Buildings

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**Introduction**

The concept of a building was first invented by Jacques Tits in order to axiomatize certain relations between finite reflection groups and Lie groups. Buildings find many applications in group theory and geometry; they are particularly relevant for studying the exceptional Lie groups. However, in this paper we are primarily interested in buildings for their own sake. We thus work quickly through the background material and examine in detail the basic facts about buildings.

This paper is divided into four sections. Section 1 consists of preliminary definitions and theorems. In particular, I introduce Coxeter groups and chamber systems, which are the basic structures involved in constructing a building. In section 2 these concepts are brought together to define a Coxeter complex. These complexes are the “building blocks” that define buildings and many results that we prove for Coxeter complexes will generalize to buildings. In section 3 I define buildings, give a number of theorems determining their basic structure, and present a few examples. Finally, section 4 outlines a connection between the geometry of buildings and linear groups.

I have chosen to follow Ronan by focusing on chamber systems rather than cell complexes as the underlying structure for buildings (see [1]). Although buildings were originally considered as cell complexes, we can develop powerful theorems much more easily by using the chamber system structure. The ideas in this paper are based on those found in books by Ronan in [1] and Brown in [2].

**1 Preliminaries**

**1.1 Chamber Systems**

Chamber systems are sets with an added partition structure. Every building has the structure of a chamber system, so the following material is crucial to understanding the work that lies ahead. This section consists mostly of elementary definitions that will be useful later on.

**Definition 1.1** Let $\Delta$ and $I$ be sets. Then $\Delta$ is a chamber system over $I$ if to each element of $I$ we associate a partition of $\Delta$. The elements of a chamber system $\Delta$ are called chambers.

Two chambers $x, y$ are $i$-adjacent (for some $i \in I$) if $x$ and $y$ belong to the same class of the partition associated with $i$. We write this equivalence relation as $x \sim_i y$. 

Example: Let $\Delta = \{1, 2, 3, 4\}$, and let $I = \{i, j\}$. Then, if we associate $i \leftrightarrow \{1, 4\}\{2, 3\}$ and $j \leftrightarrow \{1, 3, 4\}\{2\}$ (or any two other partitions), we give $\Delta$ the structure of a chamber system over $I$. In this example, 1 is $i$-adjacent to 1 and 4, and is $j$-adjacent to 1, 3, and 4. $\square$

A gallery is a sequence of chambers $(a_0, a_1, a_2, \ldots, a_k)$ such that, for every $c$ with $1 \leq c \leq k$, we have

$$\exists i \in I \text{ s.t. } a_{c-1} \sim_i a_c$$

In this paper, we always require that consecutive entries of a gallery are different (some authors call these non-stuttering galleries). The type of a gallery is any word $i_1i_2\ldots i_k$ (in elements of $I$) such that $a_{c-1} \sim_{i_c} a_c$, and the length of such a gallery is $k$, the length of the corresponding word. A subgallery of a gallery $\gamma$ is a subsequence $(a_c, a_{c+1}, \ldots, a_d)$ contained in $\gamma$.

A gallery from $x$ to $y$ is a gallery such that $a_0 = x$ and $a_k = y$. If there is a gallery from $x$ to $y$ for every pair of chambers $x, y \in \Delta$, then we say that $\Delta$ is connected. A gallery from $x$ to $y$ is minimal if its length is minimal among all galleries connecting $x$ to $y$; note that any subgallery of a minimal gallery is also minimal.

Example: Let $\Delta = \{1, 2, 3, 4\}$, $I = \{i, j\}$, and associate $i \leftrightarrow \{1, 4\}\{2, 3\}$ and $j \leftrightarrow \{1, 3, 4\}\{2\}$ as before. Then, $(1, 3, 2)$ is a gallery from 1 to 2 of type $ji$. Another gallery from 1 to 2 is $(1, 4, 1, 3, 2)$. This gallery has several different types; one of them is $ijji$. The first gallery is minimal, the second is not. $\square$

Often it will be useful to consider the action of $I$ restricted to a subset. For $J \subset I$, we call a gallery from $x$ to $y$ a $J$-gallery if it has a type consisting of elements in $J$, and we say that $x$ and $y$ are $J$-connected. The $J$-connected components of $\Delta$ are called $J$-residues. For example, if $i \in I$, then the $\{i\}$-residues are the classes of the partition associated to $i$ (we often omit the brackets for singletons in $I$ and write $i$-residue).

A morphism of chamber systems $\Delta$ and $\Gamma$ over the common set $I$ is a map $\omega: \Delta \to \Gamma$ that preserves $i$-adjacency. So, if $x \sim_i y$ in $\Delta$ then $\omega(x) \sim_i \omega(y)$ in $\Gamma$. It’s clear that any morphism also preserves galleries and connectedness. We define isomorphisms and automorphisms in the natural way.

We note in passing that, given any finite chamber system, there is a canonical method of constructing a corresponding cell complex. The geometry of the complex reflects the structure of the $J$-residues of the chamber system. In fact, as mentioned above, all of the original work with buildings
was phrased in terms of cell complexes rather than chamber systems. Although this geometric intuition will appear in some of our examples, we will not need to use this formalism.

1.2 Coxeter Groups

In this section, we define Coxeter groups and present several theorems determining their structure. Buildings are constructed by “gluing together” Coxeter groups, so these structural theorems will help us develop our first results about buildings. We omit all proofs — they can be found in any standard text on Coxeter groups (for example, see [3]).

Definition 1.2 A Coxeter group $W$ is a group satisfying the following relations on a finite set of generators \{s_1, \ldots, s_n\}:

\[
W = \langle s_1, s_2, \ldots, s_n \mid (s_is_j)^{m_{ij}} = 1 \quad \forall i, j \in \{1, \ldots, n\}\rangle
\]

where the $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ are subject to the conditions:

- $m_{ij} = m_{ji}$ \quad $\forall i, j \in \{1, \ldots, n\}$

- $m_{ii} = 1$ and $m_{ij} \geq 2$ \quad for $i \neq j$

The number of generators is called the rank of the Coxeter group.

Coxeter groups should be thought of as an abstraction of finite reflection groups. Reflection groups are generated by involutions, and we can “isolate” the behavior of any two generators since the subgroup generated by them essentially acts on a plane (and is thus a dihedral action). Similarly, Coxeter groups are generated by involutions, and the subgroup generated by any two generators is dihedral (determined by the $m_{ij}$). The following two examples make this connection more explicit.

Example: Dihedral groups are precisely the finite Coxeter groups of rank 2. Let $m \in \mathbb{N}$, and consider the Coxeter group $W$ given by

\[
\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^m = 1 \rangle
\]

Then $W$ is isomorphic to the dihedral group $D_{2m}$. If we let $u$ be rotation in the plane by the angle $\pi/m$ and let $v$ be reflection across the $x$-axis, an isomorphism is given by sending $s_1 \to v$ and $s_2 \to uv$. □
Each symmetric group $S_n$ is also a Coxeter group. It will be useful to think of $S_n$ as a reflection group by considering its natural action on the standard basis of $\mathbb{R}^n$. Note that each involution $(k, k + 1)$ under this action corresponds to a reflection. Since $S_n$ is generated by involutions of this form, it is in fact a finite reflection group. It turns out that we can then identify $S_n$ as a Coxeter group by setting $s_k = (k, k + 1)$. Note that there are only $n - 1$ such involutions, so $S_n$ has rank $n - 1$ as a Coxeter group. □

As usual, it is not immediately clear whether the relations defining $W$ introduce other “hidden” relations. It turns out that $W$ behaves just like we would expect it to. In particular, no $s_i$ is equal to $s_j$ for $i \neq j$, each $s_i$ has order 2, and each product $s_is_j$ has order given by $m_{ij}$.

As it is often hard to work with the Coxeter group directly from the relations, we work in a bit more generality with words. Let $I$ be the indexing set for the generators of $W$; we will consider words of the form $f = i_1i_2\ldots i_k$ where $i_1, \ldots, i_k \in I$. We will write $s_f$ for the element of $W$ corresponding to $f$: $s_f = s_{i_1}s_{i_2}\ldots s_{i_k}$. We denote the length of the word $f$ by $\ell(f)$.

We call two words $f$ and $g$ equivalent when they represent the same element of $W$, that is, when $s_f = s_g$. Clearly, two words will be equivalent if they can be transformed into each other by a series of the following operations:

- Removing consecutive appearances of the same variable, i.e. of the form $ii$.

- Replacing a subword of length $m_{ij}$ of the form $ijij\ldots$ with the word $jiji\ldots$ of the same length.

Transformations of the first type are called reductions, those of the second type are called homotopies. A word is called reduced if no series of homotopies allow us to perform any further reductions. Although it is not obvious, it turns out that these two transformations give us enough information to completely determine the equivalency classes of words. The following lemma outlines the basic properties about these transformations.

**Lemma 1.3** Let $f$ and $g$ be words in $I$.

- If $\ell(f)$ and $\ell(g)$ have different parity (one is odd and one is even), then $s_f \neq s_g$.

- If $f$ and $g$ are reduced and $s_f = s_g$, then $f$ and $g$ are homotopic. In particular, two reduced words of different lengths represent different elements of $W$. 

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- If $f$ and $g$ are equivalent, then they can be reduced to a common word through a series of reductions and homotopies.

