1a) (3 pts) Define torsion of a regular curve in \( \mathbb{R}^3 \).

1b) (3 pts) Define Gaussian Curvature of a surface in \( \mathbb{R}^3 \).

1c) (3 pts) State the Fundamental Theorem of Curves.

1d) (3 pts) State the Fundamental Theorem of Surfaces.

1e) (3 pts) State the Gauss-Bonnet Theorem.

2) (3 pts each) TRUE-FALSE.

a. For any given smooth functions \( \kappa(s) > 0 \) and \( \tau(s) \), there exists a regular curve \( \alpha(s) \) in \( \mathbb{R}^3 \) with curvature \( \kappa(s) \) and torsion \( \tau(s) \).

b. For any given smooth functions \( E, F, G \) and \( e, f, g \), there exists a regular surface \( S \) with \( I_S = Ex^2 + 2Fxy + Gy^2 \) and \( II_S = ex^2 + 2fy^2 + gy^2 \).

c. Let \( L \) be a straight line in a surface \( S \) in \( \mathbb{R}^3 \). Then, for any \( p \in L \) the direction given by \( L \) is principal direction.

d. Let \( S \) be a disk in \( \mathbb{R}^3 \) with Gaussian curvature \( K \leq 0 \) everywhere. Then, any two different geodesics starting at the same point cannot meet again in \( S \).

e. Let \( S_1 \) and \( S_2 \) are two surfaces which are tangent to each other along a curve \( \alpha \). If \( \alpha \) is a geodesic of \( S_1 \), then it is a geodesic of \( S_2 \), too.

3a) (10 pts) Calculate the Frenet apparatus \( (T, N, B, \kappa, \tau) \) for

\[ \alpha(t) = (\cos t, \sin t, t) \quad t \in (-1, 1) \]

3b. (6 pts) Let \( S \) be the cylinder \( \{x^2 + y^2 = 1\} \) in \( \mathbb{R}^3 \). Then, the curve above \( \alpha \subset S \). Compute the normal curvature \( \kappa_n(t) \) and geodesic curvature \( \kappa_g(t) \) of \( \alpha \).

4) Let \( S \) be a surface of revolution with parametrization

\[ \varphi(u, v) = (\phi(u) \cos v, \phi(u) \sin v, u) \]

4a. (8 pts) Find the principal directions, and compute the Gaussian curvature \( K(u, v) \) for any point \( p = \varphi(u, v) \).

4b. (8 pts) Show that any meridian curve \( (v = v_0) \) is a geodesic. Show that a parallel curve \( (u = u_0) \) is a geodesic if and only if \( \varphi'(u_0) = 0 \).

5) For \( c > 0 \), define the map \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) with \( F(p) = 3p \).

5a. (8 pts) Let \( \alpha(s) \) be a regular curve in \( \mathbb{R}^3 \), and let \( \hat{\alpha}(s) = F(s) \). Compute \( \hat{\kappa}(s) \) and \( \hat{\tau}(s) \) in terms of \( \kappa(s) \) and \( \tau(s) \).
5b. (8 pts) Let $S$ be a regular surface, and let $\hat{S} = F(S)$. Compute Gaussian curvature $\hat{K}(p)$ in terms of $K(p)$.

6a) (10 pts) Let $S$ be a smooth, closed, orientable surface in $\mathbb{R}^3$. Show that the Gauss Map $N : S \to S^2$ is surjective.

6b) (8 pts) Let $K_+(p) = \max\{0, K(p)\}$. Show that $\int_S K_+ \, dA \geq 4\pi$.

7) (24 pts) Show that if all geodesics of a connected surface $S$ are plane curves, then $S$ is contained in a plane or a sphere.

[Hint: You can use the following steps.]

Step 1. If a geodesic is a plane curve, then it is a line of curvature.

Step 2. If all geodesics of a connected surface $S$ are plane curves, then every point of $S$ is an umbilical point (principal curvatures are same, $k_1 = k_2$).

Step 3. If every point of $S$ is an umbilical point, then $S$ is contained in a plane or a sphere.
2) (3 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

a. For any given smooth functions $\kappa(s) > 0$ and $\tau(s)$, there exists a regular curve $\alpha(s)$ in $\mathbb{R}^3$ with curvature $\kappa(s)$ and torsion $\tau(s)$.

TRUE. Fund. Thm. of Curve - Exists.

b. For any given smooth functions $E, F, G$ and $e, f, g$, there exists a regular surface $S$ with $I_s = Ex^2 + 2Fxy + Gy^2$ and $II_s = ex^2 + 2fy^2 + gy^2$.

FALSE. Fund. Thm. of Surf. - Gauss-Codazzi Eqs.

c. Let $L$ be a straight line in a surface $S$ in $\mathbb{R}^3$. Then, for any $p \in L$ the direction given by $L$ is principal direction.

