0. Introduction

0.1. Let $G$ be a reductive algebraic group defined over $\mathbb{Q}$, and let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$. Let $X$ be the symmetric space for $G(\mathbb{R})$, and assume $X$ is contractible. Then the cohomology (mod torsion) of the space $X/\Gamma$ is the same as the cohomology of $\Gamma$. In turn, $X/\Gamma$ will have the same cohomology as $W/\Gamma$ if $W$ is a "spine" in $X$. This means that $W$ (if it exists) is a deformation retract of $X$ by a $\Gamma$-equivariant deformation retraction, that $W/\Gamma$ is compact, and that $\dim W$ equals the virtual cohomological dimension (vcd) of $\Gamma$. Then $W$ can be given the structure of a cell complex on which $\Gamma$ acts cellularly, and the cohomology of $W/\Gamma$ can be found combinatorially.

Spines have been found for many groups $G$ (see Section 2.6 below, [A1], [MM1], and [MM2]). This paper concerns the case where $G$ is the restriction of scalars of a general linear group over a number field $k$ with ring of integers $O$. This means $G(\mathbb{Q}) = GL_n(k)$. We shall take $\Gamma$ to be a subgroup of finite index in $GL_n(O)$, or more generally in $GL_n(\mathcal{P})$, where $\mathcal{P}$ is a projective $O$-module of rank $n$. In [A3], Ash found a spine $W$ for these $G$, calling $W$ the well-rounded retract. (This generalized [Sou2].) The retract has been used in computations; see [Sou1], [AGG], [AM1]. [AM2], and [vGT]. We remark that $k = \mathbb{Q}$; $\Gamma \subseteq GL_n(\mathbb{Z})$ for $n = 2, 3, 4$ still provide our main cases of computational interest. For imaginary quadratic fields $k$ and $n = 2$, see [Men], [SV], and [V]. For real quadratic $k$ and $n = 2$, see [B].

0.2. The present paper extends [A3] to deal with the "cusps" of $X/\Gamma$ in a way that we now describe.

Borel and Serre [BS] introduced a bordification $\overline{X}$ of $X$ such that $\overline{X}/\Gamma$ is a compactification of $X/\Gamma$. The space $\overline{X}/\Gamma$ has the same homotopy type as $X/\Gamma$, and therefore has the same homotopy type as $W/\Gamma$. The boundary of $\overline{X}/\Gamma$ is a union of finitely many "faces," one for each equivalence class mod $\Gamma$ of parabolic $\mathbb{Q}$-subgroups of $G$. 

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since both are \((HK)\backslash G/\Gamma_0\). However, the points of view on the two sides of this
equation are different: \(K\backslash Y\) is a space of lattices modulo rotations, while \(X/\Gamma_0\) is a
homogeneous space modulo an arithmetic group. We now make explicit how to
go back and forth between the two sides.

Definition. A marked lattice in \(S^a\) is a function \(f : L_0 \rightarrow S^a\) of the form
\(f(x) = gx\) for \(g \in G\). We shall identify marked lattices that differ only by a
homothety, that is, \(f\) and \(hf\) for \(h \in H\).

Let \(Y' = H\backslash G\) be the set of marked lattices. A marked lattice \(f\) gives rise to
an ordinary lattice \(L \in Y\) by setting \(L = f(L_0)\). This realizes the projection
\(Y' = H\backslash G \rightarrow Y = H\backslash G/\Gamma_0\). On the other hand, \(f(x) = gx\) gives a point
\((HK)g \in X\), which realizes the projection \(Y' = H\backslash G \rightarrow X = \{(HK)g\}\). The right
action of \(g \in G\) on \(Y'\) sends \(f : x \mapsto gx\) to \(x \mapsto gg\alpha x\). The left action of \(k \in K\)
on \(Y'\) sends \(f\) to \(k \cdot f\). Diagram (2.3.1) shows these spaces.

\[\begin{align*}
Y' &= H\backslash G \\
\text{(marked lattices)}
\end{align*}\]

\[\begin{align*}
X &= HK\backslash G \\
\text{(homogeneous space)}
\end{align*}\]

\[\begin{align*}
Y &= H\backslash G/\Gamma_0 \\
\text{(lattices)}
\end{align*}\]

\[\begin{align*}
X/\Gamma_0 &= K\backslash Y \\
\text{(arithmetic quotient)}
\end{align*}\]

2.4. Let \(f\) be a marked lattice, fixed within its homothety class. Let \(L = f(L_0)\). The arithmetic minimum of \(f\) is defined to be \(m(f) = \min\{||a||: a \in L - \{0\}\}\).
This number is positive. The set of minimal vectors of \(f\) is defined to be \(M(f) = \{a \in L : ||a|| = m(L)\}\).
The definitions of \(m(f)\) and \(M(f)\) use only the image \(L\) of \(f\), so they descend to functions \(m(L)\) and \(M(L)\) on \(Y\). On the other hand, \(m(f)\) and \(M(f)\) are
\(K\)-equivariant, since \(K\) preserves \(||\cdot||\).