Any element $w \in W$ has a set of reduced words that represent it. This lemma tells us that the length of these words is uniquely determined by $w$ — we call this the length of $w$ and denote it by $\ell(w)$. It is natural to consider how to translate information about group multiplication into information about the corresponding reduced words. The next two results tell us how to turn a reduced word expression for $w$ into a reduced word expression for $sw$. The second of the two (known as the Strong Exchange Condition) is one of the fundamental results determining the behavior of Coxeter groups.

**Lemma 1.4** If $f$ is a reduced word but the word $fj$ is not, then $f$ is homotopic to a word ending with $j$.

Note that this implies also that if the word $jf$ is not reduced then $f$ is homotopic to a word beginning with $j$.

**Theorem 1.5** Let $w = s_1 \ldots s_k$. Suppose that $t = w'sw'^{-1}$ for some generator $s$ and element $w' \in W$. If $\ell(tw) < \ell(w)$, then we can express $tw$ as the product

$$tw = s_1 \ldots s_j^* \ldots s_k$$

where $s_j^*$ represents the omission of the factor $s_j$. If $s_1 \ldots s_k$ is a reduced expression for $w$, then our choice of $s_j$ is unique.

By lemma 1.4, if $w$ is reduced and $\ell(s_jw) < \ell(w)$, then we can find an expression for $w$ as a product of generators that begins with $s_i$. So we can simply remove the first factor $s_i$ from $w$ to get a reduced expression for $s_iw$. In addition, the Strong Exchange Condition implies that a subword of a reduced word is also reduced.

## 2 Coxeter Complexes

There is a natural way of defining a chamber system on the set of elements of a Coxeter group $W$. Let $I$ be the set of indices of the generators of $W$. We then make $W$ a chamber system over $I$ by defining the $i$-residues as left-cosets of the subgroup $\{1, s_i\}$. That is, two distinct elements $x$ and $y$ in $W$ will be $i$-adjacent iff $x = ys_i$. Any such chamber system is known as a Coxeter complex. We often denote a Coxeter complex by the same letter $W$ that we use for the associated Coxeter group.
Example: Consider the Coxeter complex $D_6$; this is a chamber structure on the dihedral group

$$D_6 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^3 = 1 \rangle$$

The indexing set for this chamber system is $I = \{1, 2\}$. The partition associated to $1 \in I$ is $\{1, s_1\}\{s_2, s_2s_1\}\{s_1s_2, s_1s_2s_1\}$. The complex $D_6$ is connected; for example, $(1, s_1, s_1s_2, s_1s_2s_1)$ is a gallery from 1 to $s_1s_2s_1$ of type $1, 2, 1$. □

In a Coxeter complex, a gallery from $x$ to $y$ is a sequence of chambers $(x, xs_1, xs_2s_1, \ldots, y)$. So, a gallery simply represents a product of generators. This means that questions about galleries reduce to analyzing how words correspond to elements of the Coxeter group.

The following proposition outlines some useful facts about galleries. The first follows from the fact that distinct chambers $x$ and $y$ are $i$-adjacent if and only if $x = ys_i$. The rest follow immediately from lemma 1.3 and our results about words in Coxeter groups.

Proposition 2.1 Consider a Coxeter complex $W$.

- Any gallery in $W$ has a unique type.
- There is a gallery of type $f$ from $x$ to $y$ iff $s_f = x^{-1}y$. Thus, all galleries from $x$ to $y$ have equivalent types.
- A gallery is minimal iff its type is a reduced word. Two minimal galleries from $x$ to $y$ have homotopic types.

The automorphisms of a Coxeter complex $W$ are exactly the maps $\omega$ given by left-multiplication by elements of $W$: clearly any such map preserves $i$-adjacency. Conversely, the image of 1 $\in W$ under any automorphism determines the image of every other element, so if $\omega(1) = w$ then $\omega$ is left-multiplication by $w$.

There are other ways of constructing Coxeter complexes without considering the group $W$. Often the close links between chamber systems and simplicial complexes will provide us with geometrical examples.

Example: Let $V$ be an $n$-dimensional vector space with a basis $\{v_1, \ldots, v_n\}$. A flag of $V$ is any sequence of proper subspaces ordered by containment:

$$E_1 \subset E_2 \subset \ldots \subset E_k$$

A maximal flag is a sequence of length $n - 1$. 

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Let \( \Delta \) be the set of all maximal flags such that each subspace \( E_i \) is the span of a subset of our basis vectors \( \{ v_1, \ldots, v_n \} \). To each flag

\[
E_1 \subset E_2 \subset \cdots \subset E_{n-1}
\]

we associate an ordered sequence of the basis vectors \( \{ v_{k_1}, v_{k_2}, \ldots, v_{k_n} \} \) such that \( E_i \) is the span of the first \( i \) vectors in the sequence (and \( v_{k_n} \) is the only basis vector not contained in \( E_{n-1} \)). Since each sequence of basis vectors yields a maximal flag, the two concepts are entirely equivalent.

We can identify \( \Delta \) as a Coxeter complex as follows. There is a natural right action of \( S_n \) on \( \Delta \), which we describe by giving the action on sequences of basis vectors. We let \( w \in S_n \) acting on \( \{ v_{k_1}, \ldots, v_{k_n} \} \) result in the sequence \( \{ v_{k_w(1)}, \ldots, v_{k_w(n)} \} \).

We give \( \Delta \) the structure of a chamber system over a set of cardinality \( n - 1 \) as suggested by the action of \( S_n \): we let two flags \( E_1 \subset \cdots \subset E_{n-1} \) and \( F_1 \subset \cdots \subset F_{n-1} \) be \( i \)-adjacent when \( E_j = F_j \) for all \( j \neq i \) (and so the \( i \)th and \( (i + 1) \)th vectors in the corresponding sequences are switched). This system is isomorphic to the Coxeter complex given by \( S_n \). If we distinguish a particular flag \( E_1 \subset \cdots \subset E_{n-1} \) then the isomorphism is given by

\[
w \in S_n \iff (E_1 \subset \cdots \subset E_{n-1})w
\]

\( \square \)

### 2.1 Reflections

We define a \textit{reflection} in a Coxeter group \( W \) as a conjugate of one of our generators: a reflection \( r \) is \( ws_iw^{-1} \) for some \( w \in W \) and some \( s_i \) in our set of generators. In the case of a finite reflection group, this definition of a reflection is the natural one. In this section we analyze the automorphisms of \( W \) (as a Coxeter complex) that are given by left-multiplication by reflections. As a result, we will derive some useful facts about minimal galleries in Coxeter complexes.

For a given reflection \( r \), we define a \textit{wall} \( \Psi_r \) to be the set of residues fixed by left-multiplication by \( r \). In general, this set of residues may be very complicated. However, we know that there will be at least one \( i \)-residue contained in any wall: if \( r = ws_iw^{-1} \), then \( r \) fixes the \( i \)-residue \( \{ w, ws_i \} \), so \( \{ w, ws_i \} \in \Psi_r \). Conversely, each \( i \)-residue is fixed by a unique reflection, so there is a bijection between walls and reflections.
We say that a gallery $\gamma$ crosses the wall $\Psi_r$ each time the gallery has two adjacent chambers ($a_c, a_{c+1}$) which are interchanged by the reflection $r$. Note that $\gamma$ may cross $\Psi_r$ more than once; we define the crossing number of $\gamma$ over $\Psi_r$ to be the number of adjacent pairs interchanged by $r$.

**Theorem 2.2** Let $\gamma$ and $\mu$ be galleries from $x$ to $y$. For any wall $\Psi$, the crossing numbers of $\gamma$ and $\mu$ over $\Psi$ have equal parity.

**Proof:** We know that the types of $\gamma$ and $\mu$ are equivalent words. Theorem 1.3 states that any two equivalent words can be reduced to a common one through reductions and homotopies, so it suffices to show that these transformations do not change the parity of the crossing number of $\gamma$ over any wall $\Psi$. In particular, we can limit ourselves to the subgalleries where the transformation takes place (since the set of walls crossed outside this subgallery does not change).

First, consider a subgallery $\gamma$ of the form $(w, ws_i, w)$. If we let $r$ denote the reflection $ws_iw^{-1}$, then $\gamma$ crosses over $\Psi_r$ twice (and does not cross any other wall). Reducing $\gamma$ to the subgallery $(w)$ reduces the number of crossings over $\Psi_r$ by two and so preserves the parity of the crossing numbers.

Now, consider homotopies of a gallery $\gamma$. For notational convenience, we let $p_{ij}(c)$ denote the sequence $s_is_js_is_j...$ of length $c$. The homotopy transformation takes

$$\gamma = (w, ws_i, ws_is_j, ..., wp_{ij}(m_{ij}))$$

into the homotopic gallery

$$\mu = (w, ws_j, ws_js_i, ..., wp_{ji}(m_{ij}))$$

The adjacent pairs in $\gamma$ have the form $(wp_{ij}(k), wp_{ij}(k + 1))$ for $0 \leq k < m_{ij}$. Let $\Psi_k$ be the (unique) wall crossed at the $k$th step, and denote the corresponding reflection by $r_k$. Then:

$$r_kwp_{ij}(k) = wp_{ij}(k + 1)$$

If $k$ is even, we multiply both sides on the right by $p_{ji}(m_{ij} - 1)$; if odd we multiply by $p_{ij}(m_{ij} - 1)$. Either way:

$$r_kwp_{ji}(m_{ij} - k - 1) = wp_{ij}(m_{ij} - k)$$

Note that $(wp_{ji}(m_{ij} - k), wp_{ji}(m_{ij} - k))$ is an adjacent pair in the subgallery $\mu$. Thus, there is a bijective correspondence between walls crossed in $\gamma$ and those crossed in $\mu$. \(\square\)
2.2 Roots

Theorem 2.2 implies that for any chamber $x$ and wall $\Psi$ we can divide $W$ into two pieces: those chambers $y$ for which galleries from $x$ to $y$ cross $\Psi$ an even number of times, and those for which galleries cross $\Psi$ an odd number of times. In fact, this division is independent of $x$, as demonstrated in the following theorem.

