Does Tail Dependence Make A Difference In the Estimation of Systemic Risk? $\Delta CoVaR$ and $MES$

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Abstract

The notion of systemic risk is based on the interdependence of financial institutions and the financial market in the context of extreme tail events. This paper develops a common framework based on a copula model to estimate several popular return-based systemic risk measures: Delta Conditional Value at Risk ($\Delta CoVaR$) and its modification ($\Delta CoVaR_{\leq}$); and Marginal Expected Shortfall ($MES$) and its extension, systemic risk measure ($SRISK$). By eliminating the discrepancy of the marginal distribution, copula models provide the flexibility to concentrate only on the effects of dependence structure on the systemic risk measure. We estimate the systemic risk contributions of four financial industries consisting of a large number of institutions for the sample period from January 2000 to December 2010. First, we found that the linear quantile regression estimation of $\Delta CoVaR$, proposed by Adrian and Brunnermeier (AB hereafter) (2011), is inadequate to completely capture the non-linear contagion tail effect, which tends to underestimate systemic risk in the presence of lower tail dependence. Second, $\Delta CoVaR$ originally proposed by AB (2011) is in conflict with dependence measures. By comparison, the modified version of $\Delta CoVaR_{\leq}$ put forward by Girardi et al. (2011) and $MES$, proposed by Acharya et al. (2010), are more consistent with dependence measures, which conforms with the widely held notion that stronger dependence strength results in higher systemic risk. Third, $\Delta CoVaR_{\leq}$ is observed to have a strong correlation with tail dependence. In contrast, $MES$ is found to have a strong empirical relationship with firms’ conditional CAPM $\beta$. $SRISK$, however, provides further connection with firms’ level characteristics by accounting for information on market capitalization and liability. This stylized fact seems to imply that $\Delta CoVaR_{\leq}$ is more in line with the “too interconnected to fail” paradigm, while $SRISK$ is more related to the “too big to fail” paradigm. In contrast, $MES$ offers a compromise between these two paradigms.

Keywords: Conditional Value at Risk, Marginal Expected Shortfall, Dependence Structure, Systemic Risk, Copula

JEL Classifications: C21, G11, G32, G38

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1 Introduction

The financial crisis that occurred during 2007/2009 has spurred intensified research attempting to understand the interdependence of risk factors and the fragility of the financial system. In the past few years, the distress associated with some individual financial institutions has spread throughout the entire financial system. An individual financial institution is considered of systemic importance if its distress or failure increases the likelihood that other firms will go bankrupt and threatens the stability of the entire system, either due to its size or its interconnectedness with the rest of the financial industry. Basel III had proposed a capital surcharge for Systemically Important Financial Institutions (SIFIs). In December 2011, the Fed introduced such a surcharge for eight banks: Bank of America, Bank of New York Mellon, Citigroup, Goldman Sachs, JPMorgan, Morgan Stanley, State Street and Wells Fargo. As a result, the questions of how to accurately quantify the risk of spill-over effects and how to identify systemically important financial institutions have become crucial for macro-prudential regulators and supervision authorities.

Value at Risk (VaR) and Expected Shortfall (ES), the most widely-used risk measures by financial institutions and academic researchers, have been criticized for their failure to take into account the escalated risk of spill-over effects among financial institutions during episodes of financial crisis. Ang, Chen and Xing (2006) documented that conditional correlations between assets are much higher in downturns of the financial market. More recently, Brunnmeier and Petersen (2008) show that the negative feedback of a “loss spiral” or “margin spiral” leads to the joint depression of asset prices. Systemic risk arises because of increasing co-movement and linkages among financial institutions’ assets and liabilities during a crisis. Therefore micro-prudential regulation, which historically focused on a bank’s risk in isolation, is not adequate to contain the contagion effects of an extreme tail event.

However, the co-movement of downside risk for financial institutions during a financial crisis cannot be completely captured by Pearson’s correlation $\rho$, which only provides the linear dependence but disregards nonlinear dependence between asset prices. It is widely accepted that the linear correlation $\rho$ is inadequate to characterize the full dependence structure of non-normally distributed random variables such as the daily returns of financial institutions. Figure 1 displays the scatter plot of real data for AIG and the Dow Jones Financial Index Return (a proxy for the financial system), and the simulated data from a bivariate normal distribution based on the first two moments of actual data. Even though their first two moments (including correlation $\rho$) are identical, their respective behaviors in the tail are quite different. The exceedance correlation and quantile dependence both lie outside the 95% confidence interval for a joint normal distribution, which implies the presence of nonlinear dependence in the tail. Therefore, even though the average linear dependence strength $\rho$ between real and simulated data is controlled to be the same, their tail dependence is quite different as shown in the lower panels of Figure 1.

1 Federal Reserve Governor Daniel Tarullo defined it: “Financial Institutions are systemically important if the failure of the firm to meet its obligations to creditors and customers would have significant adverse consequences for the financial system and the broader economy.”

2 Extreme tail event is defined as the days when the return of financial institutions or the market drop below a certain extreme threshold value $r_{tT} < F^{-1}_r(\tau)$

3 Exceedance correlation of $X_1$ and $X_2$ (Ang & Chen 2001) is defined as $\text{corr}(X_1, X_2|X_1 \leq F^{-1}_1(u), X_2 \leq F^{-1}_2(u))$ in the left tail and $\text{corr}(X_1, X_2|X_1 \geq F^{-1}_1(u), X_2 \geq F^{-1}_2(u))$ in the right tail. Analogously, quantile dependence $= P(X_2 \leq F^{-1}_2(u)|X_1 \leq F^{-1}_1(u))$ in the left tail and $P(X_2 \geq F^{-1}_2(u)|X_1 \geq F^{-1}_1(u))$ in the right tail.
In other words, the co-movements between financial institutions and the market under conditions of distress (tail dependence) cannot be measured by their co-movements under normal times ($\rho$). As systemic risk mainly studies the extent to which extreme values tend to occur together, what really matters for the magnitude of systemic risk is the dependence between the tail risk of individual institutions and the financial system instead of the characteristics of the marginal distribution for individual firms.

There is an extensive and growing body of literature proposing alternative approaches to quantifying the systemic risk of financial institutions. Huang et al. (2009) used credit default swaps (CDS) and equity price correlations to construct a systemic risk indicator: distress insurance premium (DIP), a measure computed under risk-neutral probability. Zhou (2010) studied systemic importance measures under the multivariate extreme value theory (EVT) framework and found that the “too big to fail” argument is not always valid. Billo et al. (2010) used principal component analysis and a Granger-Causality test to measure the interconnectedness among the returns of hedge funds, banks, brokers and insurance companies. Eliana et al. (2012) recently used extreme value theory to investigate the extreme tail dependence among stock prices of US bank holding companies and found that it is the extreme tail dependence that makes a difference in the measurement of systemic risk. A good survey of systemic risk measures can be found in Dimitrios et al. (2012).

The most direct measure of systemic risk is simply the joint distribution of negative outcomes for a collection of systemically important financial institutions. Two leading metrics for return-based estimation of systemic risk are CoVaR, proposed by Adrian and Brunnermeier (2011), and Marginal Expected Shortfall (MES), proposed by Acharya et al. (2010), which estimate the contribution of a single institution to the overall systemic risk of the
financial system during a tail event. More specifically, Adrian et al. (2011) defined CoVaR as the VaR of the financial system when an individual financial institution is in financial distress. They further defined an institution’s contribution to systemic risk of the financial system as the difference between CoVaR when the firm is, or is not, in financial distress. Acharya et al. (2010), Brownless and Engle (2011), among others, introduced the novel concepts of systemic risk measure (SRISK) and marginal expected shortfall (MES) to estimate the systemic risk exposure of an institution. MES is the expected loss an equity investor would experience if the overall market is in the left tail. SRISK extends the MES in order to take into account both the liability and size of institutions. Over the past few years, dozens of research papers and media coverage have discussed, implemented and generalized these systemic risk measures. Hundreds of financial institutions in the world have been assessed for their systemic importance based on these measures. Sylvain et al. (2012) proposed a theoretical and empirical comparison of Marginal Expected Shortfall (MES), Systemic Risk Measure (SRISK) and ∆CoVaR. They assumed that the time varying correlation coefficient completely captures the dependence between the firm and market returns, and investigated under what conditions these different systemic risk measures converge in identifying systemic important financial institutions (SIFIs).

The notion of systemic risk is based on the interdependence of financial institutions and the financial system, especially in the context of extreme tail events. In this paper, we study these most widely used market-return-based systemic risk measures: MES and ∆CoVaR in the framework of a wide range of copula models, which is very straightforward in accommodating the nonlinear pattern of dependence structure. We aim to determine if these measures of systemic risk are consistent with the measures of tail dependence. Formally, the lower tail dependence (LTD) is defined as

\[ LTD = \lim_{u \to 0^+} P(X_2 \leq F_2^{-1}(u)|X_1 \leq F_1^{-1}(u)) \]

Where \( F^{-1}(u) \) is the inverse CDF at quantile \( u \). Therefore, we want to investigate if the higher value of dependence strength in the lower tail dependence (LTD) between financial firms and the market indicates the higher systemic risk: ∆ CoVaR or MES. Georg et al. (2012) investigated the dependence consistency of CoVaR mainly with respect to its statistical properties. They found that CoVaR based on a stress event \( X \leq VaR_x(\alpha) \) is consistent with a dependence measure of correlation \( \rho \). By contrast, the most widely used CoVaR proposed by AB (2011), which is conditional on stress event \( X = VaR_x(\alpha) \), does not monotonically increase with the linear correlation \( \rho \).

In contrast, our paper provides a more thorough empirical study for a panel of 64 top US financial institutions over the period from January 2000 to December 2011. Our empirical analysis delivers the following main results:

**First**, we found that the linear quantile regression estimation of ∆CoVaR proposed by AB (2011) is inadequate to completely take into account the non-linear contagion tail effect, which tends to underestimate systemic risk in the presence of lower tail dependence. **Second**, ∆CoVaR, originally proposed by AB (2011), is in conflict with dependence measures in terms of correlation \( \rho \), Kendall \( \tau \) or lower tail dependence (LTD). Stronger dependence strength could instead lead to lower systemic risk, which contradicts the common view that higher dependence leads to higher systemic risk. In comparison, the modified version of ∆CoVaR, based on the stress condition \( X \leq VaR_x(\alpha) \), is more consistent with dependence measures. However, it tends to converge as dependence strength (correlation \( \rho \) or Kendall \( \tau \)) is high during a financial crisis, when it is most desirable for supervision authorities to identify systemically important financial institutions (SIFIs). **MES**, on the other hand, provides a moderately better response to the dependence structure if the heterogeneity of the marginal distribution can be eliminated. **Third**, ∆CoVaR is observed to have a strong correlation with tail dependence. In contrast, MES is found to have a strong empirical relationship with firms’ conditional CAPM \( \beta \). **SRISK**, however, provides further connection with firms’ level characteristics by accounting for information on market capitalization and liability. This stylized fact seems to imply that ∆CoVaR is more in line with the “too interconnected to fail” paradigm, and SRISK is

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5For online computation of systemic risk measures, see the Stern-NYU’s V-Lab initiative at http://vlab.stern.nyu.edu/welcome/risk/.
more related to the “too big to fail” paradigm. In contrast, $MES$ is a compromise between these two paradigms. More specifically, $MES$ is a risk exposure measure. Firms’ specific marginal heterogeneity, such as volatility, and the dependence structure between firms and market jointly determine the value of $MES$. $\Delta CoVaR^{\leq}$, however, is only related to the dependence between firms and the market.

The remainder of this paper is organized as follows: Section 2 introduces the definition of Delta Conditional Value at Risk ($\Delta CoVaR$) and its modification ($\Delta CoVaR^{\leq}$); and Marginal Expected Shortfall ($MES$) and its extension, systemic risk measure ($SRISK$). Then we discuss their estimation in the common framework of a copula model. Section 3 discusses the consistency of all these systemic risk measures with the dependence measures (linear correlation $\rho$, Kendall correlation and lower tail dependence). Section 4 describes the estimation strategy for both the marginal and copula models. Section 5 presents the data and empirical studies for a group of 64 top US financial institutions from January 2000 to December 2011. Section 6 concludes.
2 Methodology

In this section, we introduce a formal definition of systemic risk measures proposed by Adrian and Brunnermeier (2011) and Acharya et al. (2010). Let us assume a financial system composed of a large number of institutions. Denote by \( r_{mt} \) and \( r_{it} \) the daily return for the market, i.e. the financial system and firm \( i \) on time \( t \), respectively. The market return can be considered as the portfolio of all firms’ returns

\[
r_{mt} = \sum_{i=1}^{N} \omega_{it} r_{it}
\]

where \( \omega_{it} \) denotes the weight (market capitalization) of firm \( i \) in the portfolio at time \( t \).

2.1 Definitions

2.1.1 \( \Delta CoVaR \)

Paralleling the definition of Value at Risk (VaR), the conditional Value at Risk (CoVaR) is defined as the expected maximal loss of a certain portfolio at some confidence interval (\( \tau \) quantile) given another portfolio at the same time experiences expected maximal loss. Formally, the CoVaR of financial institution \( i \) corresponds to the Value at Risk of the financial system \( m \) conditioning on the occurrence of tail event \( r_{it} = VaR_{\tau}^{i} \) for the institution \( i \).

\[
Pr\left(r_{mt} \leq CoVaR_{m|i}(\tau, \tau^{*}) | r_{it} = VaR_{\tau}^{i}\right) = \tau
\]

When \( \tau^{*} = \tau \), we simply denote \( CoVaR_{m|i}(\tau, \tau) = CoVaR_{m|r_{it}=VaR_{\tau}^{i}} \). The Value at Risk of the financial institution \( i \): \( VaR_{\tau}^{i} \) is defined as the \( \tau^{*} \) quantile of its loss probability distribution

\[
Pr\left(r_{it} \leq VaR_{\tau}^{i}\right) = \tau^{*}
\]

Further, AB (2011)\(^6\) proposed \( \Delta CoVaR \) as the difference between \( CoVaR_{\tau}^{m|r_{it}=VaR_{\tau}^{i}} \) when the financial institution \( i \) is in distress and \( CoVaR_{\tau}^{m|r_{it}=Median} \) when the financial institution \( i \) is in the normal state.

\[
\Delta CoVaR_{\tau}^{m|i} = CoVaR_{\tau}^{m|r_{it}=VaR_{\tau}^{i}} - CoVaR_{\tau}^{m|r_{it}=Median}
\]

\[
= CoVaR_{\tau}^{m|i}(\tau, \tau) - CoVaR_{\tau}^{m|i}(\tau, 0.5)
\]

Therefore \( \Delta CoVaR \) measures the increment of the VaR (difference between two conditional VaR), which aims to capture the contribution of a particular institution \( i \) to the tail risk of financial system as a whole.