Unless otherwise specified, we normalize \(f\) (resp., \(L\)) in its homothety class so
that \(m(f)\) (resp., \(m(L)\)) equals 1.

2.5. Definition. A marked lattice \(f\) is well rounded if \(M(f)\) spans \(S^a\) as
\(S\)-module.

By the \(\Gamma_0\)-invariance and \(K\)-equivariance noted in Section 2.4, we get a
definition of well-rounded elements in all the spaces in (2.3.1). For instance, a point in
\(X\) is well rounded if and only if, for the class of marked lattices \(f\) mod \(K\) that
represents it, one (and hence each) of the \(M(f)\) spans \(S^a\) as \(S\)-module. The definitions
in \(X/\Gamma_0\) and \(Y\) are similar.

Definition. \(W\) is the set of well-rounded elements in \(X\). We call this the well
rounded retract in \(X\). For any arithmetic group \(\Gamma \leq \Gamma_0\), \(W/\Gamma\) is called the well
rounded retract in \(X/\Gamma\).

In Section 4, we will explain why these are deformation retracts.

2.6. Example of \(GL_n(\mathbb{Z})\). As in Section 1.6, take \(k = \mathbb{Q}\) and \(G = GL_n(\mathbb{R})\). If \(L_0\) is the standard lattice \(\mathbb{Z}^n \subset \mathbb{R}^n\), then \(\Gamma_0 = GL_n(\mathbb{Z})\), and \(Y\) is the set of all
rank-\(n\) lattices in \(\mathbb{R}^n\) mod homotheties. The product \((\cdot, \cdot)\) is the standard dot
product.

Let us work out the \(GL_2(\mathbb{Z})\) case in detail. For each marked lattice \(f : L_0 \rightarrow \mathbb{R}^2\), rotate the image until \(f((1, 0))\) lies on the positive \(x\)-axis, and use a
homothety to make \(||f((1, 0))|| = 1\). This identifies \(X\) with the upper half-plane \(\mathbb{H}\)
by sending \(x = HK(\begin{pmatrix} a \\ b \end{pmatrix})\) (with \(b > 0\)) to the point \(a + bi \in \mathbb{H}\). The well-rounded retract \(W\) is the tree shown in the figure below.
The lower branches break in two infinitely many times as they approach the horizontal axis \((b = 0)\), which is not pictured.

**Figure 1**

The group \(G\) is generated by \(SL_2(\mathbb{R})\), \(H\), and \(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\). Under our identification of \(X\) with \(\mathbb{H}\), \(SL_2(\mathbb{R})\) acts on \(\mathbb{H}\) by linear fractional transformations as usual, \(H\) acts trivially, and \(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\) acts by \(a + hi \rightarrow -a + hi\). The well-known fundamental domain for the action of \(SL_2(\mathbb{Z})\) on \(\mathbb{H}\) is \(F = \{(a + bi) \in \mathbb{H} : \|a\| \leq 1/2, a^2 + b^2 \geq 1\}\), shown with hatching. A fundamental domain for \(GL_2(\mathbb{Z})\) is the right-hand half of \(F\), namely, \(F^* = \{a + bi \in F : a \geq 0\}\), shown with double hatching. The space \(X/\Gamma_0\) is homeomorphic to \(F^*\). The retract \(W/\Gamma_0\) is identified with the bottom arc of \(F^*\).

3. The Borel-Serre compactification. In this section we recall the construct-
ion of the Borel-Serre manifold with corners \(\overline{X}\), and we summarize facts about
flags and parabolic subgroups. The main goal of the section is Proposition 3.6,
which relates the geodesic action on \(X\) to the idea of rescaling a lattice in
directions given by a fixed flag \(\mathcal{F}\).
3.1. Recall that \( q_T = \{ \text{diag}(a_1, \ldots, a_n) \} \) is a maximal \( Q \)-split torus in \( G \). The set of \( Q \)-roots of \( G \) with respect to \( q_T \) is \( \{ a_i a_j^{-1} \mid 1 \leq i \neq j \leq n \} \). Fix the fundamental system of simple roots \( \Delta = \{ a_i a_j^{-1} \mid i = 1, \ldots, n - 1 \} \).

3.2. We recall some facts about parabolic \( Q \)-subgroups and flags. Let \( \{ e_1, \ldots, e_n \} \) be the standard \( k \)-basis of \( k^n \). Let \( J \subseteq \Delta \). Form a graph with vertices in \( \{ 1, \ldots, n \} \) and with an edge joining vertices \( i \) and \( i + 1 \) if and only if \( a_i a_{i+1} \in J \). The graph has \( \#(\Delta - J) + 1 \) connected components. Say the \( i \)th component in order from left to right has \( v_i \) vertices. Consider the flag \( \mathcal{F}_i = \{ 0 \subseteq v_1 \subseteq \cdots \subseteq v_i = k^n \} \) in which \( v_i \) is spanned by \( \{ e_1, \ldots, e_{i-1}, v_i \} \). The standard parabolic \( Q \)-subgroup \( P_j \) is the subgroup of \( G \) such that \( P_j(G) \) is the stabilizer of \( \mathcal{F}_i \) in \( G(Q) \). In the coordinates of Section 1.2, \( P_j \) has the usual block upper-triangular form, with diagonal blocks of sizes \( v_1 \times v_1, \ldots, v_n \times v_n \). Every parabolic \( Q \)-subgroup \( P \) of \( G \) is conjugate by some \( g \in G(Q) \) to a unique \( P_j \).

