Solutions to Calculus Review Problems

Problem 1(a) (1) Let $u = 1 + \ln(x)$. Then $du = (1/x)\ dx$ and

$$\int \frac{1}{x(\,1+\ln(x)\,)^3}\, dx = \int \frac{1}{u^3}\ du = -\frac{1}{2u^2} + C = -\frac{1}{2(1+\ln(x))^2} + C.$$

(2) Let $u = \cos(x)$ and $du = -\sin(x)\ dx$. Since $(1-\cos(x)^2)^2\cos(x)^2\sin(x)$, the integral becomes

$$-\int (1-u^2)^2u^2\ du = -\int (u^2-2u^4+u^6)\ du = -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C = -\frac{\cos(x)^3}{3} + \frac{2\cos(x)^5}{5} - \frac{\cos(x)^7}{7} + C.$$

(3) Let $u = 4x$, $du = 4\ dx$, $dv = \cos(x)\ dx$ and $v = \sin(x)$. Then

$$\int 4x\cos(x)\, dx = \int u\ dv = u\ v - \int v\ du = 4x\cos(x) - 4\int \sin(x)\ dx = 4x\cos(x) + 4\cos(x) + C$$

(4) Let $u = x$, $du = dx$, $dv = \lambda e^{-\lambda x}dx$ and $v = -e^{-\lambda x}$. Then

$$\int x\lambda e^{-\lambda x}\ dx = \int u\ dv = u\ v - \int v\ du = -xe^{-\lambda x} + \int e^{-\lambda x}\ dx = -xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda} + C, \text{ and}$$

$$\int_0^\infty x\lambda e^{-\lambda x}\ dx = \lim_{M\to\infty}\int_0^M x\lambda e^{-\lambda x}\ dx = \lim_{M\to\infty}\left[\left(-Me^{-\lambda M} - \frac{e^{-\lambda M}}{\lambda}\right) - \left(-0 - \frac{1}{\lambda}\right)\right] = \frac{1}{\lambda}.$$

(Note that $\lim_{M\to\infty} Me^{-\lambda M} = 0$ by L'Hospital's Rule.)

(5) We need to use integration-by-parts twice.

In the first case, we let $u = x^2$, $du = 2x\ dx$, $dv = \lambda e^{-\lambda x}\ dx$ and $v = -e^{-\lambda x}$. Then

$$\int u\ dv = uv - \int v\ du = -x^2e^{-\lambda x} + \int 2xe^{-\lambda x}\ dx, \text{ after simplification.}$$

Next, we use $u = 2x$, $du = 2dx$, $dv = e^{-\lambda x}\ dx$ and $v = -\frac{1}{\lambda}e^{-\lambda x}$ to find the solution to the second term. I will let you check that the following family of antiderivatives is correct:

$$\int x^2\lambda e^{-\lambda x}\ dx = \left(-x^2e^{-\lambda x} - \frac{2xe^{-\lambda x}}{\lambda} - \frac{2e^{-\lambda x}}{\lambda^2}\right) + C.$$

Now, $\int_0^\infty x^2\lambda e^{-\lambda x}\ dx = \lim_{M\to\infty}\int_0^M x^2\lambda e^{-\lambda x}\ dx$

$$= \lim_{M\to\infty}\left[\left(-M^2e^{-\lambda M} - \frac{2Me^{-\lambda M}}{\lambda} - \frac{2e^{-\lambda M}}{\lambda^2}\right) - \left(0 - 0 - \frac{2}{\lambda^2}\right)\right] = \frac{2}{\lambda^2}.$$

(Note that $\lim_{M\to\infty} M^2e^{-\lambda M} = 0$ and $\lim_{M\to\infty}\frac{2Me^{-\lambda M}}{\lambda} = 0$ by L'Hospital's rule.)

(6) Let $u = \sin(x^2)$ and $du = \cos(x^2)(2x)\ dx$.

