GALOIS REPRESENTATIONS ATTACHED TO TENSOR PRODUCTS OF ARITHMETIC COHOMOLOGY

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Abstract. We compute the action of Hecke operators on tensor products of cohomology classes of lower congruence subgroups of $\text{SL}(n, \mathbb{Z})$ in trivial weight. We use this computation to prove that if each representation in a collection of Galois representations is attached to a cohomology class of a lower congruence subgroup in trivial weight, then a sum of certain twists of the representations consistent with the main conjectures of [5, 8] is also attached to such a cohomology class.

1. Introduction

Define a Galois representation to be a continuous homomorphism $\rho : G_\mathbb{Q} \to \text{GL}(n, F)$, where $G_\mathbb{Q}$ is the absolute Galois group of $\mathbb{Q}$, and $F$ is a topological field (which may be discrete).

There are conjectures connecting certain Galois representations (with $F$ a finite field) with eigenclasses of Hecke operators acting on arithmetic cohomology [8, 5, 11]. Computational evidence for these conjectures has been given in the case $n = 3$ [5] and $n = 4$ [6], and special cases of these conjectures have been proven for reducible Galois representations [2, 3, 4].

Let $\Gamma$ be a congruence subgroup of $\text{SL}(n, \mathbb{Z})$ of level $N$. Define the Hecke algebra $K\mathcal{H}_\Gamma$ to be the commutative $K$-algebra under convolution generated by all the double cosets $T(\ell, k) = \Gamma D(\ell, k) \Gamma$ with $D(\ell, k) = \text{diag}(1, \ldots, 1, \ell, \ldots, \ell)$, such that $\ell \nmid N$.

An algebra homomorphism $\phi : K\mathcal{H}_\Gamma \to K$ will be called a $K$-Hecke packet. For example, if $W$ is an $K\mathcal{H}_\Gamma$-module, and $w \in W$ is a simultaneous eigenvector for all $T \in K\mathcal{H}_\Gamma$, then the associated eigenvalues give a $K$-Hecke packet, called a $K$-Hecke eigenpacket that “occurs” or “appears” in $W$.

Definition 1.1. Let $\phi$ be a $K$-Hecke packet, with $\phi(T(\ell, k)) = a(\ell, k)$. We define the Hecke polynomial for $\phi$ at $\ell$ to be

$$F_{\phi, \ell}(X) = \sum_{k=0}^{n} (-1)^{k} \ell^{k(k-1)/2} a(\ell, k) X^{k}$$
for any prime $\ell \nmid N$.

**Definition 1.2.** Let $\phi$ be a $K$-Hecke packet, with $\phi(T(\ell, k)) = a(\ell, k)$. We say that the Galois representation $\rho : G_Q \to \text{GL}(n, K)$ is attached to $\phi$ if for some $M \geq 1$, $\rho$ is unramified outside $MN$ and

$$\det(I - \rho(\text{Frob}_\ell)X) = F_{\phi, \ell}(X)$$

for all prime $\ell \nmid MN$.

If the Hecke packet comes from a Hecke eigenvector $w \in W$, where $W$ is a $K$-vector space on which $K\mathcal{H}_\Gamma$ acts, we will say $\rho$ is attached to $w$ and fits $W$.

(We use the arithmetic Frobenius, so that if $\omega$ is the cyclotomic character, $\omega(\text{Frob}_\ell) = \ell$.)

A theorem of Scholze [12] asserts that for any finite field $F$, any $F$-Hecke packet occurring in the homology or cohomology of a congruence subgroup of $\text{SL}(n, \mathbb{Z})$ with $F$-coefficients has a Galois representation attached. In this paper, we prove a theorem allowing us to combine Galois representations that are attached to eigenclasses of Hecke operators acting on the cohomology of lower congruence subgroups (see Definition 3.1) into a larger Galois representation that is attached to an eigenclass in the cohomology of a related lower congruence subgroup. This gives information about the relations between the Galois representations and the eigenclasses provided by Scholze’s result, and the details are consistent with the predictions of [5].

Our main theorem (Theorem 7.3) asserts that if we have a collection $\rho_i : G_Q \to \text{GL}(n_i, F)$ of Galois representations ($1 \leq i \leq t$), with each $\rho_i$ attached to a cohomology eigenclass $f_i$ on a lower congruence subgroup $\Delta_i \subset \text{SL}(n_i, \mathbb{Z})$ with trivial coefficients, then we may construct a lower congruence subgroup $\Delta_0$, and a Hecke eigenclass $h$ on $\Delta_0$ with trivial coefficients such that

$$\rho_1 \oplus \omega^{n_1} \rho_2 \oplus \cdots \oplus \omega^{n_1 + \cdots + n_t - 1} \rho_t$$

is attached to $h$.

Our proofs will build on the construction of tensor product cohomology classes in [1], and use Borel-Serre duality [9] (as improved by [10, p. 280]), and the sharbly and cosharbly complexes [1, 7].

It would be interesting (but probably quite difficult) to try to generalize the theorem to the case where the $f_i$ lie in cohomology groups with nontrivial coefficients.

We note that the main theorem does not directly prove cases of the main conjecture of [5], since the congruence subgroup used in [5] will typically not be part of a lower compatible system (as defined in Definition 3.3).

## 2. Sharbly and Cosharbly

Let $n \geq 1$ and let $\mathbb{Q}^n$ denote the vector space of $n$-dimensional column vectors with entries in $\mathbb{Q}$.

**Definition 2.1.** [1, 7] The Sharbly complex $Sh_*$ is the complex of $\mathbb{Z}\text{GL}(n, \mathbb{Q})$-modules defined as follows. As an abelian group, $Sh_k$ is generated by symbols $[v_1, \ldots, v_{n+k}]$, (which we will call basic $k$-sharblies) where the $v_i$ are nonzero vectors in $\mathbb{Q}^n$, modulo the submodule generated by the following relations:

1. $[v_{\sigma(1)}, \ldots, v_{\sigma(n+k)}] - (-1)^{\sigma} [v_{\sigma(1)}, \ldots, v_{\sigma(n+k)}]$ for all permutations $\sigma$;
2. $[v_1, \ldots, v_{n+k}]$ if $v_1, \ldots, v_{n+k}$ do not span all of $\mathbb{Q}^n$; and
The boundary map $\partial : \hat{\text{Sh}}_k \to \hat{\text{Sh}}_{k-1}$ is given by

$$\partial([v_1, \ldots, v_{n+k}]) = \sum_{i=1}^{n+k} (-1)^i [v_1, \ldots, \hat{v}_i, \ldots, v_{n+k}],$$

where as usual $\hat{v}_i$ means to delete $v_i$.

The action of $g \in \text{GL}(n, \mathbb{Q})$ on a basic sharibly is given by $g[v_1, \ldots, v_{n+k}] = [gv_1, \ldots, gv_{n+k}]$ and extended to all sharblies by linearity.

Of course, all these objects depend on $n$, which we suppress from the notation, allowing the context to determine $n$.