Let \( P_j = P_j(R) \), and \( P = P(R) \) in general.

For \( g \in G(Q) \), the group of \( Q \)-points of \( g^{-1} P g \) is the stabilizer of the flag \( \mathcal{F} = g^{-1} \mathcal{F}_j \). Any flag of the latter form is called a \( Q \)-flag. There is a one-to-one correspondence between the parabolic \( Q \)-subgroups and the \( Q \)-flags they stabilize. We write \( \mathcal{F} \supseteq \mathcal{F}_j \) if every \( v_i \) belonging to \( \mathcal{F}_j \) is also a member of \( \mathcal{F} \).

For the \( Q \)-flag \( \mathcal{F} = \{ 0 \subseteq v_1 \subseteq \cdots \subseteq v_i = k^n \} \), we set \( \# \mathcal{F} = i \). Thus \( \# \mathcal{F} = n \) when \( P \) is a Borel subgroup, \( \# \mathcal{F} = 2 \) for maximal proper \( P \), and \( \# \mathcal{F} = 1 \) when \( P = G \).

3.3. Let \( q_T \) be the subgroup of \( q_T \) in which the \( i \)th and \((i + 1)\)st diagonal entries are equal whenever \( i = \frac{i}{i} \in J \). Let \( T_j \) be the identity component of \( q_T \), and let \( A_j = H \backslash T_j \).

Let \( P = g^{-1} P g \) for \( g \in G(Q) \). There is a semidirect product decomposition \( P = M_T T U_P \). Here \( U_P \) is the unipotent radical of \( P \), \( T_P = g^{-1} T g \), and \( M_T \) is the unique Levi subgroup of \( P \) which is stable under the Cartan involution of \( G \), which fixes \( g^{-1} K g \). We set \( A_P = H \backslash T_P \).

When \( P = P \), \( A_P \) is naturally identified with a subgroup of \( A_P \) by taking the kernel of appropriate roots in \( \Delta \).

3.4. We recall the definition of the geodesic action [BS, §3]. Let \( P = g^{-1} P g \) for \( g \in G(Q) \). Let \( Z \) be the identity component of the center of \( M_T T_P \). Let \( y \in X \). There is a \( p \in P \) with \( y = HK gp \). For any \( z \in Z \), define

\[
y o z = HK g p z.
\]

The right side is independent of the choices of \( g \) and \( p \). This action of \( Z \) is the geodesic action on \( X \) associated to \( P \). It is a proper and free action, and it commutes with the ordinary action of \( P \). The group \( A_P \) is contained in \( H \backslash Z \), so it operates on \( X \) by the geodesic action.

3.5. We recall the construction of \( \tilde{X} \) [BS, §§5–7]. The roots in \( \Delta - J \) determine an isomorphism between \( A_j \) and \( (0, \infty)^{\Delta - 1} \), sending the point

\[
\text{diag}(a_1, \ldots, a_j, \ldots, a_n) \quad (a_i > 0, \text{mod } H)
\]

to \( (a_1 a_2^{-1}, \ldots, a_{n-1} a_n^{-1}) \in (0, \infty)^{n-1} \). Let \( \tilde{A}_j \) be the partial compactification of \( A_j \) corresponding to \( (0, \infty)^{\Delta - 1} \) under this isomorphism. The corner \( X(P_j) \) associated to \( P_j \) is the fiber product \( X(P_j) = X \times \tilde{A}_j \), where \( A_j \) acts on \( X \) by the geodesic action. Conjugating by an element of \( G(Q) \), we define \( \tilde{A}_P \) and a corner \( X(P) \) on \( X \times \tilde{A}_P \) for any \( P \).

Whenever \( P \subseteq P' \), there is a map \( X(P') \to X(P) \) induced by the inclusion on \( X \); we identify \( X(P') \) with its image under this map. As a set, \( \tilde{X} \) is the union of the \( X(P) \) over all \( P \). One puts on \( \tilde{X} \) the topology such that the original topology on each corner is preserved, \( \tilde{X} \) is Hausdorff, and \( \Gamma \) acts properly on \( \tilde{X} \) with a compact quotient.

Let \( e(P) \) be the set corresponding to \( X \times \tilde{A}_P \{ 0 \}^{\Delta - 1} \) in the corner \( X(P) = X \times \tilde{A}_P \). As a set, \( \tilde{X} \) is the disjoint union of \( X \) and \( e(P) \) for all proper parabolic \( Q \)-subgroups \( P \). We call \( e(P) \) the boundary face corresponding to \( P \).

Let \( \partial(P) \) be the closure of \( e(P) \) in \( \tilde{X} \). We have \( \partial(P) = \bigcup_{P' \supseteq P} e(P') \). The sets \( \partial(P_1) \) and \( \partial(P_2) \) are disjoint unless \( P_1 \cap P_2 \) is a parabolic \( Q \)-subgroup of \( P \), in which case \( \partial(P_1) \cap \partial(P_2) = \partial(P_3) \).

