Problem set #6

Due March 19, 2018

In all of the following exercises $F$ denotes $\mathbb{R}$ or $\mathbb{C}$, and $V$ and $W$ are $F$-vector spaces.

**Exercise 1.** Suppose $T : V \to W$ is a linear map, and $v_1, \ldots, v_n \in V$. For each statement give a proof or a counterexample.

(a) If $v_1, \ldots, v_n$ are linearly independent, then $T(v_1), \ldots, T(v_n)$ are linearly independent.

(b) If $T(v_1), \ldots, T(v_n)$ are linearly independent, then $v_1, \ldots, v_n$ are linearly independent.

**Exercise 2.** Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map corresponding to

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$ 

Find all solutions to $T(x) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

**Exercise 3.** Show that there is a unique linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and find the corresponding $3 \times 3$ matrix.

**Exercise 4.** Suppose $V$ is finite dimensional with basis $v_1, \ldots, v_n$. Define linear maps $T_1, \ldots, T_n \in \text{Hom}(V, F)$ as follows: if $v = c_1 v_1 + \cdots + c_n v_n$ then

$$T_1(v) = c_1, \quad \vdots, \quad T_n(v) = c_n.$$ 

Prove that $T_1, \ldots, T_n$ is a basis of $\text{Hom}(V, F)$. (Do not assume that $\text{Hom}(V, F)$ has dimension $n$. We have stated this in class, but have not yet proved it.)

**Exercise 5.** Suppose $V$ is finite dimensional and $U \subset V$ is a subspace. Show that any linear map $T : U \to W$ can be extended to a linear map defined on all of $V$. In other words, show that there is a linear map $T' : V \to W$ such that $T(u) = T'(u)$ for all $u \in U$.

**Exercise 6.** Suppose $V$ is finite dimensional and $W$ is infinite dimensional. Show that $\text{Hom}(V, W)$ is infinite dimensional.