SOCIETY OF ACTUARIES/CASUALTY ACTUARIAL SOCIETY

EXAM P PROBABILITY

EXAM P SAMPLE SOLUTIONS

Some of the questions in this study note are taken from past SOA/CAS examinations.

P-09-05 **PRINTED IN U.S.A.**

1. Solution: D

Let

G = event that a viewer watched gymnastics

B = event that a viewer watched baseball

S = event that a viewer watched soccer

Then we want to find

$$\Pr\left[(G \cup B \cup S)^c\right] = 1 - \Pr(G \cup B \cup S)$$

$$= 1 - \left[\Pr(G) + \Pr(B) + \Pr(S) - \Pr(G \cap B) - \Pr(G \cap S) - \Pr(B \cap S) + \Pr(G \cap B \cap S)\right]$$

$$= 1 - (0.28 + 0.29 + 0.19 - 0.14 - 0.10 - 0.12 + 0.08) = 1 - 0.48 = 0.52$$

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2. Solution: A

Let R = event of referral to a specialist

L = event of lab work

We want to find

P[R∩L] = P[R] + P[L] – P[R∪L] = P[R] + P[L] – 1 + P[~(R∪L)]

= P[R] + P[L] – 1 + P[~R∩~L] = 0.30 + 0.40 – 1 + 0.35 = 0.05 .

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3. Solution: D

First note

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cup B'] = P[A] + P[B'] - P[A \cap B']$$

Then add these two equations to get

$$P[A \cup B] + P[A \cup B'] = 2P[A] + (P[B] + P[B']) - (P[A \cap B] + P[A \cap B'])$$

$$0.7 + 0.9 = 2P[A] + 1 - P[(A \cap B) \cup (A \cap B')]$$

$$1.6 = 2P[A] + 1 - P[A]$$

$$P[A] = 0.6$$

4. Solution: A

For $i = 1, 2$, let

R_i = event that a red ball is drawn form urn i

B_i = event that a blue ball is drawn from urn i .

Then if x is the number of blue balls in urn 2,

$$0.44 = \Pr[(R_1 \cap R_2) \cup (B_1 \cap B_2)] = \Pr[R_1 \cap R_2] + \Pr[B_1 \cap B_2]$$
$$= \Pr[R_1]\Pr[R_2] + \Pr[B_1]\Pr[B_2]$$
$$= \frac{4}{10}\left(\frac{16}{x+16}\right) + \frac{6}{10}\left(\frac{x}{x+16}\right)$$

Therefore,

$$2.2 = \frac{32}{x+16} + \frac{3x}{x+16} = \frac{3x+32}{x+16}$$
$$2.2x + 35.2 = 3x + 32$$
$$0.8x = 3.2$$
$$x = 4$$

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5. Solution: D

Let N(C) denote the number of policyholders in classification C . Then
N(Young ∩ Female ∩ Single) = N(Young ∩ Female) – N(Young ∩ Female ∩ Married)
= N(Young) – N(Young ∩ Male) – [N(Young ∩ Married) – N(Young ∩ Married ∩ Male)] = 3000 – 1320 – (1400 – 600) = 880 .

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6. Solution: B

Let

H = event that a death is due to heart disease

F = event that at least one parent suffered from heart disease

Then based on the medical records,

$$P\left[H \cap F^c\right] = \frac{210-102}{937} = \frac{108}{937}$$
$$P\left[F^c\right] = \frac{937-312}{937} = \frac{625}{937}$$

and $P\left[H \mid F^c\right] = \dfrac{P\left[H \cap F^c\right]}{P\left[F^c\right]} = \dfrac{108}{937}\Big/\dfrac{625}{937} = \dfrac{108}{625} = 0.173$

7. Solution: D

Let

A = event that a policyholder has an auto policy

H = event that a policyholder has a homeowners policy

Then based on the information given,

$$\Pr(A \cap H) = 0.15$$

$$\Pr(A \cap H^c) = \Pr(A) - \Pr(A \cap H) = 0.65 - 0.15 = 0.50$$

$$\Pr(A^c \cap H) = \Pr(H) - \Pr(A \cap H) = 0.50 - 0.15 = 0.35$$

and the portion of policyholders that will renew at least one policy is given by

$$0.4 \ \Pr(A \cap H^c) + 0.6 \ \Pr(A^c \cap H) + 0.8 \ \Pr(A \cap H)$$

$$= (0.4)(0.5) + (0.6)(0.35) + (0.8)(0.15) = 0.53 \qquad (= 53\%)$$

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8. Solution: D

Let

C = event that patient visits a chiropractor

T = event that patient visits a physical therapist

We are given that

$$\Pr[C] = \Pr[T] + 0.14$$

$$\Pr(C \cap T) = 0.22$$

$$\Pr(C^c \cap T^c) = 0.12$$

Therefore,

$$0.88 = 1 - \Pr[C^c \cap T^c] = \Pr[C \cup T] = \Pr[C] + \Pr[T] - \Pr[C \cap T]$$

$$= \Pr[T] + 0.14 + \Pr[T] - 0.22$$

$$= 2\Pr[T] - 0.08$$

or

$$\Pr[T] = (0.88 + 0.08)/2 = 0.48$$

9. Solution: B

Let

M = event that customer insures more than one car

S = event that customer insures a sports car

Then applying DeMorgan's Law, we may compute the desired probability as follows:

$$\Pr\left(M^c \cap S^c\right) = \Pr\left[\left(M \cup S\right)^c\right] = 1 - \Pr\left(M \cup S\right) = 1 - \left[\Pr\left(M\right) + \Pr\left(S\right) - \Pr\left(M \cap S\right)\right]$$

$$= 1 - \Pr\left(M\right) - \Pr\left(S\right) + \Pr\left(S \middle| M\right)\Pr\left(M\right) = 1 - 0.70 - 0.20 + \left(0.15\right)\left(0.70\right) = 0.205$$

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10. Solution: C

Consider the following events about a randomly selected auto insurance customer:

A = customer insures more than one car

B = customer insures a sports car

We want to find the probability of the complement of A intersecting the complement of B (exactly one car, non-sports). But P ($A^c \cap B^c$) = 1 – P (A $\cup$ B)

And, by the Additive Law, P (A $\cup$ B) = P (A) + P (B) – P (A $\cap$ B).

By the Multiplicative Law, P (A $\cap$ B) = P (B | A) P (A) = 0.15 * 0.64 = 0.096

It follows that P (A $\cup$ B) = 0.64 + 0.20 – 0.096 = 0.744 and P ($A^c \cap B^c$) = 0.744 = 0.256

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11. Solution: B

Let

C = Event that a policyholder buys collision coverage

D = Event that a policyholder buys disability coverage

Then we are given that P[C] = 2P[D] and P[C $\cap$ D] = 0.15 .

By the independence of C and D, it therefore follows that

0.15 = P[C $\cap$ D] = P[C] P[D] = 2P[D] P[D] = 2$(P[D])^2$

$(P[D])^2$ = 0.15/2 = 0.075

P[D] = $\sqrt{0.075}$ and P[C] = 2P[D] = 2$\sqrt{0.075}$

Now the independence of C and D also implies the independence of C^C and D^C . As a result, we see that P[$C^C \cap D^C$] = P[C^C] P[D^C] = (1 – P[C]) (1 – P[D])

= (1 – 2$\sqrt{0.075}$) (1 – $\sqrt{0.075}$) = 0.33 .

12. Solution: E

"Boxed" numbers in the table below were computed.

	High BP	Low BP	Norm BP	Total
Regular heartbeat	0.09	0.20	0.56	0.85
Irregular heartbeat	0.05	0.02	0.08	0.15
Total	0.14	0.22	0.64	1.00

From the table, we can see that 20% of patients have a regular heartbeat and low blood pressure.

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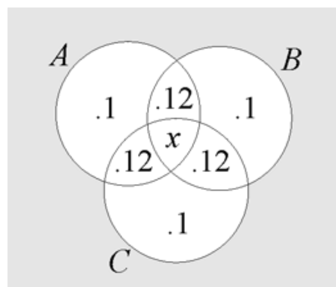
13. Solution: C

The Venn diagram below summarizes the unconditional probabilities described in the problem.



In addition, we are told that

$$\frac{1}{3} = P[A \cap B \cap C \mid A \cap B] = \frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{x}{x + 0.12}$$

It follows that

$$x = \frac{1}{3}(x + 0.12) = \frac{1}{3}x + 0.04$$

$$\frac{2}{3}x = 0.04$$

$$x = 0.06$$

Now we want to find

$$P\left[(A \cup B \cup C)^c \mid A^c\right] = \frac{P\left[(A \cup B \cup C)^c\right]}{P\left[A^c\right]}$$

$$= \frac{1 - P[A \cup B \cup C]}{1 - P[A]}$$

$$= \frac{1 - 3(0.10) - 3(0.12) - 0.06}{1 - 0.10 - 2(0.12) - 0.06}$$

$$= \frac{0.28}{0.60} = 0.467$$

14. Solution: A

$$p_k = \frac{1}{5} p_{k-1} = \frac{1}{5}\frac{1}{5} p_{k-2} = \frac{1}{5}\cdot\frac{1}{5}\cdot\frac{1}{5} p_{k-3} = ... = \left(\frac{1}{5}\right)^k p_0 \quad k \geq 0$$

$$1 = \sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k p_0 = \frac{p_0}{1-\frac{1}{5}} = \frac{5}{4} p_0$$

$p_0 = 4/5$.