Notice that the largest subgroup of \( \Gamma \) that stabilizes \( X(P) \), \( e(P) \), and \( \partial(P) \) is \( \Gamma \cap P \).

Modding out by the geodesic action of \( A_P \) gives a \( (\Gamma \cap P) \)-equivariant bundle map \( g_P : X(P) \to e(P) \). The orbits in \( X(P) \) under \( M_T U_P \) are sections of this bundle called the canonical cross sections.

3.6. We now present a central tool of the paper, an interpretation of the geodesic action in the setting of marked lattices.

Let \( F \) be a marked lattice. Let \( \mathcal{F} = \{ 0 = v_1 \subseteq \cdots \subseteq v_i = k^n \} \) be a \( Q \)-flag, with \( P = g^{-1} P g \) the parabolic \( Q \)-subgroup determined by \( \mathcal{F} \). Throughout the paper, we will use abuse notation by writing \( f(v_i \cap L_0) Q F \subseteq S^t = S^t(f(v_i)) \); this is an \( \mathfrak{h} \)-linear subspace of \( S^t \), of \( S^t \)-rank equal to \( \dim v_i \). Set \( v_i = \text{equivalent to the orthogonal complement of } (v_i, v_i) \text{ in } v_i \text{ with respect to } (,). \) We have \( S^t = v_i \oplus \cdots \oplus v_i \) as an orthogonal direct sum.

**Definition.** Let \( \mathfrak{H}^*_E \) be the group of \( R \)-linear maps \( S^t \to S^t \) that act by positive real homotheties on each summand \( v_i \) separately. Let \( \mathfrak{H}^*_E = \mathfrak{H}^*_E/H \). We call \( \mathfrak{H}^*_E \) the orthogonal scaling group with respect to \( \mathcal{F} \) and \( f \).

Note that every \( \alpha \in \mathfrak{H}^*_E \) is \( S \)-linear. The group \( \mathfrak{H}^*_E \) acts on \( Y' \) via \( f_i \mapsto \alpha \cdot f_i \). However, it does not act on \( X \). (The next proposition will show that \( \mathfrak{H}^*_E \) acts on \( Y' = H/K \) as a lift of the action of \( A_P \) on \( X = HK/G \), but this lift is only...
determined by an arbitrary choice of an element of \( K \). The choice is encoded in \( f \).

Let \( \Psi : Y' = H \backslash G \rightarrow X = H \backslash K \backslash G \) be the projection. Let \( y = \Psi(f) \). Let \( i : A_P \rightarrow \mathfrak{U}_{\mathcal{S},f} \) be the following map: if \( z \in A_P \) with \( (3.6.1) \)

\[
\begin{array}{l}
z = g^{-1} \text{diag}(a_1, \ldots, a_n, \ldots, a_1, \ldots, a_n) g \quad (a_j > 0),
\end{array}
\]

let \( i(z) \) be the element of \( \mathfrak{U}_{\mathcal{S},f} \), that (for each \( j = 1, \ldots, l \)) acts on \( \mathcal{V}_j \) by the scalar \( a_j \). It is clear that \( i \) is an isomorphism of groups.

**Proposition.** For any \( z \in A_P \), with \( i \) as above,

\[
y \circ z = \Psi(i(z) \cdot f).
\]

**Remark.** The proposition is stated in [G, (2.4)], without proof.

**Proof.** We use the notation of Section 3.4. Write \( y = HKgp \) and \( p = g^{-1}p_g \) for \( p_g \in P \). Consider the marked lattice \( f_1 : x \mapsto gpx = p_gx \). Since \( \Psi(f_1) = y = \Psi(f) \), we may choose a \( k_1 \in K \) such that \( f_1 = k_1 \cdot f \).

Write \( z \in A_P \) as \( z = g^{-1}zg \) for \( z_j = \text{diag}(a_1, \ldots, a_1, \ldots, a_1, \ldots, a_1) \in A_J \). Then

\[
y \circ z = HKgpz = HKzgpz.
\]

The marked lattice \( f_1 : x \mapsto z_Jp_Jx \) descends mod \( K \) to \( y \circ z \).

Let \( J_J \) be the flag in \( S^n \) whose \( j \)th member is the coordinate subspace spanned by \( \{e_1, \ldots, e_{n-j+1}\} \). (This \( J_J \) is essentially \( S^n \otimes_{\mathbb{R}} \mathbb{R} \).) The map \( x \mapsto gx \) carries \( \mathcal{F} \) to \( J_J \), and the further multiplication by \( p_J \) and \( z_J \) preserves \( J_J \). Thus for any \( z \in A_P \), \( f_z(V_J) \) is exactly the \( j \)th member of \( J_J \). (The idea is that, out of the whole \( K \)-equivalence class of marked lattices representing \( y \circ z \), \( f_1 \) is a representative for which the \( f_z(V_J) \) will be in standard position.) The orthogonal complement of \( \mathcal{E}_J(V_{J-1}) \) in \( f_z(V_J) \) is the coordinate subspace \( E_J \) spanned by \( \{e_{n-j+1}, \ldots, e_{n-j+1}\} \). The \( E_J \) are orthogonal, and \( z_J \) acts by the map \( a_1 : S^n \rightarrow S^n \) characterized by the property that it acts on each \( E_J \) by the scalar \( a_1 \). In short, the left-hand side of (3.6.2) is the projection under \( \Psi \) of the marked lattice given by the composition \( L_0 \xrightarrow{f_1} S^n \xrightarrow{a_1} S^n \).

