation (12) demand functions can be rewritten in the TIB demands form of equation (11).

All of the utility functions currently in the literature (cited earlier) that give rise to shape invariant demand functions are TIB. To show that demand functions of the alternative form given by equations (9), (10) and (12) exist, consider the indirect utility function

$$u = \left[\frac{1}{y - b(p, A) - m(A)} + \frac{1}{2m(A)}\right] e^{2m(A)s(p)} \tag{13}$$

for some scalar valued function $m(A)$, some function $b(p, A)$ such that $e^{b(p,A)}$ is linearly homogeneous in prices e^p, and some function $s(p)$ that is homogeneous of degree zero in prices e^p. These homogeneity restrictions are required by Assumption A1. This utility function cannot be written in the TIB form of equation (4). However, applying Roy's identity to equation (13 gives demand functions

$$w_i = \left(\frac{\partial b(p, A)}{\partial p_i} - m(A)^2 \frac{\partial s(p)}{\partial p_i}\right) + \left(\left[y - b(p, A)\right]^2 \frac{\partial s(p)}{\partial p_i}\right) \tag{14}$$

which are in the form of equation (2) and hence are shape invariant, where the first term in parentheses is $H_i(p, A)$ and the second term in parentheses is $G_i\left(y - B(p, A), p\right)$. This example corresponds to equations (9), (10) and (12) with $S(p) = s(p)$, $B(p, A) = b(p, A)$, $h(s, A) = -m(A)^2$, $g(x, s) = x^2$, and

$$c\left(s, u, A\right) = \left[ue^{-2m(A)s} - \frac{1}{2m(A)}\right]^{-1} + m\left(A\right).$$

This example also shows that it is possible to have nonlinear Engel curves that are additive in functions of A and y, since if we replace $b(p, A)$ in equation (13) with a function $b(p)$ that does not depend on A, then we obtain Engel curves of the additive form

$$w_i = H_i(A) + G_i(y) = \left[b_i - m(A)^2 s_i\right] + \left[(y - b)^2 s_i\right]$$

for constants b, s_i, and b_i, $i = 1, ..., n$. Still it remains true that nonlinear additivity in y and A is difficult to reconcile with utility maximization (though not impossible as this example shows), which was one of the original rationales for formulating shape invariant demands, as in Blundell, Duncan and Pendakur (1998).

4 Welfare Calculations and Equivalence Scales

A (logged) equivalence scale is defined as $C(p, u, A) - C(p, u, A_0)$ for some reference level of attributes A_0. An equivalence scale is defined to be IB (Independent of Base) or have ESE (Equivalence Scale Exactness) if the scale is independent of u. Consumers have IB/ESE equivalence scales if and only if they have indirect utility functions in the form of equation (3). See Lewbel (1997) for a survey of equivalence scales, and Blackorby and Donaldson (1993) and Lewbel (1989) for further definitions and details regarding IB/ESE scales. Many empirical applications of equivalence scales assume IB/ESE. A prominent example is Jorgenson and Slesnick (1993). While IB/ESE is sometimes empirically rejected, estimated equivalence scales that are not constrained to be IB/ESE are nevertheless often found to be numerically close to IB/ESE. See, e.g., Blundell and Lewbel (1991). Other tests and estimates of IB/ESE scales can be found in most of the references cited in the introduction.

It immediately follows from equations (2), (3), and (6) that if consumers have IB/ESE equivalence scales, then they must also have shape invariant Engel curves. Given demand functions (6), the TIB utility function (4) is observationally equivalent to the indirect utility function (3) that generates IB scales, so in applications if demand functions are given by equation (6) for some functions V and B, then IB/ESE equivalence scales can be constructed by making the untestable assumption that utility is given by (3) rather than (4). This is, e.g., the method used by Jorgenson and Slesnick (1993).

