Problem set #6

Due March 19, 2018

In all of the following exercises $F$ denotes $\mathbb{R}$ or $\mathbb{C}$, and $V$ and $W$ are $F$-vector spaces.

Exercise 1. Suppose $T : V \rightarrow W$ is a linear map, and $v_1, \ldots, v_n \in V$. For each statement give a proof or a counterexample.

(a) If $v_1, \ldots, v_n$ are linearly independent, then $T(v_1), \ldots, T(v_n)$ are linearly independent.

(b) If $T(v_1), \ldots, T(v_n)$ are linearly independent, then $v_1, \ldots, v_n$ are linearly independent.

Solution. (a) This is false. Take $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to be the 0 map. Then $e_1, e_2$ are linearly independent, but $T(e_1) = 0$ and $T(e_2) = 0$ are not.

(b) This is true. Assume $T(v_1), \ldots, T(v_n)$ are linearly independent. Given any linear relation

$$a_1v_1 + \cdots + a_nv_n = 0$$

we may apply $T$ to both sides to obtain $a_1T(v_1) + \cdots + a_nT(v_n) = 0$. The linear independence of $T(v_1), \ldots, T(v_n)$ now implies that $a_1 = \cdots = a_n = 0$. Thus $v_1, \ldots, v_n$ are linearly independent. \qed

Exercise 2. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be the linear map corresponding to

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Find all solutions to

$$T(x) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution. By row reduction

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 | 1 \\ 2 & 1 & 0 & 0 & 2 | 0 \\ -1 & 0 & 1 & 1 & 1 | 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1/2 & 2 | 0 \\ 0 & 1 & 0 & -1 & -2 | 0 \\ 0 & 0 & 1 & 3/2 & 3 | 1 \end{bmatrix}$$

we see that the solution set is the same as

$$x_1 + \frac{1}{2}x_4 + 2x_5 = 0$$

$$x_2 - x_4 - 2x_5 = 0$$

$$x_3 + \frac{3}{2}x_4 + 3x_5 = 1.$$
The general solution is
\[
\begin{bmatrix}
-\frac{1}{2}x_4 - 2x_5 \\
x_4 + 2x_5 \\
1 - \frac{3}{2}x_4 - 3x_5 \\
x_4 \\
x_5
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0
\end{bmatrix} + x_4 
\begin{bmatrix}
-\frac{1}{2} \\
1 \\
-\frac{3}{2} \\
1 \\
0
\end{bmatrix} + x_5 
\begin{bmatrix}
-2 \\
2 \\
-3 \\
1 \\
1
\end{bmatrix}
\]

\[\square\]

**Exercise 3.** Show that there is a unique linear map \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) satisfying
\[
T \left( \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad T \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad T \left( \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]
and find the corresponding \( 3 \times 3 \) matrix.

**Solution.** The row reduction
\[
\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
says that the vectors \( \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \), \( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \), \( \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \) are linearly independent, and so, by a result from class, there is a unique linear map \( T \) as in the statement of the exercise.

To find the corresponding matrix, use the relations
\[
\begin{align*}
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\end{align*}
\]
to compute
\[
T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \frac{1}{2} T \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) + \frac{1}{2} T \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = T \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) - \frac{1}{2} T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = T \left( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) - \frac{3}{2} T \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) + \frac{1}{2} T \left( \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}.
\]

This shows that the matrix is
\[
A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.
\]
\[\square\]
**Exercise 4.** Suppose $V$ is finite dimensional with basis $v_1, \ldots, v_n$. Define linear maps $T_1, \ldots, T_n \in \text{Hom}(V, F)$ as follows: if $v = c_1 v_1 + \cdots + c_n v_n$ then

$$T_1(v) = c_1,$$

$$\vdots$$

$$T_n(v) = c_n.$$

Prove that $T_1, \ldots, T_n$ is a basis of $\text{Hom}(V, F)$. (Do not assume that $\text{Hom}(V, F)$ has dimension $n$. We have stated this in class, but have not yet proved it.)