Therefore, P[N > 1] = 1 – P[N ≤1] = 1 – (4/5 + 4/5 · 1/5) = 1 – 24/25 = 1/25 = 0.04 .

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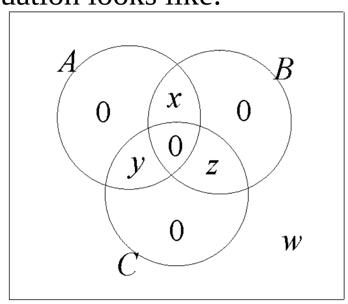
15. Solution: C

A Venn diagram for this situation looks like:



We want to find $w = 1 - (x + y + z)$

We have $x + y = \frac{1}{4}, \quad x + z = \frac{1}{3}, \quad y + z = \frac{5}{12}$

Adding these three equations gives

$$(x + y) + (x + z) + (y + z) = \frac{1}{4} + \frac{1}{3} + \frac{5}{12}$$

$$2(x + y + z) = 1$$

$$x + y + z = \frac{1}{2}$$

$$w = 1 - (x + y + z) = 1 - \frac{1}{2} = \frac{1}{2}$$

Alternatively the three equations can be solved to give x = 1/12, y = 1/6, z =1/4 again leading to $w = 1 - \left(\frac{1}{12} + \frac{1}{6} + \frac{1}{4}\right) = \frac{1}{2}$

16. Solution: D

Let N_1 and N_2 denote the number of claims during weeks one and two, respectively. Then since N_1 and N_2 are independent,

$$\Pr[N_1 + N_2 = 7] = \sum_{n=0}^{7} \Pr[N_1 = n]\Pr[N_2 = 7-n]$$
$$= \sum_{n=0}^{7} \left(\frac{1}{2^{n+1}}\right)\left(\frac{1}{2^{8-n}}\right)$$
$$= \sum_{n=0}^{7} \frac{1}{2^9}$$
$$= \frac{8}{2^9} = \frac{1}{2^6} = \frac{1}{64}$$

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17. Solution: D

Let

$O =$ Event of operating room charges

$E =$ Event of emergency room charges

Then

$$0.85 = \Pr(O \cup E) = \Pr(O) + \Pr(E) - \Pr(O \cap E)$$
$$= \Pr(O) + \Pr(E) - \Pr(O)\Pr(E) \quad (\text{Independence})$$

Since $\Pr(E^c) = 0.25 = 1 - \Pr(E)$, it follows $\Pr(E) = 0.75$.

So $0.85 = \Pr(O) + 0.75 - \Pr(O)(0.75)$

$\Pr(O)(1 - 0.75) = 0.10$

$\Pr(O) = 0.40$

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18. Solution: D

Let X_1 and X_2 denote the measurement errors of the less and more accurate instruments, respectively. If N(μ,σ) denotes a normal random variable with mean μ and standard deviation σ, then we are given X_1 is N(0, 0.0056h), X_2 is N(0, 0.0044h) and X_1, X_2 are independent. It follows that Y = $\frac{X_1 + X_2}{2}$ is N (0, $\sqrt{\frac{0.0056^2 h^2 + 0.0044^2 h^2}{4}}$) = N(0, 0.00356h) . Therefore, P[–0.005h ≤ Y ≤ 0.005h] = P[Y ≤ 0.005h] – P[Y ≤ –0.005h] = P[Y ≤ 0.005h] – P[Y ≥ 0.005h]

= 2P[Y ≤ 0.005h] – 1 = 2P$\left[Z \le \frac{0.005h}{0.00356h} \right]$ – 1 = 2P[Z ≤ 1.4] – 1 = 2(0.9192) – 1 = 0.84.

19. Solution: B

Apply Bayes' Formula. Let

$A =$ Event of an accident

$B_1 =$ Event the driver's age is in the range 16-20

$B_2 =$ Event the driver's age is in the range 21-30

$B_3 =$ Event the driver's age is in the range 30-65

$B_4 =$ Event the driver's age is in the range 66-99

Then

$$\Pr\left(B_1 \middle| A\right) = \frac{\Pr\left(A \middle| B_1\right)\Pr\left(B_1\right)}{\Pr\left(A \middle| B_1\right)\Pr\left(B_1\right) + \Pr\left(A \middle| B_2\right)\Pr\left(B_2\right) + \Pr\left(A \middle| B_3\right)\Pr\left(B_3\right) + \Pr\left(A \middle| B_4\right)\Pr\left(B_4\right)}$$

$$= \frac{(0.06)(0.08)}{(0.06)(0.08) + (0.03)(0.15) + (0.02)(0.49) + (0.04)(0.28)} = 0.1584$$

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20. Solution: D

Let

S = Event of a standard policy
F = Event of a preferred policy
U = Event of an ultra-preferred policy
D = Event that a policyholder dies

Then

$$P[U \mid D] = \frac{P[D \mid U]P[U]}{P[D \mid S]P[S] + P[D \mid F]P[F] + P[D \mid U]P[U]}$$

$$= \frac{(0.001)(0.10)}{(0.01)(0.50) + (0.005)(0.40) + (0.001)(0.10)}$$

$$= 0.0141$$

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21. Solution: B

Apply Baye's Formula:

$$\Pr\left[\text{Seri.} \middle| \text{Surv.}\right]$$

$$= \frac{\Pr\left[\text{Surv.} \middle| \text{Seri.}\right]\Pr\left[\text{Seri.}\right]}{\Pr\left[\text{Surv.} \middle| \text{Crit.}\right]\Pr\left[\text{Crit.}\right] + \Pr\left[\text{Surv.} \middle| \text{Seri.}\right]\Pr\left[\text{Seri.}\right] + \Pr\left[\text{Surv.} \middle| \text{Stab.}\right]\Pr\left[\text{Stab.}\right]}$$

$$= \frac{(0.9)(0.3)}{(0.6)(0.1) + (0.9)(0.3) + (0.99)(0.6)} = 0.29$$

22. Solution: D

Let

H = Event of a heavy smoker

L = Event of a light smoker

N = Event of a non-smoker

D = Event of a death within five-year period

Now we are given that $\Pr\left[D|L\right] = 2\ \Pr\left[D|N\right]$ and $\Pr\left[D|L\right] = \frac{1}{2}\Pr\left[D|H\right]$

Therefore, upon applying Bayes' Formula, we find that

$$\Pr\left[H|D\right] = \frac{\Pr\left[D|H\right]\Pr\left[H\right]}{\Pr\left[D|N\right]\Pr\left[N\right] + \Pr\left[D|L\right]\Pr\left[L\right] + \Pr\left[D|H\right]\Pr\left[H\right]}$$

$$= \frac{2\Pr\left[D|L\right](0.2)}{\frac{1}{2}\Pr\left[D|L\right](0.5) + \Pr\left[D|L\right](0.3) + 2\Pr\left[D|L\right](0.2)} = \frac{0.4}{0.25 + 0.3 + 0.4} = 0.42$$

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23. Solution: D

Let

C = Event of a collision

T = Event of a teen driver

Y = Event of a young adult driver

M = Event of a midlife driver

S = Event of a senior driver

Then using Bayes' Theorem, we see that

$$P[Y|C] = \frac{P[C|Y]P[Y]}{P[C|T]P[T] + P[C|Y]P[Y] + P[C|M]P[M] + P[C|S]P[S]}$$

$$= \frac{(0.08)(0.16)}{(0.15)(0.08) + (0.08)(0.16) + (0.04)(0.45) + (0.05)(0.31)} = 0.22\ .$$

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24. Solution: B

Observe

$$\Pr\left[N \ge 1 | N \le 4\right] = \frac{\Pr\left[1 \le N \le 4\right]}{\Pr\left[N \le 4\right]} = \left[\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}\right] \Big/ \left[\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}\right]$$

$$= \frac{10 + 5 + 3 + 2}{30 + 10 + 5 + 3 + 2} = \frac{20}{50} = \frac{2}{5}$$

25. Solution: B

Let Y = positive test result

D = disease is present (and ~D = not D)

Using Baye's theorem:

$$P[D|Y] = \frac{P[Y \mid D]P[D]}{P[Y \mid D]P[D] + P[Y \mid \sim D]P[\sim D]} = \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.005)(0.99)} = 0.657 .$$

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26. Solution: C

Let:

S = Event of a smoker

C = Event of a circulation problem

Then we are given that P[C] = 0.25 and $P[S \mid C] = 2\,P[S \mid C^C]$

Now applying Bayes' Theorem, we find that $P[C \mid S] = \dfrac{P[S \mid C]P[C]}{P[S \mid C]P[C] + P[S \mid C^C](P[C^C])}$

$$= \frac{2P[S \mid C^C]P[C]}{2P[S \mid C^C]P[C] + P[S \mid C^C](1 - P[C])} = \frac{2(0.25)}{2(0.25) + 0.75} = \frac{2}{2+3} = \frac{2}{5} .$$

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27. Solution: D

Use Baye's Theorem with A = the event of an accident in one of the years 1997, 1998 or 1999.

$$P[1997|A] = \frac{P[A \mid 1997]P[1997]}{P[A \mid 1997][P[1997] + P[A \mid 1998]P[1998] + P[A \mid 1999]P[1999]}$$

$$= \frac{(0.05)(0.16)}{(0.05)(0.16) + (0.02)(0.18) + (0.03)(0.20)} = 0.45 .$$

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28. Solution: A

Let

C = Event that shipment came from Company X

I_1 = Event that one of the vaccine vials tested is ineffective

Then by Bayes' Formula, $P[C \mid I_1] = \dfrac{P[I_1 \mid C]P[C]}{P[I_1 \mid C]P[C] + P[I_1 \mid C^c]P[C^c]}$