The sharibly complex gives a resolution

$$\cdots \to \text{Sh}_i \to \text{Sh}_{i-1} \to \cdots \to \text{Sh}_1 \to \text{Sh}_0 \to \text{St}(n) \to 0$$

by $\text{GL}(n, \mathbb{Q})$-modules of the Steinberg module $\text{St}$ for $\text{GL}(n)/\mathbb{Q}$ [7]. Therefore, if $K$ is a field and if $\Gamma$ is a congruence subgroup of $\text{GL}(n, Z)$, the $i$th homology of the complex $\hat{\text{Sh}}_i \otimes K$ is isomorphic to $H_i(\Gamma, \text{St}(n) \otimes K)$. The duality theorem of Borel-Serre [9], extended in [10], tells us that $H_i(\Gamma, \text{St}(n) \otimes K)$ is isomorphic to $H^{n(n-1)/2-i}(\Gamma, K)$ for any field $K$ of characteristic greater than $n + 1$ or of characteristic 0.

We now fix a field $K$ and write $\text{St}$ instead of $\text{St}(n) \otimes K$ and $\text{Sh}_i$ instead of $\text{Sh}_i \otimes K$. Let a superscript $\vee$ denote dual of $K$-vector spaces. Then

$$0 \to \text{St}^\vee \to \hat{\text{Sh}}_0^\vee \to \hat{\text{Sh}}_1^\vee \to \cdots$$

is a co-resolution of right modules. The elements of $\hat{\text{Sh}}_n^\vee$ are called $k$-cosharblies; they are $K$-linear $K$-valued antisymmetric functions on $k$-sharblies. The coboundary map in this coresolution is given by $\delta f = f \circ \partial$ (see [1, Section 1]).

The homology of $(\hat{\text{Sh}}_n)_\Gamma$ computes $H_*(\Gamma, \text{St})$. Similarly, the homology of $(\hat{\text{Sh}}_n^\vee)_\Gamma$ computes $H^*(\Gamma, \text{St}^\vee)$. By Kronecker duality, there is a perfect pairing $\langle \cdot, \cdot \rangle$ between $H_*(\Gamma, \text{St})$ and $H^*(\Gamma, \text{St}^\vee)$. The first vector space is finite dimensional by Borel-Serre duality, and hence so is the second. Given a sharibly $\Gamma$-cycle $A$ (resp. a cosharibly $\Gamma$-cocycle $f$), we will denote by $\overline{A}$ (resp. $\overline{f}$) its image in the homology of $\Gamma$ (resp. in the cohomology of $\Gamma$).

Let $1 \leq m \leq n - 1$. Denote by $\{e_1, \ldots, e_n\}$ the standard basis vectors of $\mathbb{Q}^n$, considered as column vectors. Let $W_m$ be the span of the set $\{e_1, \ldots, e_m\}$, and let $Y_m$ be the span of the set $\{e_{m+1}, \ldots, e_n\}$ in $\mathbb{Q}^n$. Using this notation, we make the following definition.

**Definition 2.2.** A basic $k$-sharibly is said to be $(i, m)$-reducible if after permutation of its vectors, it is of the form $[w_1, \ldots, w_{m+i}, y_1, \ldots, y_{n+k-m-i}]$ where $w_1, \ldots, w_{m+i}$ are vectors in $W_m$.

Given an $(i, m)$-reducible basic sharibly, $C = [w_1, \ldots, w_{m+i}, y_1, \ldots, y_{n+k-m-i}]$ we may extract the sequence $A = (w_1, \ldots, w_{m+i})$ which we will call the $W_m$-component of $C$. We will call the sequence $B = (y_1, \ldots, y_{n+k-m-i})$ the remaining component of $C$.

A $k$-sharibly will be said to be $(i, m)$-reducible if it is a sum of $(i, m)$-reducible basic $k$-sharblies.

We note that we may consider the $W_m$-component $A$ of an $(i, m)$-reducible basic sharibly as an $i$-sharibly for $\text{GL}(m)$. Also, denoting projection from $\mathbb{Q}^n \to Y_m$ by a
prime, we may view $B' = (y'_1, \ldots, y'_{n+k-m-1})$ as a $j$-sharply for $GL(n-m)$, where $j = k - i$.

3. LOWER CONGRUENCE SUBGROUPS

**Definition 3.1.** A primary lower congruence subgroup $\Delta$ of $SL(n, \mathbb{Z})$ is a subgroup of $SL(n, \mathbb{Z})$ defined by

$$\Delta = \{ g \in SL(n, \mathbb{Z}) \mid g \mod N \in P(\mathbb{Z}/N) \}$$

where $P$ is an algebraic subgroup of $GL(n)$ that is either all of $GL(n)$ or parabolic and contains the upper triangular matrices, and $N$ is a positive integer.

The **level of $\Delta$** is the integer $N$.

A lower congruence subgroup $\Delta$ of $SL(n, \mathbb{Z})$ is a finite intersection of primary lower congruence subgroups of levels $N_1, \ldots, N_k$ (with possibly varying $P$'s). The level of $\Delta$ is the least common multiple of $N_1, \ldots, N_k$.

For example, the group $\Gamma_0(n, N) \subset SL(n, \mathbb{Z})$ consisting of the matrices which modulo $N$ stabilize $(\mathbb{Z}/N)e_1$ is a lower congruence subgroup.

We now give a definition from [1], modified slightly to account for the fact that the subgroups in which we are interested lie in $SL(n)$, rather than $GL(n)$.

**Definition 3.2.** Let $n = n_0 = n_1 + \cdots + n_t$, $\Gamma_i$ a subgroup of $SL(n_i, \mathbb{Z})$ for $i = 0, \ldots, t$. Let $e_1, \ldots, e_n$ be the standard basis vectors of $U = \mathbb{Q}^n$, and let $V_i$ be the span of $\{ e_j \mid n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_i \}$. Set $F_m = V_1 + V_2 + \cdots + V_m$ and let $F$ denote the flag $(0) \subset F_1 \subset F_2 \subset \cdots \subset F_t = U$. We identify $GL(n_i, \mathbb{Z})$ with the subgroup of $GL(n, \mathbb{Z})$ which act trivially on $V_j$ for $j \neq i$ and which stabilize $V_i$.

We say that the set $\{ \Gamma_0, \Gamma_1, \ldots, \Gamma_t \}$ is compatible if

(i) $\Gamma_0 \supset \Gamma_1 \times \cdots \times \Gamma_t$;

(ii) The stabilizer of $F$ in $\Gamma_0$ projected onto $GL(n_1, \mathbb{Z}) \times \cdots \times GL(n_t, \mathbb{Z})$ and then intersected with $SL(n_1, \mathbb{Z}) \times \cdots \times SL(n_t, \mathbb{Z})$ lies in $\Gamma_1 \times \cdots \times \Gamma_t$;

(iii) If $v \in V_1 \cup V_2 \cup \cdots \cup V_t$ and $\gamma \in \Gamma_0$ and $\gamma v \in F_m$ for some $m$, then already $v \in F_m$.

We now define a specific example of a compatible set of subgroups, which we will use throughout the rest of the paper.

**Definition 3.3.** For $i = 1, \ldots, t$ let $\Delta_i$ be a lower congruence subgroup of $SL(n_i, \mathbb{Z})$ of level $N_i$. Let $N$ be divisible by the least common multiple of the $N_i$ and $n = \sum n_i$.

Set $\Delta_0 = \Delta_0(N)$ equal to the set of matrices in $SL(n, \mathbb{Z})$ which when written in $n_1, \ldots, n_t$ block form satisfy

(1) Any block below the diagonal blocks is congruent to 0 modulo $N$;

(2) The $i$-th diagonal block modulo $N$ is contained in the image of $\Delta_i$ modulo $N$.

We call the system $\Delta_0, \Delta_1, \ldots, \Delta_t$ a lower compatible system.