On the right-hand side of (3.6.2), observe that \( \Psi(a \cdot f) = \Psi(k_1 a \cdot f) = \Psi(k_1 a \cdot f_1) \). Since \( f_1 \) is a \( f_1 \)-orthogonal complement of \( k_1 \cdot f(V_{J-1}) \) in \( k_1 \cdot f(V) \), the \( f_1 \) equals \( f_1(V_{J-1}) \) in \( f_1(V) \); this says exactly that \( k_1 \cdot \mathcal{V}_J = E_J \) for all \( f_1 \). It is immediate that \( k_1 a_1 a_1 \) is the same map as \( a_1 \). Thus the right-hand side of (3.6.2) is also \( \Psi \) applied to the composition \( L_0 \xrightarrow{f_1} S^n \xrightarrow{a_1} S^n \).

**Note.** We will always normalize elements of \( A_P \) and \( \mathfrak{U}_{\mathcal{S},f} \) in their classes modulo \( H \) so that \( a_1 = 1 \).

3.7. While \( A_P \) and \( \mathfrak{U}_{\mathcal{S},f} \) act on different spaces, we use similar notation for the two actions. We write

\[
p = (p_1, \ldots, p_{l-1}) \in (0, \infty)^{l-1}
\]

for a point in either group. The \( p_j \) are the coordinates coming from the simple roots in \( J \); that is, \( p_j = a_j / a_{j+1} \) for \( j = 1, \ldots, l - 1 \). By Proposition 3.6, the point of \( \mathfrak{U}_{\mathcal{S},f} \) corresponding to \( (p_1, \ldots, p_{l-1}) \) acts on \( \mathcal{V}_J \) by the scalar

\[
a_j = (p_1 \cdots p_{j-1})^{-1} \quad (j = 2, \ldots, l).
\]

(Because of our normalizations, the action on \( \mathcal{V}_J \) is by \( 1 \).)

**Remark.** The \( p_j \)'s are good coordinates for rescaling sublattices within lattices, while the \( a_j \) are the natural coordinates for the geodesic action. Though (3.7.1) is messy, we cannot avoid using it, since part of our goal is to relate lattices and the geodesic action. The inverses in (3.7.1) arise because the well-rounded retraction goes away from the Borel-Serre boundary.

Multiplication in \( A_P \) corresponds to coordinate-wise multiplication of the \( p_j \). As in Section 3.5, the \( p_j \) extend to coordinates on \( \hat{A}_P \) with \( (p_1, \ldots, p_{l-1}) \in [0, \infty)^{l-1} \). Going to infinity in \( X(P) \) means going to zero in the \( p_j \)'s variables. We introduce a partial ordering on \( A_P \) and \( \hat{A}_P \), where \( (p_1, \ldots, p_{l-1}) \leq (p_1', \ldots, p_{l-1}') \) if and only if \( p_j \leq p_j' \) for each \( j \).

The action of \( p = (p_1, \ldots, p_{l-1}) \) on \( Y' \) is denoted \( f \mapsto p \cdot f \). This descends mod \( \mathcal{G}_0 \) to an action \( L \mapsto p \cdot L \) on \( Y \). Descending mod \( K \) as in Proposition 3.6, \( p \cdot x \) denotes the geodesic action on points \( x \) of \( X \) or \( X(\Gamma \cap P) \).

3.8. We will need to consider \( \mathbb{Q} \)-flags \( \mathfrak{M} = \{0 \subseteq M^{(1)} \subseteq M^{(2)} \subseteq \cdots \subseteq M^{(m)} = k^n\} \) in which some of the inclusions can be equalities. If \( \mathfrak{F} \) is obtained from \( \mathfrak{M} \) by replacing each string of equalities with a single space \( \mathcal{V}_J \), we call \( \mathfrak{F} \) the irredundant version of \( \mathfrak{M} \). If \( \mathfrak{M} \) is the orthogonal scaling group with respect to this flag and some \( f \), we always write \( p = (p_1, \ldots, p_{l-1}) \in \mathfrak{M} \) with reference to \( \mathfrak{F} \), not \( \mathfrak{M} \).

4. The well-rounded retraction. In this section, we define the well-rounded retraction \( r_t \). The material comes from [A3], though we emphasize different things (marked lattices, the flag of successive minima).

4.1. Let \( f \) be a marked lattice, with \( L = f(L_0) \). We normalize \( f \) up to homothety so that \( m(f) = 1 \). If \( V \subseteq k^n \) is a \( k \)-subspace, \( f(V) \otimes_{\mathbb{Q}} \mathbb{R} \) is abbreviated \( f(V) \).

