GEOMETRIC CONSISTENCY OF MANIN’S CONJECTURE

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ABSTRACT. We propose an explicit description of the exceptional set in Manin’s Conjecture. Our proposal includes the rational point contributions from any subvariety or cover with larger geometric invariants. We prove that this set is contained in a thin subset of rational points, verifying there is no counterexample to Manin’s Conjecture which arises from incompatibility of geometric invariants.

1. Introduction

Let $X$ be a geometrically integral smooth projective Fano variety over a number field $F$ and let $\mathcal{L} := \mathcal{O}_X(L)$ be an adelically metrized big and nef line bundle on $X$. Manin’s Conjecture, first formulated in [FMT89] and [BM90], predicts that the growth in the number of rational points on $X$ of bounded $\mathcal{L}$-height is controlled by two geometric constants $a(X, L)$ and $b(F, X, L)$. These constants are defined for any smooth projective variety $X$ and big and nef divisor $L$ on $X$ as

$$a(X, L) = \min\{ t \in \mathbb{R} \mid K_X + tL \in \text{Eff}^1(X) \}$$

and

$$b(F, X, L) = \text{the codimension of the minimal supported face of } \text{Eff}^1(X) \text{ containing } K_X + a(X, L)L$$

where $\text{Eff}^1(X)$ is the pseudo-effective cone of divisors of $X$. If $L$ is nef but not big, we set $a(X, L) = \infty$.

In the statement of Manin’s Conjecture an “exceptional set” of rational points must be removed in order to obtain the expected growth rate. For example, it is possible for points to grow more quickly than predicted along certain subvarieties of $X$ and such points should not be counted. More precisely, the following definition identifies the possible geometric obstructions to Manin’s Conjecture.

Definition 1.1. Let $X$ be a geometrically integral smooth projective variety over a number field $F$ and let $L$ be a big and nef divisor on $X$. A morphism of geometrically integral smooth projective varieties $f : Y \to X$ is called a breaking thin map if it satisfies the following two conditions:

1. $f$ is generically finite onto its image, and
2. $(a(Y, f^*L), b(F, Y, f^*L)) > (a(X, L), b(F, X, L))$ in the lexicographic order.

If Manin’s Conjecture is self-consistent then the exceptional set should include all subsets of the form $f(Y(F))$ where $f : Y \to X$ is a breaking thin map. Note that the point contributions from breaking thin maps need not lie on a closed set (see [BT96b] and [LR14]).
Our main theorem shows that point contributions from breaking thin maps will always be contained in a thin set as predicted by [Pey03].

**Theorem 1.2.** Let $X$ be a smooth geometrically uniruled projective variety over a number field $F$ and let $L$ be a big and nef divisor on $X$. As we vary over all breaking thin $F$-maps $f : Y \to X$, the points

$$\bigcup_f f(Y(F))$$

are contained in a thin subset of $X(F)$.

This theorem generalizes earlier partial results in [BT98b], [HTT15], [LTT18], [HJ17], [LT17b], and [Sen17b]. These papers also establish some practical techniques for computing this thin set.

In fact, we prove a more precise geometric statement (described in Section 3) which also addresses subvarieties and covers with the same $a$ and $b$ invariants as $X$. We conjecture that the exceptional set in Manin’s Conjecture always coincides with the geometrically defined thin subset of $X(F)$ described in Section 3, and we verify this in many examples where Manin’s Conjecture is known to hold.

Our approach to Theorem 1.2 can be broken down into two steps. The first step is to study the geometric behavior of the $a$ and $b$ constants over an algebraically closed field of characteristic 0. Using the minimal model program and the boundedness of singular Fano varieties, we prove a finiteness theorem for the set of breaking thin maps $f : Y \to X$.

**Definition 1.3.** Let $X$ be a smooth projective variety over a field of characteristic 0 and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Let $\pi : X \dasharrow Z$ be the Iitaka fibration associated to $K_X + a(X,L) L$. An Iitaka base change of $X$ is the normalization of a projective closure of the main component of $T \times_Z X$ for some dominant morphism $T \to Z$.

When $\kappa(K_Y + a(Y,L)L) > 0$, Manin’s Conjecture should capture the “relative” behavior of objects on fibers of the Iitaka fibration, and Iitaka base changes are naturally compatible with this philosophy.

**Theorem 1.4.** Let $X$ be a uniruled smooth projective variety over an algebraically closed field of characteristic 0. Let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. There is a finite set of breaking thin maps $\{f_i : Y_i \to X\}$ such that any breaking thin map $f : Y \to X$ will factor rationally through one of the $f_i$ after an Iitaka base change.

The second step is to derive a thinness statement over a number field $F$. Note that Theorem 1.4 only holds over an algebraically closed field, since infinitely many twists over $F$ can be identified with a single map over $\overline{F}$. Thus we must consider the behavior of rational points over all twists of a fixed map. The essential ingredient of the following theorem is the Hilbert Irreducibility Theorem proved by Serre.

**Theorem 1.5.** Let $X$ be a geometrically uniruled normal projective variety over a number field $F$. Suppose that $f : Y \to X$ is a generically finite morphism from a normal projective variety $Y$. As $\sigma$ varies over all $\sigma \in H^1(F, \text{Aut}(Y/X))$ such that $Y^\sigma$ is irreducible and

$$(a(X,L), b(F,X,L)) < (a(Y, f^*L), b(F, Y^\sigma, (f^\sigma)^*L))$$


the set
\[ Z = \bigcup_{\sigma} f^{\sigma}(Y^{\sigma}(F)) \subset X(F) \]
is contained in a thin subset of \( X(F) \).

By combining Theorem 1.5 with a weaker version of Theorem 1.4 that holds over a number field, we deduce Theorem 1.2.

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2. Geometric invariants in Manin’s Conjecture

Let \( F \) be a field of characteristic 0. A variety \( X \) defined over \( F \) is an integral separated scheme of finite type over \( F \). For such a variety, we denote its base change to an algebraic closure by \( \overline{X} \).

Definition 2.1. Let \( X \) and \( Y \) be projective varieties. A map \( f : Y \to X \) is thin if it is generically finite onto its image and admits no rational section.

If \( F \) is a number field, a thin subset of \( X(F) \) is precisely a finite union \( \bigcup_j f_j(Y_j(F)) \) where \( f_j : Y_j \to X \) are thin maps over \( F \).

In this section we recall the basic definitions and properties of the geometric invariants appearing in Manin’s Conjecture. Suppose that \( X \) is a smooth projective variety defined over \( F \). We denote the \( Néron-Severi \) space of \( \mathbb{R} \)-Cartier divisors up to numerical equivalence by \( N_1(X) \) and the space of \( \mathbb{R} \)-1-cycles modulo numerical equivalence by \( N_1(X) \). We denote the pseudo-effective cone and the nef cone of divisors by

\[ \text{Eff}^1(X), \quad \text{Nef}^1(X) \]

respectively, and the pseudo-effective cone and the nef cone of curves by

\[ \text{Eff}_1(X), \quad \text{Nef}_1(X) \]

respectively. These are are strict closed convex cones in \( N^1(X) \) and \( N_1(X) \).

Definition 2.2. Let \( X \) be a smooth projective variety defined over a field \( F \) of characteristic 0. Let \( L \) be a big and nef \( \mathbb{Q} \)-divisor on \( X \). Then we define the Fujita invariant (or \( a \)-invariant) by

\[ a(X, L) = \min \{ t \in \mathbb{R} \mid K_X + tL \in \text{Eff}^1(X) \} \]

When \( L \) is nef but not big, we formally set \( a(X, L) = +\infty \). When \( X \) is singular, we define the Fujita invariant as the Fujita invariant of the pullback of \( L \) to any smooth model. This is well-defined because of [HTT15, Proposition 2.7].
Since the cohomology of line bundles is stable under flat base change, we have \( a(X, L) = a(\overline{X}, L) \). Also, by [BDPP13] \( a(X, L) > 0 \) if and only if \( X \) is geometrically uniruled. We say that \( K_X + a(X, L)L \) is rigid, or simply that \( X \) is adjoint rigid, if \( \kappa(K_X + a(X, L)L) = 0 \). If \( a(X, L) > 0 \), then \( X \) is always covered by adjoint rigid subvarieties with the same \( a \)-value as \( X \). Indeed, one can simply take (the closure of) the fibers of the map to the canonical model of \((X, a(X, L)L)\) as constructed by [BCHM10].

**Proposition 2.3.** Let \( X \) be a smooth projective variety over a number field \( F \) and let \( L \) be a big and nef \( \mathbb{Q} \)-divisor on \( X \). We fix a finite extension \( F'/F \) and let \( Y \subset X_{F'} \) be a geometrically integral subvariety defined over \( F' \). Let \( \sigma \in \mathrm{Gal}(\overline{F}/F) \). Then we have

\[
a(Y, L|_Y) = a(\sigma(Y), L|_{\sigma(Y)}).
\]

**Proof.** Let \( \Delta_Y \) be the union of the Galois orbit of \( Y \). After applying an embedded resolution of singularities, we may assume that \( X \) and \( \Delta_Y \) are smooth. Let \( K_Y \) be the canonical divisor on \( Y \). Then we have \( \sigma(K_Y) = K_{\sigma(Y)} \). Moreover we have

\[
tL|_Y + K_Y \sim D \geq 0 \iff tL|_{\sigma(Y)} + K_{\sigma(Y)} \sim \sigma(D) \geq 0.
\]

Thus our assertion follows. \( \square \)

**Definition 2.4.** Let \( X \) be a smooth projective variety defined over a field \( F \) of characteristic 0 and let \( L \) be a big and nef \( \mathbb{Q} \)-divisor on \( X \). When \( a(X, L) < \infty \) we define the \( b \)-invariant by

\[
b(F, X, L) = \text{the codimension of the minimal supported face of } \mathbb{NH}^1(X) \text{ containing the numerical class of } K_X + a(X, L)L.
\]

By [HTT15, Proposition 2.10] we can define this invariant for singular projective varieties by pulling \( L \) back to any smooth model.

In contrast to the \( a \)-value, the \( b \)-value can change upon field extension, but it can only increase.

**Proposition 2.5.** Let \( X \) be a smooth geometrically integral projective variety defined over a number field \( F \) and let \( L \) be a big and nef \( \mathbb{Q} \)-divisor on \( X \). Let \( F'/F \) be a finite extension. Then we have

\[
b(F, X, L) \leq b(F', X_{F'}, L_{F'}).
\]

**Proof.** This follows from the fact that \( b(F, X, L) \) is given by

\[
\dim \mathbb{NH}^1(\overline{X})^\mathrm{Gal}(\overline{F}/F) \cap (K_X + a(X, L)L)^\perp.
\]

Note that the action of the absolute Galois group on \( \mathbb{NH}^1(\overline{X}) \) factors through a finite group. \( \square \)

The following useful criterion gives a geometric characterization of the \( b \)-invariant over a number field. The analogous statement over an algebraically closed field is also true; see [CFST16, Section 2] and [Sen17a, Lemma 2.11] for an in-depth discussion of this property.

**Lemma 2.6.** Let \( X \) be a smooth geometrically uniruled geometrically integral projective variety defined over a number field \( F \), and let \( L \) be a big and nef \( \mathbb{Q} \)-divisor on \( X \). Let \( \pi : X \to W \) be the Iitaka fibration for \( K_X + a(X, L)L \). Suppose that there is a non-empty
open set $W^\circ \subset W$ such that $\pi$ is well-defined and smooth over $W^\circ$. Let $w \in W^\circ$ be a closed point and fix a geometric point $\overline{w}$ above $w$. Note that $N^1(X_w)$ naturally embeds into $N^1(X_{\overline{w}})$. Then we have

$$b(F, X, L) = \dim (N^1(X_w) \cap N^1(X_{\overline{w}})_{|\pi^* (\overline{W^\circ}, \overline{w})}) / \mathbb{R} \cdot \langle E_i \rangle$$

where the $\{E_i\}$ are the classes of all irreducible divisors which dominate $W$ and which satisfy $K_X + a(X, L)L - c_i E_i \in \overline{\text{Eff}}(X)$ for some $c_i > 0$. Furthermore, there are only a finite number of such $E_i$.