2.1.2 \( \Delta CoVaR^{<} \)

Defining the financial distress being exactly at its VaR is arguably too restrictive. Girardi et al. (2011), among others, extended the definition of stress event to be below \( (r_{it} \leq VaR_{\tau}^{i}(\tau)) \) rather than exactly being at its VaR \( (r_{it} = VaR_{\tau}^{i}(\tau)) \), which allows for more severe distress event being further in the tail. Formally, the alternative

\(^6\)Adrian and Brunnermeier (2011) further introduce “Exposure CoVaR”, which reverse the conditioning set to be the tail event for market return: \( r_{mt} = VaR_{\tau}^{m} \).
CoVaR (hereafter called $CoVaR^\leq$) can be defined as:

$$Pr\left(Y \leq CoVaR_\tau | X \leq VaR_q\right) = \tau$$

$$\frac{Pr\left(Y \leq CoVaR_\tau, X \leq VaR_q\right)}{Pr\left(X \leq VaR_q\right)} = \tau$$

$$Pr\left(Y \leq CoVaR_\tau, X \leq VaR_q\right) = \tau q$$

where $q(\tau)$ is the quantile defining the tail event of the financial institution (market). Therefore the alternative CoVaR at confident interval $\tau$ can be solved from the following equation:

$$\int_{-\infty}^{CoVaR_\tau} \int_{-\infty}^{VaR_q} f(x,y) dx dy = \tau q$$

Where $f(x,y)$ is the joint density function of $X$ and $Y$.

By analogy, the modified version of Delta CoVaR: $\Delta CoVaR^\leq$ can be specified as

$$\Delta CoVaR^\leq_\tau = CoVaR^m_{it \leq VaR_{it}(\tau)} - CoVaR^{i \leq Median}_{it \leq VaR_{it}(\tau)}$$

Such a slight change of tail event from $r_{it} = VaR_{it}(\tau)$ to $r_{it} \leq VaR_{it}(\tau)$ seems to be trivial, but will prove to make a significant difference in both the estimation strategy and its statistical properties.

2.1.3 MES

Marginal Expected Shortfall ($MES$) measures the marginal contribution of firm $i$ loss to systemic loss, or the average return of firm $i$ on the $\alpha\%$ worse days when the market as a whole is in the tail of its distribution. Formally, the coherent risk of system can be measured by its conditional expected shortfall ($ES$):

$$ES_{m,t-1} = E_{t-1}(r_{mt} | r_{mt} < C) = \sum_{i=1}^{N} \omega_i E_{t-1}(r_{it} | r_{mt} < C)$$

Where $C$ is the threshold value which defines the tail event when the market return exceeds it $r_{mt} < C$. The $MES$ of firm $i$ is defined as the partial derivative of the system’s aggregate risk ($ES$) with respect to the weight of firm $i$ in the market portfolio: $\omega_i$

$$MES_{it} = \frac{\partial ES_{m,t-1}(C)}{\partial \omega_i} = E_{t-1}(r_{it} | r_{mt} < C)$$

This measures the marginal contribution of firm $i$ to the systemic risk. The higher the value of $MES$, the higher the individual contribution of the firm $i$ to the risk of the financial system as a whole. Brownlees and Engle (2011) recently show that the estimation of $MES$ can be reduced into the estimations of three components such as volatility, correlation and tail expectation. The linear market model of Brownlees and Engle (2011) can be summarized as:

$$r_{mt} = \sigma_{mt} \epsilon_{mt}$$

$$r_{it} = \sigma_{it} \rho_{it} \epsilon_{mt} + \sigma_{it} \sqrt{1 - \rho_{it}^2} \xi_{it}$$

$$\left(\epsilon_{mt}, \xi_{it}\right) \sim F$$
where the innovation process $\xi_{it}$ is uncorrelated but not independent with $\epsilon_{mt}$. Therefore the conditional $MES$ can be expressed as a function of the firm’s return volatility $\sigma_{it}$, its correlation with the market return $\rho_{it}$ and the co-movement of the tail distribution:

$$MES_{it}(C) = \sigma_{it}\rho_{it}E_{t-1}\left(\epsilon_{mt}|\epsilon_{mt} < \frac{C}{\sigma_{mt}}\right) + \sigma_{it}\sqrt{1-\rho_{it}^2}E_{t-1}\left(\xi_{it}|\epsilon_{mt} < \frac{C}{\sigma_{mt}}\right)$$

$$= \beta_{it}E_{t-1}\left(r_{mt}|r_{mt} \leq C\right) + \sigma_{it}\sqrt{1-\rho_{it}^2}E_{t-1}\left(\xi_{it}|\epsilon_{mt} < \frac{C}{\sigma_{mt}}\right)$$

Where $\beta_{it} = \rho_{it} \frac{\sigma_{it}}{\sigma_{mt}}$ measures the risk exposure of financial institution $i$ to the market based on CAPM model. Suppose that the dependence structure between the firm $i$ and the market return is completely captured by the time varying conditional correlation $\rho_{it}$ (or $\xi_{it} \perp \epsilon_{mt}$), then the second term of the above equation becomes zero, and the first component completely describes the systemic risk exposure of financial institution $i$. Therefore, the second component is dedicated to capture the non-linear dependence between the firms $i$ and the market. Browless and Engle (2011) proposed a nonparametric kernel approach to estimate the tail expectation of return between firm $i$ and market.

2.1.4 SRISK

Brownlees & Engle (2011) and Acbarya, Engle and Richardson (2012) extended $MES$ and proposed the systemic risk measures (SRISK) to account for the level of firms’ characteristics (size, leverage, etc), which corresponds to the expected capital shortfall of a financial firm if we have another financial crisis as a whole. More formally, Brownlees & Engle (2011) defined SRISK as

$$SRISK_{it} = \max\left(0; \frac{k Debt_{it}}{Equity_{it}} - (1-k)(1 - LR M E S_{it})\right)$$

where $k$ is the prudential capital ratio which we take as 8%, $Debt_{it}$ is the quarterly book value of total liabilities, $Equity_{it}$ is the market value of equity (market capitalization) today, and $LR M E S_{it}$ is long term Marginal Expected Shortfall ($MES$), which corresponds to the expected drop in equity return conditioning on the market falling by more than 40% within the next six months. Brownlees & Engle (2011) obtained $LR M E S_{it}$ by implementing a simulation exercise. Therefore SRISK can be estimated by scaling up $MES$ to account for the leverage and size of firms.

2.2 Copula Based Estimation of $\Delta CoVaR$ and $MES$

Since both systemic risk measures $\Delta CoVaR$ and $MES$ are concerned with the joint distribution of downside risk, in this paper, we study CoVaR and $MES$ in the framework of the copula model, which is well suited to accommodate the non-linear dependence of tail risk between financial institutions and the market. Copula based multivariate models provide the flexibility to specify the models for the dependence structure (copula) of multivariate random variables separately from their marginal distribution. The essence of copula based model for CoVaR and $MES$ is to model the local dependence between the lower quantile of two assets.
Theorem 1 (Sklar) (1959) Let $G$ be a joint distribution function with $n$ marginals $F_i$. Then there exists an $n$ dimension copula $C$ such that

$$G(x_1, x_2, \ldots, x_n) = C\left(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\right)$$

If all margins $F_i$ are continuous, then the copula $C$ is unique.

In contrast to the traditional dependence measure like Pearson’s correlation $\rho$ and Kendall’s $\tau$, the copula model is capable of measuring the whole dependence structure including tail dependence between random variates.

Definition 1: Tail Dependence

Let $X_1$ and $X_2$ be two random variables with CDF $F_1$ and $F_2$. Then the lower tail dependence (LTD) coefficient of two random variables $(X_1, X_2)$ is defined as

$$LTD(X_1, X_2) = \lim_{u \to 0^+} \Pr\left(X_2 < F_2^{-1}(u) | X_1 < F_1^{-1}(u)\right) = \lim_{u \to 0^+} \frac{C(u, u)}{u}$$

Analogously, the upper tail dependence (UTD) coefficient of two random variables $(X_1, X_2)$ is defined as

$$UTD(X_1, X_2) = \lim_{u \to 1^-} \Pr\left(X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)\right) = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}$$

In the following, a wide range of copula models are to be used to estimate the systemic risk measures $\Delta CoVaR$ and $MES$, and we will study their relationship with the tail dependence between the financial institutions and market.

2.2.1 $\Delta CoVaR$ Estimation

If the joint distribution $F(x, y)$ has a continuous marginal distribution, Sklar’s Theorem (1959) stated that there exists a unique copula function such that any bivariate distribution can be represented by the combination of their marginal distributions in a certain form of copula function:

$$F(x, y) = C\left(F_x(x), F_y(y); \theta\right) = C(u, v; \theta)$$

where $u$ and $v$ are the marginal distributions $F_x(x)$ and $F_y(y)$ of $x$ and $y$ respectively. Differentiating $C(u, v; \theta)$ with respect to $u$, we can obtain the conditional distribution of $y$ given $x$ (See the proof in the Appendix):

$$F(y|x) = \frac{\partial C(u, v; \theta)}{\partial u} = C_1(u, v; \theta) = C_1(F_x(x), F_y(y); \theta)$$

where $F(y|x)$ denotes the conditional distribution of $y$ given $x$ and $C_1(u, v; \theta)$ represents the partial derivative of the copula with respect to the first argument.

By defining $\tau = F(y|x)$ and inverting $C_1$ with respect to the second argument can derive the marginal distribution of $y$: $F_y(y)$ as

$$F_y(y) = C_1^{-1}(F_x(x), \tau; \theta)$$

Let $x = VaR_x(\tau)$, and $y = CoVaR_x^{y|x}$. The solution for $CoVaR$ in the context of the copula framework can be explicitly expressed as:

$$CoVaR_x^{y|x} = F_y^{-1}\left(C_1^{-1}\left(F_x(VaR_x(\tau)), \tau, \theta\right)\right)$$ (1)
The representation in Equation (1) shows the attractiveness of the copula approach for modeling the nonlinear dependence between the lower quantiles of \( x \) and \( y \). A wide range of CoVaR can be obtained by combining different marginal distributions \( F(\cdot) \) with different copula functional forms \( C(\cdot, \cdot) \). Since we aim to study the nonlinear dependence of tail risk, we restrict our attention to the different specifications of copula models to see if the nonlinear dependence structure in the tail risk plays a role in the estimation of CoVaR.

**Proposition 2.1**

Suppose \( X \) and \( Y \) are the returns of two assets with Gaussian joint distribution (Gaussian Margins and Gaussian Copula model \( C(u, v; \theta) \)). Then the closed form solution of \( CoVaR_{y|x} \) can be expressed as (See the proof in Appendix 1):

\[
CoVaR_{y|x} = \rho \sigma_y \sigma_x \frac{\mu_y}{\mu_x} \sqrt{1 - \rho^2} \Phi^{-1}(\tau) + \mu_y
\]

\[
\Delta CoVaR_{y|x} = \frac{\rho \sigma_y}{\sigma_x} (VaR_x(\tau) - VaR_x(0.5))
\]

where \( \rho \) denotes the linear Pearson correlation between \( x \) and \( y \), \( \sigma_x \) and \( \sigma_y \) represent volatility of \( x \) and \( y \) respectively, \( \mu_x \) denotes the mean of \( x \), \( \Phi^{-1} \) denotes the inverse of standard normal distribution, and \( \epsilon \) is the standardized residual with \( \epsilon \sim N(0,1) \).

It is very common to estimate CoVaR with a Gaussian joint distribution as normality is the workhorse distribution assumption in the financial literature. Under the Gaussian joint distribution, the calculation of CoVaR is very straightforward and become a trivial issue. The dependence structure between tail risks of two assets is linear and can be completely captured by the second moment of the joint distribution \( \rho \sigma_y / \sigma_x \). It is noteworthy that the linear dependence coefficient \( \rho \sigma_y / \sigma_x \) is not quantile specific, which means that the dependence structure in the lower tail is exactly the same as that in any other part of joint distribution. However, this statistical property is only specific to the joint Gaussian distribution. In general, there is no explicit reason to justify why the dependence of the tail risk at different quantiles should be identical. More often than not, the contribution of a financial institution to the systemic risk of financial market when it is in bankruptcy (\( x = VaR_x(\tau) \)) should be quite different from that when it is in the normal state (\( x = VaR_x(0.5) \)).

AB (2011) employed a linear quantile regression model to estimate CoVaR as quantile regression is robust to the unknown distribution.\(^{11} \) In this paper, we consider the quantile regression of the financial market returns \( r_{mt} \) on a particular institution’s return \( r_{it} \) at the \( \alpha \) quantile

\[
Q_{r_{mt}}(\tau) = \alpha(\tau) + \beta(\tau)r_{it}
\]

The CoVaR of financial market, conditional on the financial institution being in distress, can be defined as:

\[
CoVaR_{m|VaR_{it}(\alpha)} = \alpha(\tau) + \beta(\tau)VaR_{it}(\alpha)
\]

\(^{11}\)AB (2011) extended the quantile regression by including some additional state variables such as VIX, liquidity spread, etc. In this paper, we consider a slightly different approach to facilitate the comparison with the copula based model.
Therefore the systemic risk measure $\Delta CoVaR$ put forward by AB (2011) can be constructed as:

$$\Delta CoVaR(\tau) = \beta(\tau) \left( VaR_{it}(\tau) - VaR_{it}(0.5) \right)$$

Figure 2 displays the simulation result for the $\Delta CoVaR$ estimated by various copula based models and quantile regression models under different data generating processes. In each panel of figure, the data are generated by corresponding copula model shown in the title. All marginal distributions are set to be the standard normal distribution except for student t copula where the margin is set to student t with df=5. The parameter of each copula model is chosen such that the linear correlation $\rho$ or Kendall correlation $\tau$ is equal to the value displayed in the X-axis. Therefore the discrepancy of $\Delta CoVaR$ estimation can only be attributed to the discrepancy of dependence structure in tail risk. As we can see, when the data are generated from a joint normal distribution, quantile based and copula based estimation of $\Delta CoVaR$ fit quite well. However, as the data are simulated from a copula model with positive lower tail dependence in the downside risk (Clayton, Rotated Gumbel), quantile based estimation of systemic risk $\Delta CoVaR$, which was the most widely used estimation strategy in the recent literature, is consistently underestimated, compared with copula based estimation. This fact is not surprising, considering that the linear quantile regression still assumes linear local dependence between downside risk, which is not adequate to estimate CoVaR when the left tail of the distribution exhibits positive and nonlinear dependence between downside risk.