Now

$$P[C] = \frac{1}{5}$$

$$P[C^c] = 1 - P[C] = 1 - \frac{1}{5} = \frac{4}{5}$$

$$P[I_1 \mid C] = \binom{30}{1}(0.10)(0.90)^{29} = 0.141$$

$$P[I_1 \mid C^c] = \binom{30}{1}(0.02)(0.98)^{29} = 0.334$$

Therefore,

$$P[C \mid I_1] = \frac{(0.141)(1/5)}{(0.141)(1/5) + (0.334)(4/5)} = 0.096$$

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29. Solution: C

Let T denote the number of days that elapse before a high-risk driver is involved in an accident. Then T is exponentially distributed with unknown parameter λ. Now we are given that

$0.3 = P[T \le 50] = \int_0^{50} \lambda e^{-\lambda t}dt = -e^{-\lambda t}\Big|_0^{50} = 1 - e^{-50\lambda}$

Therefore, $e^{-50\lambda} = 0.7$ or $\lambda = -(1/50)\ln(0.7)$

It follows that $P[T \le 80] = \int_0^{80} \lambda e^{-\lambda t}dt = -e^{-\lambda t}\Big|_0^{80} = 1 - e^{-80\lambda}$

$= 1 - e^{(80/50)\ln(0.7)} = 1 - (0.7)^{80/50} = 0.435$.

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30. Solution: D

Let N be the number of claims filed. We are given $P[N = 2] = \dfrac{e^{-\lambda}\lambda^2}{2!} = 3\dfrac{e^{-\lambda}\lambda^4}{4!} = 3 \cdot P[N = 4]$ $24\lambda^2 = 6\lambda^4$

$\lambda^2 = 4 \Rightarrow \lambda = 2$

Therefore, $Var[N] = \lambda = 2$.

31. Solution: D

Let X denote the number of employees that achieve the high performance level. Then X follows a binomial distribution with parameters $n = 20$ and $p = 0.02$. Now we want to determine x such that

$$\Pr\left[X > x\right] \le 0.01$$

or, equivalently,

$$0.99 \le \Pr\left[X \le x\right] = \sum_{k=0}^{x} \binom{20}{k} (0.02)^k (0.98)^{20-k}$$

The following table summarizes the selection process for x:

x	$\Pr\left[X = x\right]$	$\Pr\left[X \le x\right]$
0	$(0.98)^{20} = 0.668$	0.668
1	$20(0.02)(0.98)^{19} = 0.272$	0.940
2	$190(0.02)^2(0.98)^{18} = 0.053$	0.993

Consequently, there is less than a 1% chance that more than two employees will achieve the high performance level. We conclude that we should choose the payment amount C such that

$$2C = 120,000$$

or

$$C = 60,000$$

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32. Solution: D

Let

X = number of low-risk drivers insured
Y = number of moderate-risk drivers insured
Z = number of high-risk drivers insured
$f(x, y, z)$ = probability function of X, Y, and Z

Then f is a trinomial probability function, so

$$\Pr\left[z \ge x+2\right] = f(0,0,4) + f(1,0,3) + f(0,1,3) + f(0,2,2)$$

$$= (0.20)^4 + 4(0.50)(0.20)^3 + 4(0.30)(0.20)^3 + \frac{4!}{2!2!}(0.30)^2(0.20)^2$$

$$= 0.0488$$

33. Solution: B

Note that

$$\Pr\left[X > x\right] = \int_x^{20} 0.005\left(20-t\right)dt = 0.005\left(20t - \frac{1}{2}t^2\right)\Big|_x^{20}$$

$$= 0.005\left(400 - 200 - 20x + \frac{1}{2}x^2\right) = 0.005\left(200 - 20x + \frac{1}{2}x^2\right)$$

where $0 < x < 20$. Therefore,

$$\Pr\left[X > 16 \middle| X > 8\right] = \frac{\Pr\left[X > 16\right]}{\Pr\left[X > 8\right]} = \frac{200 - 20\left(16\right) + \frac{1}{2}\left(16\right)^2}{200 - 20\left(8\right) + \frac{1}{2}\left(8\right)^2} = \frac{8}{72} = \frac{1}{9}$$

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34. Solution: C

We know the density has the form $C\left(10+x\right)^{-2}$ for $0 < x < 40$ (equals zero otherwise).

First, determine the proportionality constant C from the condition $\int_0^{40} f(x)dx = 1$:

$$1 = \int_0^{40} C\left(10+x\right)^{-2} dx = -C(10+x)^{-1}\Big|_0^{40} = \frac{C}{10} - \frac{C}{50} = \frac{2}{25}C$$

so $C = 25/2$, or 12.5 . Then, calculate the probability over the interval (0, 6):

$$12.5\int_0^{6}\left(10+x\right)^{-2} dx = -\left(10+x\right)^{-1}\Big|_0^{6} = \left(\frac{1}{10} - \frac{1}{16}\right)\left(12.5\right) = 0.47 \ .$$

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35. Solution: C

Let the random variable T be the future lifetime of a 30-year-old. We know that the density of T has the form $f(x) = C(10 + x)^{-2}$ for $0 < x < 40$ (and it is equal to zero otherwise). First, determine the proportionality constant C from the condition $\int_0^{40} f(x)dx = 1$:

$$1 = \int_0^{40} f(x)dx = -C(10+x)^{-1}\Big|_0^{40} = \frac{2}{25}C$$

so that $C = \dfrac{25}{2} = 12.5$. Then, calculate $P(T < 5)$ by integrating $f(x) = 12.5\,(10 + x)^{-2}$ over the interval (0.5).

36. Solution: B

To determine k, note that

$$1 = \int_0^1 k(1-y)^4\,dy = -\frac{k}{5}(1-y)^5\Big|_0^1 = \frac{k}{5}$$

k = 5

We next need to find P[V > 10,000] = P[100,000 Y > 10,000] = P[Y > 0.1]

$= \int_{0.1}^1 5(1-y)^4 dy = -(1-y)^5\Big|_{0.1}^1 = (0.9)^5 = 0.59$ and P[V > 40,000]

$= P[100{,}000\ Y > 40{,}000] = P[Y > 0.4] = \int_{0.4}^1 5(1-y)^4 dy = -(1-y)^5\Big|_{0.4}^1 = (0.6)^5 = 0.078$.

It now follows that P[V > 40,000 | V > 10,000]

$$= \frac{P[V > 40{,}000 \cap V > 10{,}000]}{P[V > 10{,}000]} = \frac{P[V > 40{,}000]}{P[V > 10{,}000]} = \frac{0.078}{0.590} = 0.132 .$$

----

37. Solution: D

Let T denote printer lifetime. Then $f(t) = \frac{1}{2} e^{-t/2}$, $0 \le t \le \infty$

Note that

$$P[T \le 1] = \int_0^1 \frac{1}{2} e^{-t/2} dt = e^{-t/2}\Big|_0^1 = 1 - e^{-1/2} = 0.393$$

$$P[1 \le T \le 2] = \int_1^2 \frac{1}{2} e^{-t/2} dt = e^{-t/2}\Big|_1^2 = e^{-1/2} - e^{-1} = 0.239$$

Next, denote refunds for the 100 printers sold by independent and identically distributed random variables $Y_1, \ldots, Y_{100}$ where

$$Y_i = \begin{cases} 200 & \text{with probability } 0.393 \\ 100 & \text{with probability } 0.239 \\ 0 & \text{with probability } 0.368 \end{cases} \qquad i = 1, \ldots, 100$$

Now $E[Y_i] = 200(0.393) + 100(0.239) = 102.56$

Therefore, Expected Refunds = $\sum_{i=1}^{100} E[Y_i] = 100(102.56) = 10{,}256$.

38. Solution: A

Let F denote the distribution function of f. Then

$$F(x) = \Pr[X \le x] = \int_1^x 3t^{-4}dt = -t^{-3}\Big|_1^x = 1 - x^{-3}$$

Using this result, we see

$$\Pr[X < 2 | X \ge 1.5] = \frac{\Pr[(X < 2) \cap (X \ge 1.5)]}{\Pr[X \ge 1.5]} = \frac{\Pr[X < 2] - \Pr[X \le 1.5]}{\Pr[X \ge 1.5]}$$

$$= \frac{F(2) - F(1.5)}{1 - F(1.5)} = \frac{(1.5)^{-3} - (2)^{-3}}{(1.5)^{-3}} = 1 - \left(\frac{3}{4}\right)^3 = 0.578$$

----

39. Solution: E

Let X be the number of hurricanes over the 20-year period. The conditions of the problem give x is a binomial distribution with n = 20 and p = 0.05 . It follows that

P[X < 2] = $(0.95)^{20}(0.05)^0 + 20(0.95)^{19}(0.05) + 190(0.95)^{18}(0.05)^2$

= 0.358 + 0.377 + 0.189 = 0.925 .

----

40. Solution: B

Denote the insurance payment by the random variable Y. Then

$$Y = \begin{cases} 0 & \text{if} \quad 0 < X \le C \\ X - C & \text{if} \quad \text{C} < X < 1 \end{cases}$$

Now we are given that

$$0.64 = \Pr(Y < 0.5) = \Pr(0 < X < 0.5 + C) = \int_0^{0.5+C} 2x\, dx = x^2\Big|_0^{0.5+C} = (0.5 + C)^2$$

Therefore, solving for C, we find $C = \pm 0.8 - 0.5$

Finally, since $0 < C < 1$, we conclude that $C = 0.3$

41. Solution: E

Let

X = number of group 1 participants that complete the study.

