Supplementary Material for "Specification Testing for Transformation Models with an Application to Generalized Accelerated Failure-time Models"

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THIS SUPPLEMENTARY MATERIAL PROVIDES THE PROOFS OF ALL THE LEMMAS IN THE TEXT.

Proof of Lemma B.1. By Lemma 10.1 in Hoderlein et al. (2014) (HSWY), $\hat{\boldsymbol{\beta}}(y|v) - \boldsymbol{\beta}(y|v) = \bar{\mathbf{S}}_b(v)^{-1}[\mathbf{V}_b(y;v) + \bar{\mathbf{B}}_b(y;v)] + O_P(\nu_{1b}^2 + \nu_{1b}b^{p+1}) = O_P(\nu_{1b} + b^{p+1})$ uniformly in $(y,v) \in \mathcal{Y}_0 \times \mathcal{V}$. The results follow from the fact hat $D_x\hat{F}_b(y|v) - D_xF(y|v) = e_1[\hat{\boldsymbol{\beta}}(y|x,z) - \boldsymbol{\beta}(y|v)]/b$. ■

Proof of Lemma B.2. The results follow from Lemma 10.5 in HSWY (2014) who prove the results based on standard arguments as used in Masry (1996), Hansen (2008), and Kong et al. (2010). ■

Proof of Lemma B.3. (a) Let $\hat{q}(y;v) \equiv \hat{r}(y;v) - r(y;v)$. Noting that $\hat{f}_c(y|v)^{-1} = f(y|v)^{-1} - [\hat{f}_c(y|v) - f(y|v)]/f(y|v)^2 + R_1(y;v)$ where $R_1(y;v) \equiv [\hat{f}_c(y|v) - f(y|v)]^2/[f(y|v)^2\hat{f}_c(y|v)]$, we have that for any $(y,v) \in \mathcal{Y}_0 \times \mathcal{V}$,

$$\begin{aligned}
\hat{q}(y;v) &= \frac{D_x\hat{F}_b(y|v)}{\hat{f}_c(y|v)} - \frac{D_xF(y|v)}{f(y|v)} = \frac{D_x\hat{F}_b(y|v) - D_xF(y|v)}{f(y|v)} + \left[\frac{1}{\hat{f}_c(y|v)} - \frac{1}{f(y|v)}\right]D_xF(y|v) + R_2(y;v) \\
&= \frac{D_x\hat{F}_b(y|v) - D_xF(y|v)}{f(y|v)} - \frac{\hat{f}_c(y|v) - f(y|v)}{f(y|v)^2}D_xF(y|v) + R_1(y;v)D_xF(y|v) + R_2(y;v) \\
&\equiv \hat{q}_1(y;v) - \hat{q}_2(y;v) + R_1(y;v)D_xF(y|v) + R_2(y;v), \text{ say,}
\end{aligned}$$

where $R_2(y;v) \equiv [\hat{f}_c(y|v)^{-1} - f(y|v)^{-1}]\left[D_x\hat{F}_b(y|v) - D_xF(y|v)\right]$. Using Lemmas B.1 and B.2, we can bound the last two terms in the last expression uniformly by $O_p(\eta_{2c}(\eta_{1b} + \eta_{2c}))$, where $\eta_{1b} \equiv \nu_{1b}b^{-1} + b^p$ and $\eta_{2c} \equiv \nu_{2c} + c^{p+1} + c^r$. In addition, uniformly in $(y,v) \in \mathcal{Y}_0 \times \mathcal{V}$,

$$\begin{aligned}
\hat{q}_1(y;v) &= [D_x\hat{F}_b(y|v) - D_xF(y|v)]/f(y|v) \\
&= b^{-1}e_1\bar{\mathbf{S}}_b(v)^{-1}[\mathbf{V}_b(y;v) + \bar{\mathbf{B}}_b(y;v)]/f(y|v) + O_P(\nu_{1b}^2b^{-1} + \nu_{1b}b^p) \\
&= b^{-1}e_1\bar{\mathbf{S}}_b(v)^{-1}\mathbf{V}_b(y;v)/f(y|v) + O_P(\nu_{1b}^2b^{-1} + b^p),
\end{aligned}$$

and

$$\begin{aligned}
\hat{q}_2(y;v) &= D_xF(y|v)[\hat{f}_c(y|v) - f(y|v)]/f(y|v)^2 \\
&= D_xF(y|v)e_2'\bar{\mathbf{S}}_c(v)^{-1}[\mathbf{V}_c^{(L)}(y;v) + \bar{\mathbf{B}}_c^{(L)}(y;v)]/f(y|v)^2 + O_P(\nu_{2c}^2 + \nu_{2c}c^{p+1}) \\
&= D_xF(y|v)e_2'\bar{\mathbf{S}}_c(v)^{-1}\mathbf{V}_c^{(L)}(y;v)/f(y|v)^2 + O_P(\nu_{2c}^2 + c^{p+1} + c^r).
\end{aligned}$$

It follows that uniformly in $(y,v) \in \mathcal{Y}_0 \times \mathcal{V}$,

$$\begin{aligned}
\hat{q}(y;v) &= b^{-1}e_1\bar{\mathbf{S}}_b(v)^{-1}\mathbf{V}_b(y;v)f(y|v)^{-1} - D_xF(y|v)e_2'\bar{\mathbf{S}}_c(v)^{-1}\mathbf{V}_c^{(L)}(y;v)f(y|v)^{-2} \\
&\quad + O_P\left(\nu_{1b}^2b^{-1} + b^p + \nu_{2c}^2 + c^{p+1} + c^r + \eta_{2c}(\eta_{1b} + \eta_{2c})\right) \\
&= b^{-1}e_1\bar{\mathbf{S}}_b(v)^{-1}\mathbf{V}_b(y;v)f(y|v)^{-1} - D_xF(y|v)e_2'\bar{\mathbf{S}}_c(v)^{-1}\mathbf{V}_c^{(L)}(y;v)f(y|v)^{-2} + O_P(\nu_{bc}).
\end{aligned}$$