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2.2.2 $\Delta CoVaR$ Estimation

With the stress condition being $X \leq VaR_x(\tau)$, the modified version of $CoVaR$ proposed by Girardi et al. (2011) can also be analyzed under the framework of the copula model. Starting from the above equation

$$\int_{-\infty}^{CoVaR_\tau} \int_{-\infty}^{VaR_q} f(x,y) dx dy = \tau q$$

The joint density function $f(x,y)$ can be substituted by the cross product of marginal density and copula density $f(x,y) = \frac{\partial^2 C(u,v)}{\partial u \partial v} f(x) f(y)$. By a simple manipulation, we have:

$$\int_{-\infty}^{CoVaR_\tau} \int_{-\infty}^{VaR_q} \frac{\partial^2 C(u,v)}{\partial u \partial v} f(x) f(y) dx dy = \tau q$$

$$\int_{0}^{F_y(CoVaR_\tau)} \int_{0}^{F_x(VaR_q)} \frac{\partial^2 C(u,v)}{\partial u \partial v} dudv = \tau q$$

(2)

Let $q = \tau$, we can solve $CoVaR_y | x \leq VaR_x(\tau)$ numerically from Equation (2). Analogously, let $q = 0.5$, $CoVaR_y | x \leq VaR_x(0.5)$ can be obtained as well. Therefore, $\Delta CoVaR$ can be estimated as

$$\Delta CoVaR_\tau = CoVaR_y | x \leq VaR_x(\tau) - CoVaR_y | x \leq VaR_{0.5}$$

Girardi et al. (2011) compared the bivariate skewed t distribution (Bauwens and Laurent 2005) with bivariate Gaussian distribution to model the density function $f(x,y)$, and they found that the former joint density outperformed the latter in the estimation of CoVaR. But they fail to explain whether this performance improvement comes from the difference in the specification of marginal distribution or the dependence structure in the tail. In our paper, we only focus on the discrepancy of dependence structure and eliminate the effect of marginal distribution as we believe that what really matters in the estimation of systemic risk is the dependence structure (especially tail dependence) rather than marginal characteristics.

Figure 3 illustrates the $CoVaR$ and $\Delta CoVaR$ under two copula models with different dependence structures in the tail of the distribution. The left panel present the Clayton copula joint distribution with lower tail dependence, while the right panel displays the joint distribution of the Gumbel copula with upper tail dependence. Suppose that bank $i$’s financial condition moves from its normal state (at Median) to its financial distress (at 1% quantile). We want to investigate what is the change of 1% quantile for the financial system. 8000 random draws are simulated from two bivariate joint distributions. The parameters of each copula model are chosen to provide a linear correlation of random variates equal to 0.8. It is not surprising that the systemic risk measure $\Delta CoVaR$ under the Clayton copula is larger than that under the Gumbel copula because the former joint distribution is characterized by the larger lower tail dependence, which results in stronger co-movement in the tail risk of financial institutions and the market.
Figure 3: This figure displays 8000 random draws from two bivariate joint distributions: Clayton-copula (left) and Gumbel-copula (right). Both margins are set to be standard Gaussian $N(0, 1)$. In each panel, the parameter is chosen such that the linear correlation $\rho$ of random variates is equal to 0.8. CoVaR is estimated at 1% quantile of financial market return. Visually, $\Delta \text{CoVaR}$ in the left panel is obviously greater than that in the right panel.

### 2.2.3 MES Estimation

Analogous to the calculation of CoVaR, the value of $MES$ is associated with the joint distribution of downside risk between the returns of the market and financial firms.

**Proposition 2.2**

Suppose $X$ and $Y$ are the returns of two assets with the marginal distributions being $X \sim F_x$ and $Y \sim F_y$. The joint distribution of $X$ and $Y$ are defined by a parametric copula $C(u, v; \theta)$. The closed form solution of marginal expected shortfall $MES = -E(X|Y < VaR_y(\tau))$ can be expressed as (see the Proof in Appendix 2):

$$MES(\tau) = -\frac{1}{\tau} \int_0^1 F_x^{-1}(u) \frac{\partial C(u, \tau; \theta)}{\partial u} du$$

where $\tau$ is the quantile percentage for the distribution of $Y$ which defines the tail event $Y < VaR_y(\tau)$.

Figure 4 compares the copula-based estimation of $MES$ with the nonparametric kernel estimation following Brownless-Engle (2011).\(^{12}\) As we can see, Brownless–Engle’s approach provides a good approximation to the estimation of $MES$ even when the data are generated by student t copula (symmetrical tail dependence) or Rotated Gumbel Copula characterized by lower tail dependence. This simulation exercise seems to justify the nonparametric kernel estimation in capturing nonlinear tail dependence.

Following the models proposed by Brownlees and Engle (2011), the return of market and financial institution $i$ is specified as

$$r_{m,t} = \sigma_{m,t} \epsilon_{m,t} \quad r_{i,t} = \sigma_{i,t} \epsilon_{i,t}$$

\(^{12}\)For the details of nonparametric kernel estimation of $MES$, see Brownless–Engle (2011)
Figure 4: This figure compares the copula-based estimation of $MES$ with the nonparametric kernel estimation following Brownless-Engle (2011). In the left panel, data are generated by the student t copula with degree of freedom $df = 5$. In the right panel, data are generated by the Rotated Gumbel Copula. Both margins are set to be student t with $df = 5$. The parameters of both copula models are selected such that the Kendall correlation of generated data is 0.5. The X-axis denotes the unconditional cut-off value $\tau$ which determines the tail event of the market return when $r_{mt} \leq VaR_{rm}(\tau)$. The simulations are implemented for 5000 times.

Proposition 2.2 implies that the marginal expected shortfall of financial institution $i$ can be written as:

$$MSE_{i,t-1}(K) = -E_{t-1}(r_{i,t}|r_{m,t} \leq K)$$

$$= -\sigma_{i,t}E_{t-1}\left(\epsilon_{i,t}|\epsilon_{m,t} \leq \frac{K}{\sigma_{m,t}}\right)$$

$$= -\frac{\sigma_{i,t}}{\tau_t} \int_0^1 F_{\epsilon_{i,t}}^{-1}(u) \frac{\partial C(u, \tau_t)}{\partial u} du$$

where the threshold value $K = r_m(\tau)$ is the unconditional quantile at $\tau\%$ of the market return $r_{mt}$, which defines the tail event when the market return exceeds K (namely, $r_{m,t} \leq K$). $\tau_t = F_{\epsilon_{m,t}}\left(\frac{K}{\sigma_{m,t}}\right)$ is the CDF of the market return evaluated at the scaled threshold value $\frac{K}{\sigma_{m,t}}$. $C(u, \tau)$ is the copula function between the innovation process $\epsilon_{i,t}$ and $\epsilon_{m,t}$.

It is noteworthy that the value of $MES$ relies not only on the dependence structure determined by copula function $C(u, v; \theta)$, but also on the marginal characteristics such as volatility $\sigma_{i,t}$ and $F_{\epsilon_{i,t}}^{-1}$. In other words, two firms with the same dependence structure ($C(u, v; \theta)$) with market but with different volatility ($\sigma_{i,t}$) would otherwise be considered as similarly systemically important, but will be treated differently by the measures of $MES$. This has been highlighted by Archaya et al. (2012) in the case of the joint Gaussian distribution where the linear correlation $\rho$ captures the full dependence structure. Therefore, unlike $\Delta CoVaR$ or $\Delta CoVaR^{\leq}$, which solely depend on dependence structure determined by the copula model, $MES$ is a firm-specific value, which is determined by the dependence structure between firms and market as well as the marginal characteristics of financial firms such as volatility $\sigma_{i,t}$. Thus, it is possible to observe a situation where a financial institution is more systemically risky according to the $MES$ measure (higher volatility), but could be less risky based on the ranking of $\Delta CoVaR^{\leq}$.
Although MES is associated with firms’ specific volatility, it still fails to account for some important firm level characteristics such as size and leverage which related directly to the distress condition of financial institutions.

2.2.4 SRISK Estimation

The discussion so far disregards the question of causality. Tail dependence only describes the extent of interconnectedness between the tail risks of financial firms and the market, but fails to indicate whether the crisis happens because the firms fail, or conversely, firms fail because of the crisis\(^{14}\). Therefore, tail dependence alone cannot accurately measure systemic risk. It is likely that a small, unlevered firm (with higher tail dependence on the market because of the fragility of a small firm per se) can appear more “dangerous” for the financial system than a big, levered one (with lower tail dependence on the market because of robustness of the big firm per se). Brownlees & Engle (2011), Acbarya, Engle and Richardson (2012) extended MES and proposed the systemic risk measures (SRISK) to account for the levels of firms’ characteristics (size, leverage, etc.). We have shown in Section 2.1.4 that SRISK is a linear function of the long run MES, liability and market capitalization

\[
SRISK_{i,t} = \max \left( 0; \ kDebt_{it} - (1 - k)(1 - LRMES_{i,t})Equity_{i,t} \right)
\]

In the followings, we elaborate on the simulation procedure and illustrate how to obtain LRMES\(_{i,t}\) in the framework of the copula model.

The following five step procedures are implemented for each “market-institution” pair.

Step 1. Draw \(S\) sequences of length \(h\) of pairs of marginal distribution \((u_{i,t}, v_{m,t})\) from the parametric copula model \(C(u, v; \theta_{it})\) for the innovation series \(\epsilon_{i,t}\) and \(\epsilon_{m,t}\)

\[
\left( u_{i,t}^{s}, v_{m,t}^{s} \right)_{t=T+1}^{T+h}, \text{ for } s = 1, 2, \ldots, S
\]

Step 2. Obtain the sequence \((\epsilon_{i,t}^{s}, \epsilon_{m,t}^{s})_{T+1}^{T+h}\) by setting \(\epsilon_{i,t}^{s} = F_{i}^{-1}(u_{i,t}^{s})\) and \(\epsilon_{m,t}^{s} = F_{m}^{-1}(v_{m,t}^{s})\), where \(F_{i}\) (\(F_{m}\)) is the empirical marginal distribution for asset \(i\) (market).

Step 3. Obtain the sequence of firm and market returns by setting \(r_{i,t}^{s} = \sigma_{it}^{s} \epsilon_{i,t}^{s}\) and \(r_{m,t}^{s} = \sigma_{mt}^{s} \epsilon_{m,t}^{s}\), where \(\sigma_{it}\) (\(\sigma_{mt}\)) is the forecast standard deviation by GJR-GARCH model for asset \(i\) (market).

Step 4. Calculate the simulated cumulative return of firm \(i\): \(R_{i,T+1:T+h}^{s}\) (analogously for the market return \(R_{m,T+1:T+h}^{s}\)) relying on the properties of logarithm returns

\[
R_{i,T+1:T+h}^{s} = \exp \left( \sum_{k=1}^{h} r_{i,T+k}^{s} \right) - 1
\]

\(^{13}\)Sylvain et al(2012) discussed the condition when systemic risk measure \(\Delta CoVaR\) and MES are consistent or converge under the assumption of joint normal distribution, where \(\Delta CoVaR\) and MES have a clean closed form solution.

\(^{14}\)Acbarya, Engle and Richardson(2012) argued that both of them are jointly endogenous variables.
Step 5. The long run MES \( \text{LRMES}_{i,T+1:T+h} \) can be obtained according to Acharya (2010):

\[
MES_{i,T+1:T+h} = \frac{\sum_{s=1}^{S} R_{i,T+1:T+h} \mathbb{1}[R_{m,T+1:T+h} < C]}{\sum_{s=1}^{S} \mathbb{1}[R_{m,T+1:T+h} < C]}
\]

where \( C \) represents the threshold value defining the systemic event.

Figure 5: This figure displays the \( \Delta \text{CoVaR}' \), \( \Delta \text{CoVaR}'' \) and \( MES (y-axis) \) estimated by Normal Copula (without tail dependence) and Student t Copula (with symmetric tail dependence). The marginal distributions are set to be Student t with \( df = 5 \). The x-axis displays linear correlation \( \rho \) which measures the average co-movement strength of the data. The parameters of copula models are chosen such that the linear correlation \( \rho \) of generated random variates is equal to the value shown in the x-axis.

3 On the Dependence Consistency of Systemic Risk: \( \Delta \text{CoVaR} \) and \( MES \)

It still remains to be determined whether the proposed return-based systemic risk measures, \( \Delta \text{CoVaR} \) and \( MES \), are consistent with the dependence measures. Intuitively, a good metric of systemic risk measure should be able to reflect the fact that stronger dependence strength (higher value of \( \rho \) or Kendall \( \tau \)) between market and firms should result in a higher value of systemic risk. In addition, given the identical value of average dependence strength (\( \rho \) or Kendall \( \tau \)), those dependence structures with lower tail dependence (e.g., Student t Copula or Rotated Gumbel Copula) should indicate higher value of systemic risk than those without tail dependence (Normal copula). In other words, a higher value of dependence strength in the lower tail of distribution should lead to a higher value of systemic risk.
higher value of systemic risk.

Figure 5 compares the $\Delta CoVaR^t$, $\Delta CoVaR^c$ and $MES$ estimated by a Normal Copula (without tail dependence) with those estimated by Student t Copula (with symmetric tail dependence). The marginal distributions for all copula models are set to be Student t with $df = 5$. Having eliminated the effect of the marginal distribution, the discrepancy of systemic risk measures, therefore, can only be attributed to the effect of dependence structures.

It is surprising to find that the behaviors of $\Delta CoVaR$ become quite strange. First, $\Delta CoVaR$ does not monotonically increase with the correlation $\rho$. It declines in the end when the strength of dependence, the correlation $\rho$, approaches a high value. Second, the lower tail dependence implied by the Student t copula does not necessarily result in a higher value of systemic risk than the Normal copula without tail dependence. When the correlation $\rho$ is sufficiently large, $\Delta CoVaR$ estimated by the Normal copula becomes significantly larger than that estimated by student t copula.

![Figure 6](image)

Figure 6: This figure displays the $\Delta CoVaR^t$, $\Delta CoVaR^c$ and $MES$ ($y$-axis) estimated by Normal Copula (without tail dependence), Rotated Gumbel Copula (with lower tail dependence) and Gumbel Copula (with upper tail dependence). The marginal distributions are set to be Student t with $df = 5$. The x-axis displays Kendall $\tau$, which measures the co-movement strength of data. The parameters of copula model are chosen such that the Kendall $\tau$ of generated random variates is equal to the value shown in the x-axis.

Figure 6 provides further evidence that $\Delta CoVaR$ is not consistent with dependence measures. As the average co-movement of data becomes stronger (Kendall $\tau$ increases), the $\Delta CoVaR$ estimated by Normal copula (without tail dependence) or Gumbel Copula (with only upper tail dependence) approaches the highest value, which contradicts the common view that stronger lower tail dependence implied by Clayton or Rotated Gumbel copula should yield a higher value of systemic risk $\Delta CoVaR$. Furthermore, the value of $\Delta CoVaR$ begins to decline when the co-movement of data becomes even stronger (Kendall $\tau$ increases), which again conflicts with the commonly-held notion that higher interconnectedness between the firm and the market should indicate a larger value of systemic risk $\Delta CoVaR$.