**Proof.** We may resolve $\pi$ to be a morphism without affecting $\pi^{-1}(W^\circ)$, and the statement for the blow-up is equivalent to the statement for the original variety. Since the generic fiber of $\pi$ is rationally connected, the Picard rank of the geometric fibers is constant over $\overline{W^\circ}$. By [And96, Théorème 5.2] we know that $N^1(X)$ will surject onto $N^1(X_{\overline{w}})_{|\pi^* (\overline{W^\circ}, \overline{w})}$ for one fiber, and hence, by constancy in the family over $\overline{W^\circ}$, for all fibers. Note that the action of $\text{Gal}(\overline{F}/F)$ on $N^1(X)$ and on $N^1(X_{\overline{w}})$ factors through a finite group. Altogether we see that the restriction map

$$N^1(X)_{\text{Gal}(\overline{F}/F)} \to N^1(X_w) \cap N^1(X_{\overline{w}})_{|\pi^* (\overline{W^\circ}, \overline{w})}$$

is surjective.

By definition $b(F, X, L)$ is the dimension of the quotient of $N^1(X)$ by all effective irreducible divisors $E$ satisfying $K_X + a(X, L)L - cE \in \overline{\text{Eff}}(X)$ for some $c > 0$. Note that $E$ lies in the kernel of the restriction map to $N^1(X_w)$ if and only if $E$ satisfies the property above and $\pi(E) \subseteq Z$. Using the surjection above, we deduce the first claim.

The geometric general fiber $X_{\overline{w}}$ is adjoint rigid with respect to $L$. If we restrict $E_i$ to $X_{\overline{w}}$, then the support must lie in the unique effective divisor equivalent to $K_{X_{\overline{w}}} + a(X, L)L|_{X_{\overline{w}}}$. Thus there are only finitely many divisors $E_i$ as in the statement, proving the second claim. $\square$

For later use we record the following corollary. Again, the analogous property is true over an algebraically closed field.

**Corollary 2.7.** Let $X$ be a smooth geometrically uniruled geometrically integral projective variety defined over a number field $F$ and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Suppose that $\pi : X \to W$ is a rational map which is well-defined and smooth over an open subset $W^\circ$ of $W$. Let $w \in W^\circ$ be a closed point such that $a(X_w, L) = a(X, L)$ and $X_w$ is adjoint rigid. We fix a geometric point $\overline{w}$ above $w$. Then we have

$$b(F, X, L) \geq \dim (N^1(X_w) \cap N^1(X_{\overline{w}})_{|\pi^* (\overline{W^\circ}, \overline{w})}) / \mathbb{R} \cdot \langle E_i \rangle$$

where the $\{E_i\}$ are the classes of all irreducible divisors which dominate $Z$ and which satisfy $K_X + a(X, L)L - c_i E_i \in \overline{\text{Eff}}(X)$ for some $c_i > 0$. Furthermore, there are only a finite number of such $E_i$.

**Proof.** We may resolve $\pi$ to be a morphism without affecting $\pi^{-1}(W^\circ)$, and the statement for the blow-up is equivalent to the statement for the original variety. We know that the general fiber of $\pi$ has the same $a$-value as $X$ and is adjoint rigid (since the Iitaka dimension increases on closed subsets). Thus for a sufficiently large ample divisor $H$ on $Z$ the Iitaka
fibration for \( K_X + a(X, L)L + \pi^*H \) is exactly the morphism \( \pi \). Applying Lemma 2.6 to the divisor \( a(X, L)L + \pi^*H \) shows that
\[
b(F, X, a(X, L)L + \pi^*H) = \dim (N^1(X_w) \cap N^1(\overline{X}_w)_{\pi^*}(|Y_{\overline{w}}, \pi|)) / \mathbb{Q}_\ell \cdot \langle E_i \rangle
\]
Note that \( b(F, X, a(X, L)L + \pi^*H) \leq b(F, X, L) \). Finally, we need to note that the exceptional divisors \( \{ E_i \} \) as in Lemma 2.6 for \( a(X, L)L + \pi^*H \) are the same as the ones described in the statement above. Indeed, both are the dominant divisorial components of the relative diminished base locus of \( K_X + a(X, L)L \) over \( W \).

2.1. Fiber dimension.

**Definition 2.8.** Let \( X \) be a smooth projective variety defined over a field \( F \) of characteristic 0. Let \( L \) be a big and nef \( \mathbb{Q} \)-divisor on \( X \). We define
\[
d(X, L) = \dim(X) - \kappa(K_X + a(X, L)L).
\]
When \( X \) is singular, we define \( d(X, L) \) by pulling \( L \) back to a smooth model. Note that \( d(X, L) \) is invariant under extension of the ground field.

**Lemma 2.9.** Let \( X \) be a smooth geometrically uniruled projective variety and let \( L \) be a big and nef \( \mathbb{Q} \)-divisor on \( X \). There is a proper closed subset \( V \subseteq X \) such that if \( f : Y \to X \) is any thin map satisfying \( a(Y, f^*L) \geq a(X, L) \) and \( d(Y, L) > d(X, L) \) then \( f(Y) \subseteq V \).

**Proof.** We may prove the theorem after passing to an algebraic closure of the ground field. By [LT17a, Theorem 3.4] it suffices to consider the case when \( a(Y, f^*L) = a(X, L) \). By combining [Bir16b] with the argument of [LT17b, Corollary 4.8] we see that it suffices to prove that \( X \) does not admit any dominant family of adjoint rigid varieties \( Y \) satisfying \( a(Y, L|_Y) = a(X, L) \) and \( \dim(Y) > d(X, L) \).

We may replace \( X \) by any birational model. In particular we may suppose that the Iitaka fibration for \( K_X + a(X, L)L \) is a morphism \( \pi : X \to W \). Thus there is an ample \( \mathbb{Q} \)-divisor \( H \) on \( W \) and an effective \( \mathbb{Q} \)-divisor \( E \) such that \( K_X + a(X, L)L \) is numerically equivalent to \( \pi^*H + E \). Suppose we have a diagram of smooth varieties
\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow q & & \downarrow \\
T & & \\
\end{array}
\]

such that \( g \) is generically finite and dominant and the fibers of \( q \) are smooth varieties \( Y_t \) satisfying \( a(Y_t, g^*L|_{Y_t}) = a(X, L) \) and \( \dim(Y_t) > d(X, L) \). We can write
\[
K_Y + a(X, L)g^*L = g^*(K_X + a(X, L)L) + R = g^*\pi^*H + (g^*E + R)
\]
for some effective divisor \( R \). Note that the restriction of \( g^*\pi^*H \) to a general fiber of \( q \) has Iitaka dimension at least 1, so that the general fiber can not possibly be adjoint rigid. \( \square \)

2.2. Face contraction. The notion of face contraction refines the \( b \)-invariant.

**Definition 2.10.** [LT17a, Definition 3.5] Let \( X \) be a smooth geometrically integral projective variety defined over a field \( F \) of characteristic 0 and let \( L \) be a big and nef \( \mathbb{Q} \)-divisor on \( X \). Let \( f : Y \to X \) be a morphism of smooth projective varieties that is generically finite onto its image. Suppose that either
• $f$ is dominant and $a(Y, f^*L) = a(X, L)$, or
• $a(Y, f^*L) = a(X, L)$, $d(Y, f^*L) = d(X, L)$, and there is a commuting diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\pi_Y \downarrow & & \pi_X \downarrow \\
T & \longrightarrow & W
\end{array}
\]

where $\pi_Y$ and $\pi_X$ are the Iitaka fibrations for the adjoint pairs and the general fiber of $\pi_Y$ maps onto a fiber contained in the smooth locus of $\pi_X$ with the same $a$-value as $X$.

Let $F_Y$ denote the face of $\text{Nef}_1(Y)$ perpendicular to $K_Y + a(Y, f^*L)f^*L$ and let $F_X$ denote the face of $\text{Nef}_1(X)$ perpendicular to $K_X + a(X, L)L$. In either setting, the pushforward map $f_* : N_1(Y) \to N_1(X)$ takes $F_Y$ into $F_X$. We say that $f$ is a face contracting morphism if $f_* : F_Y \to F_X$ is not injective.

Since the dimensions of $F_Y$ and $F_X$ are $b(F, Y, f^*L)$ and $b(F, X, L)$ respectively, a dominant breaking thin map is automatically face contracting. However, the converse is not true (see [LT17a, Example 3.7]).

**Lemma 2.11.** Let $f : Y \to X$ be a thin map of smooth geometrically integral projective varieties and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Suppose that we have a diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\phi_Y \downarrow & & \phi_X \downarrow \\
Y & \xrightarrow{f} & X
\end{array}
\]

such that $Y'$ and $X'$ are smooth and $\phi_Y$ and $\phi_X$ are birational. Suppose both $f$ and $f'$ satisfy the hypotheses of Definition 2.10. Then $f$ is face contracting with respect to $L$ if and only if $f'$ is face contracting with respect to $\phi_X^*L$.

The proof is essentially the same as the proof of [HTT15, Proposition 2.10]. We then say that any dominant map $f : Y \to X$ is face contracting if there is some birational model which satisfies Definition 2.10.

**Lemma 2.12.** Let $f : Y \to X$ be a dominant generically finite morphism of geometrically integral projective varieties and fix a big and nef $\mathbb{Q}$-divisor $L$ on $X$. Suppose there is a birational model $f' : Y' \to X'$ of $f$ with map $\phi : X' \to X$ and a diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
q \downarrow & & p \downarrow \\
T & \xrightarrow{g} & W
\end{array}
\]

satisfying the following conditions:

1. $X'$ and $Y'$ are smooth and projective,
2. $g$ is generically finite and dominant,
3. $a(Y, f^*L) = a(X, L)$ and $b(Y, f^*L) = b(X, L),$
4. $q$ is the Iitaka fibration for $K_{Y'} + a(Y', f'^*\phi^*L)f'^*\phi^*L$,


(5) the general fiber of \( p \) is adjoint rigid with the same \( a \)-value as \( X \),
(6) \( \dim(W) > \kappa(K_X + a(X, L)L) \).

Then \( f \) is face contracting.

Proof. By Lemma 2.11 we may suppose \( f' = f \). Note that any further blow-up of \( f' \) will still preserve the hypotheses on the map. Thus we may also assume that the Iitaka fibration for \( K_X + a(X, L)L \) is a morphism.

Let \( F_Y \) and \( F_X \) be the faces as in Definition 2.10. Fix an ample divisor \( H \) on \( W \). Note that \( H \) vanishes on every element of \( f_*F_Y \). However, \( H \) does not vanish on every element of \( F_X \) by assumption. Thus \( f_*F_Y \subsetneq F_X \), and since the \( b \)-values are equal \( f \) must be face contracting. \( \square \)

### 3. A conjectural description of exceptional sets in Manin’s conjecture

Let \( F \) be a number field and suppose that we have a geometrically rationally connected and geometrically integral smooth projective variety \( X \) defined over \( F \) carrying a big and nef line bundle \( \mathcal{L} = \mathcal{O}_X(L) \) with an adelic metrization on \( X \). Manin’s conjecture predicts the asymptotic formula for the counting function of rational points of bounded height associated to \( \mathcal{L} \) after removing an exceptional thin set. Originally [BM90] and its refinement [Pey95] predicted that the exceptional set for Manin’s Conjecture consisted of points on a proper closed subset. However there are now many counterexamples to these two versions of Manin’s Conjecture ([BT96b], [EJ06], [Els11], [BL17], and [LR14]). These counterexamples arise from geometric obstructions; for example, it is possible that as we vary over breaking thin maps \( f : Y \to X \) the union of the sets \( f(Y(F)) \) is Zariski dense. [Pey03] was the first to modify the conjecture by proposing that the exceptional set in Manin’s Conjecture is contained in a thin set.

In this section, we propose a conjectural description of the exceptional thin set in general. Suppose that \( X \) is a geometrically uniruled and geometrically integral smooth projective variety over \( F \). Without loss of generality we may assume that the Iitaka fibration \( \pi : X \to W \) for \( K_X + a(X, L)L \) is a morphism.

Let \( Z_0 \) be the set of rational points contained in the union of \( \mathcal{B}_x(L) \) and a proper closed subset \( \pi^{-1}V \) where \( V \subset W \) is a proper closed subset such that over \( W^\circ = W \setminus V \), \( \pi \) is smooth. Note that \( Z_0 \) consists of points on a proper closed subset of \( X \).