**Theorem 2.3** Let $\Psi$ be a wall with associated reflection $r$.

1. Any minimal gallery $\gamma$ crosses $\Psi$ at most one time.

2. Define a relation $\ast_{\Psi}$ on the set of chambers by setting $x \ast_{\Psi} y$ when no minimal gallery from $x$ to $y$ crosses $\Psi$. Then $\ast_{\Psi}$ is an equivalence relation partitioning $W$ into two sets.

**Proof:**

1. Assume $\gamma = (a_0, \ldots, a_k)$ crosses the wall $\Psi$ more than once, say at $(a_c, a_{c+1})$ and $(a_d, a_{d+1})$. The automorphism given by left-multiplication by $r$ transforms the subgallery from $a_{c+1}$ to $a_d$ into a gallery from $a_c$ to $a_{d+1}$ of the same length. This implies that $\gamma$ is not minimal.

2. Note that by part 1 and theorem 2.2, either every minimal gallery from $x$ to $y$ crosses $\Psi$ once, or no minimal gallery from $x$ to $y$ crosses $\Psi$. It is now easy to see that $\ast_{\Psi}$ is an equivalence relation; the only non-trivial property is transitivity, which is a consequence of theorem 2.2. Since $x \not\ast_{\Psi} y$ and $y \not\ast_{\Psi} z$ together imply that there is a gallery from $x$ to $z$ crossing $\Psi$ twice, and hence $x \ast_{\Psi} z$, there are only two equivalency classes.

\[ \Box \]

We call each equivalency class of $\ast_{\Psi}$ a **root**. Note that different walls correspond to different pairs of roots: given a pair of roots, there is some $i$-residue intersecting both roots, and then the unique reflection preserving this residue determines the wall. Thus, pairs of roots and walls (and hence reflections) are in one-to-one correspondence. We say that the two roots forming a pair are **opposite** each other. This terminology is derived from the action of the corresponding reflection: it simply swaps the two roots.

A subset $X \subseteq W$ is called **convex** if, for any two chambers $x, y \in X$, every minimal gallery from $x$ to $y$ is entirely contained in $X$. Theorem 2.3 implies that roots are convex. Of course, if $R$ is a root such that $x \in R$ but
y \not\in R$, then any gallery from $x$ to $y$ must cross the wall determined by $R$ (after all, this is how we defined $R$). We can combine these observations in the following theorem:

**Theorem 2.4** Let $\gamma$ be a gallery from $x$ to $y$. Then $\gamma$ is minimal if and only if the set of walls crossed by $\gamma$ corresponds exactly to the set of roots $R$ such that $x \in R$ but $y \not\in R$.

The following corollary determines which chambers lie on minimal galleries.

**Corollary 2.5** For any two chambers $x$ and $y$, the chamber $z$ lies on some minimal gallery from $x$ to $y$ if and only if it lies in every root containing both $x$ and $y$.

**Proof:** If $z$ lies on a minimal gallery from $x$ to $y$, then by the previous theorem it must be contained in every root $R$ such that $x, y \in R$. Conversely, let $z$ be contained in every root containing both $x$ and $y$. Let $\gamma$ be a minimal gallery from $x$ to $z$ and let $\mu$ be a minimal gallery from $z$ to $y$. Then composing $\gamma$ with $\mu$ yields a gallery satisfying the conditions of theorem 2.4, and so the composition is minimal. □

**Example:** Theorem 2.4 and corollary 2.5 are particularly useful for analyzing finite Coxeter complexes. If $W$ is a finite Coxeter complex, then there are a finite number of roots in $W$. Thus, by theorem 2.4, the maximum possible distance between any two chambers is the number of roots. For any chamber $x$, we can find a chamber $y$ that achieves this maximum distance: if $y$ does not achieve the maximum, then there is some root $R$ containing both $x$ and $y$, and so the corresponding reflection $r_R$ sends $y$ to an element further away from $x$. Also, $y$ is uniquely determined: if $z$ is another chamber achieving the maximal distance, then every root that contains $y$ also contains $z$, hence a minimal gallery from $z$ to $y$ is trivial. We call this unique $y$ the opposite of $x$. Note that if $x$ and $y$ are opposite, then by corollary 2.5 every chamber in $W$ lies on some minimal gallery from $x$ to $y$. □

## 3 Buildings

As mentioned previously, buildings are essentially chamber systems that look “locally” like a Coxeter complex. There are several equivalent ways of defining buildings; we use the newer definition found in [1], and show that it is
equivalent to the original definition. Our focus in this section will be to develop some basic intuition about how buildings work.

**Definition 3.1** Let $W$ be a Coxeter group and $I$ be the indexing set of the generators of $W$. A building of type $W$ is a chamber system $\Delta$ over $I$ such that:

1. For every $i \in I$, each $i$-residue of $\Delta$ contains at least two chambers.

2. There is a distance function

\[ \delta : \Delta \times \Delta \rightarrow W \]

such that for any reduced word $f$, there is a gallery of type $f$ connecting chambers $x$ and $y$ in $\Delta$ if and only if $\delta(x, y) = s_f$.

Note in particular that the definition immediately implies

- For every $x, y \in \Delta$, there is a gallery of reduced type connecting $x$ and $y$, and hence $\Delta$ is connected.

- There is only one homotopy-equivalent class of reduced words connecting $x$ to $y$.

- Since (different) chambers $x$ and $y$ are $i$-adjacent iff $\delta(x, y) = s_i$, no two chambers are both $i$ and $j$ adjacent for $i \neq j$, and so every gallery has a unique type.

**Example:** Any Coxeter complex is a building, with distance function $\delta(x, y) = x^{-1}y$. □

In fact, if every $i$-residue of the building $\Delta$ has exactly two members, then $\Delta$ is a Coxeter complex $W$. Pick one chamber $z \in \Delta$ to represent the identity element of $W$; then each gallery of type $f$ leaving $z$ ends up at a unique chamber $z(f)$. Furthermore, if $f$ and $g$ are equivalent words then $z(f) = z(g)$. So, there is a bijective correspondence between $z(f)$ in $\Delta$ and $s_f$ in $W$. We call such a building thin. In contrast, a building for which every $i$-residue has at least three members is called thick.

Note that in general a gallery of type $f$ from $x$ to $y$ does not have to be equivalent to $\delta(x, y)$. The situation differs from that of the Coxeter complexes because an $i$-residue can have more than two elements. For example, if $(x, y, z)$ is a gallery of type $ii$, then $x$ and $z$ are $i$-adjacent, so $\delta(x, z) = s_i \neq s_is_i$. The following proposition outlines some valid parallels between the properties of the distance function $\delta$ and the distance function for Coxeter complexes.
Proposition 3.2 Let $\Delta$ be a building with chambers $x$ and $y$, and let $\gamma$ be a gallery from $x$ to $y$ of type $f$.

1. If $g$ is homotopic to $f$, then there is also a gallery of type $g$ from $x$ to $y$.

2. The gallery $\gamma$ is minimal if and only if $f$ is a reduced word.

3. If $f$ is a reduced word, then there is a unique gallery of type $f$ from $x$ to $y$.

4. If a (possibly non-reduced) word $g$ has $\delta(x,y) = s_g$, then there is a gallery of type $g$ from $x$ to $y$.

Proof:

1. It suffices to consider $f = iji\ldots$ and $g = jij\ldots$ where $\ell(f) = \ell(g) = m_{ij}$. In this case, both $f$ and $g$ are reduced, so $\delta(x,y) = s_f = s_g$ and by definition there must be a gallery of type $g$ from $x$ to $y$.

2. Clearly a gallery that is not of reduced type is not minimal. Conversely, let $\gamma$ be a gallery of reduced type $f$. If $\gamma$ were not minimal, then there would be a minimal gallery $\mu$ of shorter type $g$. Since $\mu$ is minimal, $g$ is certainly reduced. But then we should have $s_f = \delta(x,y) = s_g$ and that $\ell(g) < \ell(f)$, a contradiction.

3. Let $\gamma = (x, \ldots, z, y)$ and $\mu = (x, \ldots, z', y)$ be two galleries of reduced type $f$ from $x$ to $y$. Note that $z \sim_i y \sim_i z'$. If $z \neq z'$, then there is a gallery of type $f$ from $x$ to $z'$. This gallery is not minimal (since $\mu$ contains a shorter one), so we have a contradiction to part 2. Thus, $z = z'$, and by an inductive argument on the length of $\gamma$ we must have $\gamma = \mu$.

4. This follows from part 1 and the observation that we can insert sub-galleries of type $ii$ while preserving the starting and ending points of $\gamma$.

\[\square\]

Example: Since the simplest Coxeter groups are the dihedral groups, we should start by considering the buildings of dihedral type. We can give a nice geometrical characterization of such buildings. Let $\Delta$ be a finite building of type $D_{2m}$, and let the indexing set $I$ be denoted $\{i, j\}$. We can represent
Δ by a graph as follows. We let the vertices of our graph be in one-to-one correspondence with the set of $i$ and $j$-residues. Then, for each chamber $x$ in $\Delta$ we add an edge to our graph connecting the $i$-residue containing $x$ to the $j$-residue containing $x$. Note that the valence of any vertex will be at least 2 since each $i$ and $j$-residue has size at least 2.