Figure 7 displays various systemic risk measures against tail dependence (either lower or upper). Again it shows that $\Delta CoVaR$ fails to provide a consistent measure of systemic risk. When tail dependence (X-axis) is sufficiently

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16 When I changed the degree of freedom, or control all marginal distributions to be skewed t distribution instead, the main result remain unchanged.
high, upper tail dependence (e.g., Gumbel Copula), on the contrary, implies higher systemic risk $\Delta CoVaR$ than lower tail dependence (e.g., Rotated Gumbel Copula). Furthermore, $\Delta CoVaR$ does not monotonically increase with tail dependence, as the left panel of Figure 7 illustrates. All of these odd behaviors of $\Delta CoVaR$ estimation cast doubt on its reliability in measuring systemic risk.

Figure 7: This figure displays the $\Delta CoVaR^\infty$, $\Delta CoVaR^\leq$ and $MES$ ($y$-axis) estimated by the Copula of Student t (symmetrical tail dependence), Rotated Gumbel (lower tail dependence) and Gumbel (Upper tail dependence). All margins are Student t with df = 5. The x-axis displays the value of Lower tail dependence or Upper tail dependence implied by these copulas: \( LTD = \lim_{u \to 0^+} P(X_2 \leq F_2^{-1}(u)|X_1 \leq F_1^{-1}(u)) \) and \( UTD = \lim_{u \to 1^-} P(X_2 \geq F_2^{-1}(u)|X_1 \geq F_1^{-1}(u)) \). The parameters of copula model are chosen such that the tail dependence of generated random variates is equal to the value shown in the $x$-axis.

In contrast, $\Delta CoVaR^\leq$ and $MES$ are much more consistent with dependence measures. In other words, stronger lower tail dependence, implied by the Rotated Gumbel and Student t copula, leads to higher systemic risk without exception. In addition, the values of systemic risk measure, $\Delta CoVaR^\leq$ and $MES$, both monotonically increase with the correlation $\rho$, Kendall $\tau$ or tail dependence of the data generating process. These two facts seem to provide strong support for the reliability of using $\Delta CoVaR^\leq$ and $MES$, instead of $\Delta CoVaR$, to measure systemic risk. Finally, $MES$ seems to provide a better response to the dependence measures than $\Delta CoVaR^\leq$ in the simulation.\(^{17}\) As the correlation $\rho$, Kendall $\tau$ or tail dependence attains a high value, the systemic risks measured by $\Delta CoVaR^\leq$ seem to converge and fail to detect the discrepancy of dependence structure, which implies that $\Delta CoVaR^\leq$ would fail to identify the systematically important financial institutions (SIFIs) during financial crisis, when the correlation or tail dependence between financial institutions and market as a whole is high. By comparison, $MES$ works much better to detect the discrepancy of dependence structures even when the dependence strength attains a high value. For instance, when Kendall $\tau$ = 0.8, $MES$ can still discriminate between the value estimated by normal and that estimated by Rot-Gumbel Copula, as the right panel of Figure 6 displays.

Figures 6 and 7 further show that the upper tail dependence also affects the estimation of systemic risk, which mainly concerns downside risk co-movement in the lower tail of distribution. This provides evidence that we cannot ignore what happens in the upper tail, even if our interest lies primarily in the lower tail\(^{18}\) ($\Delta CoVaR$ and $MES$ are

\(^{17}\)As we discussed before, $MES$ is not only associated with dependence structure, but also related to firm specific characteristics such as volatility. In the simulation, we eliminate the effect of marginal distribution (controlling volatility of the data generating processes to be identical) and concentrate only on the difference of dependence structure.

\(^{18}\)Walter Distaso et al. (2010) pointed out that the variation in the upper tail dependence may affect the estimation of the conditional lower tail dependence and vice versa. Therefore they employ the symmetrized Joe-Clayton copula rather than focusing on the lower
both estimated based on the joint distribution on the lower tail). The upper tail dependence increases the market’s and firms’ propensity to flourish together, representing abnormal profit and reward in economic prosperity, which should discount systemic risk during economic recession. In other words, when we evaluate the systemic risk of a financial institution, we should not restrict our attention to what is happening during financial turbulence (lower tail), but also keep an eye on its performance during economic prosperity (upper tail).

Comparing the definition of $\Delta CoVaR_{<}$ with $MES$ in detail reveals that their conditioning events are completely different. More generally, $\Delta CoVaR_{<}$ measures the sensitivity of market return with respect to firms’ return, which eventually reduces to the estimation of tail risk for market return.\(^\text{19}\). Therefore, the heterogeneity of marginal distribution is not the major issue for the estimation of $\Delta CoVaR_{<}$. Instead what really matters is the dependence structure between market and firms return. On the other hand, $MES$ is a risk exposure measure, which depends on the linear projection of firm return onto market return. Therefore, firm-specific marginal distribution characteristics such as volatility, tail thickness and skewness, all make a difference in the estimation of $MES$. In other words, the estimation of $MES$ is determined by the dependence structures between market and firms as well as the heterogeneity of marginal characteristics for firms.

\(^\text{19}\)Adrian and Brunnermeier (2011) further introduced “Exposure CoVaR”, which reverse the conditioning information set to be the tail event for market return.
4 Estimation

4.1 Modeling the Marginal Dynamics

Even though we concentrate on the dependence structure of tail risk, this is by no means to imply that marginal distribution is of no importance. As demonstrated by Fermanian and Scaillet (2005), misspecification of the marginal distribution may lead to spurious results for the dependence measure estimation. Furthermore, the measures of systemic risk (CoVaR or MES) fundamentally depend on the estimation of marginal VaR or Expected Shortfall (ES) for either the market or firms’ return. Therefore we must first model the conditional marginal distributions.\footnote{Modeling the dependence structure of the variables usually condition on the available information, and thus lead to a study of the conditional copula, which requires the specification of models for conditional marginal distribution.}

The time series of equity data usually exhibit time varying volatility and heavy-tailedness, we model each marginal series $i$ for simplicity by a univariate AR(1) and GJR-GARCH(1,1,1) model:

$$Y_{i,t} = \phi_0^i + \phi_1^i Y_{i,t-1} + \epsilon_{i,t}$$

where $\epsilon_{i,t} = \sigma_{i,t} \epsilon_{i,t}$ and $\epsilon_{i,t} \sim iid(0,1)$

$$\sigma_{i,t}^2 = \omega^i + \alpha^i \epsilon_{i,t-1}^2 + \gamma^i \epsilon_{i,t-1}^2 I_{(\epsilon_{i,t-1}<0)} + \beta^i \sigma_{i,t-1}^2$$

where $I_{(\epsilon_{i,t-1}<0)}$ captures the leverage effect in which volatility tends to increase more with negative shocks than positive ones. We estimate the model using QML which guarantees the consistency of parameter estimates as long as the conditional variance is correctly specified.

After constructing the conditional mean and volatility model, the estimated standardized residuals can be specified as

$$\hat{\epsilon}_{i,t} = \frac{Y_{i,t} - \hat{\mu}_{i,t}}{\hat{\sigma}_{i,t}}$$

where $\hat{\mu}_{i,t}$ is the estimated conditional mean $\hat{\mu}_{i,t} = \hat{\phi}_0^i + \hat{\phi}_1^i Y_{i,t-1}$, and $\hat{\sigma}_{i,t}$ is the estimated conditional volatility following the above GJR-GARCH(1,1,1) model. Given the distribution function $F^i_\epsilon$ for $\epsilon_{i,t}$, the conditional $\tau$ quantile VaR and ES for each time series $i$ can be computed as

$$\hat{VaR}^i_t(\tau) = \hat{\mu}_{i,t} + \hat{\sigma}_{i,t} F^{-1}_\epsilon(\tau)$$

$$\hat{ES}^i_t(C) = \hat{\mu}_{i,t} + \hat{\sigma}_{i,t} E(\epsilon_{i,t}|\epsilon_{i,t} \leq \frac{C}{\hat{\sigma}_{i,t}})$$

It is generally agreed that financial time series are fat tailed and asymmetric. In order to account for both skewness and kurtosis in the estimation of the univariate distribution, the standardized innovations $\epsilon_{i,t}$ are assumed to have a univariate skewed t distribution\footnote{Girardi et al.(2011) found that the estimation of CoVaR improves in its performance when accounting for both skewness and kurtosis by assuming a skewed t distribution for margins.}: $\epsilon_{i,t} \sim f(\epsilon_{i,t}; \lambda_{i,t}, \nu_{i,t})$, where $f$ denotes the pdf of the skewed t distribution (Hansen 1994, see also Jondeau and Rockinger 2003), with $\nu_{i,t}$ being the degrees of freedom and $\lambda_{i,t}$ being the asymmetry parameter.
Figure 8: This figure displays the fitted parametric estimates of a skewed t (Hansen 1994) distribution for the daily return of DowJones Financial Index (Proxy for financial market return) and AIG.

Figure 8 displays an example of the fitted parametric estimates of skewed t distribution for Market Index (the DowJones Financial Index) and one financial institution in our sample: AIG. As we can see, the fitted density seems to provide a reasonable fit to the empirical histogram for both market index and daily return of AIG. A QQ plot shows that most of empirical observations can be captured by the estimated skewed t distribution. Having modeled the marginal distribution of standardized residuals $\epsilon_{i,t}$, the estimated probability integral transformations can be specified as

$$\hat{U}_{i,t} = \hat{F}_{skew,t}(\hat{\epsilon}_{i,t}; \hat{\nu}, \hat{\lambda})$$

4.2 Copula Models Estimation

As we have discussed before, the systemic risks are all concerned with the joint distribution of tail risk, which relies on the dependence structure of the innovation process. Therefore we need to estimate a joint distribution that allows us to capture the possible non-linear dependence across the innovation processes. A straightforward approach is to use a copula model to describe the joint distribution of innovations. With the estimation of probability integral transformations $\hat{U}_{i,t}$, the copula model for the bivariate time series of the standardized residuals can be constructed as:

$$\epsilon \sim [\epsilon_{it}, \epsilon_{jt}]^T \sim F_{\epsilon t} = C_t(U, V; \theta)$$

where $U = F_i(\epsilon_{it})$ and $V = F_j(\epsilon_{jt})$.

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22 To save space, we only display the fitted estimates of the skewed distribution for AIG. The result is quite similar for the daily return of most of other financial institutions.

23 In the robustness check, I estimate the marginal distribution $\hat{U}_{i,t}$ by empirical distribution function EDF, but the main results below remain unchange.
Assuming twice differentiability of the conditional joint distribution, the copula model as well as the conditional marginal distribution yields the decomposition for the conditional joint density function

\[ f(\epsilon_{it}, \epsilon_{jt}) = f^i(\epsilon_{it})f^j(\epsilon_{jt})c(u, v; \theta) \]

where \( u = F_i(\epsilon_{it}) \) and \( v = F_j(\epsilon_{jt}) \)

Assuming the parameters in the marginal and copula densities are independent, we can separately estimate the parameters of the copula model \( \theta \) by maximizing the log likelihood of the copula density function.

\[ \hat{\theta} = \arg\max_{\theta} \sum_{t=1}^{T} \log c(u, v; \theta) \]

where \( c(u, v; \theta) = \frac{\partial^2 C(u, v)}{\partial u \partial v} \) is the density function for the copula model.

### 4.2.1 Dynamic Copula Models

Patton (2006) and Creal (2008) et al suggested similar observation-driven dynamic copula models for which the dependence parameter is a parametric function of lagged data and an autoregressive term. In this paper, we used the "Generalized Autoregressive Score" (GAS) model (Creal et al. 2011) to estimate time-varying parameters for a wide range of copula models in the same dynamic framework. Since the parameters of copula \( \theta_t \) are often constrained to lie in a particular range, this approach applies a strictly increasing transformation (e.g., log, logistic, arctan) to the copula parameter, and then model the dynamics of transformed parameters \( f_t \) without constraints.

Let the copula model be \( C(U_t, V_t; \theta_t) \). The time varying evolution of transformed parameter \( f_t \) can be modeled as

\[ f_{t+1} = \omega + \beta f_t + \alpha I_t^{-1/2} s_t \]

Where

\[ \begin{align*}
    s_t &= \frac{\partial}{\partial \theta} \log C(U_t, V_t; \theta_t) \\
    I_t &= E_{t-1}(s_t s_t')
\end{align*} \]

where \( I_t^{-1/2} s_t \) is the standardized score of the copula log-likelihood\textsuperscript{24}, which defines a steepest ascent direction for improving the model local fit in term of likelihood. For the student t copula, the transformation function \( \rho_t = \frac{1 - \exp(-f_t)}{1 + \exp(-f_t)} \) is used to ensure that the conditional correlation \( \rho_t \) takes an value inside \((-1, 1)\).

Table 2 presents the bivariate copula models studied in this paper. Since different copula models imply different dependence structures in the tail of the distribution, it is essential to be aware how well the competing specifications of copula models are capable of accurately estimating the underlying dependence process. A very simple and reliable way to select the best fitting model is to compare the value of the log likelihood function of different copula models and choose the ones with the highest likelihood.\textsuperscript{25} The metrics that will be used to test

\textsuperscript{24}The score of the copula log-likelihood can be estimated numerically if there is no closed form solution.

\textsuperscript{25}Generally speaking, the standard likelihood ratio test can’t be performed when the models are non-nested. However, Rivers and Vuong (2002) discussed how to construct non-nested likelihood ratio tests. Therefore, non-nested models can still be compared by their log-likelihood values.
the Goodness of Fit (GoF) of copula models is Akaike’s Information Criterion (AIC) which is given as:

\[ AIC = 2k - 2 \log(\hat{c}(u, v; \hat{\theta})) \]

where \( k \) is the number of model parameters and \( \log(\hat{c}(u, v; \theta)) \) is the maximized log-likelihood at the estimate of the parameter vector \( \theta \).

### 4.3 Backtesting CoVaR

Changing the distress condition from \( r_{it} = VaR_{it}(\tau) \) to the more flexible tail event \( r_{it} \leq VaR_{it}(\tau) \) facilitates the backtest of CoVaR estimates as it is very straightforward to observe the days when institution \( i \) was in financial distress. Giulio et al. (2011), Gilbert et al. (2011) investigated the backtest of CoVaR by modifying the financial distress from an institution being exactly at its VaR to being at most at its VaR. In this section, we briefly describe the procedure of backtest for \( CoVaR^\leq \).