(b) Write $\hat{r}_0 - r_0 = \hat{r}_{01} + \hat{r}_{02}$, where $\hat{r}_{01} = \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n [\hat{r}(Y_i; X_j, Z_j) - r(Y_i; X_j, Z_j)]$ and $\hat{r}_{02} = \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n [r(Y_i; X_j, Z_j) - r_0]$. It is easy to show that $\hat{r}_{02} = O_P(n^{-1/2})$ by the Chebyshev inequality. For $\hat{r}_{01}$, we have by (a) that $\hat{r}_{01} = R_{1n} - R_{2n} + O_P(\nu_{bc})$, where $R_{1n} \equiv \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_j)^{-1} \mathbf{V}_b(Y_i; V_j) \times f_{ij}^{-1}\mathbf{1}_i$, $R_{2n} \equiv \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n D_{xij} e_2' \bar{\mathbf{S}}_c(V_j)^{-1} \mathbf{V}_c^{(L)}(Y_i; V_j) f_{ij}^{-2}\mathbf{1}_i$, $\mathbf{1}_i \equiv \mathbf{1}\{Y_i \in \mathcal{Y}_0\}$, $f_{ij} \equiv f(Y_i|V_j)$, $f_i \equiv f(Y_i|V_i)$ and $D_{xij} \equiv D_x F(Y_i|V_j)$. For R_{1n}, we have

$$
\begin{aligned}
R_{1n} &= \frac{1}{n^3}\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_j)^{-1} \mu_b(V_k - V_j) K_b(V_k - V_j) \bar{\mathbf{1}}_{Y_i}(W_k) f_{ij}^{-1}\mathbf{1}_i \\
&= \frac{1}{n^3}\sum_{i=1}^n \sum_{j=1, j\neq i}^n \sum_{k=1, k\neq j,i}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_j)^{-1} \mu_b(V_k - V_j) K_b(V_k - V_j) \bar{\mathbf{1}}_{Y_i}(W_k) f_{ij}^{-1}\mathbf{1}_i \\
&\quad + \frac{1}{n^3}\sum_{i=1}^n \sum_{j=1, j\neq i}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_j)^{-1} \mu_b(V_i - V_j) K_b(V_i - V_j) \bar{\mathbf{1}}_{Y_i}(W_i) f_{ij}^{-1}\mathbf{1}_i \\
&\quad + \frac{1}{n^3}\sum_{i=1}^n \sum_{j=1, j\neq i}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_j)^{-1} \mu_b(0) K_b(0) \bar{\mathbf{1}}_{Y_i}(W_j) f_{ij}^{-1}\mathbf{1}_i \\
&\quad + \frac{1}{n^3}\sum_{i=1}^n \sum_{k=1, i}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mu_b(V_k - V_i) K_b(V_k - V_i) \bar{\mathbf{1}}_{Y_i}(W_k) f_i^{-1}\mathbf{1}_i \\
&\quad + \frac{1}{n^3}\sum_{i=1}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mu_b(0) K_b(0) \bar{\mathbf{1}}_{Y_i}(W_i) f_i^{-1}\mathbf{1}_i \\
&\equiv R_{1n,1} + R_{1n,2} + R_{1n,3} + R_{1n,4} + R_{1n,5}.
\end{aligned}
$$

It is easy to show that $R_{1n,5} = O_P(n^{-2}b^{-d-1})$, $R_{1n,4} = O_P(n^{-3/2}b^{-1})$, $R_{1n,3} = O_P(n^{-3/2}b^{-d-1})$, and $R_{1n,2} = O_P(n^{-1}b^{-1})$. Noting that $R_{1n,1}$ is a third-order U-statistic with $E(R_{n,1}) = 0$, it is straightforward to show that $E(R_{1n,1}^2) = O(n^{-1}b^{-2} + n^{-2}b^{-d-2})$. Thus $R_{1n,1} = O_P(n^{-1/2}b^{-1})$ and $R_{1n} = O_P(n^{-1/2}b^{-1})$ as $n^{-1}b^{-d} = o(1)$. By the same token, we can show that $R_{2n} = O_P(n^{-1/2})$. It follows that $\hat{r}_0 - r_0 = O_P(\nu_{bc} + n^{-1/2}b^{-1})$.

(c) Write $\hat{r}_2(y) - r_2(y) = \hat{r}_{21}(y) + \hat{r}_{22}(y)$, where $\hat{r}_{21}(y) = \frac{1}{n}\sum_{i=1}^n [\hat{r}(y; V_i) - r(y; V_i)]$ and $\hat{r}_{22}(y) = \frac{1}{n}\sum_{i=1}^n [r(y; V_i) - r_2(y)]$. By standard chaining arguments and the exponential inequality, we can show that $\sup_{y\in\mathcal{Y}_0} \|\hat{r}_{22}(y)\| = O(n^{-1/2}\sqrt{\ln n})$. By (a), $\hat{r}_{21}(y) = \bar{r}_{21}(y) + O_P(\nu_{bc})$ uniformly in $y \in \mathcal{Y}_0$, where $\bar{r}_{21}(y) \equiv \bar{r}_{21,1}(y) - \bar{r}_{21,2}(y)$, $\bar{r}_{21,1}(y) \equiv \frac{1}{n}\sum_{i=1}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mathbf{V}_b(y; V_i) f(y|V_i)^{-1} \mathbf{1}\{y \in \mathcal{Y}_0\}$, and $\bar{r}_{21,2}(y) \equiv \frac{1}{n}\sum_{i=1}^n f(y|V_i)^{-2} D_x F(y|V_i) \times e_2' \bar{\mathbf{S}}_c(V_i)^{-1} \mathbf{V}_c^{(L)}(y; V_i) \mathbf{1}\{y \in \mathcal{Y}_0\}$. Now write $\bar{r}_{21,1}(y)$ as the summation of a first order U-statistic and a second order U-statistic: $\bar{r}_{21,1}(y) = \bar{r}_{21,11}(y) + \bar{r}_{21,12}(y)$, where