Next as \( f : Y \to X \) varies over all \( F \)-thin maps such that \( Y \) is geometrically integral and smooth, \( d(Y, f^*L) < d(X, L) \) and \((a(X, L), b(F, X, L)) \leq (a(Y, f^*L), b(F, Y, f^*L))\),

we define the set \( Z_1 \subset X(F) \) by

\[
Z_1 = \bigcup_f f(Y(F)) \subset X(F).
\]

Next as \( f : Y \to X \) varies over all \( F \)-thin maps such that \( Y \) is geometrically integral and smooth, \( d(Y, f^*L) = d(X, L) \), and either

\[
(a(X, L), b(F, X, L)) < (a(Y, f^*L), b(F, Y, f^*L)),
\]

or

\[
(a(X, L), b(F, X, L)) = (a(Y, f^*L), b(F, Y, f^*L)) \quad \text{and} \quad (a(Y, f^*L), b(F, Y, f^*L)) < (a(Z, f^*L), b(F, Z, f^*L))
\]
or the $a$ and $b$ values are equal and $f$ is face contracting, we define the set $Z_2 \subset X(F)$ by

$$Z_2 = \bigcup_f f(Y(F)) \subset X(F).$$

Finally, as $f : Y \to X$ varies over all $F$-thin maps such that $Y$ is geometrically integral and smooth, $d(Y, f^*L) > d(X, L)$ and 

$$(a(Y, f^*L), b(F, Y, f^*L)) \leq (a(Y, f^*L), b(F, Y, f^*L)),$$

we define the set $Z_3 \subset X(F)$ by

$$Z_3 = \bigcup_f f(Y(F)) \subset X(F).$$

By Lemma 2.9, $Z_3$ is contained in a proper closed subset of $X$.

For any subset $Q \subset X(F)$, we define $N(Q, L, T)$ as the number of rational points on $Q$ whose height associated to $L$ is bounded by $T$.

We propose the following additional refinement of Manin’s Conjecture by describing the exceptional thin set. A similar but weaker statement was predicted in [LT17b].

**Conjecture 3.1** (Manin’s Conjecture). Let $F$ be a number field. Let $X$ be a geometrically rationally connected and geometrically integral smooth projective variety defined over $F$ and let $L$ be a big and nef line bundle with an adelic metrization on $X$.

Let $Z$ be the union of $Z_0$, $Z_1$, $Z_2$, and $Z_3$. Suppose that $X(F)$ is not a thin set. Then we have

$$N(X(F) \setminus Z, L, T) \sim c(F, Z, L)T^{a(X,L)} \log(T)^{b(F,X,L)-1}$$

as $T \to \infty$ where $c(F, Z, L)$ is Peyre-Batyrev-Tschinkel’s constant introduced in [Pey95] and [BT98b].

**Remark 3.2.** Assuming the conjecture of Colliot-Thélène that the Brauer-Manin obstructions are the only obstructions to weak approximation for smooth geometrically rationally connected varieties, it follows that $X(F)$ is not thin as soon as there is a rational point. See the remark after Conjecture 1.4 in [BL17].

**Remark 3.3.** By [HM07] $X$ is geometrically rationally connected whenever $a(X, L) > 0$ and $K_X + a(X, L)L$ is rigid. (For a careful explanation see [LTT18, Proof of Theorem 4.5].)

**Remark 3.4.** [Pey17] formulates an appealing version of Manin’s Conjecture using the notion of freeness of a rational point. Peyre’s conjecture has some similarities with Conjecture 3.1. Let $Z^f$ denote the exceptional set as in [Pey17, Formule empirique 6.13]. [Pey17, Proposition 5.8] shows that $Z^f$ includes most points on non-free curves; comparing against [LT17a, Theorem 1.1 and Proposition 6.15] we should expect these points to account for subvarieties $Y$ with $a(Y, L) > a(X, L)$.

Nevertheless, the two proposals for the exceptional set are different. The set $Z^f$ may fail to be contained in the union of the $Z_i$: a general cubic fourfold has empty $Z_i$ but admits non-free lines so that $Z^f$ is non-empty by [Pey17, Proposition 5.8]. Conversely, the union of the $Z_i$ may fail to be contained in $Z^f$: in the example of [BT96b] the $Z_i$ contains every point on a cubic surface fiber with Picard rank $> 1$ while $Z^f$ does not (see [Pey17, Section 8.3] and particularly [Pey17, Remarque 8.9]). However, it might be possible that the difference between the two definitions is negligible when considered against the asymptotic growth rate.
The main theorem of this paper is the following theorem:

**Theorem 3.5.** Let $X$ be a geometrically uniruled geometrically integral smooth projective variety defined over a number field $F$ and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. The subsets $Z_0$, $Z_1$, $Z_2$, and $Z_3$ defined above are contained in a thin subset of $X(F)$.

**Remark 3.6.** As we vary over all dominant generically finite maps $f : Y \to X$ of degree $\geq 2$ such that

$$(a(Y, f^*L), b(Y, f^*L)) = (a(X, L), b(X, L)),$$

the set $\cup f(Y(F))$ need not lie in a thin set of rational points (see [LT17b, Example 8.7]). Thus, in the definition of $Z_2$ it is important to only consider contributions from maps which are face contracting.

**Remark 3.7.** To study Manin’s Conjecture in examples one should first calculate the sets $Z_0, Z_1, Z_2, Z_3$ of Theorem 3.5. In principle, by the Borisov-Alexeev-Borisov Conjecture ([Bir16a],[Bir16b]) this computation only involves checking the behavior of subvarieties and covers in a finite degree range. However, currently this result is ineffective and in practice there is room for vast improvement of current computational techniques. For low dimensional examples the framework established by [LTT18] and [LT17b] is often sufficient for calculating these sets.

Let us compute these exceptional sets for some examples. In all of the following examples $(X, L)$ will be adjoint rigid, so $Z_0 = B_+(L)$ and $Z_3$ will be empty and they need not be considered.

**Example 3.8** (Surfaces). Let $S$ be a smooth geometrically rational geometrically integral projective surface defined over $F$ and $L$ a big and nef $\mathbb{Q}$-divisor on $S$ such that $K_S + a(S, L)L$ is rigid. For simplicity let us suppose that the Picard rank of $S$ and the geometric Picard rank of $S$ coincide. Then by [LTT18, Proposition 5.9] and [LT17b, Theorem 1.8] $Z_1$ is contained in a proper closed subset and $Z_2$ is empty. Thus we expect that Manin’s conjecture should hold after removing points on a closed set. This version of Manin’s conjecture for geometrically rational surfaces has been confirmed for many examples, see e.g. [dlBBD07], [Bro09], [Bro10], and [dlBBP12].

**Example 3.9** (Flag varieties). Let $X$ be a geometrically integral generalized flag variety defined over $F$ with a rational point and let $L = -K_X$. Manin’s conjecture for flag varieties has been established in [FMT89] with empty exceptional set. By [Bor96], the Brauer-Manin obstructions are the only obstructions to weak approximation, so in particular $X(F)$ is not thin. Hence, $Z_1$ does not cover $X(F)$. Since $X$ is homogeneous, this implies that $Z_1$ must be empty. On the other hand, since there are no subvarieties with higher $a$-value and $X$ is simply connected, there is no dominant morphism $f : Y \to X$ such that $a(Y, -f^*K_X) = a(X, -K_X)$ and $Y$ is adjoint rigid by [Sen17b, Theorem 1.1]. Thus we conclude that $Z_2$ is also empty.

**Example 3.10** (Toric varieties). Let $X$ be a geometrically integral smooth toric variety defined over a number field $F$ and let $L$ be a big and nef divisor on $X$. Manin’s conjecture for such a variety was proved in [BT96a], [BT98a], and [Sal98] after removing rational points on the boundary. Suppose that $K_X + a(X, L)L$ is rigid. Since any $F$-torus satisfies the weak approximation property, by [Ser92, Theorem 3.5.7] we see that $X(F)$ is not thin. Since the torus part is a homogeneous space, we conclude that $Z_1$ is contained in the boundary.
By the same reasoning $Z_2$ is also contained in the boundary. So our refinement is compatible with the above results. A similar proof works for smooth equivariant compactifications of other algebraic groups and Manin’s conjecture for such varieties has been established in many cases, see e.g. [CLT02], [STBT07] and [ST16].

**Example 3.11** (Le Rudulier’s example). Let $S$ be the surface $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{Q}$ and set $X = \text{Hilb}^2(S)$. [LR14] proved Manin’s conjecture for $(X, -K_X)$. We briefly explain why her result is compatible with our refinement. We freely use the notations from [LT17b, Section 9.3]. Let $L = H_1[2] + H_2[2]$. Le Rudulier proved Manin’s conjecture for $L$ after removing rational points on $D_1, D_2, E$ and $f(W(\mathbb{Q}))$. We denote this exceptional set by $Z$. The analysis in [LT17b, Section 9.3] shows that (i) all subvarieties with higher $a$ values are contained in $D_1, D_2$, or $E$; (ii) the only thin maps $g : Y \to X$ such that the image is not contained in $D_1 \cup D_2 \cup E$, $(Y, g^*L)$ is adjoint rigid, $\dim Y < \dim X$, and $(a(X, L), b(\mathbb{Q}, X, L)) \leq (a(Y, g^*L), b(\mathbb{Q}, Y, g^*L))$ are the images of the fibers of one of the projections $\pi_i : W \to \mathbb{P}^1$. These imply that $Z_1$ is contained in $Z$. To analyze $Z_2$, we first note that the geometric fundamental group of $X \setminus (D_1 \cup D_2 \cup E)$ is $\mathbb{Z}/2\mathbb{Z}$. Thus, over $\mathbb{Q}$ there is only one possible cover $f : W \to X$ such that $a(W, -f^*K_X) = a(X, -K_X)$ and $W$ is adjoint rigid. On the other hand, by copying the argument of [LT17b, Example 8.6] in this setting we see that all nontrivial twists of $f : W \to X$ have $a, b$ values less than $a, b$ values of $X$. Thus $Z_2 = f(W(\mathbb{Q}))$ is also contained in $Z$.

The circle method has been successfully used to prove Manin’s conjecture for low degree complete intersections, e.g., [Bir62] and [BHB17]. Verifying our refinement for this class of varieties is out of reach at this moment. However, based on the properties of rational curves on low degree hypersurfaces proved by [HRS04], [BK13], [RY16], [BV17] and the connection with $a$ and $b$ invariants proved in [LT17a], we expect that $Z_1$ and $Z_2$ are empty for general smooth hypersurfaces in $\mathbb{P}^n$ of degree $\leq n - 2$ and for every smooth hypersurface in $\mathbb{P}^n$ of degree $\ll \log_2(n)$.

Note that in Conjecture 3.1 we also remove point contributions for some thin maps with $a$ and $b$ values equal to $X$. We must discount contributions from such maps in order to obtain the correct leading constant.

**Example 3.12** (Peyre’s constant). The papers [EJ06], [Els11], [BL17] give many examples of Fano varieties $X$ admitting a Zariski dense set of subvarieties with the same $a$ and $b$ values as $X$ with respect to $-K_X$. Suppose that the rational points on these subvarieties grow at the expected rate. If we include these points, [BL17, Theorem 1.2] shows that Manin’s Conjecture with Peyre’s constant will be violated for an appropriate choice of anticanonical height function. In order to obtain the correct Peyre’s constant we must remove point contributions from all such subvarieties. Theorem 3.5 shows that such points always lie in a thin set, generalizing the examples proved in [BL17].

### 4. The boundedness of accumulating maps in Manin’s conjecture

Our next goal is to prove a boundedness statement for the set of breaking thin maps. Note that each of the following subsections has a different assumption on the ground field.
4.1. **Previous results.** In this subsection we assume that our ground field is an algebraically closed field of characteristic zero. We first recall two results about boundedness of subvarieties with higher $a$-values. These results rely on the boundedness of singular Fano varieties proved by Birkar in [Bir16a] and [Bir16b].

**Theorem 4.1** ([HJ17] Theorem 1.1 and [LT17a] Theorem 3.4). Let $X$ be a smooth projective uniruled variety and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Let $V$ be the union of all subvarieties $Y$ such that $a(Y, L|_Y) > a(X, L)$. Then $V$ is a proper closed subset of $X$ and each component $V_0 \subset V$ satisfies $a(V_0, L|_{V_0}) > a(X, L)$.

**Theorem 4.2** ([LT17b]). Let $X$ be a smooth projective uniruled variety and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Then there exist a proper closed subset $V$ and finitely many families of closed subschemes $\pi_i: \mathcal{U}_i \to W_i$ where $W_i$ is projective such that

- for each $i$, the evaluation map $s_i: \mathcal{U}_i \to X$ is generically finite and dominant;

- for each $i$, a general member of $\pi_i$ is an adjoint rigid subvariety in $X$ with the same $a$-value as $X$, and;

- for any adjoint rigid subvariety $Y$ with $a(Y, L|_Y) \geq a(X, L)$, either $Y$ is contained in $V$ or $Y$ is a member of a family $\pi_i: \mathcal{U}_i \to W_i$ for some $i$.

