Problem set #1

Due January 29, 2018

Exercise 1. Prove there is no $x \in \mathbb{Q}$ satisfying $x^3 = 5$.

Solution. To get a contradiction, suppose there is an $x \in \mathbb{Q}$ satisfying $x^3 = 5$. If we write $x = a/b$ for positive integers $a$ and $b$, then

$$a^3 = 5b^3.$$

In the prime factorization of $a^3$ the prime 5 appears three times as often as it appears in the prime factorization of $a$, and in particular the number of times 5 appears in the prime factorization of $a^3$ is a multiple of 3.

By similar reasoning, the number of times 5 appears in the prime factorization of $b^3$ is also a multiple of 3, and the number of times that 5 appears in the prime factorization of $5b^3$ is therefore one more than a multiple of three. This contradicts $a^3 = 5b^3$. \hfill \Box

Exercise 2. Use induction to prove

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Solution. When $n = 1$ the desired equality is

$$1^3 = \frac{1^2 \cdot (1 + 1)^2}{4},$$

which is obvious. Now suppose we are given a $k \in \mathbb{Z}^+$ such that

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}.$$
From this we compute
\[
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k + 1)^3 = \frac{k^2(k + 1)^2}{4} + (k + 1)^3
\]
\[
= \frac{k^4 + 2k^3 + k^2}{4} + \frac{4k^3 + 12k^2 + 12k + 4}{4}
\]
\[
= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}
\]
\[
= \frac{(k^2 + 2k + 1)(k^2 + 4k + 4)}{4}
\]
\[
= \frac{(k + 1)^2(k + 2)^2}{4}.
\]

The claim now follows by induction. \(\square\)

**Exercise 3.** Derive a formula for \(\sum_{i=1}^{n} i^4\).

**Solution.** First we expand
\[
(x + 1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1.
\]
Rewrite this as
\[
(x + 1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1,
\]
let \(x\) vary over the set \(\{1, 2, \ldots, n\}\), and add together the resulting system of equations
\[
2^5 - 1^5 = 5 \cdot 1^4 + 10 \cdot 1^3 + 10 \cdot 1^2 + 5 \cdot 1 + 1
\]
\[
3^5 - 2^5 = 5 \cdot 2^4 + 10 \cdot 2^3 + 10 \cdot 2^2 + 5 \cdot 2 + 1
\]
\[
4^5 - 3^5 = 5 \cdot 3^4 + 10 \cdot 3^3 + 10 \cdot 3^2 + 5 \cdot 3 + 1
\]
\[
\vdots
\]
\[
(n - 1)^5 - (n - 2)^5 = 5 \cdot (n - 2)^4 + 10 \cdot (n - 2)^3 + 10 \cdot (n - 2)^2 + 5 \cdot (n - 2) + 1
\]
\[
n^5 - (n - 1)^5 = 5 \cdot (n - 1)^4 + 10 \cdot (n - 1)^3 + 10 \cdot (n - 1)^2 + 5 \cdot (n - 1) + 1
\]
\[
(n + 1)^5 - n^5 = 5 \cdot n^4 + 10 \cdot n^3 + 10 \cdot n^2 + 5 \cdot n + 1
\]

to get
\[
(n + 1)^5 - 1 = 5 \sum_{i=1}^{n} i^4 + 10 \sum_{i=1}^{n} i^3 + 10 \sum_{i=1}^{n} i^2 + 5 \sum_{i=1}^{n} i + n.
\]
Now substitute in the known formulas
\[
\begin{align*}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4}
\end{align*}
\]
to get
\[
(n+1)^5 - 1 = 5 \sum_{i=1}^{n} i^4 + \frac{5n^2(n+1)^2}{2} + \frac{5n(n+1)(2n+1)}{3} + \frac{5n(n+1)}{2} + n,
\]
and rewrite this as
\[
\sum_{i=1}^{n} i^4 = \frac{(n+1)^5 - 1}{5} - \frac{n^2(n+1)^2}{2} - \frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2} - \frac{n}{5}
\]
\[
= \frac{6n^5 + 15n^4 + 10n^3 - n}{30}
\]
\[
= \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}.
\]

Exercise 4. Prove that for every \( n \in \mathbb{Z}^+ \)
\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.
\]

Solution. The proof is by induction. The case \( n = 1 \) asserts that \( 1 \leq 2 - 1 \), which is obvious. Now assume we are given a \( k \in \mathbb{Z}^+ \) for which
\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}.
\]
Adding \( 1/(1+k)^2 \) to both sides results in
\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(1+k)^2} \leq 2 - \frac{1}{k} + \frac{1}{(1+k)^2}
\]
\[
\leq 2 - \frac{1}{k} + \frac{1}{k(1+k)}
\]
\[
= 2 - \frac{(1+k)}{k(1+k)} + \frac{1}{k(1+k)}
\]
\[
= 2 - \frac{k}{k(1+k)}
\]
\[
= 2 - \frac{1}{1+k}.
\]
Therefore
\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(1+k)^2} \leq 2 - \frac{1}{k+1}
\]
and we are done. \qed