$$
\begin{aligned}
\bar{r}_{21,11}(y) &\equiv \frac{1}{n^2}\sum_{i=1}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mu_b(0) K_b(0) \bar{\mathbf{1}}_y(W_i) f(y|V_i)^{-1} \mathbf{1}\{y \in \mathcal{Y}_0\}, \text{ and} \\
\bar{r}_{21,12}(y) &\equiv \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1, j\neq i}^n b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mu_b(V_j - V_i) K_b(V_j - V_i) \bar{\mathbf{1}}_y(W_j) f(y|V_i)^{-1} \mathbf{1}\{y \in \mathcal{Y}_0\}.
\end{aligned}
$$

By the exponential inequality, we can show that $\sup_{y\in\mathcal{Y}_0} \|\bar{r}_{21,11}(y)\| = O(n^{-3/2}b^{-d-1}\sqrt{\ln n})$. For $\bar{r}_{21,12}(y)$, one can follow the proof of (A.10) in Gozalo and Linton (2001) and show that $\sup_{y\in\mathcal{Y}_0} ||\bar{r}_{21,12}(y)|| = O(n^{-1/2}b^{-1}\sqrt{\ln n})$.[1] Hence $\sup_{y\in\mathcal{Y}_0} \|\bar{r}_{21,1}(y)\| = O(n^{-1/2}b^{-1}\sqrt{\ln n})$. Similarly, $\sup_{y\in\mathcal{Y}_0} \|\bar{r}_{21,2}(y)\| = O(n^{-1/2}\sqrt{\ln n})$. Thus $\sup_{y\in\mathcal{Y}_0} \|\hat{r}_2(y) - r_2(y)\| = O_P(\nu_{bc} + n^{-1/2}b^{-1}\sqrt{\ln n})$. ■

[1] If we ignore the boundary points, we can write $\bar{\mathbf{S}}_b(v) = f(v)\mathbb{S} + b\mathbb{V}(v) + o(b)$ uniformly in v in the interior of $\mathcal{V}$, where

Proof of Lemma B.4. Recall $\mathbf{1}_i \equiv \mathbf{1}\{Y_i \in \mathcal{Y}_0\}$. Let $f_i \equiv f(Y_i|V_i)$ and $D_{xi} \equiv D_x F(Y_i|X_i, Z_i)$. By Lemma B.3(a), $\hat{r}_i - r_i = b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mathbf{V}_b(Y_i; V_i) f_i^{-1} \mathbf{1}_i - D_{xi} e_2' \bar{\mathbf{S}}_c(V_i)^{-1} \mathbf{V}_c^{(L)}(Y_i; V_i) f_i^{-2} \mathbf{1}_i + O_P(\nu_{bc})$. It follows that

$$
\begin{aligned}
\Gamma_{2n,1} &= b^{\frac{d}{2}+2} \sum_{i=1}^{n} \|(\hat{r}_i - r_i) \circ r_0\|^2 a_i \\
&= b^{\frac{d}{2}+2} \sum_{i=1}^{n} \left\| \left(b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mathbf{V}_b(Y_i; V_i) f_i^{-1} - D_{xi} e_2' \bar{\mathbf{S}}_c(V_i)^{-1} \mathbf{V}_c^{(L)}(Y_i; V_i) f_i^{-2} \right) \circ r_0 \right\|^2 a_i \\
&\quad + n b^{\frac{d}{2}+2} O_P\left(\nu_{bc}^2 + \nu_{bc}\left(\nu_{1b} b^{-1} + b^p + \nu_{2c} + c^{p+1} + c^r\right)\right) \\
&= \bar{\Gamma}_{2n,1} + o_P(1),
\end{aligned}
$$

where $\bar{\Gamma}_{2n,1} \equiv b^{\frac{d}{2}+2} \sum_{i=1}^{n} \left\| \left(b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mathbf{V}_b(Y_i; V_i) f_i^{-1} - D_{xi} e_2' \bar{\mathbf{S}}_c(V_i)^{-1} \mathbf{V}_c^{(L)}(Y_i; V_i) f_i^{-2} \right) \circ r_0 \right\|^2 a_i$,[2] and we use the fact that $\mathbf{1}_i a_i = a_i$ as $a(y; v)$ has compact support $\mathcal{Y}_0 \times \mathcal{V}_0$. Let $\zeta_k(w) \equiv \zeta_k(y; v)$ be as defined in Section 3.2. Then

$$
\bar{\Gamma}_{2n,1} = b^{\frac{d}{2}+2} \sum_{i=1}^{n} \left\| n^{-1} \sum_{k=1}^{n} \zeta_k(W_i) \circ r_0 \right\|^2 a_i = n^{-2} b^{\frac{d}{2}+2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \zeta(W_{i_1}, W_{i_2}, W_{i_3}),
$$

where $\zeta(W_{i_1}, W_{i_2}, W_{i_3}) \equiv \left(\zeta_{i_2}(W_{i_1}) \circ r_0\right)' \left(\zeta_{i_3}(W_{i_1}) \circ r_0\right) a_{i_1}$. Let $\varphi(w_{i_1}, w_{i_2}) \equiv E[\zeta(W_1, w_{i_1}, w_{i_2})]$, and $\bar{\zeta}(w_{i_1}, w_{i_2}, w_{i_3}) \equiv \zeta(w_{i_1}, w_{i_2}, w_{i_3}) - \varphi(w_{i_2}, w_{i_3})$. We can decompose $\bar{\Gamma}_{2n,1}$ as $\bar{\Gamma}_{2n,1} = \bar{\Gamma}_{2n,11} + \bar{\Gamma}_{2n,12}$, where

$$
\bar{\Gamma}_{2n,11} = n^{-1} b^{\frac{d}{2}+2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \varphi(W_{i_1}, W_{i_2}) \text{ and } \bar{\Gamma}_{2n,12} = n^{-2} b^{\frac{d}{2}+2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \bar{\zeta}(W_{i_1}, W_{i_2}, W_{i_3}).
$$