This graph will satisfy several properties. First, the diameter of the graph will be $m$, the length of the longest reduced word in $D_{2m}$. A cycle in the graph represents a gallery from $x$ to itself of type $ijij \ldots$. Let $p_{ij}(c)$ denote the product $ijij \ldots$ consisting of $c$ letters. Since $p_{ij}(k)$ is reduced for $k \leq m$, each of the first $m$ terms represent different chambers, and the distance from $x$ to the $m$th chamber is $m$. Thus, we see that any cycle will have length at least $2m$. Finally, note that there is no edge connecting an $i$-residue to another $i$-residue; we can partition the vertices into two components such that each component is disconnected (such a graph is called bipartite). So, a finite building of type $D_{2m}$ is a bipartite graph with diameter $m$ and minimal cycle length $2m$ (and such that every vertex has valence at least 2). Such a graph is called a generalized $m$-gon. It is not hard to show that conversely any generalized $m$-gon is the graph for some building of type $D_{2m}$. □

3.1 Apartments

As suggested by the properties above, the existence of a distance function implies that buildings behave like Coxeter complexes when we restrict ourselves to particular subsets. The notion of an apartment makes this connection explicit.

Let buildings $\Delta$ and $\Gamma$ be buildings of common type $W$, and let $X \subseteq \Delta$. An isometry from $X$ to $\Gamma$ is any map $\alpha : X \rightarrow \Gamma$ that preserves distances, that is, $\delta_\Gamma(\alpha(x), \alpha(y)) = \delta_\Delta(x, y)$. (Note that isometries are similar to morphisms of buildings; the difference is that we allow isometries to be defined on subsets.) We will be particularly interested in isometries from a Coxeter complex $W$ (with the usual distance function) into buildings of type $W$ — the image of such an isometry is called an apartment.

**Theorem 3.3** Let $A$ be an apartment and $x$ a chamber in $A$. For any $w \in W$ there is a unique chamber $y \in A$ such that $\delta(x, y) = w$.

**Proof:** This follows immediately from the corresponding statement for Coxeter complexes. □

If we knew how to break up buildings into apartments, we could produce many new results based on our theorems for Coxeter complexes. It seems
plausible that under certain circumstances we could take a small portion of an apartment contained in $\Delta$ and enlarge it to a full apartment. The following theorem shows that we can do this in complete generality.

**Theorem 3.4** An isometry $\alpha$ from a subset $X \subset W$ into a building $\Delta$ can be extended to an isometry from all of $W$ into $\Delta$.

**Proof:** By Zorn’s lemma, it suffices to show that if $X \neq W$ then we can extend $\alpha$ to an isometry on a strictly larger subset of $W$. We will show that we can enlarge the domain of $\alpha$ by one chamber. To choose this chamber, note that if $X \neq W$, then we can find $i$-adjacent chambers $y$ and $z$ in $W$ such that $y \in X$ but $z \notin X$. Since we can map $y$ to $1$ by an isometry, we will assume without loss of generality that $y = 1$ and $z = s_i$. We will show that we can extend $\alpha$ to $X \cup \{s_i\}$.

We divide the proof into two cases. In the first case, we assume that $s_i$ is not in the convex hull of $X$. It turns out that we can choose $\alpha(s_i)$ to be any chamber $i$-adjacent to $\alpha(1)$. In the second case, $s_i$ is in the convex hull of $X$, so it is on some path from $1$ to another chamber $x' \in X$. In this case, we must let $\alpha(s_i)$ be the appropriate chamber on a minimal gallery in $\Delta$ from $\alpha(1)$ to $\alpha(x')$.

In the first case, note that for all $x \in X$ the length of $\delta_W(s_i, x)$ is greater than the length of $\delta_W(1, x)$. In this case, pick any element $\xi \in \Delta$ that is $i$-adjacent to $\alpha(1)$ (but distinct from it). If we can show that for all $x \in X$ we have $\delta_{\Delta}(\xi, \alpha(x)) = \delta_W(s_i, x)$, then we can extend the domain of $\alpha$ to $X \cup \{s_i\}$ by sending $s_i \mapsto \xi$. Choose a reduced word $g$ such that $\delta_W(1, x) = s_g$ (and so $\delta_W(s_i, x) = s_ig$ and by assumption $ig$ is reduced). Since $\alpha$ is an isometry, we also have that $\delta_{\Delta}(\alpha(1), \alpha(x)) = s_g$. Since $g$ is reduced, there is a gallery from $\alpha(1)$ to $\alpha(x)$ of type $g$, and hence a gallery from $\xi$ to $\alpha(x)$ of type $ig$. Since $ig$ is reduced, $\delta_{\Delta}(\xi, \alpha(x)) = s_ig = \delta_W(s_i, x)$ as required.

In the second case, we assume that for some $x' \in X$ the length of $\delta_W(s_i, x')$ is less than the length of $\delta_W(1, x')$. Then $s_i$ lies on some minimal gallery of reduced type $f$ from $1$ to $x'$. Let $\gamma$ be the unique gallery (in $\Delta$) of type $f$ from $\alpha(1)$ to $\alpha(x')$ and pick $\xi$ as the chamber on this gallery adjacent to $\alpha(1)$.

We must show that for all $x \in X$ we have $\delta_W(s_i, x) = \delta_{\Delta}(\xi, \alpha(x))$. To this end, we define a new map $\beta : X \to W$ as $\beta(x) = s_i \delta_{\Delta}(\xi, \alpha(x))$. Note that $\beta$ is the composition of three maps: $\alpha$, $\delta_{\Delta}(\xi, -)$, and left-multiplication by $s_i$. The first and last maps are isometries, and the middle one does not increase distances. Therefore, $\beta$ does not increase distances.

Note that since $\xi$ and $\alpha(1)$ are $i$-adjacent in $\Delta$, we have that $\beta(x)$ is either $\delta_W(1, x) = x$ or $s_i \delta_W(1, x) = s_i x$. In particular, $\beta(1) = 1$ and $\beta(x') = x'$. 

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If there were some $x$ for which $\beta(x) = s_i x$, then $\beta$ would either increase the distance from 1 to $x$ or the distance from $x'$ to $x$ (since 1 and $x'$ are in different roots determined by the reflection $s_i$). Thus, we must have $\beta(x) = x$ for all $x \in X$. Therefore, $\delta_\Delta(\xi, \alpha(x)) = s_i \beta(x) = s_i x = \delta_W(s_i, x)$. □

Let $x$ and $y$ be any two chambers in $\Delta$, and let $w = \delta(x, y)$. Then, the map $\alpha : \{1, w\} \to \{x, y\}$ sending $1 \mapsto x$ and $w \mapsto y$ is an isometry of a subset of $W$ into $\Delta$. Thus we have the following corollary:

**Corollary 3.5** Any two chambers are contained in a common apartment.

This theorem gives credence to the idea that a building is simply a collection of Coxeter complexes joined together. In fact, the original definition of a building was phrased in terms of apartments; this suggests that it is the local similarity to Coxeter complexes that makes buildings worth examining in the first place. The next theorem outlines an alternative definition of a building that is very similar to the original.

**Theorem 3.6** Let $\Delta$ be a chamber system that contains a collection of subsystems (called apartments) such that the following conditions are met.

1. Each apartment is isomorphic to the same Coxeter complex $W$ (using the adjacency inherited from $\Delta$).

2. Given any chambers $x$ and $y$, there is an apartment containing both $x$ and $y$.

3. Let $A$ and $B$ be apartments and assume $x$ and $y$ are chambers in $A \cap B$. Then there is an isomorphism from $B$ to $A$ fixing $x$ and $y$.

4. Let $A$ and $B$ be apartments containing a chamber $x$, and assume that $L$ is an $i$-residue such that $A \cap L$ and $B \cap L$ are non-empty. Then there is an isomorphism from $B$ to $A$ fixing $x$ and mapping $B \cap L \to A \cap L$.

Then $\Delta$ is a building.

**Proof:** Since any chamber $x$ is contained in some apartment, every $i$-residue contains at least two chambers.

Given two chambers $x$ and $y$, we define $\delta(x, y)$ as the distance between $x$ and $y$ in any apartment that contains both of them; this function is well-defined by conditions 2 and 3. We must show that for any reduced word $f$, $\delta(x, y) = s_f$ iff there is a gallery of type $f$ from $x$ to $y$. First, assume $\delta(x, y) = s_f$ for a reduced word $f$. By condition 2, there is an apartment $A$
containing $x$ and $y$, and by definition of $\delta$ there is a gallery from $x$ to $y$ of type $f$ contained in $A$.

Conversely, assume that there is a gallery of reduced type $f$ from $x$ to $y$. We show by induction on $\ell(f)$ that $\delta(x, y) = s_f$. Clearly this is true if $\ell(f) = 1$. Now, let $f = i_1 \ldots i_{k+1}$ be a reduced word of length at least 2 and assume that $\gamma = (x, \ldots, z, y)$ is a gallery of type $f$. Since $f$ is reduced, the word $g = i_1 \ldots i_k$ is also reduced, and there is a subgallery of $\gamma$ of type $g$ from $x$ to $z$. By induction we have $\delta(x, z) = s_g$, so there must be an apartment $B$ containing a gallery $\mu = (x = b_0, b_1, \ldots, b_k = z)$ of type $g$.