Comparing ex-ante VaR forecasts with ex-post losses, the ”hit sequence” of violation for VaR can be defined as

\[ I_{t+1}^i = \begin{cases} 1 & \text{if } r_{it} \leq VaR_{it}(\tau) \\ 0 & \text{if } r_{it} > VaR_{it}(\tau) \end{cases} \]

Analogously, conditioning on the sub-sample \( I_{t+1}^i = 1 \) when the financial institution \( i \) being in distress, the hit variable for CoVaR can be easily constructed as

\[ I_{t+1}^{m|i} = \begin{cases} 1 & \text{if } r_{mt} \leq CoVaR_{mt}(\tau) \text{ and } r_{it} \leq VaR_{it}(\tau) \\ 0 & \text{if } r_{mt} > CoVaR_{mt}(\tau) \text{ and } r_{it} \leq VaR_{it}(\tau) \end{cases} \]

Having defined the conditional hit sequence \( I_{t+1}^{m|i} \) for \( CoVaR^\leq \), we assess the performance of \( CoVaR^\leq \) by unconditional coverage testing and independence testing proposed by Kupiec (1995) and Christoffersen (1998).

#### 4.3.1 Unconditional Coverage Testing

The hypothesis to test for unconditional coverage is

\[ H_o : E\left(I_{t+1}^{m|i} = \tau\right) \]

which is dedicated to test whether the average violation \( (r_{mt} \leq CoVaR^\leq) \) is equal to the coverage ratio \( \tau \). Kupiec (1995) proposed a likelihood ratio test on the difference between the observed and expected number of VaR exceedances. More formally, the likelihood ratio test statistic can be specified as

\[ LR_{uc}(\tau) = -2 \log \left( \frac{(1-\tau)^{N-N} \tau^N}{(1-N/T)^{T-N}(N/T)^N} \right) \sim \chi(1) \text{ under } H_o \]

\(^{26}\)Genest et al., 2008 introduced an alternative metric for a goodness-of-fit test: Cramer-von-Mises statistic, which measures the distance between the parametric copula and the empirical copula. However, Cramer-von-Mises statistic fail to account for the number of parameters estimated, which may lead to overfitting of the data.

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26Genest et al., 2008 introduced an alternative metric for a goodness-of-fit test: Cramer-von-Mises statistic, which measures the distance between the parametric copula and the empirical copula. However, Cramer-von-Mises statistic fail to account for the number of parameters estimated, which may lead to overfitting of the data.
where $T$ is the total number of observation in the sub-sample $I_{t+1} = 1$ when financial institutions are in distress, and $N$ is the number of violations for CoVaR which satisfy $I_{t+1} \times T_{t+1} = 1$, that is, when both financial firms and market are in distress. Therefore $N/T$ is the empirical hit ratio for the CoVaR in the sub-sample $I_{t+1} = 1$.

### 4.3.2 Conditional Coverage Testing

If the forecast of CoVaR is precise, we would not expect that the violation of CoVaR is clustered. In other words, the hit sequence $I_{t+1}$ should be independent over time. Christoffersen (1998) suggested the following likelihood approach for the independence test

$$LR_{cc} = 2 \log \left( \frac{(1 - \pi_{01})^{n_{00}} \pi_{01}^{n_{01}} (1 - \pi_{11})^{n_{10}} \pi_{11}^{n_{11}}}{(1 - N/T)^{N} (N/T)^N} \right) \sim \chi^2(1) \text{ under } H_o$$

where $\pi_{ij}, i, j = 0, 1$ is the transition probability. For example, conditional on today being a non-violation ($I_{t} = 0$), the probability of tomorrow being a violation ($I_{t+1} = 1$) is $\pi_{01}$, and $n_{ij}, i, j = 0, 1$ is the total number of days that state $j$ occurred after state $i$ occurred the previous days. Therefore, the transition probability can be calculated as

$$\pi_{01} = \frac{n_{01}}{n_{00} + n_{01}} \text{ and } \pi_{11} = \frac{n_{11}}{n_{10} + n_{11}}.$$  

An approximate estimates of CoVaR should satisfy at least these two backtests.

### 5 Data and Empirical Results

#### 5.1 Data

The recent financial crisis provides ample evidence of a risk spill-over effect from individual financial institutions to the whole financial industry. The sample studied in this paper is common and widely used in the recent literatures. We consider daily holding period returns for almost the same panel of financial institutions studied by Acharya et al. (2010), Brownlees and Engel (2011) and Sylvain Benoit et al. (2012) between January 3, 2000 and December 30, 2011. The sample contains almost all U.S. financial institutions with equity market capitalization in excess of 5bn USD as of end of June 2007 (92 firms in total), among which, there are about 64 companies that have been trading continuously during the whole sample period in which each institution has 3018 observations. Following Brownless and Engle (2011), the sample of institutions can be split into four groups based on their two-digit SIC classification codes (Depositories, Insurance, Broker-Dealers and Others). Table 1 presents the full list of financial institutions studied in the paper. The return on the Dow Jones U.S. Financials Index (DJUSFN) is used as a proxy for the market return of the financial system. CoVaR and MES are computed at the most common confidence interval $\tau = 5\%$. We obtain our data from Bloomsburg terminal.

Table 3 gives some descriptive statistics for the unfiltered return series over the whole sample. We can observe that the broker-dealers companies have been the riskiest groups: high volatility, large kurtosis, large VaR and Expected Shortfall (ES). The Jarque-Bera test confirms that all time series of daily returns are not normally distributed with significant excess kurtosis. The Ljung-Box test can be rejected for the market index (DowJones),
Depositories and Broker-Dealer companies, which indicates the presence of autocorrelation for these groups of financial institutions. As the characteristics of the extremes of the distribution show, the asymmetry of the distribution toward the right tail is mildly pronounced for all categories of financial firms.

Table 4 provides summary statistics on the parameter estimates (median across firms) for the institutions in various industry categories. As we can see, the difference between all categories are trivial. The individual volatility estimation displays the same persistence across all groups of financial firms. The univariate distributions have fat tails as expected. The degrees of freedom $\nu$ of skewed t distribution ranges between 3.8 to 4.1. The asymmetry parameter $\lambda$ is found to be close to zero, indicating the univariate distribution being quite symmetrical. The estimation of the dynamic student t copula model shows that all financial firms are highly correlated with the market index return, with the median linear correlation $\rho$ ranging from 0.64 to 0.77, which is consistent with the fact that the dependence between financial institutions and market is strong.

Figure 9: The upper panel of figure displays the median across firms of conditional correlation for the dynamic student t copula. The lower panel of figure presents the median across firms of lower (upper) tail dependence for mixture Copula C+G+N. The acronyms C+G+N refers to the mixture copula: Clayton+Gumbel+Normal. The parameters of mixture copula model are estimation every 1 month in the Rolling window of 24 months.

Figure 9 displays the median across firms of conditional correlation $\rho$ for dynamic student t copula estimated by ‘GAS’ model discussed in Section 4.2. The lower panel presents the median across firms of lower (upper) tail dependence for mixture Copula model Clayton+Gumbel+Normal. It is clear that the conditional correlation is higher during the financial crisis than that prior to crises, which is consistent with the well-documented empirical results that conditional correlations increase during an economics downturn. However, the rolling window estima-
tion of the mixture copula model shows that tail dependences do not display the same persistence. The lower tail dependence seems to be stronger than upper tail dependence only after or in the early stage of financial crisis, which calls for further scrutiny.

Having set up the margin and copula model, we next estimate $\Delta CoVaR$ and $MES$ under different copula models and check if dependence structure in the tail risk between financial firm and market makes a difference in the measures of systemic risk.

Figure 10: This figure displays the median across the firms of the time series estimates of $\Delta CoVaR$, $\Delta CoVaR^\leq$ and $MES$ under copulas models characterized by different tail dependence properties. The acronyms Rot-Gumbel refers to Rotated Gumbel Copula. The sample period runs from 2007/01/02 to 2010/12/31.
5.2 Empirical Results

5.2.1 Estimation of Systemic Risk under Different Copula Models

Figure 10 displays the median across the firms of the time series estimates of three systemic risk measures $\Delta CoVaR$, $\Delta CoVaR^\leq$ and $MES$ estimated based on different copula models which are characterized by the different properties of tail dependence. Normal copula based estimation fails to take into account tail dependences. Rotated-Gumbel Copula based estimation, however, is supposed to account for lower tail dependences of financial data. Gumbel copula based estimation, on the contrary, is assumed to accommodate upper tail dependence. All series of estimates peak at the end of 2008 or the beginning of 2009 and then decay slowly to the lower level comparable to the one before the crisis. This pattern of time evolution is perfectly consistent with the occurrence of the financial crisis as of September 17, 2008 when Lehman Brothers filed for bankruptcy after the financial support offered by Federal Reserve Bank was stopped. The upper panel of figure 10 shows clearly that the systemic risk measure for $\Delta CoVaR$ fails to reflect the fact that higher lower tail dependence tends to result in higher systemic risk as the normal copula based estimation of $\Delta CoVaR$, which disregards the tail dependence property of data, could be larger than that estimated by rotate Gumbel copula characterized by lower tail dependence. By comparison, the systemic risk measures $\Delta CoVaR^\leq$ and $MES$, which are displayed in the middle and lower panels of figure 10, seem to succeed in indicating the importance of tail dependence in the estimation of systemic risk. Accounting for lower tail dependence (Rotated Gumbel) always leads to a larger value of $\Delta CoVaR^\leq$ and $MES$.

Figure 11 shows the cross sectional relationship of the average systemic risk measures which is defined as:

$$\Delta CoVaR_i = \frac{1}{T} \sum_{t=1}^{T} \Delta CoVaR_{i,t} \quad \Delta CoVaR^\leq_i = \frac{1}{T} \sum_{t=1}^{T} \Delta CoVaR^\leq_{i,t} \quad \text{and} \quad MES_i = \frac{1}{T} \sum_{t=1}^{T} MES_{i,t}$$

The x-axis displays the estimations of these systemic risks based on the normal copula model, which is well-known to disregard the tail dependence. The y-axis show the estimations which take into account either symmetric tail dependence on both sides (student t copula) or tail dependence only on left size (Rotated Gumbel Copula). Again it indicates that all systemic risk measures except for $\Delta CoVaR$ are consistent with dependence measures in the cross sectional estimation. In other words, taking into account lower tail dependence would result in higher estimates of systemic risk measures for $\Delta CoVaR^\leq$ and $MES$ (all values lie above the diagonal line). However, the same situation is not observed in the upper panel of Figure 11, as the values of the cross sectional estimation for $\Delta CoVaR$ could lie below the diagonal line, which indicates that, normal copula based estimation could, on the contrary, leads to the larger value of the systemic risk measure $\Delta CoVaR$ than those taking into account tail dependence. It is noteworthy that, accounting for lower tail dependence seems to merely scale up all values of $\Delta CoVaR^\leq$ and $MES$, but does not significantly change the ranking of systemic importance for financial institutions. This is indicated by the strong cross sectional link in Figure 11, which implies that the values on the y-axis and x-axis increase or decrease concurrently.

Note that all of these copula based estimations may be misspecified without model selection. It is well known that the daily return of financial assets are characterized by the tail dependence on both sizes of distribution. Here we just illustrate how the outcomes could differ if tail dependence on either size is or is not accounted for.
Figure 11: This figure displays the cross sectional relationship of systemic risk measure $\Delta \text{CoVaR}$, $\Delta \text{CoVaR}^\alpha$ and $\text{MES}$ estimated with Gaussian copula(x-axis) and Student t, Rotated Gumbel copula(y-axis) from left to right, Where $\Delta \text{CoVaR}^\alpha = \frac{1}{T} \sum_{t=1}^{T} \Delta \text{CoVaR}_i^\alpha$ and $\text{MES}_i = \frac{1}{T} \sum_{t=1}^{T} \text{MES}_i^t$. Each point represents an financial institution listed in the table 1. The diagonal solid line represents the equal value corresponding to x-axis and y-axis. The acronyms "St" refers to Student t copula based estimation. "Rot-Gumbel" represents rotated Gumbel copula based estimation. The sample period covered from 2000/01/03 to 2010/12/31.

Since different copula models result in different estimates of the systemic risk measure, it has yet to be determined which copula model fits the data best. Table 5 provide a Goodness of Fit test for various copula models based on Akaike's Information Criteria (AIC). As we can see, the dynamic copula model always outperforms its corresponding static model. This fact is not surprising as it is widely accepted that financial data exhibit time varying evolutions in distributions. In addition, the dynamic student t copula model out-performs all other parametric models with no exception, including some mixture copula models, which implies that the joint distribution of daily returns between financial firms and market exhibit tail dependence on both sizes of distribution. Note that AIC or the maximal log-likelihood value only provides the average goodness of fit for the distribution as a whole. Higher AIC does not necessarily imply better goodness of fit in each area of distribution, especially in the tail of distribution, on which the estimation of systemic risk primarily concentrates. Therefore it is essential to test the performance of systemic risk estimation, especially in the tails of the distribution.
Table 6 presents the summary statistics of the P-value in the backtest for the CoVaR₈ estimation by different copula models with different marginal distributions. The testing results show that all estimates of CoVaR₈ satisfy the conditional coverage property (P-value for LRindp are all larger than 10%), which indicates that the probability of violation \( r_{m,t} \leq CoVaR_m(\tau) \) in the next period when financial firm \( i \) is in distress (namely, \( r_{i,t} \leq VaR_i(\tau) \)) doesn’t depend on the violation today. However, the unconditional coverage testing \( L_{ucp} \) do fail for some estimates. The first two columns of Table 6 demonstrates that the specification of margin is essential for the unconditional coverage property. Even though the dependence structure remains unchanged (normal copula), changing marginal distribution from normal to skewed t distribution alone would significantly improve unconditional coverage testing (the p-value of \( LR_{ucp} \) increase significantly). This result is consistent with most empirical results that VaR estimation based on Gaussian distribution often potentially underestimates downside risk. Girardi et al. (2011) found that the CoVaR₈ estimation based on the bivariate joint skewed t distribution significantly outperforms the estimation based on the bivariate Gaussian distribution. As we can see now, this improvement of performance mainly comes from the specification of marginal distribution (from normal to skewed t) rather than the change of dependence structure. Second, comparing the last four columns of table 6 reveals that the specification of dependence structure does make a difference in the estimation of CoVaR₈. The last column of table 6 presents the estimation of CoVaR₈ based on the Gumbel copula model which is characterized by the upper tail dependence. As shown, disregarding the property of lower tail dependence in the data would cause the estimations of CoVaR₈ for the 50% financial institutions in our sample to fail the unconditional coverage test. In contrast, taking into account tail dependence on both sizes of distribution (e.g., student t copula) would produce the most accurate estimation of CoVaR₈. As the third column displays, the CoVaR₈ estimation based on the skewed t margin and student t copula perform best with respect to the P values for both the unconditional and conditional coverage tests. This fact is also in line with the empirical result that student t copula in our sample data has best goodness of fit compared with other copula models, (As Table 5 shows).

