Problem set #3

Due February 12, 2018

Exercise 1. For each of the following subsets $U \subset \mathbb{C}^3$, determine whether or not $U$ is a subspace:

(a) $U = \{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 0\}$
(b) $U = \{(x_1, x_2, x_3) : x_1 + 2x_2 + 3x_3 = 4\}$
(c) $U = \{(x_1, x_2, x_3) : x_1x_2x_3 = 0\}$
(d) $U = \{(x_1, x_2, x_3) : x_1 = 5x_3\}$
(e) $U = \{(x_1, x_2, x_3) : x_1^3 = x_3^3\}$

Solution.

(a) This is a subspace: certainly $(0, 0, 0) \in U$. If $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ are in $U$ then

$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0 + 0 = 0,$$

which shows that $(x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U$. Finally, if $(x_1, x_2, x_3) \in U$ and $c \in \mathbb{C}$, then

$$(cx_1) + 2(cx_2) + 3(cx_3) = c \cdot (x_1 + 2x_2 + 3x_3) = c \cdot 0 = 0.$$ 

This proves that $c \cdot (x_1, x_2, x_3) \in U$.

(b) This is not a subspace, as the zero vector $(0, 0, 0) \in \mathbb{R}^3$ is not in $U$.

(c) This is not a subspace, as it isn’t closed under addition: the vectors $(1, 0, 0)$ and $(0, 1, 1)$ are in $U$, but their sum $(1, 1, 1)$ isn’t.

(d) This is a subspace. Clearly $0 \in U$. If $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ are in $U$ then

$$x_1 + y_1 = 5x_3 + 5y_3 = 5(x_3 + y_3),$$

which shows that

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U.$$ 

Finally, if $(x_1, x_2, x_3) \in U$ and $c \in \mathbb{C}$, then $cx_1 = 5cx_3$. This proves that $c \cdot (x_1, x_2, x_3) \in U$.

(e) This is not a subspace, as it isn’t closed under addition: if we set $\xi = e^{2\pi i/3}$ then $\xi^3 = 1$. Thus both $(\xi, 0, 1)$ and $(-1, 0, -1)$ are in $U$, but the sum $(\xi - 1, 0)$ is not in $U$.

Exercise 2. Let $P$ be the $\mathbb{R}$-vector space of all polynomials with real coefficients. Show that $U = \{f \in P : f'(-1) = 3f(2)\}$ is a subspace of $P$. Remark: the notation $f'(x)$ means the derivative of $f(x)$, in the usual sense of calculus.
Solution. First, let \( f(x) = 0 \) be the zero polynomial. Then \( f'(x) = 0 \), and hence
\[
f'(-1) = 0 = 3 \cdot 0 = f(2).
\]
Thus \( f(x) \in U \). Now we check closure under addition. If \( f_1(x), f_2(x) \in P \) and we set \( g(x) = f_1(x) + f_2(x) \) then
\[
g'(-1) = f_1'(-1) + f_2'(-1) = 3f_1(2) + 3f_2(2) = 3g(2).
\]
This proves that \( g(x) \in P \). Finally, check closure under scalar multiplication. Suppose \( f(x) \in P \) and \( c \in \mathbb{R} \). If we set \( g(x) = cf(x) \) then
\[
g'(-1) = cf'(-1) = cf(2) = g(2).
\]
This proves that \( g(x) \in P \). \( \square \)

Exercise 3. Prove that the only subspaces of \( \mathbb{R}^1 \) are the zero subspace and all of \( \mathbb{R}^1 \).

Solution. Suppose \( U \subset \mathbb{R} \) is a nonzero subspace. Pick any nonzero vector \( u \in U \). As \( U \) is closed under scalar multiplication, we must have
\[
x = \frac{x}{u} \cdot u \in U
\]
for any real number \( x \). This proves that \( U \) is all of \( \mathbb{R} \). \( \square \)

Exercise 4.

(a) Let \( U = \{(x, y, y - x) \in \mathbb{R}^3\} \). Find a subspace \( W \subset \mathbb{R}^3 \) such that \( \mathbb{R}^3 = U \oplus W \).