Let $A$ be any apartment containing $x$ and $y$. The apartments $B$ and $A$ share a common chamber $x$ and both intersect the $i_{k+1}$-residue containing $z$ and $y$. By condition 4, there is an isomorphism $\alpha$ from $B$ to $A$ fixing $x$ and sending $z$ to a chamber $i_{k+1}$-adjacent to $y$. Therefore, either $\alpha(z) = z$ or $\alpha(z)$ is the other element of $A$ that is $i_{k+1}$-adjacent to $z$. In the latter case, $(\alpha(\mu), y)$ is a gallery from $x$ to $y$ of reduced type $f$ contained in $A$. So, $\delta(x, y) = s_f$ and we have proved it for this case.

We show that the other case $\alpha(z) = y$ leads to a contradiction. First, note that $b_1$ is $i_1$-adjacent to $\alpha(b_1)$ (since both are $i_1$-adjacent to $x$). Also we know that $z \neq y$ (since $\gamma$ is a gallery). Assume for a moment that $b_1 \neq \alpha(b_1)$; then there are two different galleries from $b_1$ to $y$ of the form:

$$(b_1, \alpha(b_1), \alpha(b_2), \ldots, \alpha(b_k) = y)$$
$$(b_1, b_2, \ldots, b_k = z, y)$$

The first has type $g$ and the second has type $h = i_2 \ldots i_k i_{k+1}$. Since both these words are reduced, by induction we must have $s_g = \delta(b_1, y) = s_h$ and so $g$ and $h$ are homotopic. But then $f = g i_{k+1} = h i_{k+1}$ is not reduced, a contradiction. So we must have $b_1 = \alpha(b_1)$. Using the same reasoning inductively, $b_c = \alpha(b_c)$ for $0 \leq c \leq k$. This would imply that $z = b_k = \alpha(b_k) = y$, again a contradiction to the fact that $\gamma$ is a gallery. □

We call any set of apartments that satisfies the conditions of theorem 3.6 a system of apartments. In general a building may have many systems of apartments. However, for buildings that have the type of a finite Coxeter complex (called spherical buildings), there is a unique system of apartments — we give a proof in section 3.2.

The following theorem shows that for any building the set of all apartments forms a system, so that every building has at least one system of apartments.
Theorem 3.7 Let $\Delta$ be a building. For any pair of apartments $A, A'$ with a chamber $x$ in common, there is a unique isomorphism $\beta$ from $A$ to $A'$ fixing $x$. This $\beta$ also fixes $A \cap A'$. Furthermore, let $L$ be a $J$-residue for some $J \subseteq I$. If $L$ has non-empty intersections with $A$ and $A'$, then $\beta$ maps $A \cap L \rightarrow A' \cap L$.

Proof: The identity map from $A \cap A'$ into $\Delta$ is of course an isometry. By theorem 3.4, this map can be extended into an isomorphism $\alpha$ from $A$ into some apartment $B \subset \Delta$. Note that $\alpha$ fixes $A \cap A'$. Similarly, the identity map also induces an isometry $\alpha'$ from $A'$ into $B$ fixing $A \cap A'$. Then the required isometry is $\beta = \alpha'^{-1} \circ \alpha$. Since any isomorphism of apartments is determined by the image of one element, this is the unique isomorphism fixing $x$. Finally, since $\alpha$ and $\alpha'$ map the intersections with $L$ to the same subset of $B$, $\beta$ maps $A \cap L$ into $A' \cap L$. □

We can now provide another example of a class of buildings.

Example: Let $V$ be a vector space of finite dimension $n$ (over an arbitrary field). We will show that the set of all maximal flags of $V$ is a building of type $S_n$. Recall that if we fix a basis $v_1, \ldots, v_n$ of $V$ then we can identify the set of maximal flags spanned by the basis as a Coxeter complex of type $S_n$ — these will be the apartments of our building. We should then define two maximal flags $E_1 \subset \ldots \subset E_{n-1}$ and $F_1 \subset \ldots \subset F_{n-1}$ to be $i$-adjacent if $E_j = F_j$ for all $j \neq i$. However, it’s not immediately obvious that this system satisfies the properties of a building. Rather than analyzing a distance function directly we will find a system of apartments and use theorem 3.6.

First, we must show that any two chambers are contained in a common apartment. That is, we must show that for any two maximal flags $E_1 \subset \ldots \subset E_{n-1}$ and $F_1 \subset \ldots \subset F_{n-1}$ there is a set of basis vectors $V = \{v_1, \ldots, v_n\}$ such that each $E_j$ and $F_j$ has a subset of $X$ as a basis. This process is straight-forward — it is in fact a close analog of the proof of the Jordan-Holder theorem.

For convenience let $E_0 = F_0 = 0$ and $E_n = F_n = V$. Define $\sigma: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ by letting $\sigma(k)$ be the smallest index $l$ such that $F_l \cap E_k \notin E_{k-1}$. This is equivalent to requiring that $E_{k-1}$ and $F_l/F_{l-1}$ together generate the subspace $E_k$. We will use this fact to generate an appropriate basis.

For $1 \leq k \leq n$, define $v_k$ inductively as follows. Let $v_1$ be any vector generating $E_1$. Now, assume that the vectors $v_1, \ldots, v_{k-1}$ are a basis for $E_{k-1}$. Consider the space $E_k \cap F_{\sigma(k)}$. Certainly

$$\dim((E_k \cap F_{\sigma(k)})/(E_{k-1} \cap F_{\sigma(k)})) \leq \dim(E_k/E_{k-1}) = 1$$
Since the subspace $E_k/E_{k-1}$ is contained both in $E_k$ and in $F_{\sigma(k)}$, the dimension in above equation is precisely 1. In particular, there is a non-zero vector $v_k \in E_k \cap F_{\sigma(k)}$ such that $v_1, \ldots, v_k$ is a basis for $E_k$. By repeating this process, we get a basis of $V$ such that each $E_i$ is spanned by the subset $\{v_1, \ldots, v_i\}$ of the basis. Furthermore, $F_i$ is spanned by the set $\{v_j | \sigma(j) \leq l\}$. Note that $\sigma$ is injective, that is, $\sigma(k) \neq \sigma(k')$ for all $k \neq k'$; if it were not, then $F_k/F_{k-1}$ would be two-dimensional, contradicting maximality of the flag. Since the $v_j$ are linearly independent, the span of $\{v_j | \sigma(j) \leq l\}$ is a subspace of $F_i$ that has the proper dimension, so it must be all of $F_i$.

We must also show that there are isomorphisms between apartments satisfying the appropriate conditions. Note that if a chamber $E_1 \subset \ldots \subset E_{n-1}$ is contained in the apartment $A$ defined by $v_1, \ldots, v_n$, we can reorder the basis so that $v_1, \ldots, v_i$ span $E_i$. If the apartments $A$ and $A'$ contain a common chamber $z$, we can reorder the bases for each in a way compatible with $z$. Let $\{v_1, \ldots, v_n\}$ and $\{v'_1, \ldots, v'_n\}$ be these bases, and let $\alpha$ be the map taking $v_i \mapsto v'_i$. This is an isomorphism (as it clearly preserves $i$-adjacency for each $i$). Furthermore, any chamber in $A \cap A'$ is fixed by $\alpha$, and if $L$ is an $i$-residue of $\Delta$ intersecting both $A$ and $A'$ then $\alpha$ maps $L \cap A \mapsto L \cap A'$. Thus $\Delta$ is a building by theorem 3.6. □

### 3.2 Roots and Galleries

Theorems 3.4 and 3.6 indicate that there are sufficiently many apartments in a building to force the structure to be similar to Coxeter complexes. The following theorems analyze this similarity in more detail.

**Theorem 3.8** All apartments in $\Delta$ are convex.

**Proof:** Let $x$ and $y$ be two chambers contained in an apartment $A$, and let $\gamma = (a_0, a_1, \ldots, a_k)$ be a gallery from $x$ to $y$. Assume that $\gamma \notin A$; then there is an $i$-adjacent pair $(a_c, a_{c+1})$ such that $a_c \in A$ but $a_{c+1} \notin A$. Let $b$ be the chamber in $A$ that is $i$-adjacent to $a_c$. Now, for each $j$ with $0 \leq j < k$, let $a'_j$ be the unique chamber in $A$ such that $\delta(b, a'_j) = \delta(b, a_j)$. Clearly $a'_j$ is adjacent to $a'_{j+1}$. Furthermore, $a'_j = a_j$ for all $a_j \in A$. So, the $a'_j$ form a sequence of adjacent elements that starts at $a_0$ and ends up at $a_k$. Since $a'_c = a'_{c+1}$, this sequence will yield a gallery of length less than that of $\gamma$. Thus $\gamma$ cannot be minimal. □

A root of $\Delta$ is any root of one of its apartments (so it is a subset of $\Delta$ contained in an apartment that splits that apartment in the appropriate
Theorem 3.9 Let $\Delta$ be a thick building. Then any root in $\Delta$ is the intersection of all apartments containing it.

**Proof:** Let $R$ be a root of some apartment $A$. We must show that for any chamber $y$ not in $R$, there is some apartment $B$ such that $R \subset B$ but $y \notin B$. Clearly, it suffices to show this for chambers in $A \setminus R$. In fact, we can further limit ourselves to chambers $y \in A \setminus R$ that are adjacent to some chamber $x$ in $R$. For let $z \in A \setminus R$ be arbitrary, and let $\gamma$ be a minimal gallery from $z$ to some chamber in $R$. Then $\gamma$ must contain a chamber $y \notin R$ that is adjacent to a chamber in $R$. If we can show that there is an apartment $B$ such that $R \subset B$ but $y \notin B$, then $z \notin B$ since apartments are convex.