**Proof.** By Theorem 4.1, we only need to take care of subvarieties $Y$ with $a(Y, L) = a(X, L)$. Combining the BAB conjecture proved in [Bir16a] and [Bir16b] with the proof of [LT17b, Corollary 4.8], we know that the set of such $Y$ which are not contained in $B_+(L)$ is parametrized by a bounded subset of Chow($X$). The existence of a proper closed subset and finitely many families satisfying the second condition follows from this statement. Thus we only need to show that these families admit evaluation maps which are generically finite. This follows from the proof of [LT17b, Proposition 4.14].

We will also need two results useful for understanding dominant breaking thin maps.

**Theorem 4.3.** [Sen17b, Corollary 2.8] Let $X$ be a smooth projective uniruled variety and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Suppose that $f: Y \to X$ is a generically finite cover with $Y$ smooth and with $a(Y, f^*L) = a(X, L)$. Suppose $R_i$ is a component of the ramification divisor $R$ on $Y$ which dominates the base of the Iitaka fibration for $K_Y + a(Y, f^*L)f^*L$ and whose image $B_i$ is a component of the branch divisor $B$ on $X$. Then

$$a(B_i, L|_{B_i}) > a(X, L).$$

**Proposition 4.4.** [Sen17b, Proposition 2.15] Let $X$ be a smooth projective uniruled variety and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Then there exists a constant $M$ only depending on $X$ and $L$ such that for any thin map $f: Y \to X$ such that $a(Y, L) = a(X, L)$ and $(Y, f^*L)$ is adjoint rigid, we have

$$\deg(f: Y \to f(Y)) \leq M.$$

4.2. **Finiteness of covers.** In this subsection we work over the base field $\mathbb{C}$.

**Definition 4.5.** A good family of adjoint rigid varieties is a morphism $p: \mathcal{U} \to W$ of smooth quasi-projective varieties and a relatively big and nef $\mathbb{Q}$-divisor $L$ on $\mathcal{U}$ satisfying the following properties:

1. The map $p$ is projective, surjective, and smooth with irreducible fibers.
(2) The $a$-value $a(U_w, L)$ is constant for the fibers $U_w$ over closed points and $K_{U_w} + a(U_w, L)L$ is rigid for each fiber.

(3) Let $Q$ denote the union of all divisors $D$ in fibers $U_w$ such that $a(D, L) > a(U_w, L)$. Then $Q$ is closed and flat over $W$ and $p: U \setminus Q \to W$ is a topologically locally trivial fibration.

A base change of a good family is defined to be the good family induced via base change by a map $g: T \to W$. We say that $p$ has a good section if there is a section $W \to U \setminus Q$, i.e. there is a section avoiding $Q$.

A good morphism of good families is a diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{f} & U \\
\downarrow & & \downarrow \pi \\
T & \xrightarrow{g} & W
\end{array}
\]

and a relatively big and nef $\mathbb{Q}$-divisor $L$ on $U$ such that $p$ and $q$ are good families of adjoint rigid varieties (with respect to $L$ and $f^*L$ respectively), the relative dimensions of $p$ and $q$ are the same, and $a(Y_t, f^*L) = a(U_w, L)$ for any fiber $Y_t$.

**Lemma 4.6.** Let $p: X \to Z$ be a morphism of projective varieties with $X$ smooth and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Suppose that $p$ has connected fibers and that the general fiber of $p$ is adjoint rigid. Then there is a non-empty open subset $W \subset Z$ such that $p: p^{-1}(W) \to W$ is a good family of adjoint rigid varieties.

**Proof.** Let $W$ denote an smooth open subset of $Z$ over which $p$ is smooth. We construct the family by repeatedly shrinking $W$. After shrinking $W$, by invariance of log plurigenera we may assume that the $a$-invariant is constant and that all fibers over $W$ are adjoint rigid.

For some sufficiently ample divisor $H$ on $Z$ we have $a(X, L + p^*H) = a(X_w, L)$ for a fiber $X_w$ and the Iitaka fibration for $K_X + a(X, L + p^*H)(L + p^*H)$ is the map $p$. Let $Q_+$ denote the closed set which is the union of all subvarieties $Y$ of $X$ satisfying $a(Y, L + p^*H) > a(X_w, L)$. Using a DCC argument we may also ensure that $Q_+$ does not change if we rescale $H$ by a constant $> 1$. Then after further increasing $H$, we may ensure that for each component of $Q_+$ the map to $Z$ factors rationally through the Iitaka fibration defined by the adjoint divisor. After shrinking $W$, this guarantees that for any component of $Q_+$ the $a$-invariant does not change upon intersecting with a fiber. Set $Q$ to be the codimension 1 components of $Q_+$. By shrinking $W$ we may ensure that $p: p^{-1}(W) \cap Q \to W$ is equidimensional. Note that this set now coincides with $Q$ as defined in Definition 4.5. By [Ver76, Corollaire 5.1] after further shrinking $W$ we may guarantee that $p: p^{-1}(W) \setminus Q \to W$ is topologically trivial and $Q$ is flat over $W$.

Suppose that $p: U \to W$ is a good family of adjoint rigid varieties. Let $\mathcal{V} \subset U$ be the complement of the set $Q$ as in Definition 4.5. Suppose that $p$ admits a good section. Then using the fibration exact sequence as in [Shi, Proposition 5.5.1] we obtain

\[\pi_1(\mathcal{V}) = \pi_1(U_w) \times \pi_1(W)\]

for a fiber $\mathcal{V}_w = \mathcal{V} \cap U_w$.

**Lemma 4.7.** Let $p: U \to W$ be a good family of adjoint rigid varieties with a good section. Let $F$ denote a fiber of $p$ and let $\mathcal{V} \subset U$ be the complement of the set $Q$. 


Fix a subgroup \( \Xi \subset \pi_1(\mathcal{V}_w) \). Consider the set of subgroups
\[
\{ G \subset \pi_1(\mathcal{V}) \mid G = \Xi \rtimes H \text{ for some } H \subset \pi_1(W) \}.
\]
This set contains a unique maximal subgroup \( \Upsilon = \Xi \rtimes N \) where \( N \) is the normalizer of \( \Xi \) in \( \pi_1(W) \). Furthermore \( \Upsilon \) is stable under base change: for any morphism \( g : T \to W \) from a smooth variety \( T \), the corresponding subgroup for the family \( \mathcal{U} \times_W T \) is the preimage of \( \Upsilon \) under the natural map \( \pi_1(\mathcal{V} \times_W T) \to \pi_1(\mathcal{V}) \).

Proof. It is clear that \( \Upsilon \) is actually a subgroup and that if \( G = \Xi \rtimes H \) is any other subgroup of the desired form, then \( H \) must be contained in \( N \).

We need to show that \( \Upsilon \) is stable under base change. Let \( g : T \to W \) be a morphism from a smooth variety \( T \) inducing \( g_* : \pi_1(T) \to \pi_1(W) \). It suffices to show that the normalizer of \( \Xi \) in \( \pi_1(T) \) is exactly \( g_*^{-1}N \). To see this, recall from [Shi, Section 5] that the action of a loop \( \gamma \in \pi_1(W) \) on \( \pi_1(\mathcal{V}_w) \) can be computed using the restriction of the family to \( \gamma \). In particular, any loop in \( \pi_1(T) \) will act in the same way as its image in \( \pi_1(W) \), proving the theorem.

Note that since the fundamental group of an algebraic variety is finitely generated, there will only be finitely many subgroups of a given finite index. Thus, if a subgroup \( \Xi \) as above has finite index, its normalizer has finite index in \( \pi_1(W) \) and the maximal subgroup \( \Xi \rtimes N \) also has finite index.

**Lemma 4.8.** Let \( p : \mathcal{U} \to W \) be a good family of adjoint rigid varieties with a good section. There is a finite set of dominant generically quasi-finite good morphisms of good families \( \{ f_i : \mathcal{Y}_i \to \mathcal{U} \} \) with family maps \( q_i : \mathcal{Y}_i \to T_i \) and a closed proper subset \( D \subset W \) such that the following holds. Suppose that \( q : \mathcal{Y} \to T \) is a good family of adjoint rigid varieties admitting a good morphism \( f : \mathcal{Y} \to \mathcal{U} \). Then either \( \tilde{f}(\mathcal{Y}) \) is contained in \( p^{-1}D \), or there is a base change \( \tilde{q} : \tilde{\mathcal{Y}} \to \tilde{T} \) of \( q \) such that the induced \( \tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{U} \) factors rationally through the map \( f_i \) for some \( i \) and the induced map \( \tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}_i \) is generically a good morphism of good families such that a general fiber of \( \tilde{q} : \tilde{\mathcal{Y}} \to \tilde{T} \) is birational to a fiber of \( q_i : \mathcal{Y}_i \to T_i \).

Proof. Let \( \mathcal{V} \) denote the open subset of \( \mathcal{U} \) given by removing the set \( Q \) as in Definition 4.5. By [Shi, Proposition 5.5.1], we know that \( \pi_1(\mathcal{V}) = \pi_1(\mathcal{V}_w) \rtimes \pi_1(W) \). Since \( \mathcal{U}_w \) is adjoint rigid, by [Sen17b] there are only finitely many covers of \( \mathcal{U}_w \) which have the same \( a \)-value and are adjoint rigid and each is ramified along the set \( Q \cap \mathcal{U}_w \). In other words, there is a finite set of finite index subgroups \( \Xi_j \subset \pi_1(\mathcal{V}_w) \) such that for some fiber of \( p \) the corresponding étale cover has a projective closure with the properties above. For each such \( \Xi_j \), consider the corresponding subgroup \( \Upsilon_j = \Xi_j \rtimes N \) as in Lemma 4.7. Note that \( \Upsilon_j \) will always be a subgroup of finite index (as remarked earlier).

Let \( \mathcal{E}_j \) denote the corresponding étale cover of \( \mathcal{V} \). Note that \( \mathcal{E}_j \) admits a morphism to the étale cover \( R_j \to W \) defined by \( N \) and the fibers of this map are irreducible. Let \( r_j : \tilde{E}_j \to R_j \) be the resolution of a completion of \( \mathcal{E}_j \) to a projective family over \( R_j \). There is an open set \( R_j^c \subset R_j \) over which \( r_j \) has smooth irreducible fibers and the \( a \)-value and Iitaka dimension of the fibers is constant. We enlarge \( D \) by adding the image of \( R_j \setminus R_j^c \) for each \( j \). If the remaining fibers \( \tilde{E}_j^c \to R_j^c \) are adjoint rigid and have the same \( a \)-value as the fibers of \( p \), then after possibly shrinking \( R_j^c \) further we obtain a good family of adjoint rigid varieties \( q_j : \mathcal{Y}_j \to T_j \). In this case we include \( q_j \) in our set of families and further enlarge \( D \) by adding
the image of $R_j \setminus T_j$. If the remaining fibers fail to be adjoint rigid or fail to have the same $a$-value as the fibers of $p$, we simply ignore the family $\tilde{E}_j^i \rightarrow R_j^i$.

We have now constructed a finite set $\{f_i : Y_i \rightarrow U\}$ of good morphisms of good families and a set $D$. We will show that this set satisfies the condition in the statement of the theorem. Suppose we have a morphism $f : Y \rightarrow U$ as in the statement of the theorem. After shrinking $T$ and performing a base change (which we absorb in the notation) we may suppose that $q : Y \rightarrow T$ admits a good section. By taking the section general and further shrinking $T$, we may also suppose that its $f$-image in $U$ is disjoint from $Q$. Let $\mathcal{V}_Y$ denote the open subset obtained by removing the closed subset $Q_Y$ as in Definition 4.5. Thus for a fiber $\mathcal{V}_{Y,t}$ of $q$ we have

$$\pi_1(\mathcal{V}_Y) = \pi_1(\mathcal{V}_{Y,t}) \times \pi_1(T).$$

Note however that this semidirect product structure need not be compatible with the semidirect product structure of $\pi_1(\mathcal{V})$.