Of course, $\Delta_0(N)$ depends on $\Delta_1, \ldots, \Delta_t$, even though we omit them from the notation.

**Proposition 3.4.** Given lower congruence subgroups $\Delta_i$ of level $N_i$, as above, and an integer $N$ divisible by all the $N_i$, the subgroup $\Delta_0(N)$ is a lower congruence subgroup.
Proof. For each parabolic subgroup \( P \) used in the definition of \( \Delta_i \), let \( N_P \) be the corresponding level, and let \( \hat{P} \) be the parabolic subgroup that is \((n_1, \ldots, n_k)\)-block upper triangular, with \( \text{GL}(n_j) \) in the \( j \)-th diagonal block for \( j \neq i \), and \( P \) in the \( i \)-th block. Let \( \hat{P} \) be the primary lower congruence subgroup defined by \( \hat{P} \) modulo \( N_P \). Let \( Q \) be the parabolic subgroup of block upper triangular matrices, and let \( \hat{Q} \) be the primary lower congruence subgroup corresponding to \( Q \) modulo \( N \). Set \( \Delta = \hat{Q} \cap \bigcap_{P,i} \hat{P} \). Then \( \Delta \) is a lower congruence subgroup of level \( N \). We will show that \( \Delta = \Delta_0 \).

Given \( M \in \Delta \), it is clear that \( M \) satisfies condition (1) of Definition 3.3. Further, the \( i \)-th diagonal block of \( M \) modulo \( N \) is in \( \Delta_i \) modulo \( N \), since reducing it further modulo \( N_P \) (which divides \( N \)) yields a matrix in \( \hat{P}(\mathbb{Z}/N_P) \) for each \( P \) defining \( \Delta_i \). Hence \( \Delta \subseteq \Delta_0(N) \).

Given a matrix \( M \in \Delta_0(N) \), condition (1) implies that \( M \) lies in \( \hat{Q} \). Also, since the \( i \)-th diagonal block of \( M \) lies in \( \Delta_i \) modulo \( N \), we see that it must lie in \( P \) modulo \( N_P \) (since any element of \( \Delta_i \) lies in \( P \) modulo \( N_P \) and \( N_P(N) \) for each \( P \) used in defining \( \Delta_i \). Hence \( M \) lies in \( \hat{P} \). Therefore, \( M \in \Delta \), so \( \Delta_0(N) \subseteq \Delta \). \( \square \)

Lemma 3.5. Given lower congruence subgroups \( \Delta_i \) of level \( N_i \), as above, and an integer \( N \) divisible by all the \( N_i \), the system \( \Delta_0(N), \Delta_1, \ldots, \Delta_i \) defined in Definition 3.3 is compatible if \( N > 1 \).

Proof. Property (i) of a compatible system is clear. For property (ii) , we note that an element of the stabilizer of \( \mathcal{F} \) is block upper triangular. The diagonal blocks must all have determinant \( \pm 1 \). If they all have determinant 1, they must (by part (2) of the definition of \( \Delta_0(N) \)) be congruent modulo \( N \) to an element of \( \Delta_i \). However, one checks easily that any matrix of determinant 1 that is congruent to an element of \( \Delta_i \) modulo \( N \) actually lies in \( \Delta_i \).

For (iii), suppose \( v \in V_r, \gamma \in \Delta_0(N) \) and \( \gamma v \in F_m \). We may assume that \( v \neq 0 \). We must show that \( r \leq m \). Suppose to the contrary that \( r > m \). In block diagonal form write \( \gamma = (\gamma_{ij}) \) and \( v = (v_i) \). Then \( v_i = 0 \) if \( i \neq r \), \( v_r \neq 0 \) and \( (\gamma v)_j = \gamma_{jr}v_r \). Now \( \gamma v \in F_m \) implies that \( \gamma_{jr}v_r = 0 \) for all \( j > m \). In particular \( \gamma_{rr}v_r = 0 \). But \( \gamma_{rr} \) has a nonzero determinant (as may be observed by reducing \( \gamma \) modulo \( N \)) and this gives the desired contradiction. \( \square \)

4. Review and modification of the unstable construction

We will now review the main construction of [1]. Because we are using shar-blies and coshar-blies over \( \mathbb{Q} \) (to allow the computation of Hecke operators), and because we have changed the definition of compatibility, we must slightly alter the construction.

By induction, we only need to consider the case of a compatible system with two blocks. Let \( t = 2 \), and set \( n = n_1 + n_2 \). Let \( \Delta_1 \) be a lower congruence subgroup in \( \text{SL}(n_1, \mathbb{Z}) \) and \( \Delta_2 \) be a lower congruence subgroup in \( \text{SL}(n_2, \mathbb{Z}) \). Fix \( \Delta_0(N) = \Delta_0 \) in \( \text{SL}(n, \mathbb{Z}) \) as in Definition 3.3. Then \( \Delta_0, \Delta_1, \Delta_2 \) is a compatible system. Recall that \( V_1 \) is the span of the first \( n_1 \) standard basis vectors, and \( V_2 \) is the span of the remaining standard basis vectors, and let \( U = \mathbb{Q}^n \).

Definition 4.1. With respect to the compatible system \( \Delta_0(N), \Delta_1, \Delta_2 \):

(1) A subset \( M \) of \( U \) is pliable if there exists \( \gamma \in \Delta_0(N) \) such that \( \gamma M \) spans \( V_1 \) over \( \mathbb{Q} \). If so, we say \( \gamma \) plies \( M \).


(2) If \( M = \{m_1, \ldots, m_r\} \) is a sequence of vectors in \( U \), set
\[
\mathcal{P}(d, M) = \{S \subseteq \{1, \ldots, r\} | \{m_i \mid i \in S\} \text{ is pliable and } |S| = d\}.
\]

For \( k = i + j \), we now give a construction of a \( k \)-cosharply for \( GL(n) \) from an
\( i \)-cosharply and a \( j \)-cosharply.

**Definition 4.2.** Given the compatible system \( \Delta_0(N), \Delta_1, \Delta_2 \), assume that \( N \geq 3 \).
Let \( f \) be an \( i \)-cosharply cocycle for \( \Delta_1 \) and let \( g \) be a \( j \)-cosharply cocycle for \( \Delta_2 \).
Then \( f \) is a function of \( n_1 + i \) vectors in \( V_1 \) and \( g \) is a function of \( n_2 + j \) vectors in \( V_2 \).
We extend \( g \) to be 0 if any vector in its argument is 0. We denote by a prime
the natural projection of \( U \) to \( V_2 \). Define \( h = h_{f,g} \) by extending linearly the map
\( h : (Q^n)^{n_1+i+j} \to Q \) given by
\[
h(m_1, \ldots, m_{n+i+j}) = \sum_S (-1)^s f(g \sigma_{m_{\sigma(1)}}, \ldots, g \sigma_{m_{\sigma(n_1+i)}})g((g \sigma_{m_{\sigma(n_1+i+1)}})', \ldots, (g \sigma_{m_{\sigma(n)})})'
\]
where \( S \) runs over \( \mathcal{P}(n_1 + i, \{m_1, \ldots, m_{n+i+j}\}) \), and for each \( S \) we choose a permutation \( \sigma_S \) such that \( \{\sigma_S(1), \ldots, \sigma_S(n_1 + i)\} = S \) and a \( \gamma_S \in \Delta_0(N) \) such that
\( \gamma_S \) plies \( \{\sigma_{m_1}, \ldots, \sigma_{m_{n+i+j}}\} \).