Consider $\bar{\Gamma}_{2n,12}$ first. Write $E(\bar{\Gamma}_{2n,12}^2) = n^{-4} b^{d+4} \sum_{i_1,\ldots,i_6}^{n} E\left[\bar{\zeta}(W_{i_1}, W_{i_2}, W_{i_3}) \bar{\zeta}(W_{i_4}, W_{i_5}, W_{i_6})\right]$. Noting that $E\left[\bar{\zeta}(W_{i_1}, w_{i_2}, w_{i_3})\right] = E\left[\bar{\zeta}(w_{i_1}, W_{i_2}, w_{i_3})\right] = E[\bar{\zeta}(w_{i_1}, w_{i_2}, W_{i_3})] = 0$, $E[\bar{\zeta}(W_{i_1}, W_{i_2}, W_{i_3}) \bar{\zeta}(W_{i_4}, W_{i_5}, W_{i_6})] = 0$ if there are more than three distinct elements in $\{i_1, \ldots, i_6\}$. With this, it is easy to show that $E(\bar{\Gamma}_{2n,12}^2) = O(n^{-1} b^{d+4} (b^{-4-3d} + c^{-3(d+1)})) = o(1)$. Hence $\bar{\Gamma}_{2n,12} = o_P(1)$ by the Chebyshev inequality.

For $\bar{\Gamma}_{2n,11}$, we have $\bar{\Gamma}_{2n,11} = n^{-1} b^{\frac{d}{2}+2} \sum_{i=1}^{n} \varphi(W_i, W_i) + 2n^{-1} b^{\frac{d}{2}+2} \sum_{1 \leq i < j \leq n} \varphi(W_i, W_j) \equiv \mathbb{B}_{1n} + \mathbb{V}_{1n}$, say, where $\varphi(W_i, W_j) = \int \zeta(w, W_i, W_j) \, dF(w) = \int \left(\zeta_i(w) \circ r_0\right)' \left(\zeta_j(w) \circ r_0\right) a(w) \, dF(w)$, and $\mathbb{B}_{1n}$ and $\mathbb{V}_{1n}$ contribute to the asymptotic bias and variance of $\bar{\Gamma}_{2n,11}$, respectively. Note that as $\mathbb{V}_{1n}$ is a second-order degenerate U-statistic, we can easily verify that all the conditions of Theorem 1 of Hall (1984) are satisfied and a central limit theorem applies to it: $\mathbb{V}_{1n} \xrightarrow{d} N(0, \sigma_0^2)$, where $\sigma_0^2 = \lim_{n\to\infty} \sigma_n^2$ and $\sigma_n^2 = 2b^{d+4} E[\varphi(W_1, W_2)]^2$. Thus $\Gamma_{2n,11} - \mathbb{B}_{1n} \xrightarrow{d} N(0, \sigma_0^2)$.

Lastly, noting that $E|\mathbb{B}_{1n}| = b^{\frac{d}{2}+2} O\left(b^{-d-2} + c^{-d-1}\right)$, we have $\mathbb{B}_{1n} = O_P(b^{\frac{d}{2}+2}(b^{-d-2} + c^{-d-1}))$ by Markov inequality. ■

Proof of Lemma B.5. (a) Note that $\Gamma_{2n,2} \leq n b^{\frac{d}{2}+2} \|\hat{r}_0 - r_0\|^2 \bar{\Gamma}_{2n,2}$ where $\bar{\Gamma}_{2n,2} \equiv n^{-1} \sum_{i=1}^{n} \|r_i\|^2 a_i$. By Assumptions C.2(ii) and C.3(i), the compact support of a, and Markov inequality, $\bar{\Gamma}_{2n,2} = O_P(1)$. Using this and Lemma B.3(b) we have $\Gamma_{2n,2} = n b^{\frac{d}{2}+2} O_P\left(\nu_{bc}^2 + n^{-1} b^{-2}\right) O_P(1) = o_P(1)$.

$\mathbb{S}$ and $\mathbb{V}$ are defined as M and V in Li et al. (2003, p. 617). Following the proof of Lemma A.3 in their paper, one can show that $\bar{r}_{21,12}(y) = O(n^{-1/2})$ elementwise by using the degeneracy of the second order U-statistic defined analogously to $\bar{r}_{21,12}(y)$ but with $\bar{\mathbf{S}}_b(V_i)^{-1}$ replaced by its leading term $f(V_i)^{-1} \mathbb{S}^{-1}$. But their argument breaks down when v takes values on the boundary of $\mathcal{V}$.

[2] Write $\bar{\Gamma}_{2n,1} = b^{\frac{d}{2}+2} \sum_{i=1}^{n} \|(\xi_{1ni} - \xi_{2ni}) \circ r_0\|^2 a_i$, where $\xi_{1ni} \equiv b^{-1} e_1 \bar{\mathbf{S}}_b(V_i)^{-1} \mathbf{V}_b(Y_i; V_i) f_i^{-1}$ and $\xi_{2ni} \equiv D_{xi} e_2' \bar{\mathbf{S}}_c(V_i)^{-1} \mathbf{V}_c^{(L)}(Y_i; V_i) f_i^{-2}$. By straightforward moment calculations, we can show that ξ_{1ni} contributes to both the asymptotic bias and variance of the test statistic whereas ξ_{2ni} only contributes to the asymptotic bias.