Fix a general fiber $U_w$ of $p$, let $t$ be a point in $T$ lying over $w$, and let $\Xi_j$ denote the subgroup defined by the image of $\pi_1(f^{-1}(\mathcal{V}))$ in $\pi_1(\mathcal{V}_w)$. Note that $\pi_1(T)$ maps into $\pi_1(\mathcal{V})$ by composing the good section of $q$ with the $f$-image of $\mathcal{V}$ (recall that by construction the $f$-image of the good section avoids $Q$). Consider the finite index subgroup $M \subset \pi_1(T)$ which is the pullback of $\Xi_j \times N$. Let $\tilde{q} : \tilde{Y} \rightarrow \tilde{T}$ be defined by the base-change of $q$ over the cover of $T$ defined by $M$. Since $\tilde{f}^{-1}(\mathcal{V}) \rightarrow \tilde{T}$ is a locally topologically trivial fibration, by using the same good section constructed above we have an identification

$$\pi_1(\tilde{f}^{-1}(\mathcal{V})) = \pi_1(\tilde{f}^{-1}(\mathcal{V})) \times \pi_1(T).$$

Note that every element in $\tilde{f}_*\pi_1(\tilde{f}^{-1}(\mathcal{V}))$ will be a product of an element in $\Xi_j \times \{1\} \subset \pi_1(\mathcal{V})$ with an element in $\tilde{f}_*\pi_1(\tilde{T})$, so by construction this set is contained in $\Xi_j \times N$. By the lifting property, the map from $\tilde{f}^{-1}(\mathcal{V}) \rightarrow \mathcal{V}$ factors through one of the covers defined by $\Xi_j \times N$. For any fiber of $\tilde{q}$, if the corresponding fiber of $r_j$ is irreducible then it must be adjoint rigid and have the same $a$-value as a fiber of $p$. Thus the map $\tilde{f}$ will either factor rationally through one of the $f_j$ or will map into $p^{-1}D$.

As a consequence, we can again apply the argument of Lemma 4.8 to the preimage of any component of the closed set $D$ constructed there. Arguing by Noetherian induction, we conclude:

\textbf{Theorem 4.9.} Let $p : U \rightarrow W$ be a good family of adjoint rigid varieties with a good section. There is a finite set of generically quasi-finite good morphisms of good families $\{f_i : Y_i \rightarrow U\}$ with family maps $q_i : Y_i \rightarrow T_i$ such that the following holds. Suppose that $q : Y \rightarrow T$ is a good family of adjoint rigid varieties admitting a good morphism $f : Y \rightarrow U$. Then there is a base change $\tilde{q} : \tilde{Y} \rightarrow \tilde{T}$ of $q$ such that the induced $\tilde{f} : \tilde{Y} \rightarrow U$ factors rationally through the map $f_i$ for some $i$ and a general fiber of $\tilde{q}$ is birational to a fiber of $q_i$.

As a consequence, we prove a finiteness statement for breaking thin maps.

\textbf{Theorem 4.10.} Let $X$ be a uniruled smooth projective variety and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. There is a finite set of thin maps $\{f_i : Y_i \rightarrow X\}$ with $a(Y_i, f_i^*L) \geq a(X, L)$ satisfying the following property. For any thin map $f : Y \rightarrow X$, after an Iitaka base change to obtain a variety $\tilde{Y}$ the induced map $\tilde{f} : \tilde{Y} \rightarrow X$ will either factor rationally through some
\( f_j \) or will have image contained in \( B_+(L) \). Furthermore, in the first case we have

\[
a(Y, f^*L) = a(\tilde{Y}, \tilde{f}^*L) \leq a(Y_j, f_j^*L)
\]

and if equality of \( a \)-values is achieved then

\[
b(Y, f^*L) \leq b(\tilde{Y}, \tilde{f}^*L) \leq b(Y_j, f_j^*L).
\]

Note that Theorem 1.4 follows immediately from this theorem.

**Proof.** Repeating the argument of Theorem 4.2, there is a locally closed subset of \( \text{Chow}(X) \) parametrizing adjoint rigid subvarieties of \( X \) with \( a \)-value at least as large as \( X \) which are not contained in \( B_+(L) \). Using Lemma 4.6 and a Noetherian induction argument, we can stratify this subset of \( \text{Chow}(X) \) and resolve singularities to construct a finite set of good families which together parametrize (birational models of) all adjoint rigid varieties of \( X \) with \( a \)-value at least as large as \( X \) which are not contained in \( B_+(L) \).

After a finite base change we may ensure each family has a rational section. For each, we can shrink the base to ensure it has a good section and split off the complement as a new good family of adjoint rigid varieties. Using a Noetherian induction argument, we construct a finite collection of good families with good sections. To each such family we apply Theorem 4.9. The result is a finite collection of good families \( \{q_i : \mathcal{Y}_i \rightarrow T_i\} \) with maps \( g_i : \mathcal{Y}_i \rightarrow X \). For each such family, we make an additional base change to ensure that \( g_i \) is not birational and to kill the monodromy action of \( \pi_1(T_i) \) on the \( \text{Néron-Severi} \) group of a general fiber (and absorb this base change into the notation).

We define the thin maps \( \{f_i : Y_i \rightarrow X\} \) as follows. For each \( \mathcal{Y}_i \) set \( D_i \) to be the closure of \( g_i(\mathcal{Y}_i) \). If \( a(D_i, L|_{D_i}) \) agrees with the \( a \)-value of the fibers of \( \mathcal{Y}_i \), then \( [LT17b, \text{Proposition 4.14}] \) shows that \( g_i \) is generically quasi-finite. We differentiate the \( \mathcal{Y}_i \) satisfying this property by calling them “allowable families.” If \( \mathcal{Y}_i \) is not allowable, i.e. \( a(D_i, L|_{D_i}) \) is larger than the \( a \)-values of the fibers of \( \mathcal{Y}_i \), then \( D_i \) must be a proper closed subvariety of \( X \) by Theorem 4.1 and we add the inclusion \( D_i \hookrightarrow X \) as one of our families.

Now suppose \( f : Y \rightarrow X \) is any thin map satisfying \( a(Y, f^*L) \geq a(X, L) \) and whose image is not contained in \( B_+(L) \). After resolving we may suppose \( Y \) is smooth and admits a morphism \( q : Y \rightarrow T \) corresponding to the Itakura fibration for \( K_Y + a(Y, f^*L)f^*L \). We may shrink \( T \) to suppose the family is a good family. The map \( f : Y \rightarrow X \) will yield a map \( T \rightarrow \text{Chow}(X) \) whose image is contained in the locus parametrizing adjoint rigid subvarieties with \( a \)-value at least as large as \( a(X, L) \). Thus after shrinking \( T \) further, we will obtain a good map of good families \( Y \rightarrow \mathcal{U}_j \) for some \( j \). Following through the construction of the \( \mathcal{Y}_i \) above, Theorem 4.9 shows that after perhaps shrinking \( T \) further and making an additional base change the map \( Y \rightarrow \mathcal{U}_j \) will factor rationally through some \( \mathcal{Y}_j \). If \( \mathcal{Y}_j \) is an allowable family, then \( f \) will factor rationally through the corresponding \( Y_j \). When \( \mathcal{Y}_j \) is not allowable, then \( f \) factors through the inclusion \( D_j \hookrightarrow X \).

We next prove the inequalities for \( a \)-values. Suppose first that the family \( \mathcal{Y}_j \) constructed above from \( Y \) is an allowable family. Since the \( a \)-values of \( Y \) and \( \tilde{Y} \) are the same as the \( a \)-value of a general fiber of their Itakura fibrations, clearly \( Y \) and \( \tilde{Y} \) have the same \( a \)-values.

Next, note that a general fiber of the Itakura fibration for \( \tilde{Y} \) maps birationally onto a fiber of \( \mathcal{Y}_j \) under its family map. Since the \( a \)-invariant is constant for the fibers of \( q_j \) (since it is a good family), we see that \( Y_j \) is dominated by subvarieties with the same \( a \)-value as \( \tilde{Y} \). Thus
\[ a(\tilde{Y}, \tilde{f}^*L) \leq a(Y_j, f_j^*L) \] if \( \mathcal{Y}_j \) is not allowable, then as explained before the map \( f \) factors through the inclusion \( D_j \hookrightarrow X \) where \( D_j \) has higher \( a \)-value.

We next prove the inequalities for \( b \)-values. Equality of \( a \)-values will only occur when \( \mathcal{Y}_j \) is an allowable family. Note that \( Y \) and \( \tilde{Y} \) have the same general fibers for their Iitaka fibration, so the \( b \)-values are determined by the monodromy action and by the restriction of horizontal rigid components. It is clear that the monodromy action can only decrease upon an Iitaka base change proving the first inequality. Next, note that a general fiber of the Iitaka fibration for \( \tilde{Y} \) maps birationally onto a fiber of \( Y_j \) under its family map. Recall that the monodromy action is trivial on the smooth fibers of \( Y_j \) by construction. Furthermore, by ([Nak04, III.1.10 Proposition]) each irreducible component of the adjoint rigid divisor on a fiber of \( Y_j \) is the restriction of a different irreducible component of the relative adjoint rigid divisor. Thus by (the analogue over \( \overline{C} \) of) Corollary 2.7 we have

\[ b(\tilde{Y}, \tilde{f}^*L) \leq b(\tilde{Y}_t, \tilde{f}^*L|_{\tilde{Y}_t}) = b(Y_{j,t}, f_j^*L|_{Y_{j,t}}) \leq b(Y_j, f_j^*L) \]

where \( \tilde{Y}_t \) and \( Y_{j,t} \) denote fibers of the good families. \( \square \)

4.3. Other base fields. In this subsection we extend the results from the previous section to other base fields.

Let \( F \) be an arbitrary algebraically closed field of characteristic 0. First suppose that we have an embedding \( F \hookrightarrow \mathbb{C} \). Suppose that \( p: \mathcal{U} \to W \) is a family of adjoint rigid varieties whose base change to \( \mathbb{C} \) is a good family with a good section. Note that the set \( Q_{\mathbb{C}} \subset \mathcal{U}_{\mathbb{C}} \) as in Definition 4.5 descends to \( F \). Set \( \mathcal{V} = \mathcal{U}\setminus \mathcal{Q} \). Fix a fiber \( \mathcal{U}_w \); we claim that

\[ \pi_1(\mathcal{V}) = \pi_1^\text{et}(\mathcal{V} \cap \mathcal{U}_w) \rtimes \pi_1^\text{et}(\mathcal{W}) \]

Indeed, since fundamental groups over \( \mathbb{C} \) are finitely generated, semidirect products commute with profinite completions so we can use the comparison theorem of étale and topological fundamental groups over \( \mathbb{C} \). Furthermore, subgroups of \( \pi_1^\text{et}(\mathcal{V}) \) of finite index are in bijection with subgroups of \( \pi_1(\mathcal{V}_{\mathbb{C}}) \) of finite index. Thus one can repeat the arguments of Section 4.2 using the algebraic fundamental group. If \( F \) is an arbitrary algebraically closed field of characteristic 0, one can find a subfield \( F' \subset F \) which admits an embedding \( F' \hookrightarrow \mathbb{C} \) such that all objects of interest are defined over \( F' \).

Next we discuss the case when our ground field \( F \) is a number field. We start with a lemma we will use frequently throughout the paper.

**Lemma 4.11** ([Che04]). Let \( f: Y \dashrightarrow X \) be a dominant generically finite rational map between normal projective varieties defined over a number field \( F \). Then there exists a birational modification \( f': Y' \to X \) of \( f \) such that \( Y' \) is smooth and projective and \( \text{Bir}(\overline{Y'}/\overline{X}) = \text{Aut}(\overline{Y}/\overline{X}) \).

Furthermore, if we fix a big and nef \( \mathbb{Q} \)-divisor \( L \) on \( X \), then we may assume that the Iitaka fibration for \( K_{Y'} + a(Y', f'^*L) \) is a morphism.

In particular, any twist of \( f: Y \dashrightarrow X \) is birational to a twist of \( f': Y' \to X \).

**Proof.** We first replace \( Y \) by a normal birational model which admits a morphism to \( X \). We then replace \( Y \) by its Stein factorization, so we may assume \( \text{Bir}(\overline{Y}/\overline{X}) = \text{Aut}(\overline{Y}/\overline{X}) \). Let \( F'/F \) be a finite Galois extension such that all automorphisms in \( G = \text{Aut}(\overline{Y}/\overline{X}) \) are defined over \( F' \). Then \( G \rtimes \text{Gal}(F'/F) \) acts on \( Y_{F'} \). We resolve singularities equivariantly and take
the quotient by the Galois group $\text{Gal}(F'/F)$ to obtain a smooth variety $Y'$ satisfying the desired condition on automorphism groups.

We still must prove the last statement. Since the space of sections is preserved by changing the ground field, by [BCHM10] the pair $(Y', a(Y', f^*L)f'^*L)$ admits a canonical model $\pi : Y' \to T$. Choose the same field extension $F'/F$. Then the morphism $\pi_{F'} : Y_{F'} \to T_{F'}$ is equivariant for the group $G \rtimes \text{Gal}(F'/F)$. Thus we may take another equivariant resolution and quotient by the Galois action to ensure that $\pi$ is a morphism. □

**Definition 4.12.** Fix a number field $F$. A good family of adjoint rigid varieties over $F$ is an $F$-morphism $p : U \to W$ of smooth quasi-projective varieties and a relatively big and nef $\mathbb{Q}$-divisor $L$ on $U$ such that the base-change to the algebraic closure is a good family of adjoint rigid varieties over each component of the base.