**Lemma 4.3.** The \( h \) defined in Definition 4.2 is independent of the choices of \( \sigma_S \)
and \( \gamma_S \) and is a \( k \)-cosharply.

**Proof.** Since \( f \) and \( g \) are antisymmetric, we see that the choice of a \( \sigma_S \) for each set
\( S \) does not change the value of \( h \).

We now show that the choices of \( \gamma_S \) do not change the value of \( h \). For simplicity
of notation, suppose that \( \sigma_S \) is the identity. Suppose \( \gamma_S \) and \( \delta_S \) are two elements
of \( \Delta_0(N) \) that ply \( \{m_1, \ldots, m_{n+i+j}\} \). We claim that
\[
f(g \sigma_{m_1}, \ldots, g \sigma_{m_{n+i+1}})((g \sigma_{m_{n+i+1}})', \ldots, (g \sigma_{m_{n}})')
\]
and
\[
f(\delta \sigma_{m_1}, \ldots, \delta \sigma_{m_{n+i+1}})((\delta \sigma_{m_{n+i+1}})', \ldots, (\delta \sigma_{m_{n}})').
\]
are equal. To see this, let \( w_r = g \sigma_{m_r} \) for \( r = 1, \ldots, n + i + j \), and let \( \epsilon = \delta \sigma_{\gamma_S^{-1}} \).
Since \( \gamma_S^{-1} \) maps \( V_1 \) to the span of \( \{m_i \mid i \in S\} \), and \( \delta \sigma \) maps this span back to \( V_1 \), we see that \( \epsilon \) stabilizes \( V_1 \). Hence, \( \epsilon \) is block upper triangular. If we denote the block
diagonal components of \( \epsilon \) by \( \epsilon_1 \) and \( \epsilon_2 \), we have (by property (2) of Definition 3.3)
that each \( \epsilon_i \) is congruent modulo \( N \) to a matrix in \( \Delta_i \). Since \( N \geq 3 \) and \( \det(\epsilon_i) \) is a
unit in \( \mathbb{Z} \), and matrices in \( \Delta_i \) have determinant 1, we see that \( \det(\epsilon_i) = 1 \). Finally,
since the \( \Delta_i \) are defined solely by congruence conditions modulo divisors of \( N \)
and the requirement of being in \( SL(n_i, \mathbb{Z}) \), we deduce that \( \epsilon_1 \) and \( \epsilon_2 \) are in \( \Delta_1 \) and \( \Delta_2 \),
respectively.

Now since \( f \) and \( g \) are invariant under \( \Delta_1 \) and \( \Delta_2 \), respectively, we see that
\[
f(g \sigma_{m_1}, \ldots, g \sigma_{m_{n+i+1}})((g \sigma_{m_{n+i+1}})', \ldots, (g \sigma_{m_{n+i+j}})')
\]
\[
= f(w_1, \ldots, w_{n+i+1})g(w_{n+i+1}', \ldots, w_{n+i+j}')
\]
\[
= f(\epsilon_1 w_1, \ldots, \epsilon_1 w_{n+i+1})g(\epsilon_2(w_{n+i+1})', \ldots, \epsilon_2(w_{n+i+j})')
\]
\[
= f(\epsilon w_1, \ldots, \epsilon w_{n+i+1})g((\epsilon w_{n+i+1})', \ldots, (\epsilon w_{n+i+j})')
\]
\[
= f(\delta \sigma_{m_1}, \ldots, \delta \sigma_{m_{n+i+1}})((\delta \sigma_{m_{n+i+1}})', \ldots, (\delta \sigma_{m_{n+i+j}})')
\]
as desired.
To see that \( h \) is a cosharbly, we must also check that if \( m_1, \ldots, m_n \) do not span \( U \) then \( h(m_1, \ldots, m_n) = 0 \). This follows exactly as in [1, p. 336]. \( \square \)

We now wish to show that if \( f \) and \( g \) are cosharbly cocycles, then \( h_{fg} \) is a cosharbly cocycle, and that if \( f \) and \( g \) are not coboundaries, then \( h_{fg} \) is not a coboundary. Clearly a \( \Delta \)-invariant cosharbly cocycle \( f \) is not a coboundary if and only if there is a sharbly \( \Delta \)-cycle \( A \) such that \( f(A) \neq 0 \). Hence, given an \( i \)-sharblies \( \Delta_1 \)-cycle \( A \) with \( f(A) \) nonzero, and a \( j \)-sharblies \( \Delta_2 \)-cycle \( B \) with \( g(B) \) nonzero, we wish to construct a \( k \)-sharblies \( \Delta_0 \)-cycle \( C \) with \( h(C) \) nonzero.

**Definition 4.4.** Let \( A = \sum_w r_w[v] \) and \( B = \sum_w s_w[w] \) be respectively an \( i \)-sharblies for \( GL(n_1) \) and a \( j \)-sharblies for \( GL(n_2) \). A lift of \( B \) will be any \( B_s = \sum_w s_w[w_s] \) where the projection of each vector in \( w_s \) equals the corresponding vector in \( w \).

Given a lift \( B_s \) of \( B \), we define the \( i + j \)-sharblies for \( GL(n) \)

\[
A \otimes B_s = \sum_{v, w} r_v s_w[v, w_s],
\]

(where \( v \) can be viewed in \( V_1 \) or in \( U \) since we have embedded \( V_1 \) into \( U \) as the span of \( e_1, \ldots, e_{n_1} \).) We call \( A \otimes B_s \) a tensor sharbly.

We will also set \( B_t = \sum_w s_w[w_t] \) to be a specific lift of \( B \), namely the one where each vector in \( w_t \) has block components 0 and the corresponding vector in \( w \).

**Proposition 4.5.** Let \( \Delta_0, \Delta_1, \Delta_2 \) be a lower compatible system of groups. If \( A \) is an \( i \)-sharblies \( \Delta_1 \)-cycle, and \( B \) is a \( j \)-sharblies \( \Delta_2 \)-cycle, then the tensor sharblies \( A \otimes B_t \) is an \( i + j \)-sharblies \( \Delta_0 \)-cycle.

**Proof.** This is proved in [1, p. 335]. \( \square \)

**Proposition 4.6.** Let \( \Delta_0, \Delta_1, \Delta_2 \) be a lower compatible system of groups. Let \( f \) be an \( i \)-sharblies cocycle for \( \Delta_1 \) and let \( g \) be a \( j \)-sharblies cocycle for \( \Delta_2 \). Then \( h_{fg} \) is an \( i + j \)-sharblies \( \Delta_0 \)-cocycle.