(b) Noting that $\hat{r}_1(x) - r_1(x) = \frac{1}{n}\sum_{i=1}^n [\hat{r}(Y_i; x, Z_i) - r(Y_i; x, Z_i)] + \frac{1}{n}\sum_{i=1}^n [r(Y_i; x, Z_i) - r_1(x)]$, we have $b^{\frac{d}{2}+2}\sum_{i=1}^n \|[\hat{r}_1(X_i) - r_1(X_i)] \circ r_{2i}\|^2 a_i = R_{3n} + R_{4n} + 2R_{5n}$, where

$$\begin{aligned}
R_{3n} &\equiv \frac{b^{\frac{d}{2}+2}}{n^2}\sum_{i=1}^n \left\| \sum_{j=1}^n [\hat{r}(Y_j; X_i, Z_j) - r(Y_j; X_i, Z_j)] \circ r_{2i} \right\|^2 a_i, \\
R_{4n} &\equiv \frac{b^{\frac{d}{2}+2}}{n^2}\sum_{i=1}^n \left\| \sum_{j=1}^n [r(Y_j; X_i, Z_j) - r(X_i)] \circ r_{2i} \right\|^2 a_i, \text{ and} \\
R_{5n} &\equiv \frac{b^{\frac{d}{2}+2}}{n^2}\sum_{i=1}^n \left\| \sum_{j=1}^n [\hat{r}(Y_j; X_i, Z_j) - r(Y_j; X_i, Z_j)] \circ r_{2i} \right\| \left\| \sum_{k=1}^n [r(Y_k; X_i, Z_k) - r(X_i)] \circ r_{2i} \right\| a_i.
\end{aligned}$$

By Lemma B.3(a) we can readily show that $R_{3n} = \mathbb{B}_{2n} + o_P(1)$. We further decompose $\mathbb{B}_{2n}$ as $\mathbb{B}_{2n} = \mathbb{B}_{2n,1} + \mathbb{B}_{2n,2}$, where

$$\begin{aligned}
\mathbb{B}_{2n,1} &\equiv \frac{b^{\frac{d}{2}+2}}{n^4}\sum_{i_1=1}^n\sum_{i_2=1}^n\sum_{i_3=1}^n\sum_{i_4=1}^n \left(\zeta_{i_3}(Y_{i_2}; X_{i_1}, Z_{i_2}) \circ r_{2i_1}\right)' \left(\zeta_{i_3}(Y_{i_4}; X_{i_1}, Z_{i_4}) \circ r_{2i_1}\right) a_{i_1} \text{ and} \\
\mathbb{B}_{2n,2} &\equiv \frac{b^{\frac{d}{2}+2}}{n^4}\sum_{i_1=1}^n\sum_{i_2=1}^n\sum_{i_3=1}^n\sum_{i_4=1}^n\sum_{i_5=1, i_5\neq i_3}^n \left(\zeta_{i_3}(Y_{i_2}; X_{i_1}, Z_{i_2}) \circ r_{2i_1}\right)' \left(\zeta_{i_5}(Y_{i_4}; X_{i_1}, Z_{i_4}) \circ r_{2i_1}\right) a_{i_1}.
\end{aligned}$$

By direct moment calculations and the Chebyshev inequality, we can show that $\mathbb{B}_{2n,2} = O_P(b^{\frac{d_z}{2}} + b^{\frac{d}{2}+2}c^{-\frac{d_x}{2}})$ which is $o_P(1)$ under Assumption A.7 if $d_z > 0$, and that

$$\mathbb{B}_{2n,1} = \bar{\mathbb{B}}_{2n} = \frac{1}{n^4} b^{\frac{d}{2}+2}\sum_{i_1=1}^n\sum_{i_3=1}^n \left\| \sum_{i_2=1}^n \zeta_{i_3}(Y_{i_2}; X_{i_1}, Z_{i_2}) \circ r_{2i_1} \right\|^2 a_{i_1} = O_P(b^{\frac{d}{2}+2}(b^{-d_x-2} + c^{-d_x})).$$

It follows that $R_{3n} = \bar{\mathbb{B}}_{2n} + o_P(1)$ if $d_z > 0$. By Markov inequality, $R_{4n} = O_P(b^{\frac{d}{2}+2}) = o_P(1)$. By Cauchy-Schwarz inequality, $R_{5n} \leq \{R_{3n}R_{4n}\}^{1/2} = O_P(\{[b^{\frac{d}{2}+2}(b^{-d_x-2} + c^{-d_x}) + 1]b^{\frac{d}{2}+2}\}^{1/2}) = o_P(1)$. This completes the proof of part (b).

(c) Noting that $\hat{r}_2(y) - r_2(y) = \frac{1}{n}\sum_{i=1}^n [\hat{r}(y; X_i, Z_i) - r(y; X_i, Z_i)] + \frac{1}{n}\sum_{i=1}^n [r(y; X_i, Z_i) - r_2(y)]$, by Cauchy-Schwarz inequality we have $\Gamma_{2n,4} \leq 2R_{6n} + 2R_{7n}$, where $R_{6n} \equiv \frac{b^{\frac{d}{2}+2}}{n^2}\sum_{i=1}^n || \sum_{j=1}^n r_{1i} \circ [\hat{r}(Y_i; X_j, Z_j) - r(Y_i; X_j, Z_j)]||^2 a_i$ and $R_{7n} \equiv \frac{b^{\frac{d}{2}+2}}{n^2}\sum_{i=1}^n || \sum_{j=1}^n r_{1i} \circ [r(Y_i; X_j, Z_j) - r_2(Y_i)]||^2 a_i$. By Markov inequality $R_{7n} = O_P(b^{\frac{d}{2}+2}) = o_P(1)$. For R_{6n} we can first apply Lemma B.3 to show that $R_{6n} = \bar{R}_{6n} + o_P(1)$, where

$$\bar{R}_{6n} \equiv \frac{b^{\frac{d}{2}+2}}{n^2}\sum_{i=1}^n \left\| \sum_{j=1}^n r_{1i} \circ \left[b^{-1} e_1 \bar{\mathbf{S}}_b(V_j)^{-1} \mathbf{V}_b(Y_i; V_j) f_{ij}^{-1} - f_{ij}^{-2} D_{xij} e_2' \bar{\mathbf{S}}_c(X_j, Z_j)^{-1} \mathbf{V}_c^{(L)}(Y_i; V_j) \right] \right\|^2 a_i,$$