Now, let $y \in A \setminus R$ be $i$-adjacent to $x \in R$. Since $\Delta$ is thick, there is a chamber $y'$ not in $A$ that is $i$-adjacent to $x$ and $y$. If we can show that $R \cup \{y\}$ is isometric to $R \cup \{y'\}$, then theorem 3.4 will show that $R$ is contained in an apartment $A'$ that does not contain $y$, and the result will follow. Consider an arbitrary chamber $z \in R$. Since $x$ and $y$ are $i$-adjacent in $A$, any root of $A$ containing $z$ and $y$ also contains $x$. Thus by theorem 2.5 there is a minimal gallery $\gamma$ in $A$ from $z$ to $y$ that goes through $x$. The type of $\gamma$ is a reduced word $f$. We can find a gallery of type $f$ from $z$ to $y'$ by changing the last entry of $\gamma$ to $y'$. Since $f$ is reduced, this gallery is minimal, so $R \cup \{y\}$ is isometric to $R \cup \{y'\}$. □

Proposition 3.10 The intersection of all apartments containing $x$ and $y$ is precisely the set of all chambers that lie on a minimal gallery from $x$ to $y$.

**Proof:** Let $L$ denote the intersection of all apartments containing $x$ and $y$ and $M$ denote the set of chambers that lie on a minimal gallery from $x$ to $y$. Since apartments are convex, any apartment containing $x$ and $y$ also contains all chambers on minimal galleries from $x$ to $y$. Thus, $M \subseteq L$.

Conversely, we show that $L \subseteq M$. Let $A$ be an apartment containing $x$ and $y$, and let $R$ be the set of roots of $A$ containing both $x$ and $y$ ($R$ may be empty). Since apartments are convex, $M \subseteq A$, and so by corollary 2.5

$$M = \bigcap_{R \in R} R \cap A$$

Each $R$ is the intersection of all apartments containing it. Each apartment containing $R$ also contains $x$ and $y$, so $L \subseteq R$ for every $R \in R$. Since we also know that $L \subseteq A$, it follows that $L \subseteq M$ and we have equality. □

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Recall that a spherical building is one which has the type of a finite Coxeter complex, and in any such building there is a maximal distance between chambers.

**Corollary 3.11** If $\Delta$ is a spherical building, and the distance between chambers $x$ and $y$ is maximal, then there is a unique apartment containing $x$ and $y$.

**Proof:** Let $A$ be an apartment containing $x$ and $y$. Then $x$ and $y$ are opposite in $A$; in particular, any chamber $z \in A$ lies on a minimal gallery from $x$ to $y$. Then the intersection of all apartments containing $x$ and $y$ is $A$, so no other apartment contains $x$ and $y$. $\square$

We now can show that there is a unique system of apartments $\mathcal{A}$ for any spherical building $\Delta$. Given any apartment $A$ and chamber $x \in A$ there is a unique $y$ opposite to $x$ in $A$ that achieves the maximum distance. Conversely, let $x$ and $y$ be a pair of chambers achieving this maximum distance. There is some apartment $A \in \mathcal{A}$ containing $x$ and $y$; by corollary 3.11, this apartment is unique. So, the set of apartments in $\mathcal{A}$ is determined completely by the set of pairs of opposite points.

### 4 Automorphisms and BN-pairs

In this section we explore in more depth a natural connection between thick buildings and linear algebraic groups. In particular, we will develop a correlation between groups with a specific structure (called $BN$-pairs) and thick buildings.

Let $G$ be a group of automorphisms of a building $\Delta$. If $X$ is a subset of $\Delta$, we use the standard notation $G_X$ to denote the stabilizer of $X$ in $G$. The group $G$ is called **strongly transitive** if:

- For any $w \in W$, $G$ is transitive on pairs of chambers $(x, y)$ such that $\delta(x, y) = w$.
- There is an apartment $A$ such that $G_A$ is transitive on chambers in $A$.

Note that every strongly transitive group is transitive on the chambers of $\Delta$.

**Theorem 4.1** The group $G$ is strongly transitive on $\Delta$ if and only if for some system of apartments $\mathcal{A}$ we have that $G$ is transitive on pairs $\{(x, A) \mid A \in \mathcal{A}, x \in A\}$.
Proof: Let $G$ be strongly transitive and let $A$ be an apartment such that $G_A$ is transitive on chambers in $A$. We first show that the orbit of $A$ under $G$ yields a system of apartments. Clearly, since each $g \in G$ is an isomorphism, $gA$ is an apartment of the same type as $A$. Now, let $x'$ and $y'$ be any two chambers; we show there is some $gA$ containing both of them. Let $x, y$ be a pair of chambers in $A$ such that $\delta(x, y) = \delta(x', y')$. By strong transitivity, there is a $g$ such that $gx = x'$ and $gy = y'$; then $gA$ contains $x'$ and $y'$.

Finally, let $A$ and $gA$ be apartments containing a common chamber $x$. Since $x \in gA$, we have $g^{-1}x \in A$. Strong transitivity implies that there is some automorphism $h \in G_A$ sending $x$ to $g^{-1}x$. Then $gh$ is an isomorphism from $A$ to $gA$ fixing $x$. Since it preserves the $i$-adjacency structure, $gh$ fixes all chambers in $A \cap gA$ and maps intersections of residues in the appropriate way. Thus, the orbit of $A$ under $G$ forms a system of apartments.

We also must show that $G$ acts transitively on the pairs $(x, gA)$. Given chambers $y \in g_1A$ and $z \in g_2A$, we can map

$$(y, g_1A) \rightarrow g_1^{-1} (g_1^{-1}y, A) \rightarrow (g_2^{-1}z, A) \rightarrow g_2 (z, g_2A)$$

where the second map is given by the transitivity of $G_A$ on chambers in $A$.

Conversely, assume that $G$ acts transitively on pairs $(x, A)$ such that $A$ is in some system of apartments $\mathcal{A}$ and $x \in A$. Then in particular $G_A$ is transitive on chambers in $A$. Also, let $(x, y)$ and $(x', y')$ be pairs of chambers such that $\delta(x, y) = \delta(x', y')$. Let $A$ be an apartment containing $x$ and $y$, and $A'$ an apartment containing $x'$ and $y'$. Then there is a $g$ mapping $(x, A)$ into $(x', A')$. Since $g$ preserves $i$-adjacency, we also have $gy = y'$. Thus, $G$ is transitive on such pairs of chambers, and hence is strongly transitive. \qed

Note that we don’t necessarily know that $G$ will be transitive on all pairs of chambers and apartments $(x, A)$ such that $x \in A$. Strong transitivity is a weaker condition — we allow any system of apartments.

4.1 Strongly Transitive Group Structure

In this section, we analyze the structure of strongly transitive groups. Let $G$ be a group with a strongly transitive action on a building $\Delta$. We choose an apartment $A$ such that $G_A$ acts transitively on $A$ and a chamber $x \in A$. For the rest of this section, we let $\mathcal{A}$ denote the orbit of $A$ under $G$, $B$ denote the stabilizer of the distinguished chamber $x$, and $N$ denote the stabilizer of the distinguished apartment $A$. Our goal is to use the subgroups $B$ and $N$ to derive information about the structure of $G$.

Recall that every isomorphism of a Coxeter complex is given by left-multiplication and so is determined by the image of a single element. Since
$N$ acts transitively on chambers in $A$, it maps surjectively onto the group of isomorphisms of $A$ with kernel $B \cap N$. Therefore $N/(B \cap N)$ is isomorphic to the Coxeter group $W$.

Since $G$ is transitive on chambers in $\Delta$, cosets of $B$ in $G$ are in bijective correspondence with chambers in $\Delta$. Under this correspondence, the apartment $A$ consists of the cosets $NB$. Note that every element of $N$ that maps onto $w \in W$ under the quotient map is in the same coset of $B \cap N$, and hence the same coset of $B$. Thus, we can write $wB$ for this coset without ambiguity; it represents the unique chamber $y$ in $A$ such that $\delta(x, y) = w$.

The following powerful lemma gives a group theoretic interpretation of the distance function.

**Lemma 4.2** Let $\Delta$ be a building and $G$ a group with a strongly-transitive action on $\Delta$. Let $x$ be a chamber in the distinguished apartment $A$ and let $B = G_x$. Then for any $g \in G$

$$\delta(x, gx) = w \iff g \in BwB$$

**Proof:** Assume that $\delta(x, gx) = w$. Let $A'$ be an apartment in $\mathcal{A}$ (the orbit of $A$ under $G$) containing $x$ and $gx$. Then, there is a $b \in B$ such that $bA' = A$. Thus, $bgx$ is a chamber in our distinguished apartment $A$ with $\delta(x, bgx) = w$, so $bg \in wB$. Conversely, assume $g \in BwB$. Then, $gx = bwBx = bwx$ for some $b \in B$, and so $\delta(x, gx) = \delta(b^{-1}x, wx) = w$. $\square$

This simple lemma has many implications for the structure of $G$. First, we see that $BwB \cap Bw'B = \emptyset$ when $w \neq w'$. Furthermore, we have

$$G = \bigsqcup_{w \in W} BwB = BNB$$

which is known as the Bruhat decomposition of $G$.