Theorem 1 and Corollary 1 show that not all shape invariant demand functions come from TIB utility functions, so tests of shape invariance such as Pendakur (1998) cannot be used by themselves to test for TIB preferences. However, the next section provides simple empirical tests that can rule out the non-TIB cases of shape invariance, and hence can be used to test for all the observable implications of preferences that posess IB/ESE equivalence scales.

5 Convenient Testable Implications

Theorem 1 and Corollary 1 have some easily tested implications for Engel curves, described by the following Corollaries. Let $w_i = \omega_i(y, A)$ denote the consumer's Engel curves for each good $i = 1, ..., n$ in some price regime.

COROLLARY 2: Let Assumption A1 hold. A necessary condition to have shape invariant Engel curves not come from TIB utility is

$$\omega_i(y, A) = h_i(A) + \lambda_i G\left[y - b(A)\right] \tag{15}$$

for some functions $h_i(A)$, $b(A)$, and $G(x)$ where $G(x)$ is nonlinear in x, and some constants λ_i, for goods $i = 1, ..., n$.

COROLLARY 3: Let Assumption A1 hold. Assume $\omega_i(y, A)$ is differentiable in y. If the consumer has shape invariant Engel curves and there exists any pair of goods i and j such that $\left[\partial\omega_i(y, A)/\partial y\right] / \left[\partial\omega_j(y, A)/\partial y\right]$ is not constant (i.e., not independent of y and A), then the consumer has TIB utility.

Given shape invariance, the conditions in Corollaries 2 and 3 are sufficient to insure that utility is TIB, but they are stronger than necessary, since they do not impose the restrictions on price effects required by Theorem 1 and Corollary 1.

As noted earlier, and exemplified by equation (7), TIB utility places very few constraints on the Engel functions $G_i(x)$ in equation (1). In contrast, by Corollary 2, nonTIB shape invariance requires $G_i(x) = \lambda_i G(x)$, meaning that every good i must have the same $G_i(x)$ function up to scale. Equivalently, NonTIB shape invariance requires that for every household type A, the Engel curves for every good must be linear in the same function $G\left[y - B(A)\right]$, and it requires that all Engel curves have the same first derivatives up to scale.

Engel curves in the form of equation (15) where $G\left[y - B(A)\right]$ is linear in y are known as the Working (1943) and Leser (1963) model. Some popular demand systems have Working-Leser Engel curves, including Deaton and Muellbauer's (1980) Almost Ideal Demand System and Jorgenson, Lau, and Stoker's (1982) Indirect Translog Demand System. Although Working-Leser demands can satisfy the requirement for nonTIB shape invariance that $G_i(x) = \lambda_i G(x)$, it is still the case that all such models are TIB, because NonTIB also requires that $G(x)$ be nonlinear.

Given semiparametric or nonparametric estimates of Engel curves $\omega_i(y, A)$, the derivative condition in Corollary 3 could be directly tested using, e.g., Haag and Hoderlein (2005) or any of the many other testing procedures cited there. One could separately nonparametrically test for shape invariance itself using Pendakur (1998). Alternatively, one could test semiparametric estimates of general shape invariant engel curves using, e.g., Blundell, Duncan, and Pendakur (1998), against semiparametric estimates of the more restrictive model of equation (15).

Extending Gorman's (1981) results on exactly aggregable demand systems, Lewbel (1991) defines the rank of any demand system (aggregable or not) as the dimension of the space spanned by its Engel curves, holding A fixed. Equivalently, demand rank equals the rank of the matrix Ξ consisting of elements ξ_{ij} when demand functions are written in the form

$$w_i = \sum_{j=1}^{J} \xi_{ij}(p, A)\, \varphi_j(p, y, A) \tag{16}$$

for some integer J and some functions ξ_{ij} and φ_j. For the special case where $\varphi_j(p, y, A)$ is constrained to not depend on p (which are called exactly aggregable demands) Gorman's (1981) shows that the maximum possible rank of Ξ is three. Without the exact aggregability restriction, demands can have any rank up to $n-1$ where n is the number of goods. Rank greater than $n-1$ is not possible because Ξ has n rows and budget shares summing to one imposes a linear constraint on those rows. Lewbel (1991) shows that the rank of a demand system equals the minimum number of functions of prices that are needed to express the cost or indirect utility function.