Y = number of group 2 participants that complete the study.

Now we are given that X and Y are independent.

Therefore,

$$P\left\{\left[(X \geq 9) \cap (Y < 9)\right] \cup \left[(X < 9) \cap (Y \geq 9)\right]\right\}$$

$$= P\left[(X \geq 9) \cap (Y < 9)\right] + P\left[(X < 9) \cap (Y \geq 9)\right]$$

$$= 2P\left[(X \geq 9) \cap (Y < 9)\right] \qquad \text{(due to symmetry)}$$

$$= 2P\left[X \geq 9\right] P\left[Y < 9\right]$$

$$= 2P\left[X \geq 9\right] P\left[X < 9\right] \qquad \text{(again due to symmetry)}$$

$$= 2P\left[X \geq 9\right]\left(1 - P\left[X \geq 9\right]\right)$$

$$= 2\left[\binom{10}{9}(0.2)(0.8)^9 + \binom{10}{10}(0.8)^{10}\right]\left[1 - \binom{10}{9}(0.2)(0.8)^9 - \binom{10}{10}(0.8)^{10}\right]$$

$$= 2[0.376][1 - 0.376] = 0.469$$

----

42. Solution: D

Let

I_A = Event that Company A makes a claim

I_B = Event that Company B makes a claim

X_A = Expense paid to Company A if claims are made

X_B = Expense paid to Company B if claims are made

Then we want to find

$$\Pr\left\{\left[I_A^C \cap I_B\right] \cup \left[(I_A \cap I_B) \cap (X_A < X_B)\right]\right\}$$

$$= \Pr\left[I_A^C \cap I_B\right] + \Pr\left[(I_A \cap I_B) \cap (X_A < X_B)\right]$$

$$= \Pr\left[I_A^C\right]\Pr\left[I_B\right] + \Pr\left[I_A\right]\Pr\left[I_B\right]\Pr\left[X_A < X_B\right] \qquad \text{(independence)}$$

$$= (0.60)(0.30) + (0.40)(0.30)\Pr\left[X_B - X_A \geq 0\right]$$

$$= 0.18 + 0.12\Pr\left[X_B - X_A \geq 0\right]$$

Now $X_B - X_A$ is a linear combination of independent normal random variables.

Therefore, $X_B - X_A$ is also a normal random variable with mean

$$M = E\left[X_B - X_A\right] = E\left[X_B\right] - E\left[X_A\right] = 9{,}000 - 10{,}000 = -1{,}000$$

and standard deviation $\sigma = \sqrt{\operatorname{Var}(X_B) + \operatorname{Var}(X_A)} = \sqrt{(2000)^2 + (2000)^2} = 2000\sqrt{2}$

It follows that

$$\begin{aligned}\Pr\left[X_B - X_A \ge 0\right] &= \Pr\left[Z \ge \frac{1000}{2000\sqrt{2}}\right] \qquad (Z \text{ is standard normal})\\ &= \Pr\left[Z \ge \frac{1}{2\sqrt{2}}\right]\\ &= 1-\Pr\left[Z < \frac{1}{2\sqrt{2}}\right]\\ &= 1-\Pr\left[Z < 0.354\right]\\ &= 1-0.638 = 0.362\end{aligned}$$

Finally,

$$\begin{aligned}\Pr\left\{\left[I_A^C \cap I_B\right]\cup\left[\left(I_A \cap I_B\right)\cap\left(X_A < X_B\right)\right]\right\} &= 0.18+(0.12)(0.362)\\ &= 0.223\end{aligned}$$

---

43. Solution: D

If a month with one or more accidents is regarded as success and *k* = the number of failures before the fourth success, then *k* follows a negative binomial distribution and the requested probability is

$$\begin{aligned}\Pr\left[k \ge 4\right] = 1-\Pr\left[k \le 3\right] &= 1-\sum_{k=0}^{3}\binom{3+k}{k}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^k\\ &= 1-\left(\frac{3}{5}\right)^4\left[\binom{3}{0}\left(\frac{2}{5}\right)^0+\binom{4}{1}\left(\frac{2}{5}\right)^1+\binom{5}{2}\left(\frac{2}{5}\right)^2+\binom{6}{3}\left(\frac{2}{5}\right)^3\right]\\ &= 1-\left(\frac{3}{5}\right)^4\left[1+\frac{8}{5}+\frac{8}{5}+\frac{32}{25}\right]\\ &= 0.2898\end{aligned}$$

Alternatively the solution is

$$\left(\frac{2}{5}\right)^4+\binom{4}{1}\left(\frac{2}{5}\right)^4\frac{3}{5}+\binom{5}{2}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2+\binom{6}{3}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^3 = 0.2898$$

which can be derived directly or by regarding the problem as a negative binomial distribution with

i) success taken as a month with no accidents

ii) *k* = the number of failures before the fourth success, and

iii) calculating $\Pr\left[k \le 3\right]$

44. Solution: C

If k is the number of days of hospitalization, then the insurance payment $g(k)$ is

$$g(k) = \begin{cases} 100k & \text{for } k = 1, 2, 3 \\ 300 + 50(k-3) & \text{for } k = 4, 5. \end{cases}$$

Thus, the expected payment is $\sum_{k=1}^{5} g(k)\, p_k = 100\, p_1 + 200\, p_2 + 300\, p_3 + 350\, p_4 + 400\, p_5$

$= \left(100x5 + 200x4 + 300x3 + 350x2 + 400x1\right) =$

$\frac{1}{15}(100 + 200\times 2 + 300\times 3 + 350\times 4 + 400\times 5) = 220$

----

45. Solution: D

Note that $E\left(X\right) = \int_{-2}^{0} -\frac{x^2}{10}dx + \int_{0}^{4}\frac{x^2}{10}dx = -\frac{x^3}{30}\Big|_{-2}^{0} + \frac{x^3}{30}\Big|_{0}^{4} = -\frac{8}{30} + \frac{64}{30} = \frac{56}{30} = \frac{28}{15}$

----

46. Solution: D

The density function of T is

$$f\left(t\right) = \frac{1}{3}e^{-t/3} \quad , \quad 0 < t < \infty$$

Therefore,

$$\begin{aligned} E\left[X\right] &= E\left[\max\left(T, 2\right)\right] \\ &= \int_0^2 \frac{2}{3}e^{-t/3}dt + \int_2^\infty \frac{t}{3}e^{-t/3}dt \\ &= -2e^{-t/3}\big|_0^2 - te^{-t/3}\big|_2^\infty + \int_2^\infty e^{-t/3}dt \\ &= -2e^{-2/3} + 2 + 2e^{-2/3} - 3e^{-t/3}\big|_2^\infty \\ &= 2 + 3e^{-2/3} \end{aligned}$$

47. Solution: D

Let T be the time from purchase until failure of the equipment. We are given that T is exponentially distributed with parameter λ = 10 since 10 = E[T] = λ . Next define the payment

P under the insurance contract by $P = \begin{cases} x & \text{for } 0 \le T \le 1 \\ \frac{\text{x}}{2} & \text{for } 1 < T \le 3 \\ 0 & \text{for } T > 3 \end{cases}$

We want to find x such that

$1000 = \mathrm{E}[\mathrm{P}] = \int_0^1 \frac{x}{10}\, e^{-t/10}\, dt + \int_1^3 \frac{x}{2}\frac{1}{10} e^{-t/10}\, dt = -xe^{-t/10}\Big|_0^1 - \frac{x}{2}e^{-t/10}\Big|_1^3$

$= -x\, e^{-1/10} + x - (x/2)\, e^{-3/10} + (x/2)\, e^{-1/10} = x(1 - ½\, e^{-1/10} - ½\, e^{-3/10}) = 0.1772x$.

We conclude that x = 5644 .

----

48. Solution: E

Let X and Y denote the year the device fails and the benefit amount, respectively. Then the density function of X is given by

$$f(x) = (0.6)^{x-1}(0.4) \quad , \quad x = 1, 2, 3...$$

and

$$y = \begin{cases} 1000(5-x) & \text{if} \quad x = 1,2,3,4 \\ 0 & \text{if} \quad x > 4 \end{cases}$$

It follows that

$$E[Y] = 4000(0.4) + 3000(0.6)(0.4) + 2000(0.6)^2(0.4) + 1000(0.6)^3(0.4)$$
$$= 2694$$

----

49. Solution: D

Define $f(X)$ to be hospitalization payments made by the insurance policy. Then

$$f(X) = \begin{cases} 100X & \text{if} \quad X = 1,2,3 \\ 300 + 25(X-3) & \text{if} \quad X = 4,5 \end{cases}$$

and

$$E\left[f(X)\right] = \sum_{k=1}^{5} f(k)\Pr\left[X = k\right]$$

$$= 100\left(\frac{5}{15}\right) + 200\left(\frac{4}{15}\right) + 300\left(\frac{3}{15}\right) + 325\left(\frac{2}{15}\right) + 350\left(\frac{1}{15}\right)$$

$$= \frac{1}{3}\left[100 + 160 + 180 + 130 + 70\right] = \frac{640}{3} = 213.33$$

----

50. Solution: C

Let N be the number of major snowstorms per year, and let P be the amount paid to the company under the policy. Then Pr[N = n] = $\frac{(3/2)^n e^{-3/2}}{n!}$, n = 0, 1, 2, . . . and

$$P = \begin{cases} 0 & \text{for } N = 0 \\ 10,000(N-1) & \text{for } N \geq 1 \end{cases}.$$

Now observe that E[P] = $\sum_{n=1}^{\infty} 10,000(n-1)\frac{(3/2)^n e^{-3/2}}{n!}$

= 10,000 $e^{-3/2}$ + $\sum_{n=0}^{\infty} 10,000(n-1)\frac{(3/2)^n e^{-3/2}}{n!}$ = 10,000 $e^{-3/2}$ + E[10,000 (N – 1)]

= 10,000 $e^{-3/2}$ + E[10,000N] – E[10,000] = 10,000 $e^{-3/2}$ + 10,000 (3/2) – 10,000 = 7,231 .