(b) Let \( U = \{(x, x, x, y, y) \in \mathbb{R}^5\} \). Find nonzero subspaces \( W_1, W_2 \subset \mathbb{R}^5 \) such that \( \mathbb{R}^5 = U \oplus W_1 \oplus W_2 \).

Solution. (a) There are many possible solutions. In fact, you can take any line in \( \mathbb{R}^3 \) that intersects \( U \) only at \( 0 \). For example
\[
W = \{(a, 0, 0) \in \mathbb{R}^3\}.
\]
Then any vector in \( \mathbb{R}^3 \) decomposes as
\[
(x, y, z) = (z - y, y, z) + (x - z + y, 0, 0) \in U + W,
\]
and so \( \mathbb{R}^3 = U + W \). On the other hand any vector in \( U \cap W \) has the form
\[
(x, y, y - x) = (a, 0, 0).
\]
This implies that \( x = a, \ y = 0, \) and \( y - x = 0 \), from which it is clear that \( a = x = y = 0 \).
Thus \( U \cap W = 0 \), and we conclude that \( \mathbb{R}^3 = U \oplus W \).

(b) There are many possible solutions. One is
\[
W_1 = \{(0, b, c, 0, 0) \in \mathbb{R}^5\}, \quad W_2 = \{(0, 0, 0, d, 0) \in \mathbb{R}^5\}.
\]
Then any vector in \( \mathbb{R}^5 \) decomposes as
\[
(x_1, x_2, x_3, x_4, x_5) = (x_1, x_1, x_5, x_5) + (0, x_2 - x_1, x_3 - x_1, 0) + (0, 0, 0, x_4 - x_5, 0),
\]
which shows that $\mathbb{R}^5 = U + W_1 + W_2$. To see that the sum is direct, suppose
\[(0, 0, 0, 0, 0) = (x, x, y, y) + (0, b, c, 0, 0, 0) + (0, 0, 0, d, 0).\]
This is equivalent to
\[(0, 0, 0, 0) = (x + b, x + c, y + d, y)\]
and so
\[0 = x, \quad 0 = x + b, \quad 0 = x + c, \quad 0 = y + d, \quad 0 = y.\]
It is easy to see from these that $x = y = b = c = d = 0$. \hfill \Box

**Exercise 5.** Let $V$ be an $F$-vector space with subspaces $U_1, U_2 \subset V$.

(a) If there is a subspace $W \subset V$ such that $V = U_1 \oplus W$ and $V = U_2 \oplus W$, does it follow that $U_1 = U_2$? Prove or provide a counterexample.

(b) If $U_1 \cup U_2$ is a subspace, does it follow that either $U_1 \subset U_2$ or $U_2 \subset U_1$? Prove or provide a counterexample.

**Solution.** (a) This is false. For example in $V = \mathbb{R}^2$ we can take
\[W = \{(x, 0) \in \mathbb{R}^2\}, \quad U_1 = \{(0, y) \in \mathbb{R}^2\}, \quad \{(x, x) \in \mathbb{R}^2\}.\]
Then $\mathbb{R}^3 = U_1 \oplus W$ and $U_2 \oplus W$, but clearly $U_1 \neq U_2$.

(b) This is true. Assume that $U_1 \cup U_2$ is a subspace. If $U_1 \subset U_2$ then we are done, so we may assume that $U_1 \not\subset U_2$. Pick some $u_1 \in U_1$ such that $u_1 \not\in U_2$. To prove that $U_2 \subset U_1$, suppose $u_2 \in U_2$. As $u_1, u_2 \in U_1 \cup U_2$ and $U_1 \cup U_2$ is closed under addition, we also have
\[u_1 + u_2 \in U_1 \cup U_2.\]
This means that either $u_1 + u_2 \in U_1$ or $u_1 + u_2 \in U_2$. The second case cannot occur, as then
\[u_1 = (u_1 + u_2) - (u_2) \in U_2\]
contradicting our choice of $u_1$. Thus the first case must occur: $u_1 + u_2 \in U_1$. But this implies that
\[u_2 = (u_1 + u_2) - u_1 \in U_1,\]
which proves that $U_2 \subset U_1$. \hfill \Box

**Exercise 6.** Let $P$ be the space of all polynomials with real coefficients. We say that $f \in P$ is even if $f(-x) = f(x)$, and odd if $f(-x) = -f(x)$. Show that
\[U_0 = \{ f \in V : f \text{ is even} \}, \quad U_1 = \{ f \in V : f \text{ is odd} \}\]
are subspaces of $P$, and that $P = U_0 \oplus U_1$.