COROLLARY 4: Let Assumption A1 hold. All shape invariant demand systems arising from nonTIB utility functions have rank two.

All nonTIB demand functions, which are given by equation (14), can be written in the form of equation (16) with $\varphi_1(p, y, A) = 1$ and $\varphi_2(p, y, A) = \left[y - b(p, A)\right]^2$, so $J = 2$, implying rank at most two. In sharp contrast, shape invariant demand systems arising from TIB utility functions can have any rank up to the maximum $n-1$, as can be seen by applying Lewbel (1991) to the example of equation (7) when the $\Gamma_j(x)$ functions are constructed to be linearly independent of each other. Alternatively, applying Roy's identity to equation (7) yields demand functions in the form of equation (16) with $\varphi_j(p, y, A)$ functions proportional to the Γ_j functions and their derivatives, which provides enough distinct φ_j and ξ_{ij} functions to achieve the maximum possible rank of $n-1$.

The set of rank two demands is more general than the class of Engel curves given by Corollaries 2 and 3. However, most empirical studies of rank have found that consumers demand functions have rank higher than two. See, e.g., Lewbel (1991) and Hausman, Newey, and Powell (1995). As long as rank really is greater than two, shape invariant demands must be TIB.

6 Conclusions

Shape invariant demand functions are widely used in empirical work. This paper characterized the set of all regular preferences and associated demand functions that give rise to shape invariant Engel curves. In empirical applications, shape invariant Engel curves, equation (1), have always been modeled as deriving from TIB utility functions, equation (4). This paper shows that there exist shape invariant demand functions that do not come from TIB utility, although these exceptional nonTIB models are quite restrictive. Simple tests on Engel curves can rule out these exceptions. More specific tests based on demand functions could also be constructed, essentially by semiparametrically estimating shape invariant demand functions, equation (2), and testing for the restrictions listed in Corollary 1.

In addition to convenience in model specification, these results have important implications for IB/ESE equivalence scales, which are widely used in empirical work for social welfare evaluations. TIB utility corresponds to all of the observable implications of preferences that permit the application of IB/ESE equivalence scales. If empirical demands violate the nonTIB restrictions of Theorem 2, e.g., by Corollary 4 if budget shares are not rank two, then shape invariance implies TIB utility.

7 Proofs

LEMMA 1: Let Assumption A1 hold. Utility is TIB if and only if budget share demands have the form

$$w_i = \frac{\partial B(p, A)}{\partial p_i} + G_i \left[y - B(p, A), p \right] \tag{17}$$

PROOF OF LEMMA 1: If the utility function is TIB, so equation (4) holds, then equation (17) follows from equation (6) with the function G_i defined by

$$G_i(y, p) = -\frac{\partial V(y, p)/\partial p_i}{\partial V(y, p)/\partial y}. \tag{18}$$

For the converse, for any fixed reference level A_0 of A define $\widetilde{B}(p, A) = B(p, A) - B(p, A_0)$ and define $\widetilde{G}_i$ by

$$\widetilde{G}_i(y, p) = \frac{\partial B(p, A_0)}{\partial p_i} + G_i \left[y - B(p, A_0), p \right]$$

Then equation (17) can be written as

$$w_i = \frac{\partial \widetilde{B}(p, A)}{\partial p_i} + \widetilde{G}_i \left[y - \widetilde{B}(p, A), p \right] \tag{19}$$

and, at $A = A_0$ we obtain the demand function $w_i = \widetilde{G}_i\,(y, p)$. Let $V(y, p)$ is the indirect utility function corresponding to equation (17) or equivalently (19) when $A = A_0$. Then by Roy's identity

$$-\frac{\partial V(y, p)/\partial p_i}{\partial V(y, p)/\partial y} = \widetilde{G}_i\,(y, p)$$