$f_{ij} \equiv f(Y_i|V_j)$, and $D_{xij} \equiv D_x F(Y_i|V_j)$. Observe that $\bar{R}_{6n} \leq 2\bar{R}_{6n,1} + 2\bar{R}_{6n,2}$, where

$$\begin{aligned}
\bar{R}_{6n,1} &= \frac{b^{\frac{d}{2}+2}}{n^2}\sum_{i=1}^n \left\| \sum_{j=1}^n r_{1i} \circ \left(b^{-1} e_1 \bar{\mathbf{S}}_b(V_j)^{-1} \mathbf{V}_b(Y_i; V_j) f_{ij}^{-1} \right) \right\|^2 a_i, \text{ and} \\
\bar{R}_{6n,2} &= \frac{b^{\frac{d}{2}+2}}{n^2}\sum_{i=1}^n \left\| \sum_{j=1}^n r_{1i} \circ \left(f_{ij}^{-2} D_{xij} e_2' \bar{\mathbf{S}}_c(V_j)^{-1} \mathbf{V}_c^{(L)}(Y_i; V_j) \right) \right\|^2 a_i.
\end{aligned}$$

By straightforward but tedious moment calculations, we can show that

$$\begin{aligned} E\left(\bar{R}_{6n,1}\right) &= \frac{b^{\frac{d}{2}}}{n^4}\sum_{i=1}^{n} E\left\|\sum_{j=1}^{n}\sum_{k=1}^{n} r_{1i}\circ\left(e_1\bar{\mathbf{S}}_b\left(V_j\right)^{-1}\mu_b\left(V_k-V_j\right)K_b\left(V_k-V_j\right)\bar{\mathbf{1}}_{Y_i}\left(W_k\right)f_{ij}^{-1}\right)\right\|^2 a_i \\ &= O\left(b^{\frac{d}{2}}+n^{-1}b^{-\frac{d}{2}}+n^{-2}b^{-\frac{3d}{2}}\right)=o\left(1\right). \end{aligned}$$

Similarly, $E\left(\bar{R}_{6n,2}\right) = b^{\frac{d}{2}+2}O\left(1+n^{-1}c^{-(d+1)}+n^{-2}c^{-2(d+1)}\right) = o\left(1\right).$ Then $\bar{R}_{6n} = o_P\left(1\right)$ by Markov inequality. It follows that $R_{6n}=o_P\left(1\right)$ and $\Gamma_{2n,4}=o_P\left(1\right).$ ■

Proof of Lemma B.6. (a) Noting that $\Gamma_{2n,5}=2b^{\frac{d}{2}+2}\left(\left(\hat{r}_0-r_0\right)\circ r_0\right)'\bar{\Gamma}_{2n,5}$ where $\bar{\Gamma}_{2n,5}=\sum_{i=1}^{n}((\hat{r}_i-r_i)$ $\circ r_i)a_i$. By Lemma B.3(a), we can show that $\bar{\Gamma}_{2n,5}=\bar{\Gamma}_{2n,51}+o_P(n^{1/2}b^{-(\frac{d}{2}+1)})$, where

$$\begin{aligned} \bar{\Gamma}_{2n,51} &= \sum_{i=1}^{n}\left[\left(b^{-1}e_1\bar{\mathbf{S}}_b\left(V_i\right)^{-1}\mathbf{V}_b\left(Y_i;V_i\right)f_i^{-1}-D_{xi}e_2'\bar{\mathbf{S}}_c\left(V_i\right)^{-1}\mathbf{V}_c^{(L)}\left(Y_i;V_i\right)f_i^{-2}\right)\circ r_i\right]a_i \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\zeta_{1j}\left(Y_i;V_i\right)\circ r_i\right]a_i-\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\zeta_{2j}\left(Y_i;V_i\right)\circ r_i\right]a_i\equiv R_{8n}+R_{9n}. \end{aligned}$$

In view of $R_{8n}=\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}[b^{-1}e_1\bar{\mathbf{S}}_b\left(V_i\right)^{-1}\mu_b\left(V_j-V_i\right)K_b\left(V_j-V_i\right)\bar{\mathbf{1}}_{Y_i}\left(W_j\right)f_i^{-1}]\circ r_i\right]a_i$, it is easy to show that $E\left\|R_{8n}\right\|^2=O\left(nb^{-2}+b^{-2d-2}\right)$. Thus $R_{8n}=O_P\left(b^{-d-1}+n^{1/2}b^{-1}\right)$. Similarly, $R_{9n}=O_P(c^{-d-1}$ $+n^{1/2})$. It follows that $\bar{\Gamma}_{2n,51}=O_P(b^{-d-1}+n^{1/2}b^{-1}+c^{-d-1})$. Then by Lemma B.3(b) and the fact that $\nu_{bc}=O_P\left(n^{-1/2}b^{-1}\right)$, $\Gamma_{2n,5}=b^{\frac{d}{2}+2}O_P\left(n^{-1/2}b^{-1}\right)\left[O_P\left(b^{-d-1}+n^{1/2}b^{-1}+c^{-d-1}\right)+o_P(n^{1/2}b^{-(\frac{d}{2}+1)})\right]=$ $o_P\left(1\right).$