**Proof.** This follows from [1, pp. 336-338]. \( \square \)

**Theorem 4.7.** Let \( A \) be an \( i \)-sharblies for \( GL(n_1) \), let \( B \) be a \( j \)-sharblies for \( GL(n_2) \), and let \( B_s \) be any lift of \( B \). Suppose also that \( f \) is an \( i \)-sharblies cocycle for \( \Delta_1 \) and \( g \) is a \( j \)-sharblies cocycle for \( \Delta_2 \). Then

\[
h_{fg}(A \otimes B_s) = f(A)g(B).
\]

**Proof.** Let \( A = \sum_v r_v[v] \) and \( B = \sum_w s_w[w] \). For each \( w \), choose a lift \( w_s \) with blocks \( y_w \) and \( w \). Then

\[
A \otimes B_s = \sum_{v, w} r_v s_w[v, w_s],
\]

We compute \( h([v, w_s]) \) for each term of this sum. Let \( \gamma \in \Delta_0(N) \) and suppose that some vector in \( \gamma w_s \in V_1 \). By property (iii) of a compatible system, this implies that the corresponding vector of \( w^* \) is in \( V_1 \), which implies that the corresponding vector in \( w \) is 0. This is a contradiction (because sharblies are not allowed to have the zero vector as components), so we see that the only pliable sets of vectors among the columns of \( [v, w_s] \) are the subsets of \( v \). Hence, \( P(n_1 + i, v_1, \ldots, v_{n_1+i+j}) \) consists of the single set \( S = \{v_1, \ldots, v_{n_1+i+1}\} \) (corresponding to the pliable set of vectors \( \{v_1, \ldots, v_{n_1+1}\} \)). Therefore, when we compute \( h_{fg}([v, w_s]) \), there is only
one term in the sum, we may take \( \sigma_S = 1 \) and \( \gamma_S = 1 \), and we obtain \( h_{fg}(v, w^\alpha) = f(v)g(w) \). It follows by linearity of \( h_{fg}, f, \) and \( g \), that

\[
h_{fg}(A \otimes B^\alpha) = h_{fg}\left( \sum_{v, w} r_v s_w[v, w^\alpha] \right)
= \sum_{v, w} r_v s_w h_{fg}(v, w^\alpha)
= \sum_{v, w} r_v s_w f(v)g(w)
= \left( \sum_v r_v f(v) \right) \left( \sum_w s_w g(w) \right)
= f(A)g(B).
\]

\[\square\]

**Corollary 4.8.** In the construction of \( h_{fg} \), assume that \( f \) and \( g \) are not coboundaries. Then \( h_{fg} \) is not a coboundary.

**Proof.** Recall that a cosharbly \( \Delta \)-cocycle is a coboundary if and only if it vanishes on all sharbly \( \Delta \)-cycles.

Since \( f \) is not a coboundary, there exists an \( i \)-sharbly \( \Delta_1 \)-cycle \( A \) with \( f(A) \neq 0 \). Since \( g \) is not a coboundary, there exists a \( j \)-sharbly \( \Delta_2 \)-cycle \( B \) with \( g(B) \neq 0 \). By Proposition 4.5, \( A \otimes B^\alpha \) is an \( i + j \)-sharbly \( \Delta_0 \)-cycle. By Theorem 4.7 \( h_{fg}(A \otimes B^\alpha) = f(A)g(B) \) is nonzero, so \( h_{fg} \) is not a coboundary. \[\square\]

### 5. Hecke Operators

We have the sharbly complex \((Sh, \partial)\) and the cosharbly cocomplex \((Sh^\vee, \delta)\).

The group \( \text{GL}(n, \mathbb{Q}) \) acts on \( Sh \) by \( g[x_1, \ldots, x_k] = [gx_1, \ldots, gx_k] \). It acts on \( Sh^\vee \) by the defining \((hg)(A) = h(gA)\) for \( g \in \text{GL}(n, \mathbb{Q}) \) and \( A \) a sharbly. We then have the formula

\[ \langle hg, A \rangle = \langle h, gA \rangle \]

where \( g \in \text{GL}(n, \mathbb{Q}) \), \( h \in Sh^\vee \) and \( A \in Sh \).

Let \( \Gamma \) be a subgroup of \( \text{GL}(n, \mathbb{Q}) \). A \( \Gamma \)-sharbly is an element in the coinvariants \( H_0(\Gamma, Sh) \). A \( \Gamma \)-cosharbly is an element in the invariants \( H^0(\Gamma, Sh^\vee) \). They are paired naturally by the same pairing \( \langle \cdot, \cdot \rangle \). It is a perfect pairing.

Let \( T = \Gamma s \Gamma \) where \( s \in \text{GL}(n, \mathbb{Q}) \). Then \( T \) acts as a Hecke operator on \( \Gamma \)-sharbly \( A \) and \( \Gamma \)-cosharblies \( h \) as follows. Write \( T = \bigcup_\alpha \Gamma s_\alpha \). Then \( h|T = \sum_\alpha h s_\alpha \) and \( TA = \sum_\alpha s_\alpha A \). One easily checks as usual that these formulas give well-defined operators that do not depend on the choice of coset representatives.

We thus get an action of the Hecke operators on the coinvariant sharbly complex and the invariant cosharbly complex, and hence on the homology of these complexes. This action induces the usual Hecke operators on the (co)homology of Gamma.

One reason for using lower congruence subgroups in this paper is that the Hecke representatives \( s_\alpha \) can be taken to be the usual upper triangular ones we use for \( \text{SL}(n, \mathbb{Z}) \).
Lemma 5.1. Let $\Gamma$ be a lower congruence subgroup of $\text{SL}(n, \mathbb{Z})$. Then
\[
\Gamma D_{\ell,k} \Gamma = \coprod \Gamma s_\alpha
\]
where the $s_\alpha$ run over matrices $M$ which satisfy the following conditions:
(i) $M$ is upper triangular with 1's and $\ell$'s along the diagonal;
(ii) there are exactly $k \ell'$s on the diagonal of $M$;
(iii) the upper triangular entries of $M$ are 0 except for those $M_{ab}$ for $a < b$ where $M_{aa} = 1$ and $M_{bb} = \ell$, in which case $M_{ab}$ is an integer from 0 to $\ell - 1$ inclusive.

Proof. The lemma is well known if $\Gamma = \text{SL}(n, \mathbb{Z})$. In general, this implies that the cosets on the right are distinct. Now suppose $x \in \Gamma D_{\ell,k} \Gamma$. Then again from the level 1 case we have that $x = \delta M$ for some $\delta \in \text{SL}(n, \mathbb{Z})$ and some $M$ as described in the lemma. However, since $D_{\ell,k}$ is upper triangular (in fact it is diagonal), $x$ satisfies the same lower congruence conditions that define $\Gamma$ and since $M$ is upper triangular, $\delta$ must satisfy all the same lower triangular congruences as well. In other words, $\delta \in \Gamma$. □

Remark. If $\Gamma$ is not a lower-congruence subgroup, the conclusion of Lemma 5.1 will no longer follow. For example, if $\Gamma = \Gamma_1(N)$ for $N > 2$ it will be necessary to choose $s_\alpha$'s which are not upper triangular. Such $s_\alpha$'s would not preserve the set of $(i, m)$-reducible sharblies, and our method below would fail to compute the action of the Hecke operators on the cosharbly $h$ which we will define.

We are going to state and prove a formula relating Hecke operators in three different dimensions. Some of our previous notation left the size of the matrices unstated. Now, if necessary, indicate the size of an $m \times m$ matrix by a superscript $m$.

Let $n = n_1 + n_2$ and let $\Gamma, \Gamma_1, \Gamma_2$ be a compatible set of lower congruence subgroups as above. We have
\[
T_{\ell,k}^n = \Gamma D_{\ell,k} \Gamma = \coprod \Gamma s_{k,\alpha}
\]
with the $s_{k,\alpha}$ as described in lemma 5.1.

