Exercise 1. Find the eigenvalues of \( A = \begin{bmatrix} 7 & 4 & 4 \\ -8 & -4 & 0 \\ -8 & -4 & -5 \end{bmatrix} \in M_3(\mathbb{C}) \), and find a basis for each eigenspace. Is \( A \) is diagonalizable? If so, find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

Exercise 2. Find the eigenvalues of \( A = \begin{bmatrix} 0 & 1 & 2 \\ -5 & 5 & 3 \\ -4 & 1 & 6 \end{bmatrix} \in M_3(\mathbb{C}) \), and find a basis for each eigenspace. Is \( A \) is diagonalizable? If so, find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

Exercise 3.

(a) Suppose we have an \( n \times n \) matrix

\[
P = \begin{pmatrix} p_{11}(x) & \cdots & p_{1n}(x) \\ \vdots & \ddots & \vdots \\ p_{n1}(x) & \cdots & p_{nn}(x) \end{pmatrix}
\]

whose entries are polynomials \( p_{ij}(x) \) with coefficients in \( F \). Prove that the degree of \( \det(P) \) is at most the sum of the degrees of all of its entries.

(b) Prove that the characteristic polynomial of a matrix \( A \in M_n(F) \) is monic of degree \( n \).

Exercise 4. The trace of a square matrix \( A = (a_{ij}) \in M_n(F) \) is defined as the sum of its diagonal entries: \( \text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn} \). Prove that \( \text{trace}(AB) = \text{trace}(BA) \). Use this to prove that any two similar matrices have the same trace.

Exercise 5. Suppose \( A \in M_n(F) \).

(a) Show that there is some nonzero polynomial \( f(x) \) of degree \( \deg(f) \leq n^2 \) such that \( f(A) = 0 \). Hint: use the fact that \( M_n(F) \) is a vector space of dimension \( n^2 \).

(b) Among all nonzero monic polynomials \( f(x) \) with \( f(A) = 0 \), let \( m_A(x) \) be the one of smallest degree. The polynomial \( m_A(x) \) is the minimal polynomial of \( A \). Use the division algorithm to show that for any polynomial \( f(x) \),

\[
f(A) = 0 \iff f(x) \text{ is a multiple of } m_A(x).
\]

Exercise 6. Find the minimal and characteristic polynomials of \( \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix} \) and \( \begin{bmatrix} -13 & 5 & 5 \\ -15 & 7 & 5 \\ -15 & 5 & 7 \end{bmatrix} \), and for each matrix verify that the characteristic polynomial is a multiple of the minimal polynomial. Remark: The Cayley-Hamilton theorem asserts that the characteristic polynomial is always a multiple of the minimal polynomial.