Consider products of the form $Bs_jBwB$, where $s_j$ is a generator of $W$. This union of cosets of $B$ represents a set of chambers in $\Delta$. To determine which ones occur, we note that this double-coset is precisely all products of elements $h \in Bs_jB$ with elements $g \in BwB$. Thus, the products $hg$ represent chambers of the form $hgx$.

By lemma 4.2, $\delta(x, hx) = s_j$ and $\delta(x, gx) = w$. If we let $f$ be a reduced word such that $w = s_jf$, then there is a gallery from $x$ to $hgx$ of type $jf$. Of course, if $jf$ is reduced, then $\delta(x, hgx) = s_jw$. However, since $\Delta$ is a building, when $jf$ is not reduced we can not assume that there will be a
gallery from $x$ to $hgx$ of type $jf$. If $jf$ is not reduced, then lemma 1.4 says that $f'$ is reduced, where $f = jf'$. Then, the gallery from $x$ to $hgx$ of type $jf = jjf'$ can have two possible subgalleries corresponding to the subword $jj$. The subgallery could have the form $(x, y, x)$, in which case there would be a gallery from $x$ to $hgx$ of reduced type $f'$. Alternatively, this subgallery could be of the form $(x, y, z)$, in which case there is a gallery from $x$ to $hgx$ of reduced type $jf' = f$. So either $\delta(x, hgx) = s_jw$ or $\delta(x, hgx) = w$ and

$$Bs_jBwB \subseteq BwB \cup Bs_jwB$$

To recap: if $jf$ is reduced, then $Bs_jBwB = Bs_jwB$. Otherwise, we have $Bs_jBwB \subseteq BwB \cup Bs_jwB$. If $\Delta$ is thick, then each $j$-residue has at least three members, and so we in fact have $Bs_jBwB = BwB \cup Bs_jB$ whenever $jf$ is not reduced. In particular, if $\Delta$ is thick then $Bs_jBs_jB \neq B$.

4.2 BN-pairs

In the previous section, we obtained a list of conditions satisfied by groups having a strongly transitive action on a thick building. In this section, we analyze the problem the other way around. We will define a BN-pair as a “minimal set” of these conditions and see what we can say about groups with BN-pairs. Although there are many applications to group theory, we continue to focus on the close connections between groups with BN-pairs and thick buildings.

**Definition 4.3** Let $G$ be a group with subgroups $B$ and $N$. We say that the pair $(B, N)$ is a BN-pair for $G$ if the following conditions hold:

1. $\langle B, N \rangle = G$
2. $B \cap N \lhd N$, and $N/(B \cap N) \cong W$ for some Coxeter group $W$
3. For $w \in W$ and $s_i$ a generator of $W$, we have $Bs_iBwB \subseteq BwB \cup Bs_iwB$
4. For $s_i$ a generator of $W$, $s_iBs_iB \neq B$

Note that taking inverses in condition 3 shows $BwBs_iB \subseteq BwB \cup Bws_iB$. The definition of a BN-pair is sufficiently strong to guarantee all of the properties we saw in the previous section. In particular, we have:

**Lemma 4.4** If $BwB = Bw'B$, then $w = w'$. Thus $G = \bigsqcup_{w \in W} BwB$. 

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**Proof:** We show this by induction on $\ell(w)$. Clearly, if $w = 1$ and $BwB = Bw'B$, then $w' = 1$ as well.

Now, assume that $BwB = Bw'B$, where $\ell(w) \geq 1$. We can write $w$ as $s_iy$ with $\ell(y) < \ell(w)$. Then $y = s_iw$ and so

$$ByB \subseteq Bs_iBwB = Bs_iBw'B \subseteq Bw'B \cup Bs_iw'B$$

where the last containment follows from condition 3. By induction we know that either $y = w'$ or $y = s_iw'$. In the first case, we also have $ByB = BwB$ and so $y = w$, a contradiction. Thus, $y = s_iw'$ and $w = w'$.

Since conditions 1 and 3 together imply that $G = BNB$, the second statement follows immediately. \( \square \)

We record the work of the last section in the following theorem:

**Theorem 4.5** Let $\Delta$ be a thick building and $G$ be a group with a strongly transitive action on $\Delta$. Then for any apartment $A$ such that $G_A$ is transitive on $A$ and any chamber $x \in A$, the stabilizer $B$ of $x$ and the stabilizer $N$ of $A$ form a BN-pair for $G$.

In brief, there is a natural way to assign a BN-pair to a group having a strongly-transitive action on a thick building. It turns out that we can go the other way around as well. The following theorem shows that every group with a BN-pair has a strongly transitive action on some thick building.

**Theorem 4.6** Let $G$ be a group that has a BN-pair $(B, N)$. Then $G$ is strongly transitive on a thick building $\Delta$, where the chambers of $\Delta$ are the cosets $G/B$ and the distance function is given by

$$\delta(gB, g'B) = w \iff g^{-1}g' \in BwB$$

In addition, $N$ stabilizes an apartment containing $B$.

Before we give the proof, we first state an easy lemma that lies at the heart of the proof.

**Lemma 4.7** If $f = i_1 \ldots i_k$ is a reduced word then

$$Bs_{i_1}Bs_{i_2}B \ldots Bs_{i_k}B = Bs_fB$$
Proof: (of lemma 4.7)
We prove this by induction on \( k \). The base case is trivial. Now, assume that this holds true for all reduced words of length less than \( k \). If we let \( g \) and \( g' \) be reduced words such that \( f = i_1 g = g'i_k \) then:

\[
Bs_{i_1}B \ldots Bs_{i_k}B = Bs_{i_1}Bs_gB = Bs_{g'}Bs_{i_k}B
\]

By condition 3 for BN-pairs, the term on the left is contained in both \( Bs_gB \cup Bs_fB \) and \( Bs_{g'}B \cup Bs_fB \). By lemma 4.4 we have that \( Bs_gB \), \( Bs_{g'}B \), and \( Bs_fB \) are pairwise disjoint, and so the term on the left must be contained in \( Bs_fB \). The reverse containment is clear, and the statement follows. \( \square \)

Proof: (of theorem 4.6)
We must check that \( G/B \) satisfies the conditions for a building. Clearly every \( i \)-residue has at least two elements. We also must show that for any reduced word \( f \), \( \delta(gB, g'B) = s_f \) iff there is a gallery of type \( f \) from \( gB \) to \( g'B \).

Let \( f = i_1 \ldots i_k \) be a reduced word. If \( \gamma = (gB, \ldots, g'B) \) is a gallery of type \( f \) from \( gB \) to \( g'B \), then \( g^{-1}g' \in Bs_{i_1}B \ldots Bs_{i_k}B \). Lemma 4.7 shows that \( \delta(gB, g'B) = s_f \). Conversely, assume that \( \delta(gB, g'B) = s_f \). Lemma 4.7 shows that for some \( b_1, b_2, \ldots, b_k \in B \) we have:

\[
g'B = gb_1s_{i_1}b_2s_{i_2} \ldots b_k s_{i_k}B
\]

and we can simply read off a gallery from \( gB \) to \( g'B \). Thus \( G/B \) is in fact a building. To show that it is thick, we simply note that by condition 4 of a BN-pair \( s_iBs_iB \neq B \) and so every \( i \)-residue has at least three elements.

To show that \( N \) stabilizes an apartment, note that \( NB \) is the set of cosets \( wB \) as \( w \) ranges over all values in \( W \). This is isometric to a Coxeter complex, so \( NB \) is an apartment stabilized by \( N \). Finally, we show that \( G \) has a strongly-transitive action on \( G/B \). Since \( N \) is transitive on chambers in \( NB \), we must have that \( G_{NB} \) is transitive on \( NB \) as well. Also, let \( (gB, g'B) \) and \( (hB, h'B) \) be pairs of chambers such that \( \delta(gB, g'B) = \delta(hB, h'B) = w \). Then \( g^{-1}g' \in BwB \) and \( h^{-1}h' \in BwB \), so there is some choice of \( b_1, b_2 \in B \) such that \( g' \in gb_1wB \) and \( h' \in hb_2wB \). Then \( h_b g_1^{-1}g' \) maps \( gB \) to \( hB \) and \( g'B \) to \( h'B \). \( \square \)

Note in particular that condition 4 of the definition of a BN-pair was only used to prove that \( \Delta \) is thick. Also, we did not need to use thickness to show that conditions 1-3 hold for strongly transitive groups. Thickness is thus in some sense equivalent to condition 4.

Although the operations of theorems 4.5 and 4.6 are in a sense “opposite,” they are not true inverses of each other. There may be two pairs \( (B, N) \) and
both satisfying the properties of a BN-pair for $G$; in this case both would yield the same building under theorem 4.6 (we see an example in the following section). This ambiguity should not surprise us, as all the relevant information about the building appears in $G/B$ and $N/(B \cap N)$ — it is feasible that we can enlarge or shrink $N$ and $B \cap N$ without changing the underlying structure. The following proposition goes into more detail.

**Proposition 4.8** Let $G$ be a group with two BN-pairs $(B, N)$ and $(B, N')$ such that $N/(B \cap N) \cong N'/(B \cap N')$. If we let $T$ denote the pointwise stabilizer in $G$ of the apartment $NB = N'B$, then

$$T = \bigcap_{n \in N} nBn^{-1} = \bigcap_{n' \in N'} n'Bn'^{-1}$$

and the stabilizer of this apartment is $NT = N'T$. Furthermore, $(B, NT)$ is another BN-pair for $G$.