Let us write
\[
T_{\ell,i}^{n_1} = \Gamma_1 D_{\ell,i}^{n_1} \Gamma_1 = \coprod \Gamma_1 s_{i,\beta}^{n_1}
\]
and
\[
T_{\ell,j}^{n_2} = \Gamma_2 D_{\ell,j}^{n_2} \Gamma_2 = \coprod \Gamma_2 s_{j,\zeta}^{n_2}
\]
Then we can enumerate the set $\{s_{k,\alpha}^n\}$ of coset representatives of $T_{\ell,k}^n$ as follows, using $(n_1, n_2)$-block form:
\[
\bigcup \left\{ \begin{pmatrix} s_{i,\beta}^{n_1} & M \\ 0 & s_{j,\zeta}^{n_2} \end{pmatrix} \right\}
\]
where the union runs over pairs of coset representatives $s_{i,\beta}^{n_1}$ of $T_{\ell,i}^{n_1}$ and coset representatives $s_{j,\zeta}^{n_2}$ of $T_{\ell,j}^{n_2}$ with $0 \leq i \leq n_1$, $0 \leq j \leq n_2$, and $i + j = k$, and over all $M$ which cause the resulting matrix to satisfy the conditions of Lemma 5.1. For a given choice of $i, j, \beta, \zeta$, the possible $M$ will have $(n_1-i)j$ entries which range from 0 to $\ell - 1$ inclusive, with the remaining entries being 0.
Motivated by this enumeration of the \( s_{k,o} \)’s we make the following definition:

**Definition 5.2.** Let \( \phi, \phi_1, \phi_2 \) be characters of the Hecke algebras \( K[T_{\ell,k}] \), \( K[T_{\ell,1}] \), and \( K[T_{\ell,j}] \), respectively, such that

\[
\phi(T_{\ell,k}) = \sum_{i+j=k} \ell^{(n_1-i)_j} \phi_1(T_{\ell,1}) \phi_2(T_{\ell,j}).
\]

Then we say that \( \phi \) is Hecke-reducible into \( (\phi_1, \phi_2) \).

**Remark.** Because of the enumeration of the \( s_{k,o} \)’s above, and the way the cosharbly \( h \) is defined, we will see that the Hecke eigenvalues on \( h \) restricted to a certain set of \((i, n_1)\)-reducible sharblies define a Hecke-reducible character.

**Lemma 5.3.** If \( \phi \) is Hecke-reducible into \( (\phi_1, \phi_2) \), then the corresponding Hecke polynomials (recall Definition 1.1) satisfy the equation

\[
F_{\phi,\ell}(X) = F_{\phi_1,\ell}(X) F_{\phi_2,\ell}(\ell^{n_1} X)
\]

**Proof.** Write \( a_{\ell,k}^n = \phi(T_{\ell,k}) \) and similarly for the values of \( \phi_1 \) and \( \phi_2 \). The left hand side of the desired equation equals

\[
\sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a_{\ell,k} X^k = \sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} X^k \sum_{i+j=k} \ell^{(n_1-i)_j} a_{\ell,i} a_{\ell,j}^{n_2}
\]

whereas the right hand side equals

\[
\left( \sum_{i=0}^{n_1} (-1)^i \ell^{i(i-1)/2} a_{\ell,i} X^i \right) \left( \sum_{j=0}^{n_2} (-1)^j \ell^{j(j-1)/2} a_{\ell,j}^{n_2} X^j \right) = \sum_{k=0}^{n} (-1)^k X^k \sum_{i+j=k} \ell^{i(i-1)/2+j(j-1)/2+n_1j} a_{\ell,i} a_{\ell,j}^{n_2}.
\]

Comparing terms for a given \( k \) and a given pair \((i, j)\) with \( i + j = k \), we must prove that

\[
\ell^{k(k-1)/2+(n_1-i)_j} a_{\ell,i} a_{\ell,j}^{n_2} = \ell^{i(i-1)/2+j(j-1)/2+n_1j} a_{\ell,i} a_{\ell,j}^{n_2}.
\]

This is true because the exponents of \( \ell \) on the two sides are equal. \( \Box \)

**Corollary 5.4.** If a character \( \phi \) on the Hecke algebra \( K[\{T_{\ell,1}\}] \) is Hecke reducible into \((\phi_1, \phi_2)\), with each \( \phi_1 \), a character of \( K[\{T_{\ell,1}\}] \), and each \( \phi_2 \) has an attached Galois representation \( \sigma_j \), then \( \phi \) is attached to the Galois representation \( \rho = \sigma_1 \oplus \omega^{n_1} \sigma_2 \).

**Proof.** We have

\[
\rho(Frob_\ell) = \begin{pmatrix} \ell^{n_1} \sigma_2(Frob_\ell) & 0 \\ 0 & \sigma_1(Frob_\ell) \end{pmatrix}
\]

so that

\[
\det(I - \rho(Frob_\ell) X) = \det(I - \ell^{n_1} \sigma_2(Frob_\ell) X) \det(I - \sigma_1(Frob_\ell) X)
\]

which equals

\[
\det(I - \sigma_2(Frob_\ell)(\ell^{n_1} X)) \det(I - \sigma_1(Frob_\ell) X) = F_{\phi_2,\ell}(\ell^{n_1} X) F_{\phi_1,\ell}(X) = F_{\phi,\ell}(X).
\]

Hence, \( \phi \) is attached to \( \rho \). \( \Box \)
Theorem 6.1. Let $\Delta_1 \subset \text{SL}(n_1, \mathbb{Z})$ and $\Delta_2 \subset \text{SL}(n_2, \mathbb{Z})$ be lower congruence subgroups, and set $n = n_1 + n_2$. Choose $\Delta_0$ so that $\Delta_0, \Delta_1, \Delta_2$ is a lower compatible system. Let $f$ be an $i$-cosharply cocycle for $\Delta_1$ and let $A$ be an $i$-sharply cycle for $\Delta_1$ such that $f(A) \neq 0$. Let $g$ be a $j$-cosharply cocycle for $\Delta_2$ and let $B$ be a $j$-sharply cycle for $\Delta_2$ such that $g(B) \neq 0$. Recall that $\tilde{f}$ and $\tilde{g}$ denote the classes of $f$ and $g$ modulo coboundaries. Suppose that $\tilde{f}$ and $\tilde{g}$ are simultaneous eigenvectors of the Hecke operators, with $\tilde{f}$ affording the character $\phi_1$ and $\tilde{g}$ affording the character $\phi_2$. Define the $(i+j)$-cosharply cocycle $h = h_{fg}$ as in Definition 4.2. Then there is a Hecke-character $\phi$ such that for $C = A \otimes B_\iota$, we have

$$\langle \tilde{h}|T^n_{\ell,k}, C \rangle = \phi(T^n_{\ell,k})(\tilde{h}, C),$$

and $\phi$ is Hecke-reducible into $(\phi_1, \phi_2)$.

Proof. Let $h$ be as above, and let $C = A \otimes B_\iota$. Then $C$ is an $(i,n_1)$-reducible $k$-sharply cycle for $\Delta_0$. Define the Hecke character $\phi$ by

$$\phi(T^n_{\ell,k}) = \sum_{i+j=k} \ell^{(n_1-i)j} \phi_1(T^n_{\ell,i}) \phi_2(T^n_{\ell,j}).$$

Now let $T^n_{\ell,k} = \Delta_0 D^n_{\ell,k} \Delta_0$ be a Hecke operator. We compute

$$\langle \tilde{h}|T^n_{\ell,k}, C \rangle = \langle \tilde{h}|T^n_{\ell,k}, C \rangle.$$

Writing $T^n_{\ell,k} = \bigcup_\alpha \Delta_0 s^n_{\ell,k,\alpha}$, with each $s^n_{\ell,k,\alpha}$ upper triangular, as in lemma 5.1, we recall that the set $\{s^n_{\ell,k,\alpha}\}$ can be written as

$$\bigcup_{i+j=k} \left\{ \begin{pmatrix} s^n_{i,\beta} & M \\ 0 & s^n_{j,\zeta} \end{pmatrix} \right\},$$

where $\Delta_1 D^n_{\ell,i} \Delta_1 = \bigcup_\beta \Delta_1 s^n_{i,\beta}$, and $\Delta_2 D^n_{\ell,j} \Delta_2 = \bigcup_\zeta \Delta_2 s^n_{j,\zeta}$, and $M$ runs through all matrices that make the displayed matrix one of the coset representatives of $T^n_{\ell,k}$. Recall that for a given pair $(s^n_{i,\beta}, s^n_{j,\zeta})$ there are $\ell^{n_1-j}$ possible $M$.

