Problem set #8

Due WEDNESDAY April 4, 2018

In all of the following exercises $F$ denotes $\mathbb{R}$ or $\mathbb{C}$, and $V$ and $W$ are $F$-vector spaces. Let $P_n$ be the space of polynomials with real coefficients of degree $\leq n$.

**Exercise 1.** Suppose $X$ and $Y$ are sets, and $f : X \to Y$ and $g : Y \to X$ are functions.

(a) If $g(f(x)) = x$ for all $x \in X$, does it follow that $f$ is injective? What about surjective?
(b) If $f(g(y)) = y$ for all $y \in Y$, does it follow that $f$ is injective? What about surjective?

(Give proofs or counterexamples for all claims.)

**Solution.** (a) Suppose $g(f(x)) = x$ for all $x \in X$. The function $f$ is injective, as $f(x_1) = f(x_2) \implies g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$.

However, $f$ need not be surjective. For example, define $f : \mathbb{R} \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = (x,0)$, $g(x,y) = x$.

Then $g(f(x)) = g(x,0) = x$, but obviously $f$ is not surjective.

(b) Now suppose that $f(g(y)) = y$ for all $y \in Y$. It is clear the $f$ is surjective, because the relation $f(g(y)) = y$ already implies that every $y \in Y$ is in the image of $f$. However, $f$ need not be injective; for example we could define $g : \mathbb{R} \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ by $g(x) = (x,0)$, $f(x,y) = x$.

Then $f(g(x)) = f(x,0) = x$, but obviously $f$ is not injective. 

**Exercise 2.** Suppose $T \in \text{Hom}(V,W)$ is a linear map, and $\phi : F^m \to V$ and $\psi : F^m \to W$ are isomorphisms. Set $A = \psi^{-1} \circ T \circ \phi \in \text{Hom}(F^n,F^m)$.

(a) If $x_1, \ldots, x_s$ is a basis for $\ker(A)$, show that $\phi(x_1), \ldots, \phi(x_s)$ is a basis for $\ker(T)$.
(b) If $y_1, \ldots, y_t$ is a basis for $\text{Im}(A)$, show that $\psi(y_1), \ldots, \psi(y_t)$ is a basis for $\text{Im}(T)$.

**Solution.** (a) Using the relation $\psi \circ A = T \circ \phi$ we see that $T(\phi(x_i)) = \psi(A(x_i)) = \psi(0) = 0$. This shows that $\phi(x_1), \ldots, \phi(x_s) \in \ker(T)$. If $v \in \ker(T)$ then, using $A \circ \phi^{-1} = \psi^{-1} \circ T$, we see that $A(\phi^{-1}(v)) = \psi^{-1}(T(v)) = \psi^{-1}(0) = 0$, 


and so $\phi^{-1}(v) \in \ker(A)$. This means we may expand $\phi^{-1}(v) = c_1x_1 + \cdots + c_sx_s$ for some scalars $c_1, \ldots, c_s \in F$, and applying $\phi$ to both sides results in

$$v = c_1\phi(x_1) + \cdots + c_s\phi(x_s).$$

This proves that $x_1, \ldots, x_s$ span $\ker(T)$. To check linear independence, if $a_1\phi(x_1) + \cdots + a_s\phi(x_s) = 0$ then we must have $\phi(a_1x_1 + \cdots + a_sx_s) = 0$. As $\phi$ is injective, this implies that $a_1x_1 + \cdots + a_sx_s = 0$, and the linear independence of $x_1, \ldots, x_s$ now implies that $a_1 = 0, \ldots, a_s = 0$.

(b) We are assuming that $y_1, \ldots, y_t$ is a basis for $\text{Im}(A)$, and so there are $u_1, \ldots, u_t \in F^n$ such that $y_i = A(u_i)$. In particular, using $\psi \circ A = T \circ \phi$, we see that

$$\psi(y_i) = \psi(A(u_i)) = T(\phi(u_i)) \in \text{Im}(T).$$

On the other hand, if $w \in \text{Im}(T)$ then we may write $w = T(v)$ for some $v \in V$. Now write $v = \phi(u)$ for some $u \in F^n$, so that

$$w = T(\phi(u)) = \psi(A(u)).$$

As $A(u) \in \text{Im}(A)$ we may write $A(u) = c_1y_1 + \cdots + c_ty_t$ for some $c_1, \ldots, c_t \in F$, and now

$$w = c_1\psi(y_1) + \cdots + c_t\psi(y_t).$$

This proves that $\psi(y_1), \ldots, \psi(y_t)$ span $\text{Im}(T)$. To check linear independence, suppose

$$a_1\psi(y_1) + \cdots + a_t\psi(y_t) = 0.$$

Then $\psi(a_1y_1 + \cdots + a_ty_t) = 0$, and the injectivity of $\psi$ implies that $a_1y_1 + \cdots + a_ty_t = 0$. The linear independence of $y_1, \ldots, y_t$ now implies that $a_1 = 0, \ldots, a_t = 0$.