----

51. Solution: C

Let Y denote the manufacturer's retained annual losses.

Then $Y = \begin{cases} x & \text{for } 0.6 < x \leq 2 \\ 2 & \text{for } x > 2 \end{cases}$

and E[Y] = $\int_{0.6}^{2} x\left[\frac{2.5(0.6)^{2.5}}{x^{3.5}}\right]dx + \int_{2}^{\infty} 2\left[\frac{2.5(0.6)^{2.5}}{x^{3.5}}\right]dx = \int_{0.6}^{2} \frac{2.5(0.6)^{2.5}}{x^{2.5}}dx - \frac{2(0.6)^{2.5}}{x^{2.5}}\Bigg|_{2}^{\infty}$

$= -\frac{2.5(0.6)^{2.5}}{1.5x^{1.5}}\Bigg|_{0.6}^{2} + \frac{2(0.6)^{2.5}}{(2)^{2.5}} = -\frac{2.5(0.6)^{2.5}}{1.5(2)^{1.5}} + \frac{2.5(0.6)^{2.5}}{1.5(0.6)^{1.5}} + \frac{(0.6)^{2.5}}{2^{1.5}}$ = 0.9343 .

52. Solution: A

Let us first determine K. Observe that

$$1 = K\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) = K\left(\frac{60+30+20+15+12}{60}\right) = K\left(\frac{137}{60}\right)$$

$$K = \frac{60}{137}$$

It then follows that

$$\Pr\left[N=n\right] = \Pr\left[N=n \middle| \text{Insured Suffers a Loss}\right]\Pr\left[\text{Insured Suffers a Loss}\right]$$

$$= \frac{60}{137N}(0.05) = \frac{3}{137N} \quad , \quad N = 1,\ldots,5$$

Now because of the deductible of 2, the net annual premium $P = E\left[X\right]$ where

$$X = \begin{cases} 0 & , \text{ if } \quad N \le 2 \\ N-2 & , \text{ if } \quad N > 2 \end{cases}$$

Then,

$$P = E\left[X\right] = \sum_{N=3}^{5}(N-2)\frac{3}{137N} = (1)\left(\frac{1}{137}\right) + 2\left[\frac{3}{137(4)}\right] + 3\left[\frac{3}{137(5)}\right] = 0.0314$$

----

53. Solution: D

Let W denote claim payments. Then $W = \begin{cases} y & \text{for } 1 < y \le 10 \\ 10 & \text{for } y \ge 10 \end{cases}$

It follows that E[W] = $\int_1^{10} y\frac{2}{y^3}\,dy + \int_{10}^{\infty} 10\frac{2}{y^3}\,dy = -\frac{2}{y}\Big|_1^{10} - \frac{10}{y^2}\Big|_{10}^{\infty}$ = 2 – 2/10 + 1/10 = 1.9 .

54. Solution: B

Let Y denote the claim payment made by the insurance company.
Then

$$Y = \begin{cases} 0 & \text{with probability } 0.94 \\ \text{Max } (0, x-1) & \text{with probability } 0.04 \\ 14 & \text{with probability } 0.02 \end{cases}$$

and

$$\begin{aligned}
E[Y] &= (0.94)(0) + (0.04)(0.5003)\int_1^{15}(x-1)e^{-x/2}dx + (0.02)(14) \\
&= (0.020012)\left[\int_1^{15} xe^{-x/2}dx - \int_1^{15} e^{-x/2}dx\right] + 0.28 \\
&= 0.28 + (0.020012)\left[-2xe^{-x/2}\Big|_1^{15} + 2\int_1^{15} e^{-x/2}dx - \int_1^{15} e^{-x/2}dx\right] \\
&= 0.28 + (0.020012)\left[-30e^{-7.5} + 2e^{-0.5} + \int_1^{15} e^{-x/2}dx\right] \\
&= 0.28 + (0.020012)\left[-30e^{-7.5} + 2e^{-0.5} - 2e^{-x/2}\Big|_1^{15}\right] \\
&= 0.28 + (0.020012)\left(-30e^{-7.5} + 2e^{-0.5} - 2e^{-7.5} + 2e^{-0.5}\right) \\
&= 0.28 + (0.020012)\left(-32e^{-7.5} + 4e^{-0.5}\right) \\
&= 0.28 + (0.020012)(2.408) \\
&= 0.328 \quad \text{(in thousands)}
\end{aligned}$$

It follows that the expected claim payment is 328 .

---

55. Solution: C

The pdf of x is given by f(x) = $\dfrac{k}{(1+x)^4}$, 0 < x < ∞ . To find k, note

1 = $\displaystyle\int_0^\infty \frac{k}{(1+x)^4}dx = -\frac{k}{3}\frac{1}{(1+x)^3}\Big|_0^\infty = \frac{k}{3}$

k = 3

It then follows that E[x] = $\displaystyle\int_0^\infty \frac{3x}{(1+x)^4}dx$ and substituting u = 1 + x, du = dx, we see

E[x] = $\displaystyle\int_1^\infty \frac{3(u-1)}{u^4}du$ = $\displaystyle 3\int_1^\infty (u^{-3} - u^{-4})du = 3\left[\frac{u^{-2}}{-2} - \frac{u^{-3}}{-3}\right]_1^\infty = 3\left[\frac{1}{2} - \frac{1}{3}\right]$ = 3/2 – 1 = ½ .

56. Solution: C

Let Y represent the payment made to the policyholder for a loss subject to a deductible D.

That is $Y = \begin{cases} 0 & \text{for } 0 \le X \le D \\ x - D & \text{for } D < X \le 1 \end{cases}$

Then since E[X] = 500, we want to choose D so that

$$\frac{1}{4}500 = \int_D^{1000} \frac{1}{1000}(x-D)dx = \frac{1}{1000}\frac{(x-D)^2}{2}\bigg|_D^{1000} = \frac{(1000-D)^2}{2000}$$

$(1000 – D)^2 = 2000/4 \cdot 500 = 500^2$

$1000 – D = \pm 500$

D = 500 (or D = 1500 which is extraneous).

----

57. Solution: B

We are given that $M_x(t) = \dfrac{1}{(1-2500t)^4}$ for the claim size X in a certain class of accidents.

First, compute $M_x'(t) = \dfrac{(-4)(-2500)}{(1-2500t)^5} = \dfrac{10,000}{(1-2500t)^5}$

$$M_x''(t) = \frac{(10,000)(-5)(-2500)}{(1-2500t)^6} = \frac{125,000,000}{(1-2500t)^6}$$

Then $E[X] = M_x'(0) = 10,000$

$E[X^2] = M_x''(0) = 125,000,000$

$Var[X] = E[X^2] – \{E[X]\}^2 = 125,000,000 – (10,000)^2 = 25,000,000$

$\sqrt{Var[X]} = 5,000$.

----

58. Solution: E

Let X_J, X_K, and X_L represent annual losses for cities J, K, and L, respectively. Then $X = X_J + X_K + X_L$ and due to independence

$M(t) = E\left[e^{xt}\right] = E\left[e^{(x_J+x_K+x_L)t}\right] = E\left[e^{x_J t}\right]E\left[e^{x_K t}\right]E\left[e^{x_L t}\right]$

$= M_J(t)\, M_K(t)\, M_L(t) = (1 – 2t)^{–3}\, (1 – 2t)^{–2.5}\, (1 – 2t)^{–4.5} = (1 – 2t)^{–10}$

Therefore,

$M'(t) = 20(1 – 2t)^{–11}$

$M''(t) = 440(1 – 2t)^{–12}$

$M'''(t) = 10,560(1 – 2t)^{–13}$

$E[X^3] = M'''(0) = 10,560$

59. Solution: B

The distribution function of X is given by

$$F(x) = \int_{200}^{x} \frac{2.5(200)^{2.5}}{t^{3.5}} dt = \left. \frac{-(200)^{2.5}}{t^{2.5}} \right|_{200}^{x} = 1 - \frac{(200)^{2.5}}{x^{2.5}} \quad , \quad x > 200$$

Therefore, the p^{th} percentile x_p of X is given by

$$\frac{p}{100} = F(x_p) = 1 - \frac{(200)^{2.5}}{x_p^{\ 2.5}}$$

$$1 - 0.01p = \frac{(200)^{2.5}}{x_p^{\ 2.5}}$$

$$(1 - 0.01p)^{2/5} = \frac{200}{x_p}$$

$$x_p = \frac{200}{(1 - 0.01p)^{2/5}}$$

It follows that $x_{70} - x_{30} = \dfrac{200}{(0.30)^{2/5}} - \dfrac{200}{(0.70)^{2/5}} = 93.06$

----

60. Solution: E

Let X and Y denote the annual cost of maintaining and repairing a car before and after the 20% tax, respectively. Then Y = 1.2X and Var[Y] = Var[1.2X] = $(1.2)^2$ Var[X] = $(1.2)^2(260)$ = 374 .