One may then immediately verify that the TIB indirect utility function $V^*(y, p, A) = V(y - B(p, A), p)$ is (up to arbitrary monotonic transformation) the indirect utilty function associated with equation (19) and therefore with equation (17), because applying Roy's identity to $V^*(y, p, A)$ gives.

$$-\frac{\partial V^*(y, p, A)/\partial p_i}{\partial V^*(y, p, A)/\partial y} = -\frac{dV(y - B(p, A), p)/dp_i}{\partial V(y - B(p, A), p)/\partial y} = \frac{\partial \widetilde{B}(p, A)}{\partial p_i} + \widetilde{G}_i \left[y - \widetilde{B}(p, A), p \right]. \tag{20}$$

This shows that the demand functions (19) integrate to V^*, so by revealed preference theory and standard consumer demand duality, V^* inherits the properties of an indirect utility function from the shape invariant demand functions (e.g., homogeneity of the demand function in p and y implies homogeneity of V^*, etc.,). Note that Roy's identity could hold if V^* were a nonmonotonic transformation of the actual indirect utility function, but this case is ruled out because $V(y, p)$ itself has the properties of an indirect utility function.

PROOF OF THEOREM 1 and Corollary 1: By Shephard's lemma, $w_i = \partial C(p, u, A)/\partial p_i$, so shape invariant demands (2) have Hicksian demands of the form

$$\frac{\partial C(p, u, A)}{\partial p_i} = H_i(p, A) + G_i \left[C(p, u, A) - B(p, A), p \right] \tag{21}$$

Define $\gamma\,(p, u, A) = C(p, u, A) - B(p, A)$ and $h_i(p, A) = H_i(p, A) - \partial B(p, A)/\partial p_i$. Then equation (21) can be rewritten as

$$\frac{\partial \gamma\,(p, u, A)}{\partial p_i} = h_i(p, A) + G_i \left[\gamma\,(p, u, A), p \right]. \tag{22}$$

Without loss of generality, assume for every good i that $h_i(p, A_0) = 0$ for some constant reference level A_0. This is without loss of generality because if it did not hold then we could redefine $h_i(p, A)$ as $h_i(p, A) - h_i(p, A_0)$ and redefine $G_i(\gamma, p)$ as $G_i(\gamma, p) +$

$h_i(p, A_0)$ and the demand function (22) would be unchanged. It then follows from Lemma 1 that the utility function associated with shape invariant demands (2) will not be TIB if and only if $h_i(p, A)$ is not identically zero for any function B.

Now let $x = \gamma(p, u, A)$. Taking the derivative of equation (22) with respect to p_j for any good j gives

$$\frac{\partial^2 \gamma(p,u,A)}{\partial p_i \partial p_j} = \frac{\partial h_i(p,A)}{\partial p_j} + \frac{\partial G_i\left[\gamma(p,u,A),p\right]}{\partial \gamma(p,u,A)} \frac{\partial \gamma(p,u,A)}{\partial p_j} + \frac{\partial G_i\left[\gamma(p,u,A),p\right]}{\partial p_j}$$

$$= \frac{\partial h_i(p,A)}{\partial p_j} + \frac{\partial G_i\left[\gamma(p,u,A),p\right]}{\partial \gamma(p,u,A)} \left[h_j(p,A) + G_j(\gamma(p,u,A),p)\right] + \frac{\partial G_i\left[\gamma(p,u,A),p\right]}{\partial p_j}$$

$$= \frac{\partial h_i(p,A)}{\partial p_j} + \frac{\partial G_i(x,p)}{\partial x}\left[h_j(p,A) + G_j(x,p)\right] + \frac{\partial G_i(x,p)}{\partial p_j}$$

