Exercise 1. Find the eigenvalues of $A = \begin{bmatrix} 7 & 4 & 4 \\ -8 & -4 & -5 \end{bmatrix} \in M_3(\mathbb{C})$, and find a basis for each eigenspace. Is $A$ is diagonalizable? If so, find an invertible matrix $C$ and a diagonal matrix $D$ such that $A = CDC^{-1}$.

Solution. The matrix $A$ has characteristic polynomial

$$c_A(x) = (x + 1)^2(x - 3),$$

and so $-1$ and $3$ are the only eigenvalues. The eigenspaces are

$$E_{-1} = \text{Span}\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$E_3 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ 

These three vectors are a basis of eigenvectors, and if we set

$$C = \begin{bmatrix} -1/2 & -1/2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

then

$$A = C \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} C^{-1}.$$ 

Exercise 2. Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 & 2 \\ -5 & 5 & 3 \\ -4 & 1 & 6 \end{bmatrix} \in M_3(\mathbb{C})$, and find a basis for each eigenspace. Is $A$ is diagonalizable? If so, find an invertible matrix $C$ and a diagonal matrix $D$ such that $A = CDC^{-1}$.

Solution. First we compute the characteristic polynomial

$$c_A(x) = (x - 3)(x - 4)^2.$$ 

The eigenvalues are 3 and 4, and the corresponding eigenspaces are

$$E_3 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$
and

\[ E_4 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}. \]

As \( \dim(E_3) + \dim(E_4) < 3 \), a result from class tells us that \( A \) is not diagonalizable. \( \Box \)

**Exercise 3.**

(a) Suppose we have an \( n \times n \) matrix

\[
P = \begin{pmatrix}
  p_{11}(x) & \cdots & p_{1n}(x) \\
  \vdots & \ddots & \vdots \\
  p_{n1}(x) & \cdots & p_{nn}(x)
\end{pmatrix}
\]

whose entries are polynomials \( p_{ij}(x) \) with coefficients in \( F \). Prove that the degree of \( \det(P) \) is at most the sum of the degrees of all of its entries.

(b) Prove that the characteristic polynomial of a matrix \( A \in M_n(F) \) is monic of degree \( n \).

**Solution.**

(a) The proof is by induction on the size of the matrix. If \( P \) is a \( 1 \times 1 \) matrix the claim is obvious. Let \( d \) be the sum of all degrees of the entries of \( P \). Let \( P_{ij} \) be the matrix obtained from \( P \) by deleting the \( i \)th row and \( j \)th column, and let \( d_{ij} \) be the sum of all the degrees of entries of \( P_{ij} \). As we deleted the polynomial \( p_{ij}(x) \) when we formed \( P_{ij} \), we must have

\[ d_{ij} + \deg(p_{ij}) \leq d_{ij} + \text{(all degrees of deleted polynomials)} = d. \]

Now compute the determinant of \( P \) by expanding along the first row, so that

\[
\det(P) = p_{11}(x) \det(P_{11}) - p_{12}(x) \det(P_{12}) + \cdots \pm p_{1n}(x) \det(P_{1n}).
\]

By the induction hypothesis each term has degree at most \( \deg(p_{1j}) + d_{1j} \leq d \), and hence the same is true of \( \det(P) \).

(b) The proof is by induction. The case of a \( 1 \times 1 \) matrix is obvious.

For the \( n \times n \) case: If \( A = (a_{ij}) \) then we can compute the characteristic polynomial

\[
c_A(x) = \begin{vmatrix}
  x - a_{11} & -a_{12} & \cdots & -a_{1n} \\
  -a_{21} & x - a_{22} & \cdots & -a_{2n} \\
  \vdots & \ddots & \ddots & \vdots \\
  -a_{n1} & -a_{n2} & \cdots & x - a_{nn}
\end{vmatrix}
\]

by expanding along the top row. Using part (a) we find that

\[
c_A(x) = (x - a_{11}) \begin{vmatrix}
  x - a_{22} & -a_{22} & \cdots & -a_{2n} \\
  -a_{31} & x - a_{32} & \cdots & -a_{3n} \\
  \vdots & \ddots & \ddots & \vdots \\
  -a_{n1} & -a_{n2} & \cdots & x - a_{nn}
\end{vmatrix} + \text{(terms of degree \( \leq n - 2 \)).}
\]

By the induction hypothesis

\[
\begin{vmatrix}
  x - a_{22} & -a_{22} & \cdots & -a_{2n} \\
  -a_{31} & x - a_{32} & \cdots & -a_{3n} \\
  \vdots & \ddots & \ddots & \vdots \\
  -a_{n1} & -a_{n2} & \cdots & x - a_{nn}
\end{vmatrix}
\]

is a monic polynomial of degree \( n - 1 \), and it follows that \( c_A(x) \) is monic of degree \( n \). \( \Box \)
Exercise 4. The trace of a square matrix $A = (a_{ij}) \in M_n(F)$ is defined as the sum of its diagonal entries: $\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$. Prove that $\text{trace}(AB) = \text{trace}(BA)$. Use this to prove that any two similar matrices have the same trace.