----

61. Solution: A

The first quartile, Q1, is found by ¾ = $\int_{Q1}^{\infty}$ f(x) dx . That is, ¾ = $(200/Q1)^{2.5}$ or Q1 = 200 $(4/3)^{0.4}$ = 224.4 . Similarly, the third quartile, Q3, is given by Q3 = 200 $(4)^{0.4}$ = 348.2 . The interquartile range is the difference Q3 – Q1 .

62. Solution: C

First note that the density function of X is given by

$$f(x)=\begin{cases}\frac{1}{2} & \text{if} \quad x=1\\ x-1 & \text{if} \quad 1<x<2\\ \\ 0 & \text{otherwise}\end{cases}$$

Then

$$E(X)=\frac{1}{2}+\int_1^2 x(x-1)\,dx=\frac{1}{2}+\int_1^2\left(x^2-x\right)dx=\frac{1}{2}+\left(\frac{1}{3}x^3-\frac{1}{2}x^2\right)\Bigg|_1^2$$

$$=\frac{1}{2}+\frac{8}{3}-\frac{4}{2}-\frac{1}{3}+\frac{1}{2}=\frac{7}{3}-1=\frac{4}{3}$$

$$E\left(X^2\right)=\frac{1}{2}+\int_1^2 x^2(x-1)\,dx=\frac{1}{2}+\int_1^2\left(x^3-x^2\right)dx=\frac{1}{2}+\left(\frac{1}{4}x^4-\frac{1}{3}x^3\right)\Bigg|_1^2$$

$$=\frac{1}{2}+\frac{16}{4}-\frac{8}{3}-\frac{1}{4}+\frac{1}{3}=\frac{17}{4}-\frac{7}{3}=\frac{23}{12}$$

$$Var(X)=E\left(X^2\right)-\left[E(X)\right]^2=\frac{23}{12}-\left(\frac{4}{3}\right)^2=\frac{23}{12}-\frac{16}{9}=\frac{5}{36}$$

----

63. Solution: C

Note $Y=\begin{cases}X & \text{if} \quad 0\le X\le 4\\ 4 & \text{if} \quad 4< X\le 5\end{cases}$

Therefore,

$$E[Y]=\int_0^4\frac{1}{5}x\,dx+\int_4^5\frac{4}{5}\,dx=\frac{1}{10}x^2\Big|_0^4+\frac{4}{5}x\Big|_4^5$$

$$=\frac{16}{10}+\frac{20}{5}-\frac{16}{5}=\frac{8}{5}+\frac{4}{5}=\frac{12}{5}$$

$$E\left[Y^2\right]=\int_0^4\frac{1}{5}x^2\,dx+\int_4^5\frac{16}{5}\,dx=\frac{1}{15}x^3\Big|_0^4+\frac{16}{5}x\Big|_4^5$$

$$=\frac{64}{15}+\frac{80}{5}-\frac{64}{5}=\frac{64}{15}+\frac{16}{5}=\frac{64}{15}+\frac{48}{15}=\frac{112}{15}$$

$$\mathrm{Var}[Y]=E\left[Y^2\right]-\left(E[Y]\right)^2=\frac{112}{15}-\left(\frac{12}{5}\right)^2=1.71$$

64. Solution: A
Let X denote claim size. Then E[X] = [20(0.15) + 30(0.10) + 40(0.05) + 50(0.20) + 60(0.10) + 70(0.10) + 80(0.30)] = (3 + 3 + 2 + 10 + 6 + 7 + 24) = 55
$E[X^2]$ = 400(0.15) + 900(0.10) + 1600(0.05) + 2500(0.20) + 3600(0.10) + 4900(0.10) + 6400(0.30) = 60 + 90 + 80 + 500 + 360 + 490 + 1920 = 3500
$\text{Var}[X] = E[X^2] – (E[X])^2$ = 3500 – 3025 = 475 and $\sqrt{Var[X]}$ = 21.79 .
Now the range of claims within one standard deviation of the mean is given by [55.00 – 21.79, 55.00 + 21.79] = [33.21, 76.79]
Therefore, the proportion of claims within one standard deviation is 0.05 + 0.20 + 0.10 + 0.10 = 0.45 .

----

65. Solution: B
Let X and Y denote repair cost and insurance payment, respectively, in the event the auto is damaged. Then

$$Y = \begin{cases} 0 & \text{if } \quad x \le 250 \\ x - 250 & \text{if } \quad x > 250 \end{cases}$$

and

$$E[Y] = \int_{250}^{1500} \frac{1}{1500}(x-250)dx = \frac{1}{3000}(x-250)^2 \Big|_{250}^{1500} = \frac{1250^2}{3000} = 521$$

$$E\left[Y^2\right] = \int_{250}^{1500} \frac{1}{1500}(x-250)^2\, dx = \frac{1}{4500}(x-250)^3 \Big|_{250}^{1500} = \frac{1250^3}{4500} = 434{,}028$$

$$\text{Var}[Y] = E\left[Y^2\right] - \{E[Y]\}^2 = 434{,}028 - (521)^2$$

$$\sqrt{\text{Var}[Y]} = 403$$

----

66. Solution: E
Let X_1, X_2, X_3, and X_4 denote the four independent bids with common distribution function F. Then if we define $Y = \max(X_1, X_2, X_3, X_4)$, the distribution function G of Y is given by

$$\begin{aligned} G(y) &= \Pr[Y \le y] \\ &= \Pr\left[(X_1 \le y) \cap (X_2 \le y) \cap (X_3 \le y) \cap (X_4 \le y)\right] \\ &= \Pr[X_1 \le y]\Pr[X_2 \le y]\Pr[X_3 \le y]\Pr[X_4 \le y] \\ &= \left[F(y)\right]^4 \\ &= \frac{1}{16}(1+\sin\pi y)^4 \quad , \quad \frac{3}{2} \le y \le \frac{5}{2} \end{aligned}$$

It then follows that the density function g of Y is given by

$$g(y) = G'(y)$$
$$= \frac{1}{4}(1+\sin\pi y)^3 (\pi\cos\pi y)$$
$$= \frac{\pi}{4}\cos\pi y (1+\sin\pi y)^3 \quad , \quad \frac{3}{2} \le y \le \frac{5}{2}$$

Finally,

$$E[Y] = \int_{3/2}^{5/2} y g(y)\,dy$$
$$= \int_{3/2}^{5/2} \frac{\pi}{4} y\cos\pi y (1+\sin\pi y)^3\,dy$$

----

67. Solution: B

The amount of money the insurance company will have to pay is defined by the random variable

$$Y = \begin{cases} 1000x & \text{if} \quad x < 2 \\ 2000 & \text{if} \quad x \ge 2 \end{cases}$$

where x is a Poisson random variable with mean 0.6 . The probability function for X is

$$p(x) = \frac{e^{-0.6}(0.6)^k}{k!} \qquad k = 0,1,2,3\cdots \text{ and}$$

$$E[Y] = 0 + 1000(0.6)e^{-0.6} + 2000e^{-0.6}\sum_{k=2}^{\infty}\frac{0.6^k}{k!}$$
$$= 1000(0.6)e^{-0.6} + 2000\left(e^{-0.6}\sum_{k=0}^{\infty}\frac{0.6^k}{k!} - e^{-0.6} - (0.6)e^{-0.6}\right)$$
$$= 2000e^{-0.6}\sum_{k=0}^{\infty}\frac{(0.6)^k}{k!} - 2000e^{-0.6} - 1000(0.6)e^{-0.6} = 2000 - 2000e^{-0.6} - 600e^{-0.6}$$
$$= 573$$

$$E\left[Y^2\right] = (1000)^2(0.6)e^{-0.6} + (2000)^2 e^{-0.6}\sum_{k=2}^{\infty}\frac{0.6^k}{k!}$$
$$= (2000)^2 e^{-0.6}\sum_{k=0}^{\infty}\frac{0.6^k}{k!} - (2000)^2 e^{-0.6} - \left[(2000)^2 - (1000)^2\right](0.6)e^{-0.6}$$
$$= (2000)^2 - (2000)^2 e^{-0.6} - \left[(2000)^2 - (1000)^2\right](0.6)e^{-0.6}$$
$$= 816{,}893$$

$$\text{Var}[Y] = E\left[Y^2\right] - \{E[Y]\}^2 = 816{,}893 - (573)^2 = 488{,}564$$

$$\sqrt{\text{Var}[Y]} = 699$$

68. Solution: C

Note that X has an exponential distribution. Therefore, c = 0.004 . Now let Y denote the claim benefits paid. Then $Y = \begin{cases} x & \text{for } x < 250 \\ 250 & \text{for } x \geq 250 \end{cases}$ and we want to find m such that 0.50 $= \int_0^m 0.004 e^{-0.004x} dx = -e^{-0.004x} \Big|_0^m$ = 1 – $e^{-0.004m}$

This condition implies $e^{-0.004m} = 0.5 \Rightarrow m = 250 \ln 2 = 173.29$.