Table 7 reports the average values of three systemic risk measures by industry category from Jan 2000 to Dec 2011:

\[
\Delta CoVaR = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \Delta CoVaR_i^t \quad \text{and} \quad MES = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} MES_i^t
\]

**First,** comparing the last two columns of each panel of table indicates that, when lower tail dependence implied by student t copula is taken into account, both \( \Delta CoVaR_8 \) and \( MES \) increase while \( \Delta CoVaR \) decreases, which again verifies the previous discussion that only the former two systemic risk measures are consistent with dependence measures. **Second,** as the first two columns of panel A show, the quantile regression estimation of \( \Delta CoVaR \) proposed by AB (2011) is only comparable in magnitude to the value estimated by joint normal distribution (Norm-Norm), which tends to underestimate systemic risk in the presence of lower tail dependence. **Third,** the specification of marginal distribution seem to be much more influential in the estimation of \( \Delta CoVaR \) and \( CoVaR_8 \) than the estimation of \( MES \). Relaxing margin from Normal to Skewed t distribution yields a significant increase for the value of \( \Delta CoVaR \) as well as \( CoVaR \), while relaxing the dependence structure from Gaussian to student t only brings about a moderate increase for \( CoVaR_8 \) (Comparing the second and third column of panel B). However, the influence of Margin and Dependence structure seems to play equal role in the estimation of \( MES \) (See panel

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28Simulation results in section 2 show that as dependence strength (\( \rho \) or Kendall \( \tau \)) attain a high value, \( CoVaR_8 \) estimated by different copula models would converge. This is the reason why normal copula (second column of table 6) and rotated Gumbel (fourth column of Table 6) copula based estimations also perform pretty well in terms of both conditional and unconditional coverage tests. Therefore it is hard to detect the discrepancy of dependence structure based on the estimates of \( CoVaR_8 \).
Relaxing the margin from Norm to Skewed t distribution (the first two columns of panel C) only results in a moderate increase of value by 10.95% (from 3.38 to 3.75) for overall financial institutions. In contrast, relaxing dependence structure from Norm to Student t copula (see the last two columns of panel C) also brings about a mild increase of value by 12.80% (from 3.75 to 4.23).

Finally, Table 7 reveals that Depository institutions are the most systemically risky group based on the measure of $\Delta CoVaR^S$ or $\Delta CoVaR$, no matter what is the specification of margin and dependence structure. This findings is consistent with the results of Billo (2010) and Giulio et al.(2011) that the commercial banks have been most systemically risky due to their illiquid assets, coupled with their structure, which is not designed to withstand rapid and large loss. Summary statistics of Table 4 also reveal that the dependence (correlation $\rho$) between Depository and Market is strongest compared with other groups, which further verifies that commercial banks are more vulnerable to economics meltdown. However, based on the ranking of $MES$, Broker and Dealer become the most systemic risky group instead. This discrepancy in the ranking results from the fact that the value of $MES$ is determined not only by the dependence structure between firm and market, but also by the the firms’ individual characteristics such as volatility.

5.2.2 Which Firms are the SIFIs

The discussion so far indicates that $\Delta CoVaR^S$ and $MES$ are all consistent with dependence measures. But it has yet to be determined if these two measures of systemic risk are convergent or divergent in identifying systemic important financial institutions (SIFIs), especially with regards to the financial crisis that started in September 2008 when the supervision authorities had to determine which financial institutions to bail out based on their systemic importance.

In this paper, we define the financial crisis as the situation in which the market return exceeds its VaR at 5%, and derive the measures of $\Delta CoVaR^S$, lower tail dependence, $MES$ and $SRISK$ respectively for a list of financial institutions in our sample, which are all estimated based on the student t copula with margins being skewed t distribution.

Table 8 presents the tickers of top ten most risky financial institutions on September 17, 2008 when Lehman Brothers filed for bankruptcy after financial support facility offered by Federal Reserve stopped. Comparing the concordant pattern of each pair of risk measures in the lower panel of Table 8 reveals some interesting facts. First, we found that $\Delta CoVaR^S$ is observed to have a stronger correlation with tail dependence than other risk measures. Five of the ten financial institutions belong to the top 10 SIFIs with respect to the ranking of $\Delta CoVaR^S$ and tail dependence. In contrast, $MES$ and firms’ conditional CAPM $\beta$ are highly concordant with each other. Nine of the ten financial institutions are commonly identified as SIFIs by the ranking of $MES$ and conditional CAPM $\beta$. In addition, $SRISK$ provides a strong connection with the firm level characteristics such as leverage and market capitalization.

Table 9 reports the value of rank similarity ratio between each pair of systemic risk measure for the date in analysis which covers the periods prior to, during and post financial crisis respectively. The similarity ratio are defined as the proportion of common financial firms in each pair of rankings at a given date. For instance, a similarity ratio equal to 0.60 for the pair of systemic risk measure $\Delta CoVaR^S$ and Tail-Dependence means that among the 10 firms in the top, there are 6 firms that are commonly identified as SIFIs by these two systemic risk measure. The concordant pattern of systemic risk ranking over multiple dates indicates that the converging ranking is not specific to any particular day for each pair of $\Delta CoVaR^S$ and Tail-Dependence, $MES$ and CAPM $\beta$, or $SRISK$ and firm characteristics such as leverage. Furthermore, the similarity ratio among each pair of systemic
Figure 12: Note: The scatter plots show the cross sectional link between the time series average of the systemic risk measures displayed on the y-axis, which are all estimated by student t copula with marginal distribution being skewed t distribution. The conditional Beta is estimated as $\beta_{it} = \rho_{it} \sigma_{it} / \sigma_{mt}$. The tail dependence implied by student t copula is $\tau = 2 - 2T_{1+\nu_t} (\sqrt{1 + \nu_t} \sqrt{\frac{1-\rho_t}{1+\rho_t}})$. The $SRISK$ is calculated by the simulation exercise described in the previous section 2.2.4. The solid line in each panel is the OLS regression predicted line, which indicates the strength of cross sectional link between the two variables on the axis. Each point represents a financial institution. The estimation period is from 2004/01/02 to 2010/12/30.

Figure 12 provides further evidence for the concordant ranking of each pair of systemic risk measures. A stronger cross-sectional link can be found in the diagonal panels of Figure 12, which indicates that $\Delta CoVaR^\leq$ is more closely related to the measure of tail dependence. $MES$, however, shows a stronger cross sectional relation with conditional Beta. By comparison, $SRISK$ seems to provide closer connection with firm level characteristics such as leverage. As the first (upper left) panel of Figure 12 illustrates, the time series averages of $\Delta CoVaR^\leq$ are highly correlated with the average values of tail dependence across firms. Analogously, the middle panel shows the stronger cross sectional link between the average $MES$ and Beta. In addition, the cross sectional link between $MES$ and tail dependence is not as strong as that between $\Delta CoVaR^\leq$ and tail dependence. This observation is not surprising, as $\Delta CoVaR^\leq$ relies only on the dependence structure, while $MES$ is determined by the dependence structure as well as marginal characteristics like firms’ volatility $\sigma_{it}$, which is taken into account by the estimation of the conditional Beta $\beta_{it} = \rho_{it} \sigma_{it} / \sigma_{mt}$. The last (lower right) panel shows that $SRISK$ is more closely related to firm level information such as leverage than the other two systemic risk measures.\(^{29}\) This stylized fact seems to imply

\(^{29}\)As the value of $SRISK$ can be negative for some firms, we keep only those financial firms with positive values of $SRISK$ and thus positive contributions to the systemic risk of the financial market.
that $\Delta CoVaR^<$ is more in line with the “too interconnected to fail” paradigm, and $SRISK$ is more related to the “too big to fail” paradigm. In contrast, $MES$ offers a compromise between these two paradigms.

6 Concluding Remarks

The global financial crisis initiated in 2008 has altered the regulators’ awareness of fragility of financial system. This paper develops a common framework based on copula model to estimate several widely used return-based systemic risk measures: Delta Conditional Value at Risk ($\Delta CoVaR$) and its Modified ($\Delta CoVaR^<$), Marginal Expected Shortfall ($MES$) and its extension, systemic risk measure ($SRISK$). The nonlinear dependence of tail risk is straightforward to be accommodated in the copula model. By eliminating the discrepancy of the marginal distribution, the copula specification provides a flexibility to concentrate on the effect of the dependence structure in the estimation of systemic risk. Our empirical studies show that the nonlinear dependence of tail risk does make a difference in the estimation of $\Delta CoVaR^<$ and $MES$. Simulation exercises reveal that $\Delta CoVaR$ originally proposed by AB(2011) is in conflict with dependence measures. The modified version of $\Delta CoVaR^<$ and $MES$ is more consistent with dependence measures, which is line with economic intuition that stronger dependence strength results in higher systemic risk measures. Furthermore, we found that the linear quantile regression estimation of $\Delta CoVaR$ proposed by Adrian and Brunnermeier (2011) is inadequate to completely capture the non-linear contagion tail effect, which tends to underestimate systemic risk in the presence of lower tail dependence. We estimate the systemic risk contributions of four financial industry sample consisting of a large number of institutions for the sample period from January 2000 to December 2010, and we found that $\Delta CoVaR^<$ is observed to have a strong correlation with tail dependence. In contrast, $MES$ is found to have a strong empirical relationship with firms’ conditional CAPM $\beta$. $SRISK$, however, provides further connection with firm level characteristics by accounting for information on market capitalization and liability. This stylized fact seems to imply that $\Delta CoVaR^<$ is more in line with the “too interconnected to fail” paradigm, and $SRISK$ is more related to the “too big to fail” paradigm. In contrast, $MES$ offers a compromise between these two paradigms.
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Appendix

Proof on the relationship between conditional distribution and copula

Define \( F(y|x) \) as the conditional distribution of \( y \) given \( x \), and let \( C(u, v) \) be a copula function where \( u = F_x(x) \) and \( v = F_y(y) \). Therefore, the following proof shows that the conditional distribution \( F(y|x) \) is equal to the first order derivative of \( C(u, v) \) with respect to \( u \)

\[
F(y|x) = \Pr(Y \leq y|X = x) = \Pr\left(F_y(Y) \leq F_y(y)|F_x(X) = F_x(x)\right)
\]

Let \( F_y(Y) = V \) and \( F_x(X) = U \)

\[
= \Pr(V \leq v|U = u)
= \lim_{\Delta u \to 0^+} \frac{\Pr(V \leq v, u \leq U \leq u + \Delta u)}{\Pr(u \leq U \leq u + \Delta u)}
= \lim_{\Delta u \to 0^+} \frac{\Pr(V \leq v, U \leq u + \Delta u) - \Pr(V \leq v, U \leq u)}{\Pr(U \leq u + \Delta u) - \Pr(U < u)}
= \lim_{\Delta u \to 0^+} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u}
= \frac{\partial C(u, v)}{\partial u}
\]

Proof of Theorem 2.1

Assume a Gaussian copula model \( C(u, v; \rho) \) where \( u = \Phi(x) \) and \( v = \Phi(y) \) are standard Gaussian marginal distribution

\[
C(u, v; \rho) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))
= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} e^{\left(-\frac{(s^2 - 2\rho st + t^2)}{2(1-\rho^2)}\right)} ds dt
\]

The first order derivative of \( C(u, v; \rho) \) with respect to \( u \) can be derived as:

\[
\frac{\partial C(u, v; \rho)}{\partial u} = \frac{\partial \Phi_\rho[\Phi^{-1}(u), \Phi^{-1}(v)]}{\partial \Phi^{-1}(u)} \frac{\partial \Phi^{-1}(u)}{\partial u}
= \frac{\partial \Phi_\rho(X, Y)}{\partial X} \frac{1}{\phi(x)}
\]
Where the first component $\frac{\partial \Phi_{\rho}(X,Y)}{\partial X}$ can be further derived as:

$$\frac{\partial \Phi_{\rho}(X,Y)}{\partial X} = \frac{1}{2\pi \sqrt{1-\rho^2}} \int_{-\infty}^{Y} \exp\left\{ -\frac{(x^2 - 2\rho xt + t^2)}{2(1-\rho^2)} \right\} dt$$

$$= \frac{1}{2\pi \sqrt{1-\rho^2}} \int_{-\infty}^{Y} \exp\left\{ -\frac{(x^2 - \rho^2 x^2 + \rho^2 x^2 - 2\rho xt + t^2)}{2(1-\rho^2)} \right\} dt$$

$$= \frac{1}{2\pi \sqrt{1-\rho^2}} \int_{-\infty}^{Y} \exp\left\{ -\frac{x^2}{2} \right\} \cdot \exp\left\{ -\frac{(t - \rho x)^2}{2(1-\rho^2)} \right\} dt$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Y} \exp\left\{ -\frac{(t - \rho x)^2}{2(1-\rho^2)} \right\} dt$$

$$= \phi(x) \cdot \frac{1}{\sqrt{2\pi} \sqrt{1-\rho^2}} \int_{-\infty}^{Y} \exp\left\{ -\frac{(t - \rho x)^2}{2(1-\rho^2)} \right\} dt$$

$$= \phi(x) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Y} \exp\left\{ -\frac{(t - \rho x)^2}{2(1-\rho^2)} \right\} dt (t - \rho x) \sqrt{1-\rho^2}$$

$$= \phi(x) \cdot \Phi\left(\frac{Y - \rho X}{\sqrt{1-\rho^2}}\right)$$

$$= \phi(x) \cdot \Phi\left(\frac{\Phi^{-1}(v) - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)$$

Therefore

$$\frac{\partial C(u, v; \rho)}{\partial u} = \frac{\partial \Phi_{\rho}(X,Y)}{\partial X} \cdot \frac{1}{\phi(x)}$$

$$\frac{\partial \Phi_{\rho}(X,Y)}{\partial X} \cdot \frac{1}{\phi(x)} = \Phi\left(\frac{\Phi^{-1}(v) - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)$$

(3)

Let $\tau = \frac{\partial C(u, v; \rho)}{\partial u}$, we can solve $v$ from the above equation(4) as:

$$v = \Phi\left[\rho \Phi^{-1}(u) + \sqrt{1-\rho^2} \Phi^{-1}(\tau)\right]$$

(4)

Note that $v = F_Y(y)$ and $u = F_X(x)$.