**Exercise 5.** Let
\[
\alpha = \frac{1 + \sqrt{5}}{2} \quad \beta = \frac{1 - \sqrt{5}}{2}
\]
be the two roots of the quadratic equation \(x^2 - x - 1 = 0\). If \(n \geq 1\), prove that the \(n\)th Fibonacci number satisfies
\[
f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.
\]

**Solution.** The proof is by strong induction. Let \(P(n)\) be the statement
\[
f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.
\]
The first statement \(P(1)\) asserts that
\[
f_1 = \frac{\alpha^1 - \beta^1}{\alpha - \beta},
\]
which is true, as both sides are equal to 1. Now suppose that \(P(1), P(2), \ldots, P(k)\) are all true, so that
\[
\begin{align*}
f_1 &= \frac{\alpha^1 - \beta^1}{\alpha - \beta} \\
f_2 &= \frac{\alpha^2 - \beta^2}{\alpha - \beta} \\
\vdots \\
f_{k-1} &= \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta} \\
f_k &= \frac{\alpha^k - \beta^k}{\alpha - \beta}.
\end{align*}
\]
We must deduce that \(P(k + 1)\) is also true; i.e. that
\[
f_{k+1} = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}.
\]
The relation \(f_{k+1} = f_k + f_{k-1}\) implies
\[
\begin{align*}
f_{k+1} &= f_k + f_{k-1} \\
&= \frac{\alpha^k - \beta^k}{\alpha - \beta} + \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta} \\
&= \frac{\alpha^{k-1}(\alpha + 1) - \beta^{k-1}(\beta + 1)}{\alpha - \beta}.
\end{align*}
\]
Each of $\alpha$ and $\beta$ is a root of $x^2 - x - 1 = 0$, and hence
\[
\alpha^2 = \alpha + 1 \quad \text{and} \quad \beta^2 = \beta + 1.
\]
Thus the above expression for $f_{k+1}$ simplifies to
\[
f_{k+1} = \frac{\alpha^{k-1}(\alpha + 1) - \beta^{k-1}(\beta + 1)}{\alpha - \beta}
\]
\[
= \frac{\alpha^{k-1}(\alpha^2) - \beta^{k-1}(\beta^2)}{\alpha - \beta}
\]
\[
= \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}
\]
as desired.

\[\square\]

Exercise 6. Suppose $n \geq 0$ is an integer.

(a) Using l'Hôpital’s rule and induction, prove that
\[
\lim_{x \to \infty} \frac{x^n}{e^x} = 0.
\]
You may assume without proof that the formula is true when $n = 0$.

(b) Using induction to prove that
\[
\int_0^\infty x^n e^{-x} \, dx = n!
\]
(hint: use (a) and integration by parts).

Solution. (a) We prove the claim using induction. The base case $n = 0$ is
\[
\lim_{x \to \infty} e^{-x} = 0,
\]
which is given. Now assume that $\lim_{x \to \infty} \frac{x^k}{e^x} = 0$ for some $k \geq 0$. Applying l'Hôpital’s rule we see that
\[
\lim_{x \to \infty} \frac{x^{k+1}}{e^x} = \lim_{x \to \infty} \frac{(k+1)x^k}{e^x} = (k+1) \lim_{x \to \infty} \frac{x^k}{e^x} = (k+1) \cdot 0 = 0,
\]
and by induction we’re done.

(b) Again the proof is by induction. When $n = 0$ we compute
\[
\int_0^\infty x^0 e^{-x} \, dx = \int_0^\infty e^{-x} \, dx = -e^{-x}\bigg|_0^\infty = 0 - (-1) = 1.
\]
For the inductive step we assume that
\[
\int_0^\infty x^{n-1} e^{-x} \, dx = (n-1)!
\]
and try to prove
\[ \int_0^\infty x^n e^{-x} \, dx = n! \]  

(1)

We compute the left hand side of (1) using integration by parts. Setting
\[ u = x^n, \quad dv = e^{-x} \, dx, \quad du = nx^{n-1} \, dx, \quad v = -e^{-x} = -1/e^x, \]
we obtain
\[ \int_0^\infty x^n e^{-x} \, dx = \left. \frac{x^n}{e^x} \right|_0^\infty + n \int_0^\infty x^{n-1} e^{-x} \, dx. \]

Now, at \( x = 0 \), we can see that \( x^n e^{-x} \) is 0. Part (a) showed that
\[ \lim_{x \to \infty} \frac{x^n}{e^x} = 0, \]
and so the above simplifies to
\[ \int_0^\infty x^n e^{-x} \, dx = 0 + n \int_0^\infty x^{n-1} e^{-x} \, dx = n \cdot (n-1)! = n! \]
as desired. \( \square \)