----

69. Solution: D

The distribution function of an exponential random variable T with parameter θ is given by $F(t) = 1 - e^{-t/\theta}$, $t > 0$

Since we are told that T has a median of four hours, we may determine θ as follows:

$$\frac{1}{2} = F(4) = 1 - e^{-4/\theta}$$

$$\frac{1}{2} = e^{-4/\theta}$$

$$-\ln(2) = -\frac{4}{\theta}$$

$$\theta = \frac{4}{\ln(2)}$$

Therefore, $\Pr(T \geq 5) = 1 - F(5) = e^{-5/\theta} = e^{-\frac{5\ln(2)}{4}} = 2^{-5/4} = 0.42$

----

70. Solution: E

Let X denote actual losses incurred. We are given that X follows an exponential distribution with mean 300, and we are asked to find the 95th percentile of all claims that exceed 100 . Consequently, we want to find p_{95} such that

0.95 = $\dfrac{\Pr[100 < x < p_{95}]}{P[X > 100]} = \dfrac{F(p_{95}) - F(100)}{1 - F(100)}$ where F(x) is the distribution function of X .

Now $F(x) = 1 - e^{-x/300}$.

Therefore, 0.95 = $\dfrac{1 - e^{-p_{95}/300} - (1 - e^{-100/300})}{1 - (1 - e^{-100/300})} = \dfrac{e^{-1/3} - e^{-p_{95}/300}}{e^{-1/3}} = 1 - e^{1/3} e^{-p_{95}/300}$

$e^{-p_{95}/300} = 0.05\, e^{-1/3}$

$p_{95} = -300 \ln(0.05\, e^{-1/3}) = 999$

71. Solution: A

The distribution function of Y is given by

$$G(y) = \Pr\left(T^2 \le y\right) = \Pr\left(T \le \sqrt{y}\right) = F\left(\sqrt{y}\right) = 1 - 4/y$$

for $y > 4$. Differentiate to obtain the density function $g(y) = 4y^{-2}$

Alternate solution:

Differentiate $F(t)$ to obtain $f(t) = 8t^{-3}$ and set $y = t^2$. Then $t = \sqrt{y}$ and

$$g(y) = f\left(t(y)\right)\left|dt/dy\right| = f\left(\sqrt{y}\right)\left|\frac{d}{dt}\left(\sqrt{y}\right)\right| = 8y^{-3/2}\left(\frac{1}{2}y^{-1/2}\right) = 4y^{-2}$$

---

72. Solution: E

We are given that R is uniform on the interval $(0.04, 0.08)$ and $V = 10,000e^R$

Therefore, the distribution function of V is given by

$$F(v) = \Pr[V \le v] = \Pr\left[10,000e^R \le v\right] = \Pr\left[R \le \ln(v) - \ln(10,000)\right]$$

$$= \frac{1}{0.04}\int_{0.04}^{\ln(v)-\ln(10,000)} dr = \frac{1}{0.04}\, r\Big|_{0.04}^{\ln(v)-\ln(10,000)} = 25\ln(v) - 25\ln(10,000) - 1$$

$$= 25\left[\ln\left(\frac{v}{10,000}\right) - 0.04\right]$$

---

73. Solution: E

$$F(y) = \Pr[Y \le y] = \Pr\left[10X^{0.8} \le y\right] = \Pr\left[X \le \left(Y/10\right)^{10/8}\right] = 1 - e^{-\left(Y/10\right)^{10/8}}$$

Therefore, $f(y) = F'(y) = \frac{1}{8}\left(\frac{Y}{10}\right)^{1/4} e^{-(Y/10)^{5/4}}$

74. Solution: E

First note R = 10/T . Then

$F_R(r) = P[R \le r] = P\left[\frac{10}{T} \le r\right] = P\left[T \ge \frac{10}{r}\right] = 1 - F_T\left(\frac{10}{r}\right)$. Differentiating with respect to

r $f_R(r) = F'_R(r) = d/dr \left(1 - F_T\left(\frac{10}{r}\right)\right) = -\left(\frac{d}{dt}F_T(t)\right)\left(\frac{-10}{r^2}\right)$

$\frac{d}{dt}F_T(t) = f_T(t) = \frac{1}{4}$ since T is uniformly distributed on [8, 12] .

Therefore $f_R(r) = \frac{-1}{4}\left(\frac{-10}{r^2}\right) = \frac{5}{2r^2}$.

----

75. Solution: A

Let X and Y be the monthly profits of Company I and Company II, respectively. We are given that the pdf of X is f . Let us also take g to be the pdf of Y and take F and G to be the distribution functions corresponding to f and g . Then G(y) = Pr[Y ≤ y] = P[2X ≤ y] = P[X ≤ y/2] = F(y/2) and g(y) = G′(y) = d/dy F(y/2) = ½ F′(y/2) = ½ f(y/2) .

----

76. Solution: A

First, observe that the distribution function of X is given by

$$F(x) = \int_1^x \frac{3}{t^4}\,dt = -\frac{1}{t^3}\Big|_1^x = 1 - \frac{1}{x^3} \quad , \quad x > 1$$

Next, let X_1, X_2, and X_3 denote the three claims made that have this distribution. Then if Y denotes the largest of these three claims, it follows that the distribution function of Y is given by

$$G(y) = \Pr[X_1 \le y]\Pr[X_2 \le y]\Pr[X_3 \le y]$$

$$= \left(1 - \frac{1}{y^3}\right)^3 \quad , \quad y > 1$$

while the density function of Y is given by

$$g(y) = G'(y) = 3\left(1 - \frac{1}{y^3}\right)^2\left(\frac{3}{y^4}\right) = \left(\frac{9}{y^4}\right)\left(1 - \frac{1}{y^3}\right)^2 \quad , \quad y > 1$$

Therefore,

$$E[Y] = \int_1^\infty \frac{9}{y^3}\left(1-\frac{1}{y^3}\right)^2 dy = \int_1^\infty \frac{9}{y^3}\left(1-\frac{2}{y^3}+\frac{1}{y^6}\right)dy$$
$$= \int_1^\infty \left(\frac{9}{y^3}-\frac{18}{y^6}+\frac{9}{y^9}\right)dy = \left[-\frac{9}{2y^2}+\frac{18}{5y^5}-\frac{9}{8y^8}\right]_1^\infty$$
$$= 9\left[\frac{1}{2}-\frac{2}{5}+\frac{1}{8}\right] = 2.025 \text{ (in thousands)}$$

----

77. Solution: D

Prob. = 1– $\int_1^2 \int_1^2 \frac{1}{8}(x+y)dxdy$ = 0.625

Note

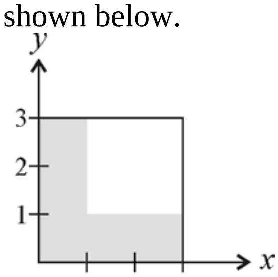
$\Pr\left[(X \le 1)\cup(Y \le 1)\right] = \Pr\left\{\left[(X > 1)\cap(Y > 1)\right]^c\right\}$ (De Morgan's Law)

$$= 1-\Pr\left[(X>1)\cap(Y>1)\right] \qquad = 1-\int_1^2\int_1^2 \frac{1}{8}(x+y)\,dxdy \qquad = 1-\frac{1}{8}\int_1^2 \frac{1}{2}(x+y)^2\Big|_1^2 dy$$
$$= 1-\frac{1}{16}\int_1^2\left[(y+2)^2-(y+1)^2\right]dy \qquad = 1-\frac{1}{48}\left[(y+2)^3-(y+1)^3\right]\Big|_1^2 = 1-\frac{1}{48}(64-27-27+8)$$
$$= 1-\frac{18}{48} = \frac{30}{48} = 0.625$$

----

78. Solution: B

That the device fails within the first hour means the joint density function must be integrated over the shaded region shown below.



This evaluation is more easily performed by integrating over the unshaded region and subtracting from 1.

$\Pr\left[(X<1)\cup(Y<1)\right]$

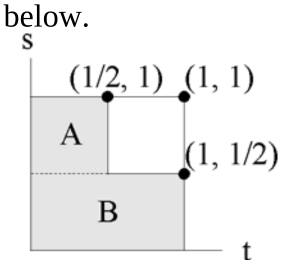
$$=1-\int_1^3\int_1^3\frac{x+y}{27}\,dx\,dy=1-\int_1^3\left.\frac{x^2+2xy}{54}\right|_1^3dy=1-\frac{1}{54}\int_1^3(9+6y-1-2y)\,dy$$

$$=1-\frac{1}{54}\int_1^3(8+4y)\,dy=1-\frac{1}{54}\left.\left(8y+2y^2\right)\right|_1^3=1-\frac{1}{54}(24+18-8-2)=1-\frac{32}{54}=\frac{11}{27}=0.41$$

---

79. Solution: E

The domain of *s* and *t* is pictured below.



Note that the shaded region is the portion of the domain of s and t over which the device fails sometime during the first half hour. Therefore,

$$\Pr\left[\left(S\le\frac{1}{2}\right)\cup\left(T\le\frac{1}{2}\right)\right]=\int_0^{1/2}\int_{1/2}^1 f(s,t)\,ds\,dt+\int_0^1\int_0^{1/2} f(s,t)\,ds\,dt$$

(where the first integral covers A and the second integral covers B).

---

80. Solution: C

By the central limit theorem, the total contributions are approximately normally distributed with mean $n\mu=(2025)(3125)=6,328,125$ and standard deviation $\sigma\sqrt{n}=250\sqrt{2025}=11,250$. From the tables, the 90$^{\text{th}}$ percentile for a standard normal random variable is 1.282 . Letting *p* be the 90$^{\text{th}}$ percentile for total contributions,

$\frac{p-n\mu}{\sigma\sqrt{n}}=1.282,$ and so $p=n\mu+1.282\,\sigma\sqrt{n}=6,328,125+(1.282)(11,250)=6,342,548$.