We have assumed that both $X$ and $Y$ follow Gaussian marginal distribution

$$Y \sim N(\mu_y, \sigma_y) \quad X \sim N(\mu_x, \sigma_x)$$
Therefore
\[ v = F_Y(y) = \Phi\left(\frac{y - \mu_y}{\sigma_y}\right) \]
\[ u = F_X(x) = \Phi\left(\frac{x - \mu_x}{\sigma_x}\right) \]

Substituting the above two equations into the equation (5) yields
\[ \Phi\left(\frac{y - \mu_y}{\sigma_y}\right) = \Phi\left[\rho\Phi^{-1}\left(\Phi\left(\frac{x - \mu_x}{\sigma_x}\right) + \sqrt{1 - \rho^2}\Phi^{-1}(\tau)\right)\right] \]
\[ \frac{y - \mu_y}{\sigma_y} = \rho\left(\frac{x - \mu_x}{\sigma_x}\right) + \sqrt{1 - \rho^2}\Phi^{-1}(\tau) \]
\[ y = \rho\frac{\sigma_y}{\sigma_x}x - \rho\frac{\sigma_y}{\sigma_x}\mu_x + \sigma_y\sqrt{1 - \rho^2}\Phi^{-1}(\tau) + \mu_y \]

Now, set both x and y to be on their tail risk, respectively. Namely, \( x = VaR^x_\tau \) and \( y = CoVaR^{y|x}_\tau \), we have
\[ CoVaR^{y|x}_\tau = \rho\frac{\sigma_y}{\sigma_x}VaR^x_\tau - \rho\frac{\sigma_y}{\sigma_x}\mu_x + \sigma_y\sqrt{1 - \rho^2}\Phi^{-1}(\tau) + \mu_y \]

Therefore there is linear dependence between \( CoVaR^{y|x}_\tau \) and \( VaR^x_\tau \). The linear dependence coefficient \( \rho\frac{\sigma_y}{\sigma_x} \) is constant over quantiles. The time variation of this coefficient can be driven by the time variation of \( \rho_{x,y,t} \) or the ratio of their volatility \( \frac{\sigma_{y,t}}{\sigma_{x,t}} \).

Following AB (2011), we can further derive \( \Delta CoVaR \) as
\[ \Delta CoVaR^{y|x}_\tau = CoVaR^{y|x}_\tau - CoVaR^{y|x}_{\tau=\text{median}} = \rho\frac{\sigma_y}{\sigma_x}(VaR^x_\tau - VaR^x_{0.5}) \]

**Proof of Theorem 2.2**

Suppose \( X \) and \( Y \) are the returns of two assets with the marginal distribution being \( X \sim F_x \) and \( Y \sim F_y \). The marginal expected shortfall of \( X \) conditioning on the tail event \( Y < VaR_y(\tau) \) can be defined as \( MES = -E(X|Y < VaR_y(\tau)) \). Following the definition of expectation, we
have

\[ MES(\tau) = -E(X|Y \leq VaR_y(\tau)) \]

\[ = - \sum_{i=1}^{N} x_i Pr\left(X = x_i|Y \leq VaR_y(\tau)\right) \]

\[ = - \sum_{i=1}^{N} x_i \frac{Pr\left(X = x_i, Y \leq VaR_y(\tau)\right)}{Pr\left(Y \leq VaR_y(\tau)\right)} \]

\[ = - \sum_{i=1}^{N} x_i \frac{Pr\left(Y \leq VaR_y(\tau)|X = x_i\right)Pr(X = x_i)}{Pr\left(Y \leq VaR_y(\tau)\right)} \]

\[ = - \frac{1}{\tau} \sum_{i=1}^{N} x_i Pr\left(Y \leq VaR_y(\tau)|X = x_i\right)Pr(X = x_i) \]

(5)

Let \( Pr(Y = y_i) = v \) and \( Pr(X = x_i) = u \), we have already proved in the beginning of appendix that the conditional distribution \( F(y_i|x_i) \) is equal to the first order derivative of copula function \( C(u, v) \) with respect to \( u \)

\[ Pr\left(Y \leq y_i|X = x_i\right) = \frac{\partial C(u, v)}{\partial u} \]

Therefore

\[ Pr\left(Y \leq VaR_y(\tau)|X = x_i\right) = \frac{\partial C(u, \tau)}{\partial u} \]  

(6)

Substituting equation (6) into the equation (5) yields

\[ MES(\tau) = -E(X|Y \leq VaR_y(\tau)) \]

\[ = - \frac{1}{\tau} \sum_{i=1}^{N} x_i \frac{\partial C(u, \tau)}{\partial u} Pr(X = x_i) \]
In the case of continuous random variables, the above equation can be further simplified as:

\[ MES(\tau) = -E \left( X \mid Y \leq VaR_y(\tau) \right) \]

\[ = -\frac{1}{\tau} \sum_{i=1}^{N} x_i \frac{\partial C(u, \tau)}{\partial u} Pr(X = x_i) \]

\[ = -\frac{1}{\tau} \int_{-\infty}^{+\infty} x \frac{\partial C(u, \tau)}{\partial u} f(x) dx \]

\[ = -\frac{1}{\tau} \int_{0}^{1} F_x^{-1}(u) \frac{\partial C(u, \tau)}{\partial u} du \]
<table>
<thead>
<tr>
<th>Tickers</th>
<th>Company Names of US Financial Institutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>BAC</td>
<td>Bank of America Corp</td>
</tr>
<tr>
<td>BB&amp;T</td>
<td>BB &amp; T Corp</td>
</tr>
<tr>
<td>BKN</td>
<td>Bank of New York Mellon Corp</td>
</tr>
<tr>
<td>C</td>
<td>Citigroup Inc</td>
</tr>
<tr>
<td>CMA</td>
<td>Comerica Inc</td>
</tr>
<tr>
<td>HRAN</td>
<td>Huntington Bancshares Inc</td>
</tr>
<tr>
<td>HCBK</td>
<td>Hudson City Bankshares Inc</td>
</tr>
<tr>
<td>JPM</td>
<td>J P Morgan Chase &amp; Co</td>
</tr>
<tr>
<td>MI</td>
<td>Marsh &amp; McLeary Corp</td>
</tr>
<tr>
<td>MTB</td>
<td>M &amp; T Bank corp</td>
</tr>
<tr>
<td>NTRS</td>
<td>Northern trust Corp</td>
</tr>
<tr>
<td>NYB</td>
<td>New York Community Bancorp Inc</td>
</tr>
<tr>
<td>PBCP</td>
<td>Peoples United Financial Inc</td>
</tr>
<tr>
<td>PNC</td>
<td>P N C Financial Services grp Inc</td>
</tr>
<tr>
<td>RF</td>
<td>Regions Financial Corp. New</td>
</tr>
<tr>
<td>SNV</td>
<td>Synovus Financial Corp.</td>
</tr>
<tr>
<td>STI</td>
<td>Suntrust Bank Inc</td>
</tr>
<tr>
<td>STT</td>
<td>State Street Corp</td>
</tr>
<tr>
<td>USB</td>
<td>U S Bancorp Del</td>
</tr>
<tr>
<td>WFC</td>
<td>Wells Fargo &amp; Co. New</td>
</tr>
<tr>
<td>ZION</td>
<td>Zions Bancorp</td>
</tr>
<tr>
<td>AIG</td>
<td>American International Group Inc</td>
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<tr>
<td>ALL</td>
<td>Allstate Corp</td>
</tr>
<tr>
<td>AQL</td>
<td>Aon Corp</td>
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<tr>
<td>BLY</td>
<td>Berkley W R Corp</td>
</tr>
<tr>
<td>BRK</td>
<td>Berkshire Hathaway Inc. Del</td>
</tr>
<tr>
<td>CB</td>
<td>Chubb Corp</td>
</tr>
<tr>
<td>CNA</td>
<td>C N A Financial Corp</td>
</tr>
<tr>
<td>CVH</td>
<td>Coventry health Care Inc</td>
</tr>
<tr>
<td>HIG</td>
<td>Hartford nancial Svcs Grp Inc</td>
</tr>
<tr>
<td>HNT</td>
<td>Health Net Inc</td>
</tr>
<tr>
<td>HUM</td>
<td>Humana Inc</td>
</tr>
<tr>
<td>LNC</td>
<td>Lincoln National Corp In</td>
</tr>
<tr>
<td>MBI</td>
<td>M B I A Inc</td>
</tr>
<tr>
<td>MMC</td>
<td>Marsh &amp; McLennan Cos Inc</td>
</tr>
<tr>
<td>PGR</td>
<td>Progressive Corp</td>
</tr>
<tr>
<td>TMK</td>
<td>Torchmark Corp</td>
</tr>
<tr>
<td>TRV</td>
<td>Travelers companies Inc</td>
</tr>
<tr>
<td>UNH</td>
<td>United Health Group Inc</td>
</tr>
<tr>
<td>UNM</td>
<td>Unum Group</td>
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</table>

Table 1: Tickers and Company Names of US Financial Institutions
<table>
<thead>
<tr>
<th>Copula</th>
<th>Parametric Form</th>
<th>LTD</th>
<th>UTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>$\Phi_\rho\left(\Phi^{-1}(u),\Phi^{-1}(v)\right)$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Student $t$</td>
<td>$t_{\nu,\rho}\left(t_{\nu}^{-1}(u),t_{\nu}^{-1}(v)\right)$</td>
<td>$2 - 2T_{1+\nu}(\sqrt{1+\nu}\sqrt{\frac{1-\rho}{1+\rho}})$</td>
<td>$2 - 2T_{1+\nu}(\sqrt{1+\nu}\sqrt{\frac{1-\rho}{1+\rho}})$</td>
</tr>
<tr>
<td>Frank</td>
<td>$-\frac{1}{\theta}\log\left(1 + \frac{e^{-\theta u} - 1}{e^{-\theta v} - 1}\right)$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Plackett</td>
<td>$\frac{1}{2}(\theta - 1)^{-1}(1 + (\theta - 1)(u + v) - [(1 + (\theta - 1)(u + v))^2 - 4\theta uv]^{1/2})$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Clayton</td>
<td>$(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$</td>
<td>$2^{-1/\theta}$</td>
<td>-</td>
</tr>
<tr>
<td>Rotated-Clayton</td>
<td>$u + v - 1 + ((1 - u)^{-\theta} + (1 - v)^{-\theta} - 1)^{-1/\theta}$</td>
<td>-</td>
<td>$2^{-1/\theta}$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$exp\left(-\left[(-\log u)^{\theta} + (-\log v)^{\theta}\right]^{1/2}\right)$</td>
<td>-</td>
<td>$2 - 2^{1/\theta}$</td>
</tr>
<tr>
<td>Rotated-Gumbel</td>
<td>$u + v - 1 + exp\left(-\left[(-\log u)^{\theta} + (-\log v)^{\theta}\right]^{1/2}\right)$</td>
<td>$2 - 2^{1/\theta}$</td>
<td></td>
</tr>
<tr>
<td>Joe-Clayton</td>
<td>$1 - \left(1 - [(1 - \bar{u})^\gamma + (1 - \bar{v})^{-\gamma} - 1]^{-1/\gamma}\right)^{1/\kappa}$</td>
<td>$2^{-1/\gamma}$</td>
<td>$2^{-1/\gamma}$</td>
</tr>
</tbody>
</table>
Table 3: Summary Statistics for the unfiltered return series

<table>
<thead>
<tr>
<th></th>
<th>DowJones Index</th>
<th>Depository</th>
<th>Insurance</th>
<th>Broker-Dealer</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0116</td>
<td>0.0396</td>
<td>0.0557</td>
<td>0.0472</td>
<td>0.0572</td>
</tr>
<tr>
<td>Std.dev</td>
<td>2.1113</td>
<td>2.8772</td>
<td>2.9194</td>
<td>4.4298</td>
<td>3.4317</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.3022</td>
<td>0.7225</td>
<td>0.4246</td>
<td>1.3455</td>
<td>0.7839</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>14.107</td>
<td>23.0947</td>
<td>27.9047</td>
<td>31.3129</td>
<td>25.5438</td>
</tr>
<tr>
<td>5%-VaR(left)</td>
<td>-2.7719</td>
<td>-3.9201</td>
<td>-4.0153</td>
<td>-4.8486</td>
<td>-3.8385</td>
</tr>
<tr>
<td>5%-VaR(right)</td>
<td>2.7517</td>
<td>4.6253</td>
<td>4.5193</td>
<td>5.5541</td>
<td>4.0362</td>
</tr>
<tr>
<td>5%-ES(left)</td>
<td>-0.2496</td>
<td>-0.3258</td>
<td>-0.3275</td>
<td>-0.3693</td>
<td>-0.3864</td>
</tr>
<tr>
<td>5%-ES(right)</td>
<td>0.2655</td>
<td>0.3616</td>
<td>0.3591</td>
<td>0.4283</td>
<td>0.4352</td>
</tr>
</tbody>
</table>

This table provides summary statistics on the daily holding period return of some major US financial institutions and DowJones U.S. Financial Index for the period from 2000/01/03 until 2011/12/30. For each category, we report the average value across all institutions about the mean, the standard deviation, the skewness, the kurtosis, the 5% VaR and ES for the left and right tail of distribution. The second last row gives the median over all institutions of p-value of Jarque-Bera test for normality, and the last row reports the median of p-value for the Ljung-Box test with \( m = \log(T) \) lags.

Table 4: Summary Statistics on Parameter Estimates (medians)

<table>
<thead>
<tr>
<th></th>
<th>DowJones Index</th>
<th>Depository</th>
<th>Insurance</th>
<th>Broker-Dealer</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: GARCH Model ( \sigma_t^2 = \omega + \alpha e_{t-1}^2 + \gamma e_{t-1} I(e_{t-1} &lt; 0) + \beta \sigma_{t-1}^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \omega )</td>
<td>0.0103</td>
<td>0.0228</td>
<td>0.0339</td>
<td>0.0413</td>
<td>0.0591</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.0174</td>
<td>0.0363</td>
<td>0.0344</td>
<td>0.0176</td>
<td>0.0332</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.1039</td>
<td>0.0866</td>
<td>0.0932</td>
<td>0.0891</td>
<td>0.0670</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.9291</td>
<td>0.9190</td>
<td>0.9148</td>
<td>0.9366</td>
<td>0.9224</td>
</tr>
<tr>
<td>Panel B: Skewed t distribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \nu )</td>
<td>4.1092</td>
<td>3.9022</td>
<td>4.0798</td>
<td>3.8338</td>
<td>3.7901</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>-0.0106</td>
<td>0.0038</td>
<td>0.0135</td>
<td>0.0015</td>
<td>0.0047</td>
</tr>
<tr>
<td>Panel C: 'GAS' Dynamic Student t Copula Model (median of means)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho )</td>
<td>-</td>
<td>0.7728</td>
<td>0.6451</td>
<td>0.7634</td>
<td>0.6353</td>
</tr>
<tr>
<td>df</td>
<td>-</td>
<td>7.5087</td>
<td>8.3618</td>
<td>6.9992</td>
<td>7.6626</td>
</tr>
</tbody>
</table>

This table provides summary statistics on parameter estimates for all industry categories. Panel A reports the median across the institutions of the parameter estimates of the GJR-GARCH model for standardized residuals. Panel B reports the median across firms of the parameter estimates of the skewed t distribution for innovations. Panel C reports the median across firms of the mean over time of parameters for the "Generalized Autoregressive Score" student t copula.
Table 5: AIC for the Different Copula Models

The metric for Goodness of Fit (GoF) test of Copula model will be Akaike’s Information Criterion (AIC).