Recall that \( Y' = H \backslash G \) is the space of marked lattices (Section 2.3). For \( i = 1, \ldots, n \), let

\[
Y'_i = \{ f \in Y' | \text{rank}_S(S \cdot M(f)) \geq i \}.
\]
Definition. Let $W_p$ be the set of all marked lattices $f$ in $W$ such that for all $j = 1, \ldots, l$, 
$$S \cdot (M(L) \cap f(V_j)) = f(V_j).$$
 ($W_p$ was called $W'$ in the introduction.) Geometrically, $W_p$ consists of the lattices $L = f(L_0)$ whose intersection with each subspace $f(V_j)$ is well rounded in that subspace. By the definition of the cell structure on $W$ [A3, pp. 466–467], $W_p$ is a closed subcomplex of $W$ and is stable under $\Gamma \cap P$. A cell $V$ in $W_p$ is said to respect $\mathcal{F}$. If $\mathcal{F} = \gamma \mathcal{F}'$ for $\gamma \in \Gamma$, then the images of $W_p$ and $W_p$ in $W/\Gamma$ are equal. The $\mathcal{F}$-flags fall in only finitely many equivalence classes mod $\Gamma$, so only finitely many distinct subcomplexes of $W/\Gamma$ arise in this way.

If $\mathcal{M}$ is a $\mathbb{Q}$-flag with possible equalities (Section 3.8), and $\mathcal{F}$ is the irredundant version of $\mathcal{M}$, let $W_{\mathcal{M}} = W_p$.

6.2. The following is immediate from the definitions in Section 4.1.

Lemma. If $f$ has flag of successive minima $\mathcal{M}$, then $r(f) \in W_{\mathcal{M}}$.

6.3. Example of $\text{GL}_2(\mathbb{Q})$. Refer again to Figure 1. For the standard flag $\mathcal{F} = \{0 \subseteq \mathbb{Q} \cdot (1,0) \subseteq \mathbb{Q} \cdot 1\}$ with associated $P$, the space $W_p$ is the horizontal sequence of arcs forming the top of $W$. The boundary face $e(P)$ in $X$ (not shown) is the horizontal line at $b = \infty$. The geodesic action by $A_P$ flows straight up, pushing $X$ upward to converge to $e(P)$. The well-rounded retraction pulls the region above $W$ straight down onto $W_p$. For torsion-free $\Gamma$, $e(P)/(\Gamma \cap P)$ and $W_p/(\Gamma \cap P)$ are homeomorphic to circles, and the well-rounded retraction induces the obvious isomorphism between the cohomology of these spaces. (The situation is more complicated for $n > 2$, when the boundary faces are no longer disjoint.)

Remark. For small $n$, the $W_p/(\Gamma \cap P)$ are readily computable; see Section 10.6.

7. Neighborhoods of infinity in $X$. We now prove our main results, explaining the connection between the well-rounded retraction $r$, the geodesic action, and the topology of the Borel-Serre boundary. Sections 7.1–7.4 give a chain of propositions that allow us to define a tubular neighborhood $\tilde{N}_p$ of $e(P)$ (called $\tilde{N}$ and $e$ in the introduction). In Sections 7.5 and 7.6 we prove that $r: X \to W$ has the continuous extension $\tilde{r}: X \to W$, where (for all $\mathcal{F}$) $\tilde{r}$ is constant on the fibers of $\tilde{N}_p$ and carries $\tilde{N}_p$ onto $W_p$.

7.1. Fix a marked lattice $f$, with $L = f(L_0)$. As in Section 4.1, let $\mathcal{M} = \{0 \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(n)} = k^*\}$ be the flag of successive minima for $f$. Let $\mathcal{F}$ be the irredundant version of $\mathcal{M}$, associated with $P$. Let $\mathcal{U} = \mathcal{U}_{P,f}$. Let $f'' = r(f)$ be the image of $f$ under the well-rounded retraction.

The idea of the following lemma is that if $f$ is close enough to $e(P)$, then the well-rounded retraction carries the whole orthant $\{p \in \mathcal{U} \mid p \leq 1\} \cdot f$ to a single point of $W$. The lemma says that, to know that $f$ is “close enough” to $e(P)$, it suffices to know that $f''$'s flag of successive minima is the flag $\mathcal{F}$ corresponding to $P$ (or at least contains this $\mathcal{F}$; see Section 7.2).

Lemma. Let $f, \mathcal{M}, \mathcal{F}, P,$ and $\mathcal{U}$ be as above. For any $p = (p_1, \ldots, p_{l-1}) \in \mathcal{U}$ with $p \leq 1, \ldots, 1$, $f'' = p \cdot f$ satisfies $r(f'') = r(f') = f''$. The common image $f''$ will lie in $W_p = W_{\mathcal{M}}$.

Proof. It suffices to prove the lemma when $p = (1, \ldots, 1, \rho_1, 1, \ldots, 1)$ for a fixed $\rho_1 < 1$; the general case follows by applying this for each $j$. Let $i$ be the unique index such that $f'_j = M^{(i)}$ and $M^{(i)} \subseteq M^{(i+1)}$.