Solution. Write $A = (a_{ij})$ and $B = (b_{ij})$ for the entries of $A$ and $B$. If we let $X = AB$ and $Y = BA$ then $X$ and $Y$ have entries

$$x_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$$

and

$$y_{ij} = b_{i1}a_{1j} + \cdots + b_{in}a_{nj}.$$ 

Therefore

$$\text{trace}(AB) = \text{trace}(X) = x_{11} + x_{22} + \cdots + x_{nn}$$

$$= (a_{11}b_{11} + \cdots + a_{1n}b_{n1})$$

$$+ (a_{21}b_{12} + \cdots + a_{2n}b_{n2})$$

$$\vdots$$

$$+ (a_{n1}b_{1n} + \cdots + a_{nn}b_{nn})$$

$$= \sum_{k,\ell} a_{k\ell}b_{\ell k}.$$ 

and

$$\text{trace}(BA) = \text{trace}(Y) = y_{11} + y_{22} + \cdots + y_{nn}$$

$$= (b_{11}a_{11} + \cdots + b_{1n}a_{n1})$$

$$+ (b_{21}a_{12} + \cdots + b_{2n}a_{n2})$$

$$\vdots$$

$$+ (b_{n1}a_{1n} + \cdots + b_{nn}a_{nn})$$

$$= \sum_{k,\ell} b_{k\ell}a_{\ell k}.$$ 

These formulas for $\text{trace}(AB)$ and $\text{trace}(BA)$ are the same (the two sums are the same, just with the terms rearranged).

For the final claim about similar matrices having the same trace: suppose $A$ and $B$ are similar. Thus there is an invertible matrix $D$ such that $A = DBD^{-1}$. From what we proved above

$$\text{trace}(A) = \text{trace}(DBD^{-1}) = \text{trace}(D^{-1}DB) = \text{trace}(B).$$ 

Exercise 5. Suppose $A \in M_n(F)$.

(a) Show that there is some nonzero polynomial $f(x)$ of degree $\deg(f) \leq n^2$ such that $f(A) = 0$. Hint: use the fact that $M_n(F)$ is a vector space of dimension $n^2$.

(b) Among all nonzero monic polynomials $f(x)$ with $f(A) = 0$, let $m_A(x)$ be the one of smallest degree. The polynomial $m_A(x)$ is the minimal polynomial of $A$. Use the division algorithm to show that for any polynomial $f(x)$,

$$f(A) = 0 \iff f(x) \text{ is a multiple of } m_A(x).$$
Solution. (a) Recall that $M_n(F)$ is a vector space of dimension $n^2$. The list $I, A, A^2, \ldots, A^{n^2}$ has $n^2 + 1$ elements, and so these matrices must be linearly dependent. In other words, we can find a nontrivial linear relation

$$a_{n^2}A^{n^2} + a_{n^2-1}A^{n^2-1} + \cdots + a_1A + a_0I = 0.$$ 

In other words, $A$ is a zero of the polynomial

$$f(x) = a_{n^2}x^{n^2} + a_{n^2-1}x^{n^2-1} + \cdots + a_1x + a_0 = 0$$

(b) First suppose that $f(x)$ is a multiple of $m_A(x)$, so that $f(x) = m_A(x)q(x)$ for some polynomial $q(x)$. Setting $x = A$ shows that

$$f(A) = m_A(A)q(A).$$ 

As $m_A(A) = 0$, by definition of minimal polynomial, we find that $f(A) = 0$.

Now suppose that $f(A) = 0$. By the division algorithm there are polynomials $q(x)$ and $r(x)$ such that

$$f(x) = m_A(x)q(x) + r(x)$$

and $\deg(r(x)) < \deg(m_A(x))$. Setting $x = A$ shows that $0 = r(A)$. Among all nonzero polynomials having $A$ as a zero, $m_A(x)$ has the smallest possible degree. As $r(x)$ is a polynomial of lower degree having $A$ as a zero, the only possibility is that $r(x) = 0$. Thus $f(x) = m_A(x)q(x)$.

Exercise 6. Find the minimal and characteristic polynomials of $\begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$ and $\begin{bmatrix} -13 & 5 & 5 \\ -15 & 7 & 5 \\ -15 & 5 & 7 \end{bmatrix}$, and for each matrix verify that the characteristic polynomial is a multiple of the minimal polynomial. Remark: The Cayley-Hamilton theorem asserts that the characteristic polynomial is always a multiple of the minimal polynomial.

Solution. The minimal and characteristic polynomials of $A = \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$ are

$$m_A(x) = x^2 - 9x + 26$$
$$c_A(x) = x^2 - 9x + 26,$$

and so certainly $c_A(x)$ is a multiple of $m_A(x)$.

The minimal and characteristic polynomials of $A = \begin{bmatrix} -13 & 5 & 5 \\ -15 & 7 & 5 \\ -15 & 5 & 7 \end{bmatrix}$, are

$$m_A(x) = x^2 + x - 6$$
$$c_A(x) = x^3 - x^2 - 8x + 12,$$

and

$$c_A(x) = m_A(x) \cdot (x - 2).$$