\[ AIC := 2k - 2\log(\hat{c}(u, v; \hat{\theta})) \]

Comparing Dependence Structures Using Information Criteria (AIC)

<table>
<thead>
<tr>
<th>Sample (2000/01/03 ~ 2011/12/31)</th>
<th>Q5</th>
<th>Q25</th>
<th>MEAN</th>
<th>Median</th>
<th>Q75</th>
<th>Q95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>-3159.956</td>
<td>-2555.641</td>
<td>-1862.686</td>
<td>-1830.318</td>
<td>-1189.603</td>
<td>-479.192</td>
</tr>
<tr>
<td>Clayton</td>
<td>-2501.409</td>
<td>-2118.555</td>
<td>-1534.261</td>
<td>-1539.268</td>
<td>-989.298</td>
<td>-406.144</td>
</tr>
<tr>
<td>Frank</td>
<td>-3042.834</td>
<td>-2528.719</td>
<td>-1838.441</td>
<td>-1822.108</td>
<td>-1176.598</td>
<td>-428.260</td>
</tr>
<tr>
<td>RotGumbel</td>
<td>-3070.431</td>
<td>-2590.530</td>
<td>-1849.739</td>
<td>-1837.672</td>
<td>-1181.537</td>
<td>-473.883</td>
</tr>
<tr>
<td>G+RG+N</td>
<td>-3310.359</td>
<td>-2755.918</td>
<td>-1954.758</td>
<td>-1917.125</td>
<td>-1255.380</td>
<td>-516.206</td>
</tr>
<tr>
<td>C+G+F</td>
<td>-3273.078</td>
<td>-2755.918</td>
<td>-1954.758</td>
<td>-1917.125</td>
<td>-1255.380</td>
<td>-516.206</td>
</tr>
<tr>
<td>Dynamic Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Student</td>
<td>-3424.752</td>
<td>-2859.831</td>
<td>-2084.023</td>
<td>-2098.774</td>
<td>-1348.755</td>
<td>-563.794</td>
</tr>
</tbody>
</table>

The acronyms "SJC" refers to symmetrized Joe-Clayton Copula, which was proposed by Andrew Patton (2006). Dynamic Copula model was estimated following the "Generalized Autoregressive Score" (GAS) model suggested by Creal et al (2011), which was discussed in section 3.2. The acronyms "G+RG+N" refers to mixture copula model of Gumbel+Rotated Gumbel+Normal. "G+RG+F": Gumbel+Rotated Gumbel+Frank. "C+G+F": Clayton+Gumbel+Frank. and "C+G+N" : Clayton+Gumbel+Normal

Table 6: Summary Statistics of P-Value For Unconditional Coverage and Independence Test for CoVaR Estimates

\[ r_{i,t} \leq VaR_i^c(\tau) \]
\[ CoVaR^c(\tau) \]
\[ \tau = 0.05 \]

<table>
<thead>
<tr>
<th>Margin-Copula</th>
<th>Margin-Copula</th>
<th>Margin-Copula</th>
<th>Margin-Copula</th>
<th>Margin-Copula</th>
<th>Margin-Copula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Norm - Norm</td>
<td>Skewt - Norm</td>
<td>Skewt - T</td>
<td>Skewt - RotGumbel</td>
<td>Skewt - Gumbel</td>
<td></td>
</tr>
<tr>
<td>Q5 LR_{ucp}</td>
<td>0.000</td>
<td>0.072</td>
<td>0.114</td>
<td>0.044</td>
<td>0.000</td>
</tr>
<tr>
<td>LR_{indep}</td>
<td>0.130</td>
<td>0.179</td>
<td>0.194</td>
<td>0.185</td>
<td>0.135</td>
</tr>
<tr>
<td>Q10 LR_{ucp}</td>
<td>0.000</td>
<td>0.167</td>
<td>0.184</td>
<td>0.048</td>
<td>0.001</td>
</tr>
<tr>
<td>LR_{indep}</td>
<td>0.138</td>
<td>0.216</td>
<td>0.235</td>
<td>0.215</td>
<td>0.142</td>
</tr>
<tr>
<td>Q25 LR_{ucp}</td>
<td>0.000</td>
<td>0.349</td>
<td>0.418</td>
<td>0.090</td>
<td>0.004</td>
</tr>
<tr>
<td>LR_{indep}</td>
<td>0.273</td>
<td>0.296</td>
<td>0.340</td>
<td>0.341</td>
<td>0.201</td>
</tr>
<tr>
<td>Q50 LR_{ucp}</td>
<td>0.002</td>
<td>0.680</td>
<td>0.705</td>
<td>0.514</td>
<td>0.050</td>
</tr>
<tr>
<td>LR_{indep}</td>
<td>0.537</td>
<td>0.390</td>
<td>0.411</td>
<td>0.473</td>
<td>0.633</td>
</tr>
<tr>
<td>Q75 LR_{ucp}</td>
<td>0.015</td>
<td>0.842</td>
<td>0.856</td>
<td>0.808</td>
<td>0.157</td>
</tr>
<tr>
<td>LR_{indep}</td>
<td>0.747</td>
<td>0.479</td>
<td>0.544</td>
<td>0.617</td>
<td>0.830</td>
</tr>
<tr>
<td>Q95 LR_{ucp}</td>
<td>0.189</td>
<td>0.942</td>
<td>0.973</td>
<td>0.956</td>
<td>0.275</td>
</tr>
<tr>
<td>LR_{indep}</td>
<td>0.928</td>
<td>0.681</td>
<td>0.663</td>
<td>0.721</td>
<td>0.935</td>
</tr>
</tbody>
</table>

LR_{ucp} refers to Kupiec’s (1995) test statistics for unconditional coverage testing. LR_{indep} is Christofersen’s (1998) test statistics for Independence testing. The acronyms Q5 denotes the 5% quantile of the summary statistics of P-Value.
Table 7: **Comparison of ∆CoVaR and MES Measures** (2000/01/03 – 2011/12/30)

<table>
<thead>
<tr>
<th>Distress Definition</th>
<th>( r_{i,t} = \text{VaR}^i_t(\tau) )</th>
<th>( \Delta \text{CoVaR} = \text{VaR}^i_t(\tau) \Delta \text{CoVaR}^i(\tau) )</th>
<th>( \tau = 0.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A</strong></td>
<td>AB(quantile) Margin-Copula</td>
<td>Margin-Copula Margin-Copula</td>
<td>Margin-Copula</td>
</tr>
<tr>
<td>Overall</td>
<td>1.73</td>
<td>1.68</td>
<td>2.28</td>
</tr>
<tr>
<td>Depository</td>
<td>1.95</td>
<td>1.92</td>
<td>2.52</td>
</tr>
<tr>
<td>Insurance</td>
<td>1.55</td>
<td>1.46</td>
<td>2.06</td>
</tr>
<tr>
<td>Broker-Dealer</td>
<td>1.68</td>
<td>1.70</td>
<td>2.33</td>
</tr>
<tr>
<td>Others</td>
<td>1.67</td>
<td>1.61</td>
<td>2.22</td>
</tr>
</tbody>
</table>

AB(quantile) refer to the estimation of ∆CoVaR by quantile regression following Adrian and Brunnermeier (2011). Norm-Norm represents bivariate normal joint distribution. As proposition 1 shows, the close form solution in this case is \( \Delta \text{CoVaR} = \rho \sigma_m F^{-1}_r(\tau) \).

<table>
<thead>
<tr>
<th>Distress Definition</th>
<th>( r_{i,t} \leq \text{VaR}^i_t(\tau) )</th>
<th>( \Delta \text{CoVaR} = \text{VaR}^i_t(\tau) \Delta \text{CoVaR}^i(\tau) )</th>
<th>( \tau = 0.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel B</strong></td>
<td>Margin-Copula Margin-Copula</td>
<td>Margin-Copula Margin-Copula</td>
<td>Margin-Copula</td>
</tr>
<tr>
<td>Overall</td>
<td>1.16</td>
<td>1.81</td>
<td>1.93</td>
</tr>
<tr>
<td>Depository</td>
<td>1.27</td>
<td>2.00</td>
<td>2.02</td>
</tr>
<tr>
<td>Insurance</td>
<td>1.05</td>
<td>1.64</td>
<td>1.84</td>
</tr>
<tr>
<td>Broker-Dealer</td>
<td>1.17</td>
<td>1.85</td>
<td>1.95</td>
</tr>
<tr>
<td>Others</td>
<td>1.13</td>
<td>1.76</td>
<td>1.92</td>
</tr>
</tbody>
</table>

\( MES = E(r_{it}|r_{mt} \leq q_{0.05}) \)

<table>
<thead>
<tr>
<th><strong>Panel C</strong></th>
<th>Margin-Copula Margin-Copula</th>
<th>Margin-Copula Margin-Copula</th>
<th>Margin-Copula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall</td>
<td>3.38</td>
<td>3.75</td>
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<tr>
<td>Depository</td>
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<td>3.90</td>
<td>4.27</td>
</tr>
<tr>
<td>Insurance</td>
<td>2.80</td>
<td>3.13</td>
<td>3.63</td>
</tr>
<tr>
<td>Broker-Dealer</td>
<td>4.26</td>
<td>4.78</td>
<td>5.37</td>
</tr>
<tr>
<td>Others</td>
<td>3.57</td>
<td>3.91</td>
<td>4.48</td>
</tr>
</tbody>
</table>

Norm-Norm represents bivariate normal joint distribution. The close form solution in this case is \( MES = E(r_{it}|r_{mt} \leq q_{0.05}) = \rho \frac{\sigma_i}{\sigma_m} E(r_{mt}|r_{mt} \leq q_{0.05}) \).
### Table 8: Systemic Risk Rankings: $\Delta CoVaR^S$, Tail Dependence, $MES$, $\beta$ and $SRISK$

<table>
<thead>
<tr>
<th>Rank</th>
<th>$\Delta CoVaR^S$</th>
<th>Tail-Dep $\beta$</th>
<th>MES</th>
<th>LEV</th>
<th>MV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>NYX</td>
<td>JPM</td>
<td>AIG</td>
<td>AIG</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>BAC</td>
<td>C</td>
<td>LEH</td>
<td>LEH</td>
<td>JPM</td>
</tr>
<tr>
<td>3</td>
<td>JPM</td>
<td>BAC</td>
<td>WB</td>
<td>WB</td>
<td>BAC</td>
</tr>
<tr>
<td>4</td>
<td>C</td>
<td>TROW</td>
<td>FNM</td>
<td>FNM</td>
<td>AIG</td>
</tr>
<tr>
<td>5</td>
<td>GNW</td>
<td>BEN</td>
<td>FRE</td>
<td>FRE</td>
<td>FNM</td>
</tr>
<tr>
<td>6</td>
<td>AIZ</td>
<td>LM</td>
<td>BAC</td>
<td>BAC</td>
<td>MS</td>
</tr>
<tr>
<td>7</td>
<td>TROW</td>
<td>AXP</td>
<td>C</td>
<td>C</td>
<td>FRE</td>
</tr>
<tr>
<td>8</td>
<td>AXP</td>
<td>PFG</td>
<td>ABK</td>
<td>ABK</td>
<td>GS</td>
</tr>
<tr>
<td>9</td>
<td>TMK</td>
<td>PRU</td>
<td>MS</td>
<td>MS</td>
<td>MER</td>
</tr>
<tr>
<td>10</td>
<td>FNF</td>
<td>UNM</td>
<td>MBI</td>
<td>SLM</td>
<td>WB</td>
</tr>
</tbody>
</table>

Notes: The upper panel displays the ranking of the top 10 financial institutions based on the estimation of $\Delta CoVaR^S$, Tail Dependence (Tail-Dep), $MES$, conditional CAPM $\beta$ and $SRISK$ which are all estimated by the student t copula with marginal distribution being skewed student t distribution. The last two columns display the firms' characteristics: Leverage (LEV) and Market Capitalization (MV). The ranking is implemented on September 17, 2008 when Lehman Brothers filed for bankruptcy.

### Table 9: Rank similarity ratio for the top 10 most risky Institutions

<table>
<thead>
<tr>
<th>Pairs</th>
<th>$\Delta CoVaR^S$ vs Tail-Dep</th>
<th>$\Delta CoVaR^S$ vs MES</th>
<th>$\Delta CoVaR^S$ vs $\beta$</th>
<th>$\Delta CoVaR^S$ vs LEV</th>
<th>$\Delta CoVaR^S$ vs $S$RISK</th>
<th>$MES$ vs Tail-Dep</th>
<th>$MES$ vs $\beta$</th>
<th>$MES$ vs LEV</th>
<th>$MES$ vs $S$RISK</th>
<th>$SRISK$ vs Tail-Dep</th>
<th>$SRISK$ vs $\beta$</th>
<th>$SRISK$ vs LEV</th>
<th>$SRISK$ vs $S$RISK</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta CoVaR^S$ vs Tail-Dep</td>
<td>0.60</td>
<td>0.60</td>
<td>0.70</td>
<td>0.50</td>
<td>0.50</td>
<td>0.70</td>
<td>0.60</td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>MES vs Tail-Dep</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>SRISK vs Tail-Dep</td>
<td>0.40</td>
<td>0.30</td>
<td>0.40</td>
<td>0.40</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>$\Delta CoVaR^S$ vs $\beta$</td>
<td></td>
<td></td>
<td>0.90</td>
<td>1.00</td>
<td>1.00</td>
<td>0.90</td>
<td>0.90</td>
<td></td>
<td></td>
<td>0.90</td>
<td>0.90</td>
<td>0.90</td>
<td>0.90</td>
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<tr>
<td>MES vs $\beta$</td>
<td></td>
<td></td>
<td></td>
<td>0.90</td>
<td>0.90</td>
<td>0.90</td>
<td>0.90</td>
<td></td>
<td></td>
<td>0.90</td>
<td>0.90</td>
<td>0.90</td>
<td>0.90</td>
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<tr>
<td>SRISK vs $\beta$</td>
<td></td>
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<td></td>
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<td>0.30</td>
<td>0.30</td>
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<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>$\Delta CoVaR^S$ vs LEV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>MES vs LEV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>SRISK vs LEV</td>
<td>0.80</td>
<td>0.60</td>
<td>0.60</td>
<td>0.50</td>
<td>0.50</td>
<td>0.70</td>
<td>0.30</td>
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<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>$\Delta CoVaR^S$ vs $S$RISK</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
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<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>MES vs $S$RISK</td>
<td>0.30</td>
<td>0.30</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.30</td>
<td>0.30</td>
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<td></td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Notes: The table presents the rank similarity measure between each pair of systemic risk measure displayed in the first column. The similarity ratio measure is simply defined as the proportions of top 10 risky firms that are simultaneously identified as SIFIs by the two rankings at a given date. The acronyms “Tail-Dep” and “LEV” represent tail dependence and Leverage, respectively.