Since $x = 0$ implies $u = 0$ and $x = \sqrt{\pi/2}$ implies $u = 1$, we can use substitution in the definite integral to get

$$\int_0^{\sqrt{\pi/2}} x\ \cos(x^2)\sin(x^2)\, dx = \frac{1}{2}\int_0^1 u\ du = \left[\frac{u^2}{4}\right]_0^1 = \frac{1}{4}.$$

Problem 1(b) We can split the integral at $t = 0$ (or at any fixed $t = a$), giving

$$\int_{x^2}^{x^3} e^{\cos(t)}\,dt = \int_{x^2}^{0} e^{\cos(t)}\,dt + \int_{0}^{x^3} e^{\cos(t)}\,dt = -\int_{0}^{x^2} e^{\cos(t)}\,dt + \int_{0}^{x^3} e^{\cos(t)}\,dt.$$

The fundamental theorem of calculus and the chain rule now imply that the derivative is

$$\frac{d}{dx}\int_{x^2}^{x^3} e^{\cos(t)}\,dt = -\frac{d}{dx}e^{\cos(x^2)} + \frac{d}{dx}e^{\cos(x^2)} = -e^{\cos(x^2)}(2x) + e^{\cos(x^3)}(3x^2).$$

Problem 1(c) (1) Let $u = \ln(x)$, $du = \frac{1}{x}\,dx$. Since $\int \frac{1}{u}\,du = \ln|u| + C$,

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{M\to\infty}\left[\ln(\ln(M)) - \ln(\ln(2)\right] = \infty.$$ Thus, the integral diverges.

(2) Since $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$, and since

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{M\to-\infty}\int_M^0 \frac{1}{1+x^2}\,dx = \lim_{M\to-\infty}[0 - \arctan(M)] = \frac{\pi}{2} \text{ and}$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{M\to\infty}\int_0^M \frac{1}{1+x^2}\,dx = \lim_{M\to\infty}[\arctan(M) - 0] = \frac{\pi}{2},$$

we know that

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Thus, the doubly-improper integral converges to π.

(3) Let $u = 1 + x^2$, $du = 2x\,dx$. Since $\int \frac{1}{2u}\,du = \frac{1}{2}\ln|u| + C$, and since

$$\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{M\to\infty}\int_0^M \frac{x}{1+x^2}\,dx = \lim_{M\to\infty}\left[\frac{1}{2}\ln(1+M^2) - 0\right] = \infty,$$

we already know that the doubly-improper integral diverges without checking the integral over the negatives.

Footnote on the last two parts of Problem 1(c):

Convergence in both cases requires convergence on $(-\infty, a]$ and convergence on $[a, \infty)$ for any $a \in \mathbf{R}$. I used $a = 0$ in both cases. In #1(c)(2), both integrals converge, and thus the integral over the entire real line converges. The situation was different in #1(c)(3): once we knew that one of the two parts diverges, we know that the integral over the entire real line diverges.

Problem 2(a). (1) The integral test can be used to determine that this series diverges. The work needed to justify this answer was done in Problem 1(c)(1).

(2) The limit comparison test can be used to determine that this series converges. Let

$$a_n = \frac{5n+3}{2n^4 - 6n},\quad b_n = \frac{1}{n^3}.\quad \text{Since } \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{5}{2},$$ and the series with terms b_n converges,

we know that the series with terms a_n converges as well. [*Note* that the series with terms b_n converges since it is a p-series with $p = 3$.]

(3) We can write the series as follows:

$$\sum_{n=1}^{\infty}(5^{-n} + 4/\sqrt{n}) = \left(\sum_{n=1}^{\infty}\frac{1}{5^n}\right) + 4\left(\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\right) = \frac{1}{4} + 4\left(\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\right).$$

The first term is a convergent geometric series (with $a = 1/5$, $r = 1/5$), while the second term is a multiple of a divergent p-series (with $p = 1/2$). Thus, the entire series diverges.

Problem 2(b). (1) This geometric series with $a = 1$ and $r = 0.8$ converges to 5.