We note that since each $s^n_{k,\alpha}$ is upper triangular, it preserves the space $W_{n_1}$, so its action on $A$ takes $A$ to an $i$-sharply for $\text{GL}(n)$.

One computes easily that if

$$s^n_{k,\alpha} = \begin{pmatrix} s^n_{i,\beta} & * \\ 0 & s^n_{j,\zeta} \end{pmatrix},$$

then

$$s^n_{k,\alpha} C = s^n_{i,\beta} A \otimes s^n_{k,\alpha}(B_\iota),$$
where we note that $s^n_{k,a} B_1$ is a lift of $s^n_{\lambda} B$. Applying Theorem 4.7, we find that

$$\langle h|T^n_{\ell,k}C \rangle = \langle h,T^n_{\ell,k}C \rangle$$

$$= \sum_{\alpha} \langle h,s^n_{k,a} C \rangle$$

$$= \sum_{i+j=k} \ell^{(n-i)j} \sum_{\beta} \sum_{\zeta} \langle f,s^n_{i,j} A \rangle \langle g,s^n_{j} B \rangle$$

$$= \sum_{i+j=k} \ell^{(n-i)j} \{ f,T_{\ell,i} A \langle g,T_{\ell,j} B \rangle$$

$$= \sum_{i+j=k} \ell^{(n-i)j} \{ f,T_{\ell,i} A \langle g,T_{\ell,j} B \rangle$$

$$= \sum_{i+j=k} \ell^{(n-i)j} \phi_1(T_{\ell,i}) \phi_2(T_{\ell,j}) \langle f,A \rangle \langle g,B \rangle$$

$$= \sum_{i+j=k} \ell^{(n-i)j} \phi_1(T_{\ell,i}) \phi_2(T_{\ell,j}) \langle h,C \rangle$$

$$= \langle \phi(T^n_{\ell,k}) h,C \rangle.$$ 

The theorem follows. \qed

We now show that the previous theorem suffices to prove that there is a $k$-cosharbly cocycle that, modulo coboundaries, is an eigenvector of all the Hecke operators affording the Hecke character $\phi$ defined above.

**Lemma 6.2.** Let $K$ be a field, $H$ a commutative $K$-algebra, and $A$ a left $H$-module. Assume that $\langle \mu|T,a \rangle = \langle \mu, Ta \rangle$, for all $\mu \in A^\vee$, $a \in A$, $T \in H$.

Let $\chi : H \rightarrow K$ be a character and suppose that there are nonzero elements $\mu_0 \in A^\vee$ and $a_0 \in A$ such that $\langle \mu_0|T,a_0 \rangle = \chi(T) \langle \mu_0,a_0 \rangle$ for every $T \in H$. Then for every $x$ in the cyclic submodule $H a_0$ of $A$,

$$\langle \mu_0|T,x \rangle = \chi(T) \langle \mu_0,x \rangle.$$ 

**Proof.** It suffices to prove the displayed formula when $x = T'a_0$ for some $T' \in H$. Then

$$\langle \mu_0|T,T'a_0 \rangle = \langle \mu_0|T'T,a_0 \rangle$$

$$= \chi(T'T) \langle \mu_0,a_0 \rangle$$

$$= \chi(T) \chi(T') \langle \mu_0,a_0 \rangle$$

$$= \chi(T) \langle \mu_0|T',a_0 \rangle$$

$$= \chi(T) \langle \mu_0,T'a_0 \rangle.$$ 

\qed

**Corollary 6.3.** Under the conditions of the lemma, if in addition $A$ is finite-dimensional over $K$, then $A$ and $A^\vee$ each contain an $H$-eigenvector with system of eigenvalues given by the character $\chi$.

**Proof.** The displayed formula shows that the restriction $\mu_0|_{H a_0}$ is an $H$-eigenvector in $(H a_0)^\vee$ with system of eigenvalues given by the character $\chi$. The canonical surjective projection $A^\vee \rightarrow (H a_0)^\vee$ is a map of $H$-modules, so the second assertion follows from finite-dimensionality and Jordan canonical form. Then the first assertion follows by taking the transpose of the Jordan canonical forms. \qed
7. Application

**Theorem 7.1.** Let $K$ be a field of characteristic greater than $n_1 + n_2 + 1$ or of characteristic 0. Let $\Delta_0, \Delta_1, \Delta_2$ be a lower compatible system of groups inside $\text{SL}(n, \mathbb{Z}), \text{SL}(n_1, \mathbb{Z}),$ and $\text{SL}(n_2, \mathbb{Z}),$ respectively. Assume that $H^i(\Delta_1, F)$ has a Hecke eigenvector affording the Hecke character $\phi_1,$ and that $H^j(\Delta_2, F)$ has a Hecke eigenvector affording the Hecke character $\phi_2.$ Then $H^{n_1 n_2+i+j}(\Delta_0, F)$ has a Hecke eigenvector affording the character $\phi$ that is Hecke reducible into $(\phi_1, \phi_2).$

**Remark.** Note that the “top dimension” for cohomology of subgroups of $\text{SL}(n, \mathbb{Z})$ is $n(n - 1)/2.$ In order for this to work at all, it is necessary that $n_1 n_2 + i + j \leq (n_1 + n_2)(n_1 + n_2 - 1)/2$ whenever $i \leq n_1(n_1 - 1)/2$ and $j \leq n_2(n_2 - 1)/2.$ Since

$$\frac{(n_1 + n_2)(n_1 + n_2 - 1)}{2} - n_1 n_2 = \frac{n_1(n_1 - 1)}{2} + \frac{n_2(n_2 - 1)}{2} \geq i + j$$

this will be true. We remark that this implies that “top-dimensional” eigenvectors combine into “top-dimensional” eigenvectors.