So by Young's Theorem

$$\frac{\partial h_i(p,A)}{\partial p_j} + \frac{\partial G_i(x,p)}{\partial x}\left[h_j(p,A) + G_j(x,p)\right] + \frac{\partial G_i(x,p)}{\partial p_j} = \tag{23}$$

$$\frac{\partial h_j(p,A)}{\partial p_i} + \frac{\partial G_j(x,p)}{\partial x}\left[h_i(p,A) + G_i(x,p)\right] + \frac{\partial Gj(x,p)}{\partial p_i}.$$

This is essentially Slutsky symmetry corresponding to $\gamma(p, u, A)$ instead of to the log cost function $C(p, u, A)$. Now take equation (23), and subtract from it the same expression evaluated at the reference level $A_0 \neq A$. Recalling that $h_k(p, A_0) = 0$ for $k = i, j$, this gives

$$\frac{\partial h_i(p,A)}{\partial p_j} + \frac{\partial G_i(x,p)}{\partial x} h_j(p,A) = \frac{\partial h_j(p,A)}{\partial p_i} + \frac{\partial G_j(x,p)}{\partial x} h_i(p,A). \tag{24}$$

Next take equation (24), and subtract from it the same expression evaluated at some reference level $x_0 \neq x$. This gives

$$\left[\frac{\partial G_i(x,p)}{\partial x} - \frac{\partial G_i(x_0,p)}{\partial x}\right] h_j(p,A) = \left[\frac{\partial G_j(x,p)}{\partial x} - \frac{\partial G_j(x_0,p)}{\partial x}\right] h_i(p,A). \tag{25}$$

Since x_0 is a constant, we may define $\kappa_k(p) = \partial G_k(x_0, p)/\partial x_0$ for $k = i, j$. Then equation (25) is

$$\left[\frac{\partial G_i(x,p)}{\partial x} - \kappa_i(p)\right] h_j(p,A) = \left[\frac{\partial G_j(x,p)}{\partial x} - \kappa_j(p)\right] h_i(p,A). \tag{26}$$

This equation can only hold if either $h_k(p, A) = 0$ for $k = i, j$, or

$$\frac{\partial G_k(x, p)}{\partial x} = \kappa_k(p) \quad \text{for } k = i, j \tag{27}$$

or, for some functions $g^*(x, p)$ and $h(p, A)$ that do not vary by i or j, and for some functions $\lambda_i(p)$ and $\lambda_j(p)$

$$\frac{\partial G_k(x, p)}{\partial x} - \kappa_k(p) = \lambda_k(p) g^*(x, p), \quad h_k(p, A) = \lambda_k(p) h(p, A) \quad \text{for } k = i, j \tag{28}$$

We can rule out $h_i(p, A) = 0$ for all A, since then by the argument earlier the utility function would be TIB. We can also rule out equation (27) as follows. Equation (27) requires that $G_k(x, p)$ be linear in x. Having budget shares that are linear in x requires an indirect utility functions that is ordinally equivalent to $u = e^{\theta(p,A)} \left[y - b(p, A) \right]$ for some functions θ and b (see, e.g., Muellbauer 1976 for a proof). By Roys identity the demand functions arising from this utility function have the form

$$w_i = \frac{\partial b(p, A)}{\partial p_i} + \frac{\partial \theta(p, A)}{\partial p_i} \left[y - b(p, A) \right]$$

which are shape invariant only if $\partial \theta(p, A) / \partial p_i$ does not depend on A, and in that case they are by Lemma 1 derived from TIB utility.