The marked lattices $f'$ and $f''$ agree on $L_0 \cap M^{(0)}$, since $p$ acts by 1 in that region. For $x \in L_0$ with $x \notin M^{(i)}$, no vector $f'(x)$ is shorter than the corresponding vector $f(x)$, since $p$ magnifies lengths in the directions perpendicular to $f(M^{(i)})$ by $\rho_1^{-1} \geq 1$. And $f(x)$ cannot be a minimal vector of $f$, since the minimal vectors lie in $f(M^{(i)})$, which is contained in $f(M^{(0)})$. It follows that the well-rounded retractions for $f$ and $f''$ will proceed exactly the same up through the end of the $i$th step; that is, if $\mu_j$ and $\mu'_j$ (for $f$ and $f''$, resp.) are as in Section 4.1, then

$$\mu_j = \mu'_j \quad \text{for all } i < j.$$ 

Set $g = r^{(i)} \circ \cdots \circ r^{(1)}(f)$ and $g' = r^{(i)} \circ \cdots \circ r^{(1)}(f'')$. It follows from (3.7.1) and (7.1.1) that $g' = (1, 1, \ldots, 1, \rho_1^{-1}, 1, \ldots, 1) \cdot g$. The $i$th stage of the retraction will, by definition, carry $g$ to

$$1, \ldots, 1, \rho_1^{-1}, 1, \ldots, 1 \cdot g$$

and carry $g'$ to

$$1, \ldots, 1, \rho_1^{-1}, 1, \ldots, 1 \cdot g'.$$

However, by the uniqueness statement in (4.1.1), $\mu_i^{-1}$ must equal $\mu'_i - \rho_i$. Thus the marked lattices (7.1.2) and (7.1.3) will coincide. The $r^{(i+1)}$ and later stages of the retraction will agree on this common lattice, so the final values $r(f)$ and $r(f'')$ are equal. By Section 6.2, $f'' \in W_p$. □

7.2. We record a corollary of the previous lemma. The idea is that if $f''$'s flag of successive minima refines the flag $\mathcal{F}'$ for some $P'$, then $f$ is “close enough” to $e(P')$. After all, if $f$ is close enough to $e(P)$, it should be close enough to any larger Borel-Serre boundary component $e(P')$ that has $e(P)$ in its closure.

Corollary. Let the notation be as in Section 7.1. Let $\mathcal{F}'$ be any $\mathbb{Q}$-flag with $\mathcal{F}' \subseteq \mathcal{F}$, and let $\mathcal{W}'$ be the orthogonal scaling group for $\mathcal{F}'$ and $f$. Then for any $P \in \mathcal{W}'$ with $P \leq (1, \ldots, 1), f : P \cdot f$ and $f$ map to the same point under the well-rounded retraction. This point lies in $W_p$.
Thus $q_P \circ R_1$ affords a $(\Gamma \cap P)$-equivariant homotopy between $\delta \circ P_{\delta(P)}$ and the identity. □

Remark. Formula (8.5.3) defines a well-rounded retraction within $\delta(P)$ itself, carrying $\delta(P)$ onto a subset of $\delta(P)$ homeomorphic to $W_{\mathcal{F}}$. We know that $\delta(P)/(\Gamma \cap P)$ is a compactification of a bundle whose base is a locally symmetric space (perhaps with finite quotient singularities) for a group of lower rank, and whose fiber is an arithmetic quotient of $U_P$. Accordingly, there is a cellular quotient map from $W_{\mathcal{F}}/(\Gamma \cap P)$ to the well-rounded retract in the lower-rank locally symmetric space. This map may be constructed by dualizing the techniques of [M2]; see Section 10.7.

9. The main results. In Section 9.5 we establish our main result, as we construct the spectral sequence described in the introduction. The bulk of the proofs resides in Sections 9.2–9.4. We set up two spectral sequences, one (9.2.1) computing the cohomology of the Borel-Serre boundary $\partial X/\Gamma$, and the other (9.3.1) based on the subcomplexes $W_{\mathcal{F}}/(\Gamma \cap P)$ in the well-rounded retract $W/\Gamma$. Because the map $\bar{\varepsilon}$ gives homotopy equivalences on $\overline{X}/\Gamma$ and all its faces, it induces an isomorphism between these spectral sequences, and between their abutments. We set up canonical maps $H^*(\overline{X}/\Gamma) \to (9.2.1)$ and $H^*(W/\Gamma) \to (9.3.1)$. Finally, we show that the map $H^*(W/\Gamma) \to (9.3.1)$ computes the canonical restriction map $H^*(\overline{X}/\Gamma) \to H^*(\partial X/\Gamma)$.

9.1. Let $A^{p,q}$ be any first-quadrant double complex. There are two standard filtrations of the double complex, $\{A^{p,q} \mid q \geq q_0\}$ and $\{A^{p,q} \mid p \geq p_0\}$. The spectral sequences these give are called, respectively, the type-I and type-II sequences for $A^{p,q}$. Our main results involve type-II sequences. The cohomology of the single complex associated to $A^{p,q}$ is called the abutment of (either) spectral sequence.