FALSE. $S = \{x^2+y^2 = 1\}$ is a ruled surface.

d. Let $S$ be a surface in $\mathbb{R}^3$ with Gaussian curvature $K \leq 0$ everywhere. Then, any two different geodesics starting at the same point cannot meet again in $S$.

TRUE. $\int K dA + \theta_1 + \theta_2 \leq 2\pi$ $\ni \frac{\theta_2}{\theta_1} \leq 0$

e. Let $S_1$ and $S_2$ are two surfaces which are tangent to each other along a curve $\alpha$. If $\alpha$ is a geodesic of $S_1$, then it is a geodesic of $S_2$, too.

TRUE. $v_n \parallel N \Rightarrow v_n \parallel N$.
3a) (10 pts) Calculate the Frenet apparatus (T, N, B, κ, τ) for
\[ α(t) = (\cos t, \sin t, t) \quad t \in (-1, 1) \]

Frenet by arc length:
\[
\alpha'(s) = \left( \frac{\cos t}{\sqrt{1 + t^2}}, \frac{\sin t}{\sqrt{1 + t^2}}, \frac{1}{\sqrt{1 + t^2}} \right)
\]

\[
\frac{d}{ds} \alpha'(s) = \left( \frac{d}{ds} \frac{\cos t}{\sqrt{1 + t^2}}, \frac{d}{ds} \frac{\sin t}{\sqrt{1 + t^2}}, \frac{d}{ds} \frac{1}{\sqrt{1 + t^2}} \right) = \left( -\frac{\sin t}{\sqrt{1 + t^2}}, \frac{\cos t}{\sqrt{1 + t^2}}, -\frac{1}{2(1 + t^2)^{3/2}} \right)
\]

\[
\kappa = \frac{1}{2}
\]

\[
N = \left( -\frac{\cos t}{\sqrt{1 + t^2}}, -\frac{\sin t}{\sqrt{1 + t^2}}, 0 \right)
\]

\[
B = \left( \frac{\cos t}{\sqrt{1 + t^2}}, \frac{\sin t}{\sqrt{1 + t^2}}, 0 \right)
\]

3b. (6 pts) Let \( S \) be the cylinder \( \{x^2 + y^2 = 1\} \) in \( \mathbb{R}^3 \). Then, the curve above \( α \subset S \). Compute the normal curvature \( κ_n(t) \) and geodesic curvature \( κ_g(t) \) of \( α \).

\[
\mathcal{L} = \mathcal{L}(\alpha(t)) = (\cos t, \sin t, t) \quad \Rightarrow \quad \mathcal{N} = \Psi_{\mathcal{L}} \psi_{\mathcal{N}} = (\cos t, \sin t, 0)
\]

\[
\Rightarrow \quad \alpha'' \parallel \mathcal{N} \Rightarrow k = k_n \quad \text{and} \quad k_g = 0
\]

\[
= \frac{1}{2}
\]
4) Let \( S \) be a surface of revolution with parametrization
\[
\varphi(u, v) = (\phi(u) \cos v, \phi(u) \sin v, u)
\]

4a. (8 pts) Find the principal directions, and compute the Gaussian curvature \( K(u, v) \) for any point \( p = \varphi(u, v) \).

\[
\begin{align*}
\varphi_u &= \left\langle \phi'(u) \cos v, \phi'(u) \sin v, 1 \right\rangle, \quad |\varphi_u| = \sqrt{1 + \phi'(u)^2}, \quad E = 1 + \phi'(u)^2, \\
\varphi_v &= \left\langle -\phi'(u) \sin v, \phi'(u) \cos v \right\rangle, \quad |\varphi_v| = |\phi'(u)|, \quad F = 0, \\
\varphi_{uv} &= \left\langle -\phi''(u) \sin v - \phi'(u) \phi'(u) \cos v, \phi''(u) \cos v - \phi'(u) \phi'(u) \sin v \right\rangle, \quad N = \frac{\phi'(u)}{\sqrt{1 + \phi'(u)^2}}, \\
\end{align*}
\]

\[
\begin{align*}
\varphi_{uu} &= \left\langle \phi''(u) \cos v, \phi''(u) \sin v, 0 \right\rangle, \\
\varphi_{uv} &= \left\langle -\phi''(u) \sin v, \phi''(u) \cos v \right\rangle, \\
\varphi_{vv} &= \left\langle -\phi'(u) \phi'(u) \sin v, -\phi'(u) \phi'(u) \cos v \right\rangle \Rightarrow \quad N = \frac{\phi'(u)}{\sqrt{1 + \phi'(u)^2}}, \\
\end{align*}
\]

\[
K = \frac{eg - f^2}{E^2 F^2} = \phi''(u)^2
\]

4b. (8 pts) Show that any meridian curve \( u = u_0 \) is a geodesic.