We are therefore left with equation (28) as the necessary conditions for shape invariant demands to not be TIB. Integrating the expression for $G_k(x, p)$ in (28) with respect to x gives, for some function $G(x, p)$ that does not vary by i or j, and for some function $\delta_k(p)$ for $k = i, j$,

$$G_k(x, p) = \lambda_k(p) G(x, p) + \kappa_k(p) x + \delta_k(p), \quad h_k(p, A) = \lambda_k(p) h(p, A) \quad \text{for } k = i, j \tag{29}$$

Subsituting equation (29) into equation (24) gives

$$\frac{\partial \left[\lambda_i(p) h(p, A) \right]}{\partial p_j} + \left(\lambda_i(p) \frac{\partial G(x, p)}{\partial x} + \kappa_i(p) \right) \left[\lambda_j(p) h(p, A) \right] = \tag{30}$$

$$\frac{\partial \left[\lambda_j(p) h(p, A) \right]}{\partial p_i} + \left(\lambda_j(p) \frac{\partial G(x, p)}{\partial x} + \kappa_j(p) \right) \left[\lambda_i(p) h(p, A) \right]$$

which simplifies to

$$\frac{\partial \lambda_i(p)}{\partial p_j} + \lambda_i(p) \left(\frac{\partial h(p, A)}{\partial p_j} \frac{1}{h(p, A)} - \kappa_j(p) \right) = \frac{\partial \lambda_j(p)}{\partial p_i} + \lambda_j(p) \left(\frac{\partial h(p, A)}{\partial p_i} \frac{1}{h(p, A)} - \kappa_i(p) \right). \tag{31}$$

Difference this with respect to A, using $h(p, A_0) = 0$ to obtain

$$\lambda_i(p)\left(\frac{\partial h(p, A)}{\partial p_j}\frac{1}{h(p, A)}\right) = \lambda_j(p)\left(\frac{\partial h(p, A)}{\partial p_i}\frac{1}{h(p, A)}\right). \tag{32}$$

Subtracting (32) from (31) gives

$$\frac{\partial \lambda_i(p)}{\partial p_j} - \lambda_i(p)\kappa_j(p) = \frac{\partial \lambda_j(p)}{\partial p_i} - \lambda_j(p)\kappa_i(p) \tag{33}$$

Now observe that for any function $\sigma(p)$ and any nonzero function $\tau(p)$ we may define $\overline{g}(x, p) = \left[G(x, p) - \sigma(p)\right]/\tau(p)$ and by equation (29) we obtain, for $k = i, j$,

$$G_k(x, p) = \lambda_k(p)\left[\tau(p)\overline{g}(x, p) + \sigma(p)\right] + \kappa_k(p)x + \delta_k(p). \tag{34}$$

Let $\widetilde{g}(u, p) = \overline{g}\left[\gamma(p, u, A_0), p\right]$. Evaluate equation (22) at $A = A_0$, substituting in equation (34) to get

$$\frac{\partial \gamma(p, u, A_0)}{\partial p_k} = \lambda_k(p)\left[\tau(p)\widetilde{g}(u, p) + \sigma(p)\right] + \kappa_k(p)\gamma(p, u, A_0) + \delta_k(p). \tag{35}$$

Take the derivative of this expression with respect to u, and divide the result by $\partial\gamma(p, u, A_0)/\partial u$ to get

$$\frac{\partial \ln\left[\partial\gamma(p, u, A_0)/\partial u\right]}{\partial p_k} = \lambda_k(p)\tau(p)\varsigma(p, u) + \kappa_k(p). \tag{36}$$

where

$$\varsigma(p, u) = \frac{\partial\widetilde{g}(u, p)/\partial u}{\partial\gamma(p, u, A_0)/\partial u} \tag{37}$$

noting that $\partial\gamma(p, u, A_0)/\partial u > 0$ because $\partial\gamma(p, u, A_0)/\partial u = \partial C(p, u, A_0)/\partial u$ for cost function C. By equation (36),

$$\frac{\partial\left[\ln\left[\partial\gamma(p, u_1, A_0)/\partial u\right] - \ln\left[\partial\gamma(p, u_0, A_0)/\partial u\right]\right]}{\partial p_k} = \lambda_k(p)\tau(p)\left[\varsigma(p, u_1) - \varsigma(p, u_0)\right]. \tag{38}$$