(b) Write $\Gamma_{2n,6}=-2r_0'\bar{\Gamma}_{2n,6}$ where $\bar{\Gamma}_{2n,6}\equiv b^{\frac{d}{2}+2}\sum_{i=1}^{n}\left(\left(\hat{r}_i-r_i\right)\circ\left(\hat{r}_{1i}-r_{1i}\right)\circ r_{2i}\right)a_i$. Then $\bar{\Gamma}_{2n,6}=$ $R_{10n}+R_{11n}$, where $R_{10n}\equiv n^{-1}b^{\frac{d}{2}+2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left(\hat{r}_i-r_i\right)\circ\left[\hat{r}\left(Y_j;X_i,Z_j\right)-r\left(Y_j;X_i,Z_j\right)\right]\circ r_{2i}\right)a_i$ and $R_{11n}\equiv n^{-1}b^{\frac{d}{2}+2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left(\hat{r}_i-r_i\right)\circ\left[r\left(Y_j;X_i,Z_j\right)-r_1\left(X_i\right)\right]\circ r_{2i}\right)a_i$. Using Lemma B.3, we can show that $R_{10n}=\bar{R}_{10n}+o_P\left(1\right)$, where

$$\begin{aligned} \bar{R}_{10n} &= n^{-1}b^{\frac{d}{2}+2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\left[b^{-1}e_1\bar{\mathbf{S}}_b\left(V_i\right)^{-1}\mathbf{V}_b\left(Y_i;V_i\right)f_i^{-1}-D_{xi}e_2'\bar{\mathbf{S}}_c\left(V_i\right)^{-1}\mathbf{V}_c^{(L)}\left(Y_i;V_i\right)f_i^{-2}\right]\right. \\ &\quad \left.\circ\left[b^{-1}e_1\bar{\mathbf{S}}_b\left(X_i,Z_j\right)^{-1}\mathbf{V}_b\left(Y_j;X_i,Z_j\right)f_{ji}^{-1}-D_{xi}e_2'\bar{\mathbf{S}}_c\left(X_i,Z_j\right)^{-1}\mathbf{V}_c^{(L)}\left(Y_j;X_i,Z_j\right)f_{ji}^{-2}\right]\circ r_{2i}\right\}a_i\mathbf{1}_i\mathbf{1}_j \\ &= n^{-3}b^{\frac{d}{2}+2}\sum_{i=1}^{n}\left[\sum_{l=1}^{n}\zeta_l\left(W_i\right)\circ\sum_{j=1}^{n}\sum_{k=1}^{n}\zeta_k\left(Y_j;X_i,Z_j\right)\circ r_2\left(Y_i\right)\right]a_i. \end{aligned}$$

Noting that $E\left\|\bar{R}_{10n}\right\|^2=O\left(b^{d+4}\left(b^{-2d_x-4}+c^{-2d_x}\right)\right)$, we have $\bar{R}_{10n}=O_P(b^{(d_z-d_x)/2}+b^{\frac{d}{2}+2}c^{-d_x})$ which is $o_P\left(1\right)$ under Assumption A.7 if $d_z>d_x$ and otherwise not. Hence $r_0'R_{10n}=\mathbb{B}_{3n}+o_P\left(1\right)$ and $\mathbb{B}_{3n}=$ $O_P(b^{(d_z-d_x)/2}+b^{\frac{d}{2}+2}c^{-d_x})$. For R_{11n}, we apply Cauchy-Schwarz inequality to obtain $R_{11n}\leq\left\{\alpha_{1n}\alpha_{2n}\right\}^{1/2}$, where $\alpha_{1n}\equiv b^{\frac{d}{2}+2}\sum_{i=1}^{n}\left\|\left(\hat{r}_i-r_i\right)\circ r_{2i}\right\|^2a_i$ and $\alpha_{2n}\equiv n^{-1}b^{\frac{d}{2}+2}\sum_{i=1}^{n}\left\|\sum_{j=1}^{n}\left[r\left(Y_j;X_i,Z_j\right)-r_1\left(X_i\right)\right]\right\|^2a_i$. Analogously to the determination of the probability order of $\Gamma_{2n,1}$, we can show that $\alpha_{1n}=O_P(b^{\frac{d}{2}+2}(b^{-d-2}+$ $c^{-d-1}))$. Next, $\alpha_{2n}=O_P(b^{\frac{d}{2}+2})$ by Markov inequality. It follows that $R_{11n}=O_P(b^{-\frac{d}{4}}+b^{\frac{d}{4}+1}c^{-\frac{d+1}{2}})O_P(b^{\frac{d}{4}+1})$ $=O_P(b+b^{\frac{d}{2}+2}c^{-\frac{d+1}{2}})=o_P\left(1\right)$ and $\Gamma_{2n,6}=-2\mathbb{B}_{3n}+o_P\left(1\right).$

(c) Write $\Gamma_{2n,7}=-2r_0'\bar{\Gamma}_{2n,7}$ where $\bar{\Gamma}_{2n,7}\equiv b^{\frac{d}{2}+2}\sum_{i=1}^{n}[(\hat{r}_i-r_i)\circ r_{1i}\circ(\hat{r}_{2i}-r_{2i})]a_i$. We further decompose $\bar{\Gamma}_{2n,7}$ as $\bar{\Gamma}_{2n,7}=R_{12n}+R_{13n}$, where $R_{12n}\equiv n^{-1}b^{\frac{d}{2}+2}\sum_{i=1}^{n}\sum_{j=1}^{n}\{(\hat{r}_i-r_i)\circ r_{1i}\circ[\hat{r}\left(Y_i;V_j\right)-r\left(Y_i;V_j\right)]\}a_i$

and $R_{13n} \equiv n^{-1} b^{\frac{d}{2}+2} \sum_{i=1}^{n} \sum_{j=1}^{n} \{(\hat{r}_i - r_i) \circ r_{1i} \circ [r(Y_i; V_j) - r_2(Y_i)]\} a_i$. Following the analysis of R_{10} and R_{11n}, we can readily show that $R_{sn} = o_P(1)$ for $s = 12, 13$. It follows that $\Gamma_{2n,7} = o_P(1)$.

