Exercise 1. Directly from the definition $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ of the $2 \times 2$ determinant, prove the following.

(a) $\det(A) = \det(A^T)$

(b) $\det(AB) = \det(A) \det(B)$

(c) $A$ is invertible if and only if $\det(A) \neq 0$, in which case $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Solution. The first follows from $\det \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \det \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

The second follows from

$$
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \det \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} = (aa' + bc')(cb' + dd') - (ca' + dc')(ab' + bd') = (aca'b' + ada'd' + bcb'c' + bdc'd') - (aca'b' + bca'd' + adb'c' + bdc'd') = (ada'd' + bcb'c') - (bca'd' + adb'c') = (ad - bc)(a'd' - b'c') = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \det \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.
$$

For the third part, first assume that $A$ is invertible. Taking the determinant of both sides of $AA^{-1} = I$ shows that $\det(A) \det(A^{-1}) = 1$, and hence $\det(A) \neq 0$. Conversely, suppose that $\det(A) \neq 0$. This allows us to define

$$
B = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

Now direct calculation gives

$$
AB = \frac{1}{\det(A)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

and hence $B = A^{-1}$. This proves both that $A$ is invertible, and that

$$
A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$
Exercise 2.

(a) If $A \in M_n(F)$ and $c$ is a scalar, how are $\det(cA)$ and $\det(A)$ related?

(b) Suppose $A \in M_{m \times n}(F)$. Prove that $A$ is surjective if and only if $A^T$ is injective.

(c) Assuming $A \in M_n(F)$ is invertible, prove that $(A^{-1})^T = (A^T)^{-1}$, and that $\det(A^{-1}) = \det(A)^{-1}$.

Solution. (a) Suppose $A$ has columns

$$A = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}.$$ 

Then, using linearity in each column,

$$\det(cA) = \begin{vmatrix} cv_1 & \ldots & cv_n \end{vmatrix} = c^n \det(A).$$

(b) If we view $A : F^n \to F^m$ and $A^T : F^m \to F^n$, then

$A^T$ is injective $\iff$ $\dim(\text{Im}(A^T)) = m$ $\iff$ $\text{rank}(A^T) = m$ $\iff$ $\text{rank}(A) = m$ $\iff$ $A$ is surjective.

(c) For the first claim we use

$$I = I^T = (A \cdot A^{-1})^T = (A^{-1})^T \cdot A^T$$

to see that $(A^{-1})^T$ is the inverse of $A^T$. In other words, $(A^{-1})^T = (A^T)^{-1}$.

For the second claim, recall that $\det(AB) = \det(A) \det(B)$ for any $n \times n$ matrices $A$ and $B$. In particular,

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A) \det(A^{-1}).$$

Dividing both sides by $\det(A)$ implies that $\det(A)^{-1} = \det(A^{-1})$.

Exercise 3. Compute the determinants

$$\det \begin{bmatrix} 0 & 1 & 2 \\ 2 & 6 & -1 \\ 3 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 2 & 0 & 2 & -4 \\ 12 & 6 & 6 & 1 \\ 0 & -1 & 4 & 5 \\ 3 & 2 & 3 & 2 \end{bmatrix}.$$

Solution. The usual methods show that

$$\det \begin{bmatrix} 0 & 1 & 2 \\ 2 & 6 & -1 \\ 3 & 0 & 4 \end{bmatrix} = -47,$$

$$\det \begin{bmatrix} 2 & 0 & 2 & -4 \\ 12 & 6 & 6 & 1 \\ 0 & -1 & 4 & 5 \\ 3 & 2 & 3 & 2 \end{bmatrix} = -232.$$ 

Exercise 4. Prove or find counterexamples:

$$\text{rank}(BA) \leq \text{rank}(B), \quad \text{rank}(CB) \leq \text{rank}(B)$$

for any matrices $A$, $B$, and $C$ for which the products are defined.
Solution. If we view $A : F^n \to F^m$, $B : F^m \to F^\ell$, and $C : F^\ell \to F^k$, then

$$\text{Im}(BA) \subset \text{Im}(B), \quad \ker(B) \subset \ker(CB).$$

As rank is the dimension of the image, the first inclusion implies $\text{rank}(BA) \leq \text{rank}(B)$. The second inclusion implies that $\dim(\ker(B)) \leq \dim(\ker(CB))$. As

$$m = \dim(\ker(B)) + \dim(\text{Im}(B))$$
$$m = \dim(\ker(CB)) + \dim(\text{Im}(CB)),$$

we must therefore have $\text{rank}(CB) \leq \text{rank}(B)$.

Exercise 5. Show that

$$\det \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = (z - x)(z - y)(y - x).$$

Solution. Using row operations we compute

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$
$$= (y-x)(z-x) \begin{vmatrix} 1 & x^2 \\ 0 & 1 + x \\ 0 & 1 + z \end{vmatrix}$$
$$= (y-x)(z-x) \begin{vmatrix} 1 & x^2 \\ 0 & 1 + x \\ 0 & 0 & z-y \end{vmatrix}$$
$$= (y-x)(z-x)(z-y).$$

Exercise 6. Suppose $V$ is finite dimensional and $T : V \to V$ is a linear map. If $A$ is the matrix of $T$ with respect to some basis $e_1, \ldots, e_n \in V$, and $B$ is the matrix of $T$ with respect to another basis $f_1, \ldots, f_n \in V$, show that there is an invertible matrix $C$ with $B = CAC^{-1}$, and deduce from this that $\det(A) = \det(B)$.

Solution. Let $\alpha : F^n \to V$ be the isomorphism sending the standard basis vectors to $e_1, \ldots, e_n$. Let $\beta : F^n \to V$ be the isomorphism sending the standard basis vectors to $f_1, \ldots, f_n$. The matrices $A$ and $B$ then correspond to the linear maps

$$A = \alpha^{-1} \circ T \circ \alpha : F^n \to F^n$$
$$B = \beta^{-1} \circ T \circ \beta : F^n \to F^n$$

respectively. If we let $C \in M_n(F)$ be the matrix of the linear map $C = \beta^{-1} \circ \alpha$, then

$$CAC^{-1} = (\beta^{-1} \circ \alpha) \circ (\alpha^{-1} \circ T \circ \alpha) \circ (\alpha^{-1} \circ \beta) = \beta^{-1} \circ T \circ \beta = B.$$