for any utility levels u_1 and u_0. Now $\varsigma(p, u_1) - \varsigma(p, u_0)$ must be nonzero for some u_1 and u_0, since otherwise for some function $\widetilde{r}(p)$

$$\frac{\partial\widetilde{g}(u, p)}{\partial u} = \widetilde{r}(p)\frac{\gamma(p, u, A_0)}{\partial u}$$

which requires for some function $\widetilde{s}(p)$ that $\widetilde{g}(u, p) = \widetilde{s}(p) + \widetilde{r}(p)\gamma(p, u, A_0)$, so $\overline{g}(x, p) = \widetilde{s}(p) + \widetilde{r}(p)x$, which we can rule out because it implies G_k is linear in x and so comes from TIB utility.

Equation (38) holds for any choice of functions $\tau(p) \neq 0$, so let

$$\tau(p) = \left[\varsigma(p, u_1) - \varsigma(p, u_0)\right]^{-1}$$

Also, define the function $S(p)$ by

$$S(p) = \ln\left[\partial\gamma(p, u_1, A_0)/\partial u\right] - \ln\left[\partial\gamma(p, u_0, A_0)/\partial u\right]$$

With these definitions, equation (38) becomes

$$\frac{\partial S(p)}{\partial p_k} = \lambda_k(p) \tag{39}$$

for $k = i, j$. Substituting equation (39) into equation (33) gives

$$\frac{\partial^2 S(p)}{\partial p_i \partial p_j} - \frac{\partial S(p)}{\partial p_i}\kappa_j(p) = \frac{\partial^2 S(p)}{\partial p_j \partial p_i} - \frac{\partial S(p)}{\partial p_i}\kappa_i(p)$$

which requires that, for some function $t(p)$,

$$\kappa_k(p) = \frac{\partial S(p)}{\partial p_k} t(p) \tag{40}$$

Substituting equations (39) and (40) into equation (35) gives

$$\frac{\partial\gamma(p, u, A_0)}{\partial p_k} = \frac{\partial S(p)}{\partial p_k}\left[\tau(p)\widetilde{g}(u, p) + \gamma(p, u, A_0) + \sigma(p)\right] + \delta_k(p). \tag{41}$$

Equation (41) holds for any choice of function $\sigma(p)$, so define

$$\sigma(p) = -\left[\tau(p)\widetilde{g}(u_0, p) + \gamma(p, u_0, A_0)\right]$$

and define the function $R(p)$ by $R(p) = \gamma(p, u_0, A_0)$. It then follows from equation (41) that

$$\frac{\partial R(p)}{\partial p_k} = \delta_k(p). \tag{42}$$

Substituting (39), (40) and (42) into (34) gives

$$G_k(x, p) = \frac{\partial S(p)}{\partial p_k}\left[\tau(p)\left[\overline{g}(x, p) + x\right] + \sigma(p)\right] + \frac{\partial R(p)}{\partial p_k}. \tag{43}$$

Now define $g(x, p) = \tau(p)\left[\overline{g}(x, p) + x\right] + \sigma(p)$, and by equations (39) and (43), equation (29) becomes

$$G_k(x, p) = \frac{\partial S(p)}{\partial p_k} g(x, p) + \frac{\partial R(p)}{\partial p_k}, \quad h_k(p, A) = \frac{\partial S(p)}{\partial p_k} h(p, A) \quad \text{for } k = i, j. \tag{44}$$

Now substitute equation (44) into equation (22) to obtain

$$\frac{\partial\left[\gamma\left(p,u,A\right)-R(p)\right]}{\partial p_i}=\frac{\partial S(p)}{\partial p_i}\left[h(p,A)+g(\gamma\left(p,u,A\right),p)\right]. \qquad (45)$$