**Proof:** Since theorem 4.6 only depends on $G/B$ and $N/(B \cap N)$, it is clear that these two pairs will yield the same building, and that the stabilizer and pointwise stabilizer of the apartment have the required form. Then by theorem 4.5, $(B, NT)$ is a BN-pair for $G$. □

This proposition suggests that the BN-pair $(B, NT)$ is the most natural choice, as it most fully reflects the action of $G$ on the building. Indeed, if we add an additional axiom to our definition of a BN-pair of the form

$$B \cap N = \bigcap_{n \in N} nBn^{-1}$$

then the ambiguity is resolved.

### 4.3 The General Linear Group

In this section, we use our work with BN-pairs to analyze the general linear group. Recall that if $V$ is an $n$-dimensional vector space over any field $F$, then the set of maximal flags in $V$ forms a building of type $S_n$. This building is thick because $i$-residues correspond to the set of points on a projective line over $F$ (and so have more than 2 elements). The group $GL_n$ has a strongly transitive action on this building: we will show that it is transitive on pairs $(x, A)$ where $A$ is any apartment in $G$ and $x \in A$.

Let $A$ and $A'$ be apartments, and let $x \in A$, $x' \in A'$. Let $v_1, \ldots, v_n$ be a basis for $A$, and $u_1, \ldots, u_n$ be a basis for $A'$. We can reorder the basis vectors
in $A$ and $A'$ such that the $i$th subspace in the flags $x$ and $x'$ is spanned by the first $i$ vectors in our basis. If we let $Q$ denote the matrix with $i$th column equal to $v_i$, and $R$ denote the matrix with $i$th column equal to $u_i$, then $RQ^{-1}$ maps the pair $(x, A)$ into $(x', A')$.

By theorem 4.5, $GL_n$ should have a BN-pair. To find such a pair, we distinguish the pair $(e, A)$ where $A$ corresponds to the standard basis for $V$ and $e$ is the standard flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, e_2, \ldots, e_{n-1} \rangle$$

Then, $B = G_e$ is the set of all matrices in $GL_n$ that map each subspace $E_i = \text{Span}\{e_1, \ldots, e_i\}$ into itself. Thus, this stabilizer $B$ is the group of upper-triangular matrices. Note that two bases for $V$ will yield the same apartment iff each vector in one basis is a scalar multiple of a vector in the other. So, the stabilizer $N = G_A$ is the subgroup consisting of matrices with exactly one non-zero entry in each column and each row. Since $N$ is transitive on chambers in $A$, the pair $(B, N)$ is a BN-pair for $G$. The intersection $B \cap N$ is the set of diagonal matrices, so $N/B \cap N$ is the group of permutation matrices — the matrices with exactly one 1 in each column and each row — and can be identified naturally with $S_n$ as required.

If we let $N'$ denote the group of permutation matrices, then $(B, N')$ is another BN-pair for $GL_n$. We verify this by checking the conditions of the definition. It is clear that $B \cap N' = 1$ and so $N' \cong S_n$. Before we verify the other axioms, we examine briefly how $S_n$ acts on $GL_n$ by conjugation. Let $g$ be an arbitrary element in $GL_n$. Conjugating by $w \in S_n$ takes the $(i, j)$th entry in $g$ to the $(w(i), w(j))$th entry in $wgw^{-1}$. If we consider two entries $g_{ij}$ and $g_{ji}$ that are diagonally opposite each other, they will again be opposite in $wgw^{-1}$. Thus, conjugation by $w$ permutes the entries of $g$ on the diagonal. Also, note that $s_iBs_i \not\subset B$, since some matrix in $s_iBs_i$ has a non-zero entry in the $(i+1, i)$ position, and hence does not belong to $B$.

We next verify that $B$ and $N'$ together generate $GL_n$. Recall that $GL_n$ is generated by elementary matrices, which are matrices of one of the following forms:

- any permutation matrix
- any (invertible) diagonal matrix
- any matrix with 1’s on the diagonal and one other non-zero entry (called shear matrices)
Since permutation matrices are contained in $N'$ and diagonal matrices are in $B$, it suffices to show that $\langle B, N' \rangle$ contains every shear matrix. Note that each shear matrix is either upper or lower triangular. If $b \in B$ is an arbitrary upper triangular matrix, then conjugating $b$ by the element $(1, n)(2, n-1)(3, n-2) \ldots \in S_n$ yields a lower triangular matrix. By choosing $b$ appropriately, we can generate any lower triangular matrix in this way, so we indeed have $\langle B, N' \rangle = GL_n$.

It remains to show that for any $s_i$ and $w$

$$Bs_iBwB \subseteq BwB \cup Bs_iwB$$

This condition can be rewritten as

$$B(s_iB)(wBw^{-1}) \subseteq B \cup Bs_i$$

This means that any matrix in $s_iB$ can be reduced to one in $B$ or $Bs_i$ by left-multiplication by elements in $B$ and right-multiplication by elements in $wBw^{-1}$. We will show this by considering a particular $s_i$ and showing that it holds true regardless of our choice of $w$.

For concreteness, we consider $s_1$ — all the other cases are similar. If we let $\ast$ denote arbitrary entries in $F$, then the general matrix in $s_1B$ has shape

$$s_1b = \begin{pmatrix} 0 & \ast & \ldots & \ast \\ \ast & \ast & \ldots & \ast \\ 0 & 0 & \ast & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ast \end{pmatrix}$$

Since $B$ contains all upper triangular shear matrices, we can use left-multiplication by matrices in $B$ to row reduce our matrix $s_1b$ to the form:

$$Q = \begin{pmatrix} 0 & q_3 & 0 & \ldots & 0 \\ q_1 & q_2 & 0 & \ldots & 0 \\ 0 & 0 & \ast & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ast & \ldots \end{pmatrix}$$

We will be done if we can show that, regardless of our choice of $w$, multiplying $Q$ on the right by matrices in $wBw^{-1}$ yields either a matrix in $B$ or $Bs_i$.

Consider the $2 \times 2$ matrix in the upper left-hand corner. Note that if $q_2 = 0$ then $Q \in Bs_1$, and if $q_1 = 0$ then $Q \in B$. Otherwise, we have:

$$\begin{pmatrix} 0 & q_3 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 \\ -q_1/q_2 \end{pmatrix} = \begin{pmatrix} k & q_3 \\ 0 & q_2 \end{pmatrix} \in B$$

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\[
\begin{pmatrix}
0 & q_3 \\
q_1 & q_2
\end{pmatrix}
\begin{pmatrix}
1 & -q_2/q_1 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & q_3 \\
q_1 & 0
\end{pmatrix} \in Bs_1
\]

So, it suffices to show that \(wBw^{-1}\) contains either every shear matrix with a non-zero entry in the (2, 1) position or every shear matrix with a non-zero entry in the (1, 2) position. To this end, recall that conjugation by \(w\) takes symmetrically-opposite pairs of entries in \(b\) to symmetrically-opposite pairs. Since \((1, 2)\) and \((2, 1)\) are opposite, one of their preimages must have been in the right-upper half of \(b\), and so we can find a \(b'\) such that one of these entries is non-zero. Furthermore, we can choose \(b'\) so that all diagonal entries are 1 and all other entries in \(b'\) are 0. Then, the resulting matrix \(wb'w^{-1}\) will have the required form. Thus, the final condition of BN-pairs holds, and \((B, N')\) is indeed a BN-pair for the general linear group. This demonstrates that \(GL_n\) has two BN-pairs that are essentially equivalent.

**Areas for Further Study**

Having developed a basic theory of the structure of buildings, we now discuss briefly several possibilities for further study. Much of the theory of buildings centers around classification theorems, and the analysis we have done is adequate preparation to start work in this direction.

The most fundamental classification theorem is that of spherical buildings. This theorem describes all spherical buildings whose Coxeter groups have rank at least 3. Of course, the natural first step is to classify finite Coxeter groups. This classification is well-known; the set of all finite Coxeter groups is composed of four infinite classes along with six additional exceptional groups (or seven, depending on how you count). Thus, every spherical building will have a type that falls into one of these classes. The problem can be reduced further in the following manner. Much of the information about the structure of a building is contained in the different \(J\)-residues for \(|J| = 2\). In fact, by including a labeling scheme on the chambers, it is possible in many cases to collate all of the important information about the building into a foundation consisting of these \(J\)-residues and the labeling information. Foundations allow us to reconstruct the building by gluing together the \(J\)-residues according to an equivalency determined by the labeling. It turns out that every spherical building yields a foundation. Furthermore, these foundations each yield a unique building provided that they satisfy some conditions on their restriction to spherical buildings of rank 3. There are only a few such restrictions to consider, and analyzing these leads to a full classification.
Another important case is that of affine buildings — those whose Coxeter complexes correspond to affine reflection groups. For such buildings, each apartment will correspond to a tiling of a Euclidean vector space. Thus, we can impose a natural metric on affine buildings that is derived from this local Euclidean structure. It turns out that to each affine building we can associate a spherical “building at infinity”. By using the classification of spherical buildings, we can again reduce the problem of classifying affine buildings to a more manageable question to one about root groups and labeling schemes.

These classification theorems come in useful for working with Lie groups or manifolds: if we can construct a building based on the object in question, our classification theorem will help us determine its properties. As the close link between buildings and BN-pairs suggests, the theory of buildings can be used to derive many more useful theorems in group theory.
References


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