Let $\overline{Q}$ denote the subset of $\overline{U}$ as in Definition 4.5. Note that $\overline{Q}$ descends to $F$ by Proposition 2.3. We denote this set by $Q$. A good section of a good family over $F$ is a section avoiding $Q$.

It is natural to wonder whether one can prove a version of Theorem 4.10 over a number field which takes twists into account. We will instead prove a version of Lemma 4.8 which keeps track of the behavior of rational points. We weaken the hypotheses slightly by allowing $Y$ and the $Y_i$ to be closures of good families.

**Lemma 4.13.** Let $p : U \to W$ be a good family of adjoint rigid varieties over a number field $F$ admitting a good section $\sigma$. There is a proper closed subset $C \subseteq U$ and a finite set of dominant generically finite and proper morphisms $\{f_i : Y_i \to U\}$ defined over $F$, with projective maps $q_i : Y_i \to T_i$ which are generically good families, such that the following holds.

Suppose that $q : Y \to T$ is a projective surjective morphism of varieties over $F$ and that we have a diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & U \\
q \downarrow & & \downarrow p \\
T & \xrightarrow{g} & W
\end{array}
\]

satisfying the following properties:

1. There is some open subset $T^\circ \subset T$ such that $Y$ is a good family of adjoint rigid varieties over $T^\circ$ and the map $f : q^{-1}(T^\circ) \to U$ is a good morphism.

2. There is a rational point $y \in Y(F)$ contained in the smooth locus of $Y$ such that $f(y) \notin C$.

Then for some index $j$ there will be a twist $f_j^\sigma : Y_j^\sigma \to U$ such that $f(y) \in f_j^\sigma(\mathcal{Y}_j^\sigma(F))$. Furthermore, after a base change over $T$ the induced map $\tilde{f} : \tilde{Y} \to U$ will factor rationally through $f^\sigma_j$ and a general fiber of the structure map for $\tilde{Y}$ will map birationally to a fiber of the structure map for $\mathcal{Y}_j^\sigma$.

**Proof.** Let $Q$ denote the closed subset of $U$ as in Definition 4.12 and let $V$ denote its complement. Let $\overline{D} \subset \overline{W}$ be the proper closed subset obtained by applying Lemma 4.8 over the algebraic closure. After including its Galois conjugates, we may assume that $D$ is defined over the ground field. We then start by setting $C = p^{-1}(D) \cup Q$; we will increase $C$ later.
We next construct families $\mathcal{Y}_j$ as follows. We may suppose that $\mathcal{V}$ admits a rational point since otherwise the statement is vacuous. Let $v'$ be a point on $\mathcal{V}$ such that $w = \pi(v')$ and $v' = \sigma(w)$. As in Lemma 4.8 each $\Xi_{ij} \subset \pi_{ij}^{\text{et}}(\overline{V} \cap \overline{U}_w, v')$ gives a normalizer $N_{ij} \subset \pi_{ij}^{\text{et}}(W, w)$. We take the maximum subgroup $\tilde{N}_j \subset N_j$ such that the monodromy of $\tilde{N}_j$ on

$$\pi_1^{\text{et}}(\overline{V} \cap \overline{U}_w, v') \Bigg/ \bigcap_{g \in \pi_1^{\text{et}}(\overline{V} \cap \overline{U}_w, v')} g\Xi_j g^{-1}$$

is trivial. Then $\tilde{\Xi}_j = \Xi_j \times \tilde{N}_j$ gives us étale covers $s_j : \overline{E}_j \to \overline{V}$. Suppose the map $\overline{E}_j \to \overline{V}$ descends to a morphism $E_j \to \mathcal{V}$ over $F$ in such a way that $E_j$ admits a rational point. Choose one such $F$-model $E_j$ with a rational point and let $T_j$ denote the Stein factorization of the map $E_j \to W$. We then define $\mathcal{Y}_j$ over $F$ by taking a smooth compactification of the fibers of the map $E_j \to T_j$. By choosing the compactification appropriately we may guarantee the map $E_j \to \mathcal{V}$ extends to a map $s_j : \mathcal{Y}_j \to \mathcal{U}$.

We make a few additional changes to the family. After taking a Galois closure, we may assume that $T_j/W$ is Galois; we absorb this change into the notation. We let $T_j^\circ$ denote an open subset so that $\mathcal{Y}_j^\circ = \pi_j^{-1}(T_j^\circ)$ is a good family of adjoint rigid varieties. By applying Lemma 4.11, we may assume that $\operatorname{Bir}(\overline{Y}_j/\overline{U}) = \operatorname{Aut}(\overline{Y}_j/\overline{U})$. Note that the birational modification may force us to shrink $T_j^\circ$ but does not affect the remaining properties. After possibly shrinking further, we may guarantee that $T_j^\circ$ is étale over an open set $W^\circ$ and that $s_j : s_j^{-1}(\overline{V}^\circ) \to \overline{V}^\circ$ is étale where $\overline{V}^\circ$ denotes the preimage of $\overline{W}^\circ$. After all these changes we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{Y}_j & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \\
T_j & \longrightarrow & W
\end{array}$$

We enlarge $C$ by adding $s_j(\mathcal{Y}_j \setminus \mathcal{Y}_j^\circ) \cup s_i(\pi_i^{-1}(B_i)) \cup E_j$ where $E_j$ is the branch locus of $s_j : \mathcal{Y}_j \to \mathcal{U}$ and $B_j$ is the branch locus of $T_j \to W$.

Suppose now that $\overline{E}_j \to \overline{V}$ fails to descend to $F$ in such a way that it admits a rational point. At least over $\overline{F}$ we can repeat the construction to obtain a morphism of varieties $\overline{P}_k \to \overline{U}$ over $\overline{F}$ with structure maps $\overline{P}_k \to \overline{S}_k$. We enlarge $C$ by adding $s_k(\overline{P}_k \setminus \overline{P}_k^\circ) \cup s(\pi_1^{-1}(B_k)) \cup E_k$. By taking the union with Galois conjugates we may assume that $C$ is defined over the ground field.

Note that the families constructed here are independent of the initial choice of $v'$. Indeed, we only used $v'$ to define geometric covers over $\overline{F}$; all the other choices in the construction were obtained intrinsically from the geometry of this finite set of covers.

Now we prove the universal property for these families. Assume that $f : \mathcal{Y} \to \mathcal{U}$ is a morphism as in the statement. Our goal is to show that $f(y) \in s_f^\circ(\mathcal{Y}_j^\circ(F))$ for some twist $\sigma$. Let $t = q(y)$. We may find a generically finite surjective base change $T' \to T$ with a rational point $t' \in T'(F)$ mapping to $t$ such that $T'$ is smooth at $t'$ and the main component $\mathcal{Y}_{T'}$ admits a rational section $\tau$ such that $(y, t') = \tau(t')$. We may furthermore ensure that the image of $\tau$ intersects the smooth locus of $\mathcal{Y}_{T'}$. Let $\tilde{T} \to T'$ be the blow up at $t'$ and consider the main component $\tilde{\mathcal{Y}}_{\tilde{T}}$ of the base change by $\tilde{T}$. Let $\mathcal{Y}_{\tilde{T}}$ be a resolution of $\tilde{\mathcal{Y}}_{\tilde{T}}$ chosen in such a way that $\tilde{\mathcal{Y}}_{\tilde{T}}$ still admits a rational section $\tau$. Note that $\tau$ is well-defined.
on the generic point of the exceptional divisor lying over \( t' \). Thus by taking the image under \( \tau \) of a suitable rational point \( \tilde{t} \) in the exceptional divisor we obtain a rational point \( y' \in \tilde{Y}_{\tilde{t}} \) mapping to \( y \) and \( \tilde{t} \). Let \( v \) be the image of \( y \) in \( V \) and set \( w = \pi(v) \) and \( v' = \sigma(w) \).

Geometrically, the argument of Lemma 4.8 shows that some Iitaka base change of \( \tilde{Y}_{\tilde{t}} \) admits a rational map to \( \tilde{Y}_j \) for some \( j \) or \( \tilde{P}_k \) for some \( k \). Assume for a contradiction that it factors through \( \bar{P}_k \). We claim that if we take the Stein factorization \( Y' \) of the map of fibers \( \tilde{Y}_{\tilde{t}} \to \tilde{U}_w \) and then base change to \( F \) the result is birational to the adjoint rigid variety \( \bar{P}_{k, \pi} \)

where \( \bar{w} \) is some suitably chosen preimage of \( w \). Indeed, first choose an open subset \( \tilde{T} \) of \( T \) such that the image of this set in \( \tilde{W}_\pi \) as defined above and the \( \tau \)-image of this set lies in the preimage of \( \tilde{V}_\pi \). Let \( \tilde{T}' \) denote the étale cover of \( \tilde{Y}_\pi \) defined by the finite index subgroup of \( \pi_1(\tilde{V}_\pi) \) constructed by pulling back under \( \tau \) the subgroup of \( \pi_1(\tilde{V}_\pi) \) corresponding to the étale cover defined by \( \Xi_k \). For the open subset of \( \tilde{T} \) over which we have a good family, just as in Lemma 4.8 we know that the main component of the base change \( \tilde{Y}_\pi \) admits a rational map to \( \bar{P}_k \). Since the map \( \tilde{T}' \to \tilde{T} \) is étale, \( \tilde{Y}_\pi \) is smooth in a neighborhood of the fiber \( \tilde{Y}_{\tilde{t}} \prime \). Let \( \tilde{Y}_\pi \) denote a smooth resolution of the rational map to \( \bar{P}_k \). The fiber \( \tilde{Y}_{\tilde{t}} \prime \) maps to some fiber \( \bar{P}_{k, \pi} \). Thus the Stein factorization of \( \tilde{Y}_{\tilde{t}} \prime \to \tilde{U}_w \) is birational to \( \bar{P}_{k, \pi} \). Note that this map also factors through our original fiber \( \tilde{Y}_{\tilde{t}} \) and that the first step of this factorizing has connected fibers. Thus the Stein factorization of our original fiber over \( F \) is birational to \( \bar{P}_{k, \pi} \). Since Stein factorization commutes with base change to the algebraic closure our assertion follows.

This implies that the subgroup \( \Xi_k \) defining \( \bar{P}_k \) admits a corresponding extension \( \Xi_{\bar{k}} \subset \pi_1(\bar{V}_w, \nu') \) in the arithmetic fundamental group. We next show that this means that \( s_{k, 1}(\bar{V}) \) must descend to the ground field. Indeed, using the fact that we constructed \( \tilde{N}_k \) to have a trivial monodromy action on the cosets of the intersection of conjugates of \( \Xi_k \), one can show that \( \tilde{N}_k \cdot \Xi_k = \Xi_{\bar{k}} \cdot \tilde{N}_k \) so that one may define \( s_{k, 1}(\bar{V}) \to \bar{V} \) using \( \tilde{N}_k \cdot \Xi_{\bar{k}} \) which is an extension of \( \bar{Y}_k \). Moreover we claim that \( s_{k, 1}(\bar{V}) \) admits a fiber birational to \( \bar{Y}' \) which is an isomorphism on an open neighborhood of the image of \( y \). Indeed, since \( T_k \) corresponds to an extension of \( \tilde{N}_k \) its fundamental group comes with a splitting of the Galois group which is compatible with the splitting of \( \pi_1(\bar{V}_j, \nu) \). Since \( T_k \) is Galois over \( W \) it comes with a rational point \( t_j \) mapping to \( w \). By comparing fundamental groups, we see that the fiber over \( t_j \) is birational to the variety defined by \( \Xi_{\bar{k}} \) as claimed. We conclude that \( s_{k, 1}(\bar{V}) \) admits a rational point coming from \( y \). However, the fact that the geometric model descends to the ground field with a rational point contradicts our definition of the \( \bar{P}_k \). Thus some base change of \( \tilde{Y}_{\tilde{t}} \) admits a rational map to \( \tilde{Y}_j \) for some \( j \).