**Proof.** By Borel-Serre duality,

$$H^{n_1 n_2+i+j}(\Delta_0, F) \cong H_{(n_1 + n_2)(n_1 + n_2 - 1)/2-n_1 n_2-i-j}(\Delta_0, St)$$

$$= H_{n_1(n_1-1)/2-i+n_2(n_2-1)/2-j}(\Delta_0, St).$$

By assumption, $H^i(\Delta_1, F) \cong H_{n_1(n_1-1)/2-i}(\Delta_1, St)$ has a Hecke eigenvector $\tilde{v}$ affording the character $\phi_1.$ Similarly $H^j(\Delta_2, F) \cong H_{n_2(n_2-1)/2-j}(\Delta_2, St)$ has a Hecke eigenvector $\tilde{w}$ affording the character $\phi_2.$ By duality, there are nonzero cosharbly cocycles $f$ for $\Delta_1$ and $g$ for $\Delta_2$ that $\tilde{f}$ and $\tilde{g}$ afford the characters $\phi_1$ and $\phi_2,$ respectively. Then Theorem 6.1 combined with Corollary 6.3 yields a cosharbly $\Delta_0$-cocycle representing an element in $H_{n_1(n_1-1)/2-i+n_2(n_2-1)/2-j}(\Delta_0, St)^\vee$ affording the character $\phi = (\phi_1, \phi_2).$ Hence, there is also a cosharbly $\Delta_0$-cycle $\bar{x}$ with

$$\bar{x} \in H_{n_1(n_1-1)/2-i+n_2(n_2-1)/2-j}(\Delta_0, St)$$

affording the character $\phi.$ By Borel-Serre duality, using the fact that

$$\frac{(n_1 + n_2)(n_1 + n_2 - 1)}{2} - \frac{n_1(n_1 - 1)}{2} - \frac{n_2(n_2 - 1)}{2} = n_1 n_2$$

we see that there is a Hecke eigenvector in $H^{n_1 n_2+i+j}(\Delta_0, F)$ affording the character $\phi.$

**Theorem 7.2.** Let $F$ be a finite field of characteristic greater than $n_1 + n_2 + 1,$ and for $i = 1, 2$ let $\Delta_i$ be a lower congruence subgroup of $\text{SL}(n_i, \mathbb{Z}).$ If $\rho_i : \text{G}_Q \to \text{GL}(n, F)$ fits $H^i(\Delta_i, F),$ then $\rho_1 \otimes \omega^{n_1} \rho_2$ fits $H^{n_1 n_2+i+j}(\Delta_0, F),$ where $\Delta_0, \Delta_1, \Delta_2$ is a lower compatible system.

**Proof.** This follows immediately from Theorem 7.1 and Corollary 5.4.

We can, of course use induction to prove a theorem for arbitrarily many Galois representations fitting cohomology groups. To state the theorem, it is convenient to define the second elementary symmetric polynomial in $t$ variables,

$$s_t(x_1, \ldots, x_t) = \sum_{1 \leq i < j \leq t} x_i x_j.$$
Theorem 7.3. Let $F$ be a finite field with characteristic greater than $n_1 + n_2 + 1$. For $1 \leq i \leq t$, let $\Delta_i \subseteq \text{SL}(n_i, \mathbb{Z})$ be lower congruence subgroups, and assume that there is a Galois representation $\rho_i : G_{\overline{Q}} \to \text{GL}(n_i, F)$ fitting $H^k(\Delta_i, F)$. Choose $\Delta_0$ so that $\Delta_0, \Delta_1, \ldots, \Delta_t$ is a lower compatible system. Set

$$m_j = \sum_{i=1}^{j-1} n_i.$$  

Then the Galois representation

$$\rho = \rho_1 \oplus \omega^{m_2} \rho_2 \oplus \cdots \oplus \omega^{m_t} \rho_t$$

fits

$$H^k(\Delta_0, F),$$

where $k = s_t(n_1, \ldots, n_t) + k_1 + \cdots + k_t$.

Proof. This follows easily from Theorem 7.2 by induction, using the fact that $n_1(n_2 + \cdots + n_t) + s_{t-1}(n_2, \ldots, n_t) = s_t(n_1, n_2, \ldots, n_t)$.

\[
\Box
\]

8. Examples

Throughout this section, let $F$ be a finite field of sufficiently large characteristic $p$.

Example 8.1. Let $\rho_1 : G_{\overline{Q}} \to \text{GL}(n_1, F)$ and $\rho_2 : G_{\overline{Q}} \to \text{GL}(n_2, F)$ be Galois representations fitting $H^{n_1(n_1-1)/2}(\Delta_1, F)$ and $H^{n_2(n_2-1)/2}(\Delta_2, F)$, respectively (so both representations are attached to eigenclasses in the top-dimensional cohomology). Assume that $\Delta_0, \Delta_1, \Delta_2$ is a lower compatible system. Then $\rho_1 \oplus \omega^{n_1} \rho_2$ fits $H^{n(n-1)/2}(\Delta_0, F)$, so it is attached to a top-dimensional cohomology class.

This example would apply to two-dimensional odd irreducible Galois representations $\rho_1$ and $\rho_2$ with Serre weight $2$ that are defined over $F$. By Serre’s conjecture they would be attached to cohomology eigenclasses in $H^1(\Gamma_0(N_1), F)$ where $N_i$ is the Serre level of $\rho_i$. Then $\rho_2 \oplus \omega^2 \rho_1$ would be attached to a cohomology class in $H^6(\Gamma_0(N_1N_2), F)$, where $\Gamma_0(N_1N_2)$ is defined so that the triple $\Gamma_0(N_1N_2), \Gamma_0(N_1), \Gamma_0(N_2)$ is a lower compatible system.

Example 8.2. Let $\rho_1 : G_{\overline{Q}} \to \text{GL}(3, F)$ and $\rho_2 : G_{\overline{Q}} \to \text{GL}(3, F)$ be irreducible Galois representations, each attached to a cohomology eigenclass in $H^3(\Gamma_0(N_1), F)$ (computational examples of such representations may be found in [5]). Define $\Gamma_0(N_1N_2)$ in such a way that the triple $\Gamma_0(N_1N_2), \Gamma_0(N_1), \Gamma_0(N_2)$ is a lower compatible system. As described in [8], by Lefschetz duality, each $\rho_i$ must also be attached to a cohomology eigenclass in $H^2(\Gamma_0(N_i), F)$. Hence, applying Theorem 7.2 multiple times, we find that $\rho_1 \oplus \omega^3 \rho_2$ is attached to eigenclasses in $H^k(\Gamma_0(N_1N_2), F)$ for $k = 3^2 + 3 + 3 = 15$, $k = 3^2 + 3 + 2 = 14$, and $k = 3^2 + 3 + 2 = 13$.

Example 8.3. In [6], examples are given of Hecke eigenclasses in cohomology groups of the form $H^5(\Gamma_0(4, N), \mathbb{C})$ whose systems of Hecke eigenvalues, reduced modulo $p$ will be attached to Galois representations $\rho : G_{\overline{Q}} \to \text{GL}(4, F)$. Letting $\rho_1$ and $\rho_2$ be two such representations, with levels $N_1$ and $N_2$, and defining
\[ \Gamma_{00}(N_1, N_2) \] so that the triple \( \Gamma_{00}(N_1, N_2), \Gamma_0(N_1), \Gamma_0(N_2) \) is a lower compatible system, Theorem 7.2 demonstrates the existence of a cohomology eigenclass in \( H^{26}(\Gamma_{00}(N_1, N_2), \mathbb{C}) \) whose system of Hecke eigenvalues, reduced modulo \( p \), will have the eight-dimensional Galois representation \( \rho_1 \oplus \omega^4 \rho_2 \) attached. Note that in this case, the top degree for the cohomology of congruence subgroups of \( SL(8) \) is 28.

In all of these examples, if we assume that the predicted weights of the original Galois representations (as defined in [5]) are all \( F \), then a predicted weight of the Galois representation constructed as a direct sum can also be seen to be \( F \), so that the construction in Theorem 7.2 is in accordance with the weight prediction of the main conjecture of [5]. As mentioned earlier, however, the prediction made for the level in [5] does not match, because the groups \( \Gamma_{00}(N_1, N_2) \) are not of the form considered in [5].

**References**


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