(d) Noting that $\Gamma_{2n,8} = 2b^{\frac{d}{2}+2} (\hat{r}_0 - r_0)' \bar{\Gamma}_{2n,8}$ where $\bar{\Gamma}_{2n,8} \equiv \sum_{i=1}^{n} (r_i \circ (\hat{r}_{1i} - r_{1i}) \circ r_{2i}) a_i$. Then $\bar{\Gamma}_{2n,8} = R_{14n} + R_{15n}$, where $R_{14n} \equiv \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \{r_i \circ [\hat{r}(Y_j; X_i, Z_j) - r(Y_j; X_i, Z_j)] \circ r_{2i}\} a_i$ and $R_{15n} \equiv \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \{r_i \circ [r(Y_j; X_i, Z_j) - r_1(X_i)] \circ r_{2i}\} a_i$. By straightforward moment calculations, we can show that $R_{15n} = O_P(n^{1/2})$. By Lemma B.3(a), we can show that $R_{14n} = \bar{R}_{14n} + o_P(n^{1/2} b^{-(\frac{d}{2}+1)})$, where

$$\bar{R}_{14n} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} [r_i \circ \zeta_k(Y_j; X_i, Z_j) \circ r_{2i}] a_i.$$

Noting that $E \left\| \bar{R}_{14n} \right\|^2 = O(nb^{-2} + (b^{-d_x-2} + b^{-2d_z-2} + c^{-d_x-1} + c^{-2d_z-2}) + n^{-1}(b^{-d-2-d_z} + c^{-d-2-d_z}))$, we have $\bar{R}_{14n} = O_P(n^{1/2} b^{-1} + b^{-d_z-1} + c^{-d_z-1})$. Consequently, $\bar{\Gamma}_{2n,8} = O_P\left(n^{1/2} b^{-1} + b^{-d_z-1} + c^{-d_z-1}\right) + o_P(n^{1/2} b^{-(\frac{d}{2}+1)})$. Then by Lemma B.3(b) and the fact that $\nu_{bc} = O_P\left(n^{-1/2} b^{-1}\right)$,

$$\begin{aligned} \Gamma_{2n,8} &= b^{\frac{d}{2}+2} O_P\left(n^{-1/2} b^{-1}\right) \left[O_P\left(n^{1/2} b^{-1} + b^{-d_z-1} + c^{-d_z-1}\right) + o_P(n^{1/2} b^{-(\frac{d}{2}+1)})\right] \\ &= O_P(b^{\frac{d}{2}} + n^{-1/2} b^{\frac{d_x-d_z}{2}} + n^{-1/2} b^{\frac{d}{2}+2} c^{-d_z-1}) + o_P(1) = o_P(1). \end{aligned}$$

(e) By the Cauchy-Schwarz inequality and Lemmas B.5(a) and (c), $|\Gamma_{2n,9}| \le 2 (\Gamma_{2n,2} \Gamma_{2n,4})^{1/2} = o_P(1)$.

(f) Let $\bar{\Delta} r_{1ij} = \hat{r}(Y_j; X_i, Z_j) - r(Y_j; X_i, Z_j)$, $\Delta r_{1ij} = r(Y_j; X_i, Z_j) - r_1(X_i)$, $\bar{\Delta} r_{2ij} = \hat{r}(Y_i; X_j, Z_j) - r(Y_i; X_j, Z_j)$, and $\Delta r_{2ij} = r(Y_i; X_j, Z_j) - r_2(Y_i)$. Using $\hat{r}_{1i} - r_{1i} = \frac{1}{n} \sum_{j=1}^{n} \left(\bar{\Delta} r_{1ij} + \Delta r_{1ij}\right)$ and $\hat{r}_{2i} - r_{2i} = \frac{1}{n} \sum_{k=1}^{n} \left(\bar{\Delta} r_{2ik} + \Delta r_{2ik}\right)$, we can decompose $\Gamma_{2n,10}$ as follows:

$$\begin{aligned} \Gamma_{2n,10} &= 2n^{-2} b^{\frac{d}{2}+2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \Big\{ (\Delta r_{1ij} \circ r_{2i})' (r_{1i} \circ \Delta r_{2ik}) a_i + \left(\bar{\Delta} r_{1ij} \circ r_{2i}\right)' \left(r_{1i} \circ \bar{\Delta} r_{2ik}\right) a_i \\ &\quad + \left(\bar{\Delta} r_{1ij} \circ r_{2i}\right)' (r_{1i} \circ \Delta r_{2ik}) a_i + (\Delta r_{1ij} \circ r_{2i})' \left(r_{1i} \circ \bar{\Delta} r_{2ik}\right) a_i \Big\} \\ &\equiv 2R_{16n} + 2R_{17n} + 2R_{18n} + 2R_{19n}, \text{ say.} \end{aligned}$$

By moment calculations, $E(R_{16n}) = O(b^{\frac{d}{2}+2})$ and $E\left(R_{16n}^2\right) = O(b^{d+4})$, implying that $R_{16n} = O_P(b^{\frac{d}{2}+2}) = o_P(1)$. For R_{17n}, we can show that $R_{17n} = \bar{R}_{17n} + o_P(1)$, where

$$\bar{R}_{17n} = n^{-4} b^{\frac{d}{2}+2} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \sum_{i_4=1}^{n} \sum_{i_5=1}^{n} \left(\zeta_{i_4}(Y_{i_2}; X_{i_1}, Z_{i_2}) \circ r_{2i_1}\right)' \left(r_{1i_1} \circ \zeta_{i_5}(Y_{i_1}; X_{i_3}, Z_{i_3})\right) a_{i_1}.$$

Noting that $E\left(\bar{R}_{17n}^2\right) = O\left(b^{d+4} b^{-4}\right) + n^{-1} b^{d+4} (b^{-d_x-4} + b^{-d_z-4} + c^{-d_x} + c^{-d_z})) = o(1)$, we have $\bar{R}_{17n} = o_P(1)$. Similarly, we can show that $R_{18n} = o_P(1)$ and $R_{19n} = o_P(1)$. Consequently, $\Gamma_{2n,10} = o_P(1)$. ■

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