Next we would like to show that some twist of \( Y_j \) contains a rational point \( y_j \) mapping to \( v \). As we discussed before, the Stein factorization \( Y' \) of \( \tilde{Y}_{\tilde{t}} \to \tilde{U}_w \) is birational to an étale cover of \( V \cap \tilde{U}_w \) defined by \( \Xi_j \subset \pi_1(\tilde{V} \cap \tilde{U}_w, \nu') \). Just as in the previous paragraph, we construct a cover \( \bar{Y}_{\bar{v}} \to \bar{U} \) with the family structure \( \bar{Y}_{\bar{v}} \to T_{\bar{v}} \). This comes with a rational point \( t_j \) on \( T_{\bar{v}} \) mapping to \( v \) and the two varieties \( \bar{Y}' \) and \( \bar{Y}_{\bar{v}} \) are birational over the ground field. Thus \( \bar{Y}_{\bar{v}} \) comes with a rational point \( y_j \) mapping to \( v \), proving the first claim of Lemma 4.13.
We conclude by proving a factoring property over $F$. Define $T^\nu$ to be the main component of the base change

$$
\tilde{T} \times_\mathcal{U} \mathcal{Y}^\nu_j \rightarrow \mathcal{Y}^\nu_j
$$

where the map on the bottom is the composition of the rational section to $\tilde{\mathcal{Y}}_T$ and the map to $\mathcal{U}$. Since $Y^\sigma_j \rightarrow \mathcal{U}$ is étale on a neighborhood of the image of $\tilde{t}$ and admits a rational point mapping to the image of $\tilde{t}$, we see that $T^\nu$ admits a rational point $t^\nu$ mapping to $\tilde{t}$ and that the base change $\tilde{\mathcal{Y}}_{T^\nu}$ is smooth at any point of the fiber at $t^\nu$. Arguing as in Lemma 4.8, the map $\tilde{\mathcal{Y}}_{T^\nu} \rightarrow X$ factors rationally through the twist $\mathcal{Y}^\sigma_j$ after base changing to $\overline{F}$.

Then it follows from the lifting property that the map $\tilde{\mathcal{Y}}_{T^\nu} \rightarrow X$ factors rationally through $\mathcal{Y}^\sigma_j$ over the ground field. Indeed, by the lifting property over $\overline{F}$ one may find a rational map $\overline{h} : \tilde{\mathcal{Y}}_{T^\nu} \rightarrow \overline{\mathcal{Y}}^j$ mapping $(y^\nu, t^\nu)$ to the point $y_j$ constructed above. Let $s$ be an element of the Galois group. Then both $\overline{h}$ and $\overline{h}^s$ are lifts of the same map to $\mathcal{U}$ and they are both mapping $(y^\nu, t^\nu)$ to $y_j$. Thus $\overline{h} = \overline{h}^s$ by the uniqueness of the lift. Thus our assertion follows. \hfill $\square$

5. Twists

In this section we work over a number field. We start with a lemma encapsulating our application of Hilbert’s Irreducibility Theorem.

**Lemma 5.1.** Let $f : Y \rightarrow X$ be a surjective generically finite morphism of normal geometrically integral projective varieties defined over a number field $F$. Suppose that the extension of geometric function fields $\mathcal{F}(\overline{Y})/\mathcal{F}(\overline{X})$ is Galois with Galois group $G$. Then there is a thin set of points $Z \subset X(F)$ such that if $x \in X(F) \setminus Z$ then $f^{-1}(x)$ is irreducible and the corresponding extension of residue fields is Galois with Galois group $G$.

**Proof.** According to Lemma 4.11 there is a birational model $f' : Y' \rightarrow X$ of $f$ such that $\text{Aut}(\overline{Y}/\overline{X}) = \text{Bir}(\overline{Y}/\overline{X}) = G$. Since $Y$ and $Y'$ only differ in a closed set, it suffices to prove the statement under the additional assumption that the geometric automorphism group coincides with the geometric birational automorphism group.

Then there is a finite field extension $F'/F$ such that $\text{Aut}(\overline{Y}/\overline{X}) = \text{Aut}(Y_{F'}/X_{F'})$. Note that $f_{F'} : Y' \rightarrow X'$ is a Galois covering over an open subset $X'^\sigma$ of $X'$. Applying the Hilbert Irreducibility Theorem to this open set and adding on $(X' \setminus X'^{\sigma})(F')$, we obtain a thin subset $Z' \subset X(F')$ satisfying the desired property with respect to $F'$-points. The intersection $Z = Z' \cap X(F)$ is also thin, giving the result. \hfill $\square$

**Theorem 5.2.** Let $X$ be a geometrically uniruled normal projective variety over a number field $F$ with a big and nef $\mathbb{Q}$-divisor $L$ on $X$. Suppose that $f : Y \rightarrow X$ is a dominant generically finite morphism from a normal projective variety $Y$. As $\sigma$ varies over all $\sigma \in H^1(F, \text{Aut}(\overline{Y}/\overline{X}))$ such that $Y^\sigma$ is irreducible,

$$(a(X, L), b(F, X, L)) \leq (a(Y^\sigma, (f^\sigma)^*L), b(F, Y^\sigma, (f^\sigma)^*L)),$$
and \( f^\sigma \) is face contracting the set
\[
Z = \bigcup_{\sigma} f^{\sigma}(Y^{\sigma}(F)) \subset X(F)
\]
is contained in a thin subset of \( X(F) \).

**Proof.** We start with several simplifications. If \( X \) is not geometrically integral, then \( X(F) \) is thin since it is contained in a proper closed subset of \( X \). So we may suppose \( X \) is geometrically integral.

Suppose that \( Y \) is not geometrically integral. Then any twist \( Y^{\sigma} \) of \( Y \) which has a rational point not contained in \( \text{Sing}(Y^{\sigma}) \) must be reducible. Thus, the set \( Z \) is contained in the thin set \( (f(\text{Sing}(Y)))(F) \). So from now on we assume that \( Y \) is geometrically integral.

If \( f: Y \to X \) induces an extension of geometric function fields \( F(\overline{Y})/F(\overline{X}) \) that is not Galois, then we may conclude by [LT17b, Proposition 8.2]. So we may assume that \( f: Y \to X \) is Galois.

Suppose \( Y \) is not smooth. Choose a birational model \( f': Y' \to X \) as in Lemma 4.11. Note that the statement for \( f' \) implies the statement for \( f \). Indeed, if \( B \) denotes the locus where the rational map \( \phi : Y \dasharrow Y' \) is not defined then
\[
\bigcup_{\sigma} f^{\sigma}(Y^{\sigma}(F)) \subset \bigcup_{\tau} f'^{\tau}(Y'^{\tau}(F)) \cup f(B)(F).
\]
So from now on we assume that \( Y \) is smooth and \( G = \text{Aut}(\overline{Y}/\overline{X}) = \text{Bir}(\overline{Y}/\overline{X}) \).

Since \( f \) is dominant the only case we need to consider is when \( a(Y, f^*L) = a(X, L) \). Suppose that \( F_1/F \) is a finite extension so that \( N^1(\overline{Y}) = N^1(Y_{F_1}) \). By Lemma 5.1 there is a thin set \( Z_1 \subset X(F_1) \) such that for any point \( x \in X(F_1) \setminus Z_1 \) the fiber \( f^{-1}(x) \) is irreducible over \( F_1 \) and the corresponding extension of residue fields is Galois with Galois group \( G \). We let \( Z_1 = Z_1' \cap X(F) \) which is a thin set.

We prove that if a twist \( \sigma \) satisfies \( f^{\sigma}(Y^{\sigma}(F)) \not\subset Z_1 \) then \( b(F, Y^{\sigma}, L) \leq b(F, X, L) \) and if equality is achieved then \( f^{\sigma} \) is not face contracting.

First of all, it is easy to see that \( N^1(\overline{Y})^G \) is spanned by \( N^1(\overline{X}) \) and \( f \)-exceptional divisors. Let \( F_X \) be the minimal face of \( \overline{\text{Eff}}^1(\overline{X}) \) containing \( a(X, L) + K_X \) and \( F_Y \) be the minimal face of \( \overline{\text{Eff}}^1(\overline{Y}) \) containing \( a(X, L) + K_Y \). Since \( F_Y \) contains all \( f \)-exceptional effective divisors, we conclude that the natural map
\[
N^1(\overline{X})/\langle F_X \rangle \to N^1(\overline{Y})^G/\langle F_Y \rangle^G
\]
is surjective.

Now suppose that \( x \in X(F) \setminus Z_1 \) and that there is a point \( y \in Y^{\sigma}(F) \) with \( f^{\sigma}(y) = x \). Since a 1-cocycle corresponding to the twist \( \sigma \) must induce a surjection from \( \text{Gal}(\overline{F}/F_1) \) onto \( G \), we conclude that this Galois group acts on \( N^1(\overline{Y}^{\sigma}) \) at least as \( G \) does. Combining with our earlier discussion, we see that
\[
N^1(X)/\langle F_X \rangle \to N^1(Y^{\sigma})/\langle F_Y^{\sigma} \rangle
\]
is surjective. This implies that \( b(F, X, L) \leq b(F, Y^{\sigma}, f^{\sigma*}L) \) and if the equality is achieved, then the cover is not face contracting. Thus our assertion follows.

□
In this section we work over a number field $F$. Our next lemma shows that the $b$-value of a pair $(X, L)$ can only increase upon taking twists of an Iitaka base change. Thus in the proof of Theorem 3.5 it is harmless to replace any variety $Y$ by all twists of an Iitaka base change given by a Galois extension. This will allow us to apply the factoring results from Section 4.

**Lemma 6.1.** Let $X$ be a geometrically uniruled smooth projective variety over a number field $F$ and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Suppose that $K_X + a(X, L)L$ has positive Iitaka dimension and let $\pi : X \to Z$ denote the Iitaka fibration to a projective variety $Z$. Suppose that $g : T \to Z$ is any dominant generically finite map from a projective variety $T$ and set $Y$ to be a projective closure of the main component of $X \times_Z T$. Let $f : Y \to X$ denote the corresponding dominant map. Then every twist $f^\sigma : Y^\sigma \to X$ of $f$ with $Y^\sigma$ irreducible satisfies $b(F, Y^\sigma, f_{\sigma}\ast L) \geq b(F, X, L)$.

**Proof.** Note that in this situation we have $a(Y^\sigma, f_{\sigma}\ast L) = a(X, L)$. Since the $b$-invariant is preserved by birational equivalence, we may replace $X$ by a birational model (which we continue to call $X$ by abuse of notation) and replace $Y$ by a birational model as in Lemma 4.11 (which we continue to call $Y$ by abuse of notation) so that $\pi$ is a morphism and $\text{Bir}(\overline{Y/X}) = \text{Aut}(\overline{Y/X})$. Note that the statement for our new birational model implies the statement for our original variety.

Let $\mathcal{F}$ be the minimal supported face of $\overline{\text{Eff}}^1(X)$ containing $K_X + a(X, L)L$. Note that every $\pi$-vertical effective divisor $D$ has class contained in $\mathcal{F}$, since $K_X + a(X, L)L$ is more effective than the pullback of an ample divisor from $Z$.

We claim that $K_{Y^\sigma} + a(X, L)L_{f_{\sigma}\ast L}$ is contained in $(f_{\sigma})^{-1}\mathcal{F}$. We can write $K_{Y^\sigma} = f_{\sigma\ast}K_X + R$ where $R$ is the ramification divisor. Every component of $R$ that is not contracted by $f_{\sigma}$ must be vertical for the map $Y \to T$, so that $f_{\sigma\ast}R \in \mathcal{F}$, proving the claim. Note that $(f_{\sigma})^{-1}\mathcal{F}$ contains a unique maximal face $\mathcal{F}^\sigma$ of $\overline{\text{Eff}}^1(Y^\sigma)$ and that this face is supported. Indeed, if $C$ is a nef curve class cutting out the supported face $\mathcal{F}$, then $f_{\sigma}\ast C$ is a nef curve class cutting out this face of $\overline{\text{Eff}}^1(Y^\sigma)$. Thus

$$b(F, X, L) = \dim(N^1(X)/\text{Span}(\mathcal{F})) \leq \dim(N^1(Y^\sigma)/\text{Span}((f_{\sigma})^{-1}\mathcal{F})) \leq b(F, Y^\sigma, L).$$

Finally, we prove our main theorem.

**Proof of Theorem 3.5:** As mentioned before $Z_0$ and $Z_3$ are contained in proper closed subsets of $X$, so it suffices to consider only $Z_1$ and $Z_2$.

Recall from the proof of Theorem 4.2 that the locus in $\text{Chow}(X)$ defined by adjoint rigid varieties with $a$-value equal to $X$ is bounded and locally closed. We can then repeat the construction of Theorem 4.2 over $F$ to obtain a closed set $V$ and families $\pi_i : U_i \to W_i$ defined over $F$ which satisfy the conclusion of Theorem 4.2.

Suppose that $U_i$ is not geometrically irreducible. Then the Zariski closure $\overline{s_i(U_i(F))}$ is a proper closed subset of $X$ where $s_i : U_i \to X$ is the evaluation map. We enlarge $V$ by adding this proper closed subset to $V$.