Show that a parallel curve \( v = v_0 \) is a geodesic if and only if \( \phi'(u_0) = 0 \).

\[
\begin{align*}
\phi \text{ geodesic} \iff N_u \parallel N_S, \\
\text{a parallel curve } u = u_0 \text{ is a geodesic if and only if } \phi'(u_0) = 0.
\end{align*}
\]

\[
\begin{align*}
\varphi_u &= \left\langle \phi(u) \cos v_0, \phi(u) \sin v_0, u \right\rangle \text{ is a parallel curve}, \\
T_u \perp N_u \iff N_u \parallel N_S, \\
\varphi_v &= \left\langle -\phi'(u) \cos v, \phi'(u) \sin v \right\rangle, \\
N_{du} &= N_S = \left\langle \cos v_0, \sin v_0, 0 \right\rangle \iff \phi'(u_0) = 0.
\end{align*}
\]
5) For $c > 0$, define the map $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $F(p) = 3p$.

5a. (8 pts) Let $\alpha(s)$ be a regular curve in $\mathbb{R}^3$, and let $\hat{\alpha}(s) = F(s)$. Compute $\hat{\kappa}(s)$ and $\hat{\tau}(s)$ in terms of $\kappa(s)$ and $\tau(s)$.

\[
\hat{\alpha}(s) = 3\alpha(s) \quad \Rightarrow \quad \hat{\alpha}' = \alpha'(s) \quad \Rightarrow \quad \hat{\alpha}''(s) = \frac{1}{3} \alpha''(s)
\]

\[
\hat{\kappa}(s) = \frac{1}{3} \kappa(s) \quad \Rightarrow \quad \hat{\tau}(s) = \frac{1}{3} \tau(s)
\]

5b. (8 pts) Let $S$ be a regular surface, and let $\hat{S} = F(S)$. Compute Gaussian curvature $\hat{K}(p)$ in terms of $K(p)$.

For any $p$, let $k_1$ and $k_2$ be principal curvatures with $\hat{d}_1$ and $\hat{d}_2$ curves in principal directions.

By above $\hat{\kappa}$ and $\hat{\tau}$ are principal directions at $\hat{p} = F(p)$

\[
\hat{\kappa}_p = \hat{\kappa}_1 \hat{n}_1 \cdot \frac{k_1 k_2}{2} = \frac{K}{g} \quad \Rightarrow \quad \hat{K}_p = \frac{K}{g}
\]
6a) (10 pts) Let $S$ be a smooth, closed, orientable surface in $\mathbb{R}^3$. Show that the Gauss Map $N : S \to S^2$ is surjective.

Fix a unit vector $\nu \in S^2$. Let $P_\nu^t$ be the plane $\nu \cdot p = t$ for $t$ very large, $P_\nu^t \cap S = \emptyset$. Find the first point of contact at $p_0 \in S$.

$$= \quad \nu_{p_0} S = P_{\nu_{p_0}}^0 \quad \Rightarrow \quad N(p_0) = \nu \quad \square.$$

6b) (8 pts) Let $K_+ (p) = \max \{0, K(p)\}$. Show that $\int_S K_+ dA \geq 4\pi$.

$$\int_S K_+ dA = \int_{S^+} K dA \quad \text{where} \quad S^+ = \{ p \in S \mid K(p) > 0 \}$$

$$\int_{S^+} K dA = \int_{\text{area}} \det(N) dA = |N(S^+)|.$$  

Claim: $N : S^+ \to S^2$ surjective.

By above proof, at the first point of contact $p_0$, $S$ lies in one side of the tangent plane $\Rightarrow K(p_0) > 0 \Rightarrow N(S^+) \to S^2$ onto

$$= \quad |N(S^+)| > |S^+| = 4\pi.$$
**Bonus** (2½ pts) Show that if all geodesics of a connected surface \( S \) are plane curves, then \( S \) is contained in a plane or a sphere.

[Hint: You can use the following steps.]

**Step 1.** If a geodesic is a plane curve, then it is a line of curvature.

**Step 2.** If all geodesics of a connected surface \( S \) are plane curves, then every point of \( S \) is an umbilical point (principal curvatures are same, \( k_1 = k_2 \)).

**Step 3.** If every point of \( S \) is an umbilical point, then \( S \) is contained in a plane or a sphere.

---

**Step 1:** A geodesic \( \Rightarrow N_\nu = N_S \). 

Since \( \nu \) is a plane curve, \( N'_\perp N \Rightarrow N' \parallel \tau \Rightarrow N'_\nu(t): x(t) \cdot d' \Rightarrow \) a line of curvature. 

---

**Step 2:** Let \( p \in S \). \( \forall \nu \in T_p \) \( \exists \nu \in S \), geodesic with \( \nu(0) = p \) and \( \nu'(0) = \nu \).

By step 1:

\( \forall p \in S \), \( \forall \nu \in T_p \) \( \nu \) is a line of curvature \( \Rightarrow \lambda \nu = \nu \nu \nu' \).

\( \Rightarrow k_1 = k_2 = k_\nu \Rightarrow \) any \( p \) is umbilical.

---

**Step 3:** Do Carmo, 3.2 Prop. 4 (Page 149.)