Recall that $\gamma\left(p,u,A\right)$ was defined by $\gamma\left(p,u,A\right)=C(p,u,A)-B(p,A)$. Without loss of generality we may therefore assume $R(p)=0$, since if $R(p)\neq 0$ we could have started out by redefining $B(p,A)$ as $B(p,A)+R(p)$. Equation (45) then reduces to

$$\frac{\partial\gamma\left(p,u,A\right)}{\partial p_i}=\frac{\partial S(p)}{\partial p_i}\left[h(p,A)+g(\gamma\left(p,u,A\right),p)\right]. \qquad (46)$$

Consider taking the total derivative of $\gamma\left(p,u,A\right)$ with respect to p, holding u and A fixed. Using equation (46),

$$\begin{aligned}\frac{d\gamma\left(p,u,A\right)}{dS(p)} &= \sum_i\frac{\partial\gamma\left(p,u,A\right)}{\partial p_i}\frac{dp_i}{dS(p)}\\ &= \sum_i\frac{\partial S(p)}{\partial p_i}\frac{dp_i}{dS(p)}\left[h(p,A)+g(\gamma\left(p,u,A\right),p)\right]\\ &= \frac{dS(p)}{dS(p)}\left[h(p,A)+g(\gamma\left(p,u,A\right),p)\right]\end{aligned}$$

so

$$\frac{d\gamma\left(p,u,A\right)}{dS(p)}=h(p,A)+g(\gamma\left(p,u,A\right),p) \qquad (47)$$

which, substituted into equation (46) gives

$$\frac{\partial\gamma\left(p,u,A\right)}{\partial p_i}=\frac{d\gamma\left(p,u,A\right)}{dS(p)}\frac{\partial S(p)}{\partial p_i}$$

and therefore

$$\gamma\left(p,u,A\right)=c\left[S(p),u,A\right]$$

for some function c. With equation (47) this implies

$$\frac{dc\left[S(p),u,A\right]}{dS(p)}=h(p,A)+g\left[c\left(S(p),u,A\right),p\right].$$

This shows that the functions h and g can depend on p only through $S(p)$, so without loss of generality we may rewrite these functions as $h\left[S(p),A\right]$ and $g\left[x,S(p)\right]$, so

$$\frac{dc\left[S(p),u,A\right]}{dS(p)}=h\left[S(p),A\right]+g\left[c\left(S(p),u,A\right),S(p)\right]. \qquad (48)$$

Equation (12) then follows from the definitions of c, γ, B and ω_i. Also, equation (48) shows that the function c must be a solution to the differential equation (10), and the cost function follows from the definitions of c, γ, B.

PROOF OF COROLLARY 2: Let $p = p_0$ denote the price regime. Then equation (15) follows from equation (12) with $b(A) = B(p_0, A)$, $Gy - B(A) = gy - B(A, p_0)$, $\lambda_i = \partial S(p_0)/\partial p_i$, and

$$h_i(A) = \frac{\partial B(p, A)}{\partial p_i} + \frac{\partial S(p)}{\partial p_i} h(p, A) + \delta_i(p).$$

PROOF OF COROLLARY 3: If the consumer has shape invariance but does not have TIB utility, then by equation (15), $\left[\partial\omega_i(y, A)/\partial y\right] / \left[\partial\omega_j(y, A)/\partial y\right] = \lambda_i/\lambda_j$, which is independent of y and A.

PROOF OF COROLLARY 4: By Theorem 1, the cost functions of all nonTIB shape invariant demand systems depends on p only through two functions, S and B. Theorem 1 of Lewbel (1991) shows that the rank of a demand system equals the minimum number of functions of prices that can be used to write the cost function, so the rank of all nonTIB shape invariant demand systems must be less than or equal to two. They cannot be rank one, because Lewbel (1991) shows that all rank one demand systems are homothetic. All homothetic demand systems are special cases of demands that are linear in x, and by Corollary 1, nonTIB shape invariant demands cannot be linear in x.

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