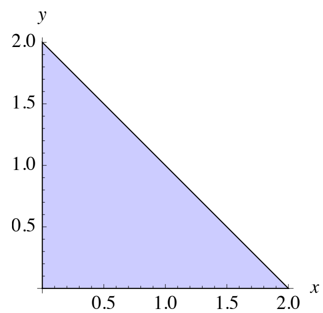
(2) This geometric series with $a = 0.8^3$ and $r = 0.8$ converges to 2.56.

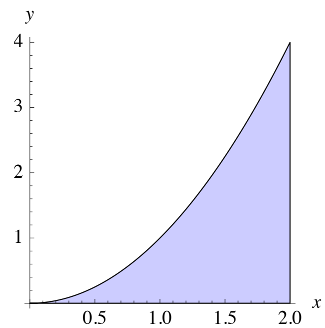
(3) This geometric series with $a = 4/5^4$ and $r = 4/5$ converges to $4/5^3 = 4/125 = 0.032$.

(4) This is the Maclaurin series for e^x. Thus, the series converges to e^x.

(5) This series converges to e^x minus the first two terms of its Maclaurin expansion: $e^x - 1 - x$.

Problem 3. Supporting graphics are as follows:





(a) The domain is $D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2 - x\}$ (*left plot*), and

$$\iint_D f \; dA = \int_0^2 \int_0^{2-x} 3(x+y) \; dy \; dx = 8, \text{ since}$$

The inside integral equals $\left[3xy + \frac{3}{2}y^2\right]_0^{2-x} = 3x(2-x) + \frac{3}{2}(2-x)^2 = 6 - \frac{3}{2}x^2$, and the outside integral has value $\int_0^2 (6 - \frac{3}{2}x^2) \; dx = \left[6x - \frac{1}{2}x^3\right]_0^2 = 12 - 4 = 8$.

(b) The domain is $D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq x^2\}$ (*right plot*), and

$$\int_0^4 \int_{\sqrt{y}}^2 \sin(x^3) \; dx \; dy = \int_0^2 \int_0^{x^2} \sin(x^3) \; dy \; dx = \frac{1}{3}(1 - \cos(8)), \text{ since}$$

the inside integral is easily seen to yield $\sin(x^3) \; x^2$, and the outside integral becomes

$$\int_0^2 \sin(x^3)x^2 \; dx = \left[-\frac{1}{3}\cos(x^3)\right]_0^2 = -\frac{1}{3}(\cos(8) - 1) = \frac{1}{3}(1 - \cos(8)).$$

[*Note* that I used $u = x^3$ and $du = 3x^2 \; dx$ to get the antiderivative.]

(c) The domain is $D = \{(x, y) \mid x > 0, \; 0 < y \leq rx\}$, and the integral is as follows:

$$\iint_D f \; dA = \int_0^\infty \int_0^{rx} e^{-x-y} \; dy \; dx = \frac{r}{1+r}, \text{ since}$$

the inside integral has value $[-e^{-x-y}]_0^{rx} = -e^{-x-rx} + e^{-x}$, and the outside integral becomes

$$\int_0^\infty (-e^{-x-rx} + e^{-x}) \; dx = -\left(\int_0^\infty e^{-x(1+r)} \; dx\right) + \left(\int_0^\infty e^{-x} \; dx\right) = -\left(\frac{1}{1+r}\right) + 1 = \frac{r}{1+r},$$

where the last two answers are justified as follows:

(1) $\displaystyle\int_0^\infty e^{-x(1+r)} \; dx = \left[-\frac{1}{1+r}e^{-x(1+r)}\right]_0^{x\to\infty} = \lim_{x\to\infty}\left(-\frac{1}{1+r}e^{-x(1+r)}\right) + \frac{1}{1+r} = 0 + \frac{1}{1+r} = \frac{1}{1+r}.$

(2) $\displaystyle\int_0^\infty e^{-x} \; dx = \left[-e^{-x}\right]_0^{x\to\infty} = \lim_{x\to\infty}\left(-e^{-x}\right) + 1 = 0 + 1 = 1.$