Exercise 1. Directly from the definition \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \) of the 2 \( \times \) 2 determinant, prove the following.

(a) \( \det(A) = \det(A^T) \)

(b) \( \det(AB) = \det(A) \det(B) \)

(c) \( A \) is invertible if and only if \( \det(A) \neq 0 \), in which case \( A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \).

Exercise 2.

(a) If \( A \in M_n(F) \) and \( c \) is a scalar, how are \( \det(cA) \) and \( \det(A) \) related?

(b) Suppose \( A \in M_{m \times n}(F) \). Prove that \( A \) is surjective if and only if \( A^T \) is injective.

(c) Assuming \( A \in M_n(F) \) is invertible, prove that \( (A^{-1})^T = (A^T)^{-1} \), and that \( \det(A^{-1}) = \det(A)^{-1} \).

Exercise 3. Compute the determinants

\[
\det \begin{bmatrix} 0 & 1 & 2 \\ 2 & 6 & -2 \\ 3 & 0 & 4 \end{bmatrix}
\quad \text{and} \quad
\det \begin{bmatrix} 2 & 0 & 2 & -4 \\ 0 & 6 & 3 & 3 \\ -1 & 4 & 3 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}.
\]

Exercise 4. Prove or find counterexamples:

\[
\text{rank}(BA) \leq \text{rank}(B), \quad \text{rank}(CB) \leq \text{rank}(B)
\]

for any matrices and \( A, B, \) and \( C \) for which the products are defined.

Exercise 5. Show that

\[
\det \begin{bmatrix} 1 & x & z^2 \\ 1 & y & z^2 \\ 1 & y^2 & z^2 \end{bmatrix} = (z - x)(z - y)(y - x).
\]

Exercise 6. Suppose \( V \) is finite dimensional and \( T : V \rightarrow V \) is a linear map. If \( A \) is the matrix of \( T \) with respect to some basis \( e_1, \ldots, e_n \in V \), and \( B \) is the matrix of \( T \) with respect to another basis \( f_1, \ldots, f_n \in V \), show that there is an invertible matrix \( C \) with \( B = CAC^{-1} \), and deduce from this that \( \det(A) = \det(B) \).