**Exercise 3.** Consider the linear map $T : P_2 \to \mathbb{R}^3$ defined by

$$T(p) = \begin{pmatrix} p(1) \\ p(2) \\ p(-1) \end{pmatrix}.$$

Compute the matrix of $T$ with respect to the bases $1, x, x^2 \in P_2$ and $e_1, e_2, e_3 \in \mathbb{R}^3$, and find bases for the kernel and image of $T$.

**Solution.** By computing

$$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e_1 + e_2 + e_3$$

$$T(x) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = e_1 + 2e_2 - e_3$$

$$T(x^2) = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = e_1 + 4e_2 + e_3$$

we see that the matrix is

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix}.$$

This matrix is invertible, and so $T$ is an isomorphism. It’s kernel is 0 and its image is $\mathbb{R}^3$. 

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Exercise 4. Define a linear map $T : P_3 \rightarrow P_3$ by

$$T(p) = p(0)x^3 + p(1)x^2 + p(1)x + p(0).$$

Compute the matrix of $T$ with respect to the basis $1, x, x^2, x^3 \in P_3$, and find bases for the kernel and image of $T$.

Solution. (a) By computing

$$T(1) = 1 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 1 \cdot x^3$$
$$T(x) = 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 0 \cdot x^3$$
$$T(x^2) = 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 0 \cdot x^3$$
$$T(x^3) = 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 0 \cdot x^3$$

we see that the matrix is

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Row reduction gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The usual analysis of the reduced row echelon form shows that

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \in \ker \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right)$$

and

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \in \text{Im} \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right)$$

are bases. Under the isomorphism $\mathbb{R}^4 \rightarrow P_3$ determined by our basis $1, x, x^2, x^3$, we obtain bases

$-x + x^2, -x + x^3 \in \ker(T)$

and

$1 + x + x^2 + x^3, x + x^2 \in \text{Im}(T)$.

Exercise 5. Compute the inverse of $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, and factor $A$ as a product of elementary matrices. Using your calculation of $A^{-1}$, solve

$$Ax = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad Ax = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
Solution. The inverse is
\[ A^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \]
from which we compute the solutions to the above equations to be
\[ A^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1 \\ -2 \end{bmatrix}, \quad A^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ -1 \end{bmatrix}, \quad A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \]

There are many factorizations into elementary matrices. By keeping track of the row reductions that take \( A \) to \( I \), we see that
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
A = I.
\]
Thus one factorization is
\[
A = \begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
1/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}^{-1}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}.
\]

Exercise 6. Suppose \( f(x) \in P_{99} \). Show that there is a polynomial \( g(x) \in P_{100} \) such that
\[ 5g''(x) + 3g'(x) = f(x), \]
and that \( g(x) \) unique up to addition of a constant polynomial.

Solution. Let \( T : P_{100} \to P_{99} \) be the linear map defined by
\[ T(g) = 5g'' + 3g'. \]

If \( g(x) \) has degree \( d \geq 1 \) then \( \deg(T(g)) = d - 1 \). Thus \( T(g) \) cannot be 0. This proves that the kernel of \( T \) consists of only the constant polynomials. In particular \( \ker(T) \) has dimension 1. But now
\[ 101 = \dim(P_{100}) = \dim(\ker(T)) + \dim(\Im(T)) = 1 + \dim(\Im(T)), \]
so
\[ \dim(\Im(T)) = 100 = \dim(P_{99}). \]
This proves that \( \Im(T) = P_{99} \), so \( T \) is surjective. Therefore, for any \( f(x) \in P_{99} \) there is a \( g(x) \in P_{100} \) such that \( T(g) = f \).

If also \( h \in P_{100} \) satisfies \( T(h) = f \), then \( T(h - g) = f - f = 0 \). Thus \( h - g \in \ker(T) \). We saw above that the kernel of \( T \) consists only of constant polynomials, and hence
\[ h = g + \text{constant}. \]