Suppose that $U_i$ is geometrically irreducible. Let us further suppose that the evaluation map $s_i : U_i \to X$ is birational. After applying a resolution, we may assume that $U_i$ is
smooth. Let $W_i^o$ be a Zariski open locus so that $\pi_i : \pi_i^{-1}(W_i^o) \to W_i^o$ is a good family of adjoint rigid varieties. Let $Q_i$ be the closed subset associated to this family and define $\mathcal{V}_i = \pi_i^{-1}(W_i^o) \setminus Q_i$. We enlarge $V$ by adding the proper closed subset $s_i(U_i \setminus \mathcal{V}_i) \cup s_i(E_i)$ where $E_i$ is the ramification divisor of $s_i$.

We will next take a couple successive base changes $W_i^\mu \to W_i$; during this operation we let $W_i^\mu_0$ denote the preimage of $W_i^\mu$. After taking a finite Galois base change $W_i^\mu_0 \to W_i$ and shrinking $W_i^\mu$, we may assume that $\pi_i : \mathcal{V}_i^\mu \to W_i^\mu$ admits a good section $\sigma_i$. Furthermore, after taking some cyclic cover, we may assume that $U_i^\mu \to W_i^\mu$ is birational to the Iitaka fibration for $(U_i^\mu, s_i^*L)$. By applying Lemma 4.13 to $\pi_i : U_i^\mu \to W_i^\mu$, we obtain families $\mathcal{Y}_{i,j} \to T_{i,j}$ such that $s_{i,j} : Y_{i,j} \to U_{i,j}$ descends to $s_{i,j} : \mathcal{Y}_{i,j} \to U_i$ with a rational point. We make a few additional changes to these families. First we replace $Y_{i,j} \to T_{i,j}$ by smooth projective closures while preserving the existence of a morphism $\pi_{i,j} : Y_{i,j} \to T_{i,j}$. Thus over some open subset $T_{i,j}$, which is the preimage of an open set $W_i^\sigma$, $\pi_{i,j}$ is a good family of adjoint rigid varieties. After killing monodromy and taking a Galois closure, we may assume that the geometric monodromy of $\pi_{i,j}$ is trivial and $T_{i,j}/W_i$ is Galois. By applying Lemma 4.11, we may assume that $\text{Bir}(\mathcal{Y}_{i,j}/\mathcal{X}) = \text{Aut}(\mathcal{Y}_{i,j}/\mathcal{X})$. By shrinking further we may ensure that there is an effective divisor numerically equivalent to $K_{\mathcal{Y}_{i,j}} + a(X, L)s_{i,j}^*L$ which does not contain any fiber over $T_{i,j}$. Note that the birational modification may force us to shrink $W_i^\sigma$ to keep the good family structure. After all these changes we have a commutative diagram

$$
\begin{align*}
\mathcal{Y}_{i,j} & \longrightarrow U_i \\
\downarrow & \\
T_{i,j} & \longrightarrow W_i
\end{align*}
$$

We enlarge $V$ by adding $s_{i,j}(\mathcal{Y}_{i,j} \setminus \mathcal{Y}_{i,j}^\sigma) \cup s_i(\pi_i^{-1}(B_{i,j})) \cup s_i(E_{i,j}) \cup s_i(Q_i) \cup s_i(C_i)$ where $E_{i,j}$ is the branch locus of $s_{i,j} : Y_{i,j} \to U_i$, $B_{i,j}$ is the branch locus of $T_{i,j} \to W_i$, and $C_i$ is the closed subset from Lemma 4.13. Before continuing, we prove that for any twist $\mathcal{Y}_{i,j}^\sigma$ over $U_i$ and for any closed point $t \in T_{i,j}^\sigma$, we have $b(F, \mathcal{Y}_{i,j}^\sigma, s_{i,j}^*L) = b(F, \mathcal{Y}_{i,j,t}^\sigma, s_{i,j,t}^*L|_{\mathcal{Y}_{i,j,t}})$ (where $\mathcal{Y}_{i,j,t}$ denotes the fiber over $t$). Let $\bar{t} \in T_{i,j}^\sigma$ denote a geometric point above $t$. By construction $\mathcal{Y}_{i,j}^\sigma$ has a birational model with a trivial geometric monodromy action, implying that the monodromy action on $N^1(\mathcal{Y}_{i,j,t})$ preserves the subspace spanned by divisors which are contracted upon the rational map to the corresponding fiber. Any such divisor will lie in the support of the rigid divisor $a(X, L)s_{i,j}^*L|_{\mathcal{Y}_{i,j,t}} + K_{\mathcal{Y}_{i,j,t}}$. Thus we see that the geometric monodromy acts trivially on $N^1(\mathcal{Y}_{i,j,t})/\oplus \mathbb{R}E_i$ where the $E_i$ are the geometric irreducible components of the rigid divisor above. By Lemma 2.6 we obtain the desired equality.

**The construction of the thin set:** We now construct a thin set $Z' \subset X(F)$. The construction involves several steps. First set $Z' = V(F)$. If the evaluation map for the family $\pi_i : U_i \to W_i$ has degree $> 1$, then we add $s_i(U_i(F))$.

Otherwise $s_i$ is birational. As $\sigma$ varies over all $\sigma \in H^1(F, \text{Aut}(\mathcal{Y}_{i,j}/\mathcal{X}))$ such that

$$(a(X, L), b(X, L)) \leq (a(\mathcal{Y}_{i,j}^\sigma, (s_{i,j})^*L), b(F, \mathcal{Y}_{i,j}^\sigma, (s_{i,j})^*L))$$
and the map is face contracting we add the set
\[ \bigcup_{\sigma} s_{i,j}^\sigma(U_{i,j}^\sigma(F)) \subset X(F) \]
to \( Z' \). Repeating this process for the finitely many \( Y_{i,j} \), we obtain a set \( Z' \) which is contained in a thin set of \( X(F) \) by Theorem 5.2. We show that \( Z_1 \) and \( Z_2 \) are contained in \( Z' \).

**The set** \( Z_1 \): Assume that \( f : Y \to X \) is a thin map such that \( Y \) is smooth and geometrically integral, \( d(Y, f^*L) < d(X, L) \), and
\[
(a(X, L), b(F, X, L)) \leq (a(Y, f^*L), b(F, Y, f^*L)).
\]

We would like to show that for any rational point \( y \in Y(F) \) the image \( f(y) \in Z' \). We may assume that \( f(y) \notin V \) since otherwise the statement is clear. Then \( a(Y, f^*L) = a(X, L) \).

Let \( \phi : Y \dashrightarrow C \) be the Iitaka fibration for \( K_Y + a(Y, f^*L)f^*L \). After replacing \( Y \) by a birational model (and taking any preimage of \( y \)), we may assume that this Iitaka fibration is a morphism. Again, if the \( a \)-values of the images of the fibers of \( \phi \) are larger than \( a(X, L) \) then \( f(Y) \subset V \), so we may suppose otherwise. Thus \( C \) admits a rational map \( g : C \dashrightarrow W_i \) for some \( i \). After some birational modification (and again taking a preimage of \( y \)), we may assume that this rational map is a morphism. Without of loss of generality we may assume that \( U_i \) is geometrically irreducible and \( s_i : U_i \to X \) is birational as otherwise the statement is clear.

Let \( c = \phi(y) \). Let \( C^\mu \subset C \times W_i, W_i^\mu \to C \) be the main component of the base change of \( W_i^\mu \to W_i \) by \( C \); note that this cover is only ramified along the preimage of \( B_{i,j} \). After replacing \( W_i^\mu \) by a twist we may assume that \( C^\mu \) comes with a rational point \( c^\mu \) mapping to \( c \). Let \( Y^\mu \) denote the base change of \( Y \) to \( C^\mu \). Let \( v \) be the image of \( y \) in \( Y^\mu \) and \( w = \pi_i(v) \) and \( v' = \sigma(w) \) for our section \( \sigma \).

Note that we have modified \( Y_{i,j} \) and \( T_{i,j} \) by a base change so they are not the same constructions as given by Lemma 4.13. Nevertheless, we can repeat the argument of Lemma 4.13 to obtain a factoring result. More precisely, applying a base change \( \tilde{C}^\mu \to C^\mu \) with a rational point \( \tilde{c}^\mu \) mapping to \( c^\mu \), we can find a resolution \( \tilde{Y}_{C^\mu} \) of the base change with a rational section which is well-defined at \( \tilde{c}^\mu \) and the image of this point is mapping to \( y \). Taking a further base change \( C^\nu \to \tilde{C}^\mu, \tilde{Y}_{C^\nu} \to X \) factors rationally through some twist \( Y_{i,j}^\nu \) over the ground field. Note that \( \tilde{Y}_{C^\nu} \) is smooth at any point on the fiber at \( c^\nu \). Indeed, \( C^\nu \) is the main component of the base change
\[
\tilde{C}^\mu \times_{U^\mu} Y_{i,j}^\sigma \to Y_{i,j}^\sigma.
\]

After applying a birational modification (and replacing \( (y', c') \) by any preimage) we may assume that the map to \( Y_{i,j}^\sigma \) is a morphism. This implies that
\[
f(y) \in s_{i,j}^\sigma(Y_{i,j}^\sigma(F)).
\]

It only remains to verify
\[
(a(X, L), b(X, L)) \leq (a(Y_{i,j}^\sigma, (s_{i,j}^\sigma)^*L), b(F, Y_{i,j}^\sigma, (s_{i,j}^\sigma)^*L)).
\]
and if equality is achieved then \( s_{i,j}^\sigma \) is face contracting. By the construction we know that the \( a \)-values are the same.

We first show that \( b(F,Y, f^*L) \leq b(F, \mathcal{Y}^\sigma_{i,j}, (s_{i,j}^\sigma)^*L) \). Let \( t \in T^\sigma_{i,j} \) be a closed point. Let \( \mathcal{Y}^\sigma_{i,j,t} \) denote the corresponding fiber of \( \pi_{i,j} \). As argued above we have an equality
\[
b(F, Y^\sigma_{i,j}, s_{i,j}^\sigma L) = b(F, \mathcal{Y}^\sigma_{i,j,t}, s_{i,j}^\sigma L_{|\mathcal{Y}^\sigma_{i,j,t}}) \]
Similarly for a general closed point \( c \in C^\mu \), by applying Lemma 6.1 and Lemma 2.6 we obtain
\[
b(F, Y, f^*L) \leq b(F, Y_{C^\mu,c}, f^{\mu*}L) \leq b(F, Y_{C^\mu,c}, f^{\mu*}L) \leq b(F, Y_{C^\mu,c}, f^{\mu*}L).
\]
Our assertion follows from the fact that \( Y_{C^\mu,c} \) is birational to \( \mathcal{Y}^\sigma_{i,j,t} \) for some \( t \).

Finally if
\[
(a(X, L), b(F, X, L)) = (a(\mathcal{Y}^\sigma_{i,j}, (s_{i,j}^\sigma)^*L), b(F, \mathcal{Y}^\sigma_{i,j}, (s_{i,j}^\sigma)^*L))
\]
we have \( d(\mathcal{Y}^\sigma_{i,j}, s_{i,j}^\sigma L) = d(Y, f^*L) < d(X, L) \). Since \( X \) is birational to \( U \), we see that \( s_{i,j}^\sigma \) is face contracting by Lemma 2.12.

**The set** \( Z_2 \): Assume that \( f : Y \to X \) is a thin map such that \( Y \) is smooth and geometrically integral, \( d(Y, f^*L) = d(X, L) \), and either
\[
(a(X, L), b(F, X, L)) < (a(Y, f^*L), b(F, Y, f^*L))
\]
or equality is achieved and \( f \) is face contracting. We would like to show that \( f(Y(F)) \subset Z_1 \).

The argument is essentially the same as for the set \( Z_1 \). The main difference is the case when the \( a \) and \( b \) values are equal. In this situation, if \( f : Y \to X \) is face contracting then we claim that the map \( s_{i,j}^\sigma : \mathcal{Y}^\sigma_{i,j} \to X \) is also face contracting. It suffices to show that the map of faces \( F_{Y} \to F_{\mathcal{Y}^\sigma_{i,j}} \) in Definition 2.10 is injective. Indeed, recall that a fiber \( Y_t \) maps birationally to a fiber of \( (\mathcal{Y}^\sigma_{i,j})_t \), so there is a natural identification \( F_Y = F_{(\mathcal{Y}^\sigma_{i,j})_t} \). Then \( F_Y \) is simply the monodromy-invariant part of the left hand side and \( F_{\mathcal{Y}^\sigma_{i,j}} \) agrees with the right hand side.

\[ \square \]

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