**Solution.** First we check that $U_0$ and $U_1$ are subspaces. The zero polynomial $f(x) = 0$ satisfies $f(-x) = 0 = f(x)$ so is even. If $f(x)$ and $g(x)$ are even then $f(-x) + g(-x) = f(x) + g(x)$, and so $f + g$ is even. Finally if $f(x)$ is even and $c \in \mathbb{R}$ then $cf(-x) = cf(x)$, so $cf(x)$ is also even. This shows that $U_0$ is a subspace, and the proof that $U_1$ is a subspace is similar.
To show that $P = U_0 \oplus U_1$, we first need to check that $P = U_0 + U_1$. Given any $f(x) \in P$, define new polynomials

$$f_0(x) = \frac{f(x) + f(-x)}{2}, \quad f_1(x) = \frac{f(x) - f(-x)}{2}.$$ 

Then

$$f_0(-x) = \frac{f(-x) + f(x)}{2} = f_0(x)$$

and

$$f_1(-x) = \frac{f(-x) - f(x)}{2} = -f_1(x).$$

This proves that $f_0(x) \in U_0$ and $f_1(x) \in U_1$. Moreover,

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f_0(x) + f_1(x)$$

shows that $P = U_0 + U_1$.

Here is a different proof of the same claim: write $f(x) = a_n x^n + \cdots + a_1 x + a_0$ and set

$$f_0(x) = \sum_{k \text{ even}} a_k x^k, \quad f_1(x) = \sum_{k \text{ odd}} a_k x^k.$$ 

Once again we have $f_0(x) \in U_0$ and $f_1(x) \in U_1$, and $f(x) = f_0(x) + f_1(x)$. Thus $P = U_0 + U_1$.

To check that the sum is direct, it now suffices to show that $U_0 \cap U_1 = 0$. Suppose

$$f(x) \in U_1 \cap U_2.$$ 

This implies that $f(x)$ satisfies both $f(-x) = f(x)$ and $f(-x) = -f(x)$, and hence

$$f(x) = f(-x) = -f(x).$$

This implies that $f(x) = 0.$
Exercise 7. Let $U = \{(x, y) : x, y \in \mathbb{R}\} \subset \mathbb{R}^3$. Find a subspace $W \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$.

Solution. There are many possible solutions. In fact, you can take $W$ to be any line in $\mathbb{R}^3$ that isn’t contained in $U$. One example would be

$$W = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}.$$  

As any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ decomposes as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ z \end{bmatrix} + \begin{bmatrix} x - y \\ 0 \\ 0 \end{bmatrix} \in U + W,$$

we have $\mathbb{R}^3 = U + W$. To check that the sum is direct it suffices, by a result proved in class, to show that $U \cap W = \{0\}$. Any vector

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in U \cap W$$

must satisfy the relations $x = y$ (because $v \in U$) and $y = z = 0$ (because $v \in W$). Thus $x = y = z = 0$, and so $v = 0$.

Exercise 8. Let $U = \{(x, x, y, y) : x, y \in \mathbb{R}\} \subset \mathbb{R}^4$. Find a subspace $W \subset \mathbb{R}^4$ such that $\mathbb{R}^4 = U \oplus W$.

Solution. There are many possible solutions. In fact, you can take $W$ to be any plane in $\mathbb{R}^4$ that intersects $U$ only at 0. One example would be

$$W = \left\{ \begin{bmatrix} s \\ 0 \\ 0 \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$  

As any vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$ decomposes as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} x_1 - x_2 \\ 0 \\ 0 \\ x_4 - x_3 \end{bmatrix} \in U + W,$$

we have $\mathbb{R}^4 = U + W$. To check that the sum is direct it suffices, by a result proved in class, to show that $U \cap W = \{0\}$. Any vector

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in U \cap W$$

must satisfy the relations $x_1 = x_2$ and $x_3 = x_4$ (because $v \in U$) and $x_2 = x_3 = 0$ (because $v \in W$). Thus $x_1 = x_2 = x_3 = x_4 = 0$, and so $v = 0$.  