**Solution.** First we show that $T_1, \ldots, T_n$ are linearly independent. Suppose

$$0 = a_1 T_1 + \cdots + a_n T_n.$$

Using

$$T_1(v_1) = 1, \quad T_2(v_1) = 0, \quad T_3(v_1) = 0, \ldots, T_n(v_1) = 0,$$

we find that

$$0 = a_1 T_1(v_1) + a_2 T_2(v_1) + a_3 T_3(v_1) + \cdots + a_n T_n(v_1) = a_1 + 0 + \cdots + 0.$$

Repeating the argument with $v_1$ replaced by $v_2$ shows that $0 = a_2$, and so on.

Next we show that $T_1, \ldots, T_n$ span $\text{Hom}(V, F)$. Fix some $T \in \text{Hom}(V, F)$ and define $a_1 = T(v_1), \ldots, a_n = T(v_n)$. The claim is that

$$a_1 T_1 + \cdots + a_n T_n = T.$$

By a result from class, it suffices to show that these two linear maps agree on the basis vectors $v_1, \ldots, v_n$. But this is clear from

$$a_1 T_1(v_1) + a_2 T_2(v_1) + a_3 T_3(v_1) + \cdots + a_n T_n(v_1) = a_1 + 0 + \cdots + 0 = T(v_1),$$

and similarly with $v_1$ replaced by $v_2$ and so on. \(\square\)

**Exercise 5.** Suppose $V$ is finite dimensional and $U \subset V$ is a subspace. Show that any linear map $T : U \to W$ can be extended to a linear map defined on all of $V$. In other words, show that there is a linear map $T' : V \to W$ such that $T(u) = T'(u)$ for all $u \in U$.

**Solution.** Fix a basis $u_1, \ldots, u_k \in U$, and extend it to a basis $u_1, \ldots, u_k, v_1, \ldots, v_\ell \in V$. By a result from class there is a unique linear map $T' : V \to W$ satisfying

$$T'(u_1) = T(u_1),$$

$$\vdots$$

$$T'(u_k) = T(u_k),$$

$$T'(v_1) = 0,$$

$$\vdots$$

$$T'(v_\ell) = 0.$$

As $T'$ and $T$ agree on a basis of $U$, they agree on all of $U$ by a result from class. \(\square\)
Exercise 6. Suppose $V$ is finite dimensional and $W$ is infinite dimensional. Show that $\text{Hom}(V, W)$ is infinite dimensional.

Solution. The assumption that $W$ is infinite dimensional implies that there is an infinite sequence of vectors $w_1, w_2, \ldots \in W$ such that $w_1, \ldots, w_n$ is linearly independent for every choice of $n$.

Fix any nonzero vector $v \in V$, and extend it to a basis $v, v_1, \ldots, v_k \in V$. For every $n \in \mathbb{Z}_{>0}$ define a linear map $T_n : V \to W$ by

$$T_n(v) = w_n, \quad T_n(v_1) = 0, \ldots, T_n(v_k) = 0.$$ 

This defines an infinite sequence $T_1, T_2, \ldots \in \text{Hom}(V, W)$.

I claim that $T_1, \ldots, T_n$ are linearly independent. Indeed, given any linear relation

$$0 = a_1 T_1 + \cdots + a_n T_n$$

we may evaluate both sides at the vector $v$ to obtain

$$0 = (a_1 T_1 + \cdots + a_n T_n)(v) = a_1 w_1 + \cdots + a_n w_n.$$ 

The linear independence of $w_1, \ldots, w_n$ now implies that $a_1, \ldots, a_n$.

The existence of the infinite sequence $T_1, T_2, \ldots \in \text{Hom}(V, W)$ proves that $\text{Hom}(V, W)$ is infinite dimensional. \qed