Throughout the paper, (co)homology groups have coefficients in any fixed abelian group.

9.2. We set up a Mayer-Vietoris spectral sequence for the cohomology of $\partial X/\Gamma$. Let $\Phi$ be a set of representatives of the $\Gamma$-equivalence classes of the $Q$-flags $\mathcal{F} = \{V_1 \subset V_2 \subset \cdots \subset V_n = k^n\}$ (see Section 3.2). There are only finitely many such equivalence classes. Let $\mathcal{D} = \{(\delta(P)/(\Gamma \cap P))_{\mathcal{F} \in \Phi}\}$ (corresponding to maximal proper parabolic $Q$-subgroups). This $\mathcal{D}$ is a cover of $\partial X/\Gamma$. By Section 3.5, the nonempty $(p+1)$-fold intersections of distinct members of $\mathcal{D}$ are exactly the sets in $\{(\delta(P)/(\Gamma \cap P))_{\mathcal{F} \in \Phi_{p+1}}\}$. This means we are in the correct setting to use the Čech cohomology techniques of [BT, §15].

For $0 \leq p \leq n - 2$, define

$$A^{p,q} = \bigoplus_{\mathcal{F} \in \Phi_{p+1}} C^q(\delta(P)/(\Gamma \cap P)),$$

where $C^q$ denotes the singular $q$-cochains. We make $A^{p,q}$ into a double complex, where the vertical maps are $(-1)^p$ times the coboundary maps, and the horizontal maps are induced as Čech cohomology from the inclusions $\hat{\varepsilon}(P) \to \hat{\varepsilon}(P')$ for $\mathcal{F}' \supseteq \mathcal{F}$, with the alternating sign convention of [BT, (15.7.1)]. Exactly as in [BT, (15.7)], we see that the $q$th row of $A^{p,q}$ is exact, except at the $p = 0$ position where the kernel equals $C^q(\partial X/\Gamma)$. Thus the type-I spectral sequence $E^{p,q}_{\mathcal{D}}$ collapses at $E_2$ to $H^*(\partial X/\Gamma)$. Therefore, the total complex of $A^{p,q}$ computes $H^*(\partial X/\Gamma)$.

Let $E^{p,q}_{\mathcal{D}}$ be the type-II spectral sequence for $A^{p,q}$. We have

$$(9.2.1) \quad E^{p,q}_{\mathcal{D}} = \bigoplus_{\mathcal{F} \in \Phi_{p+1}} H^q(\delta(P)/(\Gamma \cap P) \to H^p(\partial X/\Gamma)).$$

9.3. For $0 \leq p \leq n - 2$, define

$$W^{p,q} = \bigoplus_{\mathcal{F} \in \Phi_{p+1}} C^q(W_{\mathcal{F}}/(\Gamma \cap P)),$$

made into a double complex in the same way as $A^{p,q}$. (Recall that $W_{\mathcal{F}} \to W_{\mathcal{F}'}$ whenever $\mathcal{F}' \supseteq \mathcal{F}$.) Let $E^{p,q}_{\mathcal{D}}$ be the type-II spectral sequence for $W^{p,q}$. As above,

$$(9.3.1) \quad E^{p,q}_{\mathcal{D}} = \bigoplus_{\mathcal{F} \in \Phi_{p+1}} H^q(W_{\mathcal{F}}/(\Gamma \cap P)).$$

Let $W^*$ denote the single complex associated to $W^{p,q}$. Thus the abutment of (9.3.1) is $H^{p+q}(W^*)$.

9.4. We now define a map of double complexes. Consider the single-column double complex

$$A^{p,q} = \begin{cases} C^q(\overline{X}/\Gamma) & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

There is an obvious map $\hat{\varepsilon}^{p,q} : A^{p,q} \to A^{p,q}$, where for $p = 0$ we map $\omega \in C^q(\overline{X}/\Gamma)$ to the direct sum over $\mathcal{F} \in \Phi_0$ of the restriction of $\omega$ to $C^q(\delta(P)/(\Gamma \cap P))$.

Lemma. The map $\hat{\varepsilon}^{p,q} : A^{p,q} \to A^{p,q}$ is a chain map of double complexes.

Proof. Let $\omega \in C^q(\overline{X}/\Gamma)$, and let $\tau$ be its image in $A^{p,q}$. The only thing to check is that the horizontal arrow $\delta : A^{p,q} \to A^{p-1,q}$ carries $\tau$ to $0$. But as we have said, the kernel of $\delta$ is precisely the set of elements that come from a global cochain on $\partial X/\Gamma$. And $\tau$ does come from a global cochain, namely, the restriction of $\omega$ to $\partial X/\Gamma$. □

Considering the type-I spectral sequences, we see that $\hat{\varepsilon}^{p,q} : A^{p,q} \to A^{p,q}$ induces on the abutments the canonical restriction map $H^q(\overline{X}/\Gamma) \to H^q(\partial X/\Gamma)$.

Next, consider

$$W^{p,q} = \begin{cases} C^q(W/\Gamma) & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$