\qed
Exercise 9. Suppose \( U_1, U_2 \subset V \) are subspaces. Prove

\[
U_1 \cup U_2 \text{ is a subspace} \iff \text{either } U_1 \subset U_2 \text{ or } U_2 \subset U_1.
\]

Solution. First we prove \((\Leftarrow)\). If \( U_1 \subset U_2 \) then \( U_1 + U_2 = U_2 \), and hence \( U_1 + U_2 \) is a subspace. Similarly, if \( U_2 \subset U_1 \) then \( U_1 + U_2 = U_1 \), and hence \( U_1 + U_2 \) is a subspace.

The implication \((\Rightarrow)\) is trickier. Here is one approach: Assume that \( U_1 \cup U_2 \) is a subspace. If \( U_1 \subset U_2 \) then we are done, so we may assume that \( U_1 \not\subset U_2 \). Pick some \( u_1 \in U_1 \) such that \( u_1 \notin U_2 \). To prove that \( U_2 \subset U_1 \), suppose \( u_2 \in U_2 \). As \( u_1, u_2 \in U_1 \cup U_2 \) and \( U_1 \cup U_2 \) is closed under addition, we also have

\[
u_1 + u_2 \in U_1 \cup U_2.
\]

This means that either \( u_1 + u_2 \in U_1 \) or \( u_1 + u_2 \in U_2 \). The second case cannot occur, as then

\[
u_1 = (u_1 + u_2) - (u_2) \in U_2
\]

contradicting our choice of \( u_1 \). Thus the first case must occur: \( u_1 + u_2 \in U_1 \). But this implies that

\[
u_2 = (u_1 + u_2) - u_1 \in U_1,
\]

which proves that \( U_2 \subset U_1 \).

\(\square\)

Exercise 10. Let \( P \) be the space of all polynomials with real coefficients. We say that \( f \in P \) is \textit{even} if \( f(-x) = f(x) \), and \textit{odd} if \( f(-x) = -f(x) \). Show that

\[
U_0 = \{ f \in V : f \text{ is even} \}, \quad U_1 = \{ f \in V : f \text{ is odd} \}
\]

are subspaces of \( P \), and that \( P = U_0 \oplus U_1 \).

Solution. First we need to check that \( P = U_0 + U_1 \). Given any \( f(x) \in P \), define new polynomials

\[
f_0(x) = \frac{f(x) + f(-x)}{2}, \quad f_1(x) = \frac{f(x) - f(-x)}{2}.
\]

Then

\[
f_0(-x) = \frac{f(-x) + f(x)}{2} = f_0(x)
\]

and

\[
f_1(-x) = \frac{f(-x) - f(x)}{2} = -f_1(x).
\]

This proves that \( f_0(x) \in U_0 \) and \( f_1(x) \in U_1 \). Moreover,

\[
f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f_0(x) + f_1(x)
\]

shows that \( P = U_0 + U_1 \).

Here is a different proof of the same claim: write \( f(x) = a_n x^n + \cdots + a_1 x + a_0 \) and set

\[
f_0(x) = \sum_{k \text{ even}} a_k x^k, \quad f_1(x) = \sum_{k \text{ odd}} a_k x^k.
\]

Once again we have \( f_0(x) \in U_0 \) and \( f_1(x) \in U_1 \), and \( f(x) = f_0(x) + f_1(x) \). Thus \( P = U_0 + U_1 \).
To check that the sum is direct, it now suffices to show that \( U_0 \cap U_1 = \{0\} \). Suppose

\[ f(x) \in U_1 \cap U_2. \]

This implies that \( f(x) \) satisfies both \( f(-x) = f(x) \) and \( f(-x) = -f(x) \), and hence

\[ f(x) = f(-x) = -f(x). \]

This implies that \( f(x) = 0. \)

\[ \square \]