MATH2211 Honors Linear Algebra
Feb. 28, 2018
Exam #1

You have 50 minutes to solve all 6 problems.
Each problem is worth 10 points.
To receive full credit you must fully justify your answers.
Cheating of any kind will result in a failing grade for the course.
Good luck.

NAME ___________________________
Problem 1. Suppose \( n \geq 1 \). Prove that
\[
\frac{1 - 1}{1!} + \frac{2 - 1}{2!} + \frac{3 - 1}{3!} + \cdots + \frac{n - 1}{n!} = \frac{n! - 1}{n!}
\]

Solution: The proof is by induction. If \( n = 1 \) the claim is that
\[
\frac{1 - 1}{1!} = \frac{1! - 1}{1!},
\]
which is obviously true. Now suppose that
\[
\frac{1 - 1}{1!} + \frac{2 - 1}{2!} + \frac{3 - 1}{3!} + \cdots + \frac{n - 1}{n!} = \frac{n! - 1}{n!}
\]
for some \( n \). Adding \( \frac{n}{(n+1)!} \) to both sides results in
\[
\frac{1 - 1}{1!} + \frac{2 - 1}{2!} + \frac{3 - 1}{3!} + \cdots + \frac{n - 1}{n!} + \frac{n}{(n+1)!} = \frac{n! - 1}{n!} + \frac{n}{(n+1)!}
\]
\[
= \frac{(n+1)(n! - 1)}{(n+1)!} + \frac{n}{(n+1)!}
\]
\[
= \frac{n \cdot n! + n! - 1}{(n+1)!}
\]
\[
= \frac{(n + 1) \cdot n! - 1}{(n+1)!}
\]
as desired.
Problem 2. Let

\[ v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \]

(a) Find all solutions to

\[ x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}. \]

Solution: We need to find all solutions to the system of linear equations

\[
\begin{align*}
    x_1 + 2x_2 &= 3, \\
    x_1 + x_2 - x_3 + x_4 &= -2, \\
    -x_1 + x_2 + 3x_3 &= 0.
\end{align*}
\]

Using row reduction

\[
\begin{bmatrix}
    1 & 2 & 0 & 0 & | & 3 \\
    1 & 1 & -1 & 1 & | & -2 \\
    -1 & 1 & 3 & 0 & | & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
    1 & 0 & -2 & 0 & | & 1 \\
    0 & 1 & 1 & 0 & | & 1 \\
    0 & 0 & 0 & 1 & | & -4
\end{bmatrix}
\]

we see that the above system of equation is equivalent to

\[
\begin{align*}
    x_1 - 2x_3 &= 1, \\
    x_2 + x_3 &= 1, \\
    x_4 &= -4,
\end{align*}
\]

which has as its general solution

\[
\begin{bmatrix}
    1 + 2x_3 \\
    1 - x_3 \\
    x_3 \\
    -4
\end{bmatrix} =
\begin{bmatrix}
    1 \\
    1 \\
    0 \\
    -4
\end{bmatrix} + x_3 \cdot
\begin{bmatrix}
    2 \\
    -1 \\
    1 \\
    0
\end{bmatrix}.
\]
(b) Delete vectors from the list $v_1, v_2, v_3, v_4$ to produce a basis of $\mathbb{R}^3$, or explain why this can’t be done.

**Solution:** If you delete $v_3$ from the list you obtain a basis $v_1, v_2, v_4 \in \mathbb{R}^3$. The easiest way to see this is by the row reduction

$$
\begin{bmatrix}
1 & 2 & 0 \\
1 & 1 & 1 \\
-1 & 1 & 0
\end{bmatrix}
\mapsto
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

The reduced row echelon form has a pivot in every row and column, and so the vectors $v_1, v_2, v_4 \in \mathbb{R}^3$ are both linearly independent and span.
Problem 3. Find four complex solutions to $z^4 = -1$. Express your answers both in the form $z = re^{i\theta}$, and in the form $z = x + iy$.

Solution: Using the relation $e^{i\pi} = -1$, we see that each of

\[
e^{i\pi/4} = \frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}
\]
\[
e^{i3\pi/4} = -\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}
\]
\[
e^{i5\pi/4} = -\frac{\sqrt{2}}{2} - i \cdot \frac{\sqrt{2}}{2}
\]
\[
e^{i7\pi/4} = \frac{\sqrt{2}}{2} - i \cdot \frac{\sqrt{2}}{2}
\]

is a solution to $z^4 = -1$. 

Problem 4. Suppose $v_1, v_2, v_3, v_4 \in \mathbb{R}^4$ is a basis, and $U \subset \mathbb{R}^4$ is a subspace satisfying

$$v_1, v_2, v_3 \in U, \quad v_4 \notin U.$$ 

Show that $v_1, v_2, v_3$ span $U$.

Solution: A result from class implies that $\dim(U) \leq \dim(\mathbb{R}^4)$, and hence

$$\dim(U) \leq 4.$$ 

The dimension cannot be 4, for then a result from class would tell us that the inclusion $U \subset \mathbb{R}^4$ is actually an equality, contradicting $v_4 \notin U$. The dimension cannot be 0, 1, or 2, as $v_1, v_2, v_3 \in U$ are linearly independent.

Thus, by process of elimination,

$$\dim(U) = 3.$$ 

We now know that $v_1, v_2, v_3 \in U$ is a linearly independent list of three elements in a vector space of dimension three, and hence must be a basis. In particular, these vectors span $U$. 
Problem 5. Let $P$ be the space of all polynomials with real coefficients, and let

$$U = \left\{ f(x) \in P : \int_0^1 f(x) \, dx = 0 \right\}.$$

(a) Show that $U$ is a subspace of $P$.

Solution: Certainly $0 \in U$, as $\int_0^1 0 = 0$. If $f(x), g(x) \in U$ then

$$\int_0^1 f(x) + g(x) = \int_0^1 f(x) + \int_0^1 g(x) = 0 + 0 = 0,$$

so that $f(x) + g(x) \in U$. Finally, if $c \in \mathbb{R}$ and $f(x) \in U$ then

$$\int_0^1 cf(x) = c \int_0^1 f(x) = c \cdot 0 = 0,$$

so that $cf(x) \in U$. 
(b) Show that $U$ is infinite dimensional.

**Solution:** The subspace $U$ is infinite dimensional. Indeed, if we define

$$f_n(x) = x^n - \frac{1}{n+1}$$

then

$$\int_0^1 f_n(x) = \int_0^1 x^n - \frac{1}{n+1} = \frac{1}{n+1} - \frac{1}{n+1} = 0,$$

so $f_n(x) \in U$. It is clear that the list $f_1(x), \ldots, f_n(x)$ is linearly independent. As we are able to find linearly independent lists of arbitrary length, $U$ must be infinite dimensional.
Problem 6. Prove **ONE** of the following results from class (your choice!)

(a) If $V$ is a vector space and $v_1, \ldots, v_n \in V$, then

$v_1, \ldots, v_n$ are linearly dependent $\iff$ there is some $v_i \in \text{Span}\{v_1, \ldots, v_{i-1}\}$

(b) If $V$ is a vector space and $U, W \subset V$ are subspaces such that

$U + W = V$ and $U \cap W = 0$

then $V = U \oplus W$.

**Solution:** See your lecture notes.