---

81. Solution: C

Let $X_1, \ldots, X_{25}$ denote the 25 collision claims, and let $\bar{X} = \frac{1}{25}(X_1 + \ldots + X_{25})$. We are given that each X_i ($i = 1, \ldots, 25$) follows a normal distribution with mean 19,400 and standard deviation 5000 . As a result $\bar{X}$ also follows a normal distribution with mean 19,400 and standard deviation $\frac{1}{\sqrt{25}}(5000) = 1000$. We conclude that $P[\bar{X} > 20{,}000]$

$= P\left[\frac{\bar{X} - 19{,}400}{1000} > \frac{20{,}000 - 19{,}400}{1000}\right] = P\left[\frac{\bar{X} - 19{,}400}{1000} > 0.6\right] = 1 - \Phi(0.6) = 1 - 0.7257$

$= 0.2743$.

---

82. Solution: B

Let $X_1, \ldots, X_{1250}$ be the number of claims filed by each of the 1250 policyholders. We are given that each X_i follows a Poisson distribution with mean 2 . It follows that $E[X_i] = \mathrm{Var}[X_i] = 2$. Now we are interested in the random variable $S = X_1 + \ldots + X_{1250}$. Assuming that the random variables are independent, we may conclude that S has an approximate normal distribution with $E[S] = \mathrm{Var}[S] = (2)(1250) = 2500$.

Therefore $P[2450 < S < 2600] =$

$$P\left[\frac{2450 - 2500}{\sqrt{2500}} < \frac{S - 2500}{\sqrt{2500}} < \frac{2600 - 2500}{\sqrt{2500}}\right] = P\left[-1 < \frac{S - 2500}{50} < 2\right]$$

$$= P\left[\frac{S - 2500}{50} < 2\right] - P\left[\frac{S - 2500}{50} < -1\right]$$

Then using the normal approximation with $Z = \frac{S - 2500}{50}$, we have $P[2450 < S < 2600]$

$\approx P[Z < 2] - P[Z > 1] = P[Z < 2] + P[Z < 1] - 1 \approx 0.9773 + 0.8413 - 1 = 0.8186$.

---

83. Solution: B

Let $X_1, \ldots, X_n$ denote the life spans of the n light bulbs purchased. Since these random variables are independent and normally distributed with mean 3 and variance 1, the random variable $S = X_1 + \ldots + X_n$ is also normally distributed with mean

$$\mu = 3n$$

and standard deviation

$$\sigma = \sqrt{n}$$

Now we want to choose the smallest value for n such that

$$0.9772 \le \Pr[S > 40] = \Pr\left[\frac{S - 3n}{\sqrt{n}} > \frac{40 - 3n}{\sqrt{n}}\right]$$

This implies that n should satisfy the following inequality:

$$-2 \ge \frac{40-3n}{\sqrt{n}}$$

To find such an n, let's solve the corresponding equation for n:

$$-2 = \frac{40-3n}{\sqrt{n}}$$

$$-2\sqrt{n} = 40-3n$$

$$3n - 2\sqrt{n} - 40 = 0$$

$$\left(3\sqrt{n}+10\right)\left(\sqrt{n}-4\right) = 0$$

$$\sqrt{n} = 4$$

$$n = 16$$

----

84. Solution: B

Observe that

$$E\left[X+Y\right] = E\left[X\right]+E\left[Y\right] = 50+20 = 70$$

$$Var\left[X+Y\right] = Var\left[X\right]+Var\left[Y\right]+2\ Cov\left[X,Y\right] = 50+30+20 = 100$$

for a randomly selected person. It then follows from the Central Limit Theorem that T is approximately normal with mean

$$E\left[T\right] = 100\left(70\right) = 7000$$

and variance

$$Var\left[T\right] = 100\left(100\right) = 100^2$$

Therefore,

$$\Pr\left[T<7100\right] = \Pr\left[\frac{T-7000}{100} < \frac{7100-7000}{100}\right]$$

$$= \Pr\left[Z<1\right] = 0.8413$$

where Z is a standard normal random variable.

---

85. Solution: B

Denote the policy premium by P . Since x is exponential with parameter 1000, it follows from what we are given that E[X] = 1000, Var[X] = 1,000,000, $\sqrt{Var[X]}$ = 1000 and P = 100 + E[X] = 1,100 . Now if 100 policies are sold, then Total Premium Collected = 100(1,100) = 110,000

Moreover, if we denote total claims by S, and assume the claims of each policy are independent of the others then E[S] = 100 E[X] = (100)(1000) and Var[S] = 100 Var[X] = (100)(1,000,000) . It follows from the Central Limit Theorem that S is approximately normally distributed with mean 100,000 and standard deviation = 10,000 . Therefore,

P[S ≥ 110,000] = 1 – P[S ≤ 110,000] = 1 – $P\left[Z \le \frac{110,000-100,000}{10,000}\right]$ = 1 – P[Z ≤ 1] = 1 – 0.841 ≈ 0.159 .

---

86. Solution: E

Let $X_1, ..., X_{100}$ denote the number of pensions that will be provided to each new recruit. Now under the assumptions given,

$$X_i = \begin{cases} 0 & \text{with probability} \quad 1-0.4=0.6 \\ 1 & \text{with probability} \quad (0.4)(0.25)=0.1 \\ 2 & \text{with probability} \quad (0.4)(0.75)=0.3 \end{cases}$$

for $i = 1,...,100$. Therefore,

$$E[X_i] = (0)(0.6)+(1)(0.1)+(2)(0.3) = 0.7 ,$$

$$E\left[X_i^2\right] = (0)^2(0.6)+(1)^2(0.1)+(2)^2(0.3) = 1.3 , \text{ and}$$

$$\text{Var}[X_i] = E\left[X_i^2\right] - \left\{E[X_i]\right\}^2 = 1.3-(0.7)^2 = 0.81$$

Since $X_1, ..., X_{100}$ are assumed by the consulting actuary to be independent, the Central Limit Theorem then implies that $S = X_1 + ... + X_{100}$ is approximately normally distributed with mean

$$E[S] = E[X_1]+...+E[X_{100}] = 100(0.7) = 70$$

and variance

$$\text{Var}[S] = \text{Var}[X_1]+...+\text{Var}[X_{100}] = 100(0.81) = 81$$

Consequently,

$$\begin{aligned} \Pr[S \le 90.5] &= \Pr\left[\frac{S-70}{9} \le \frac{90.5-70}{9}\right] \\ &= \Pr[Z \le 2.28] \\ &= 0.99 \end{aligned}$$

---

87. Solution: D

Let X denote the difference between true and reported age. We are given X is uniformly distributed on (–2.5,2.5) . That is, X has pdf f(x) = 1/5, –2.5 < x < 2.5 . It follows that

μ_x = E[X] = 0

σ_x^2 = Var[X] = E[X^2] = $\int_{-2.5}^{2.5} \frac{x^2}{5} dx = \frac{x^3}{15}\Big|_{-2.5}^{2.5} \quad = \frac{2(2.5)^3}{15}$ =2.083

σ_x =1.443

Now $\bar{X}_{48}$, the difference between the means of the true and rounded ages, has a distribution that is approximately normal with mean 0 and standard deviation $\frac{1.443}{\sqrt{48}}$ = 0.2083 . Therefore,

$P\left[-\frac{1}{4} \leq \bar{X}_{48} \leq \frac{1}{4}\right] = P\left[\frac{-0.25}{0.2083} \leq Z \leq \frac{0.25}{0.2083}\right]$ = P[–1.2 ≤ Z ≤ 1.2] = P[Z ≤ 1.2] – P[Z ≤ –1.2]

= P[Z ≤ 1.2] – 1 + P[Z ≤ 1.2] = 2P[Z ≤ 1.2] – 1 = 2(0.8849) – 1 = 0.77 .

---

88. Solution: C

Let *X* denote the waiting time for a first claim from a good driver, and let *Y* denote the waiting time for a first claim from a bad driver. The problem statement implies that the respective distribution functions for *X* and *Y* are

$F(x) = 1 - e^{-x/6}$, $x > 0$ and

$G(y) = 1 - e^{-y/3}$, $y > 0$

Therefore,

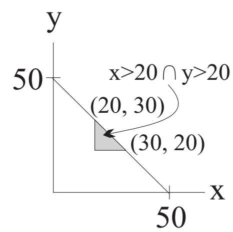
$\Pr\left[(X \leq 3) \cap (Y \leq 2)\right] = \Pr\left[X \leq 3\right]\Pr\left[Y \leq 2\right]$

$= F(3)G(2) = \left(1 - e^{-1/2}\right)\left(1 - e^{-2/3}\right) = 1 - e^{-2/3} - e^{-1/2} + e^{-7/6}$

89. Solution: B

We are given that $f(x,y) = \begin{cases} \dfrac{6}{125{,}000}(50-x-y) & \text{for } 0 < x < 50-y < 50 \\ 0 & \text{otherwise} \end{cases}$

and we want to determine P[X > 20 ∩ Y > 20] . In order to determine integration limits, consider the following diagram:



We conclude that P[X > 20 ∩ Y > 20] = $\dfrac{6}{125{,}000}\displaystyle\int_{20}^{30}\int_{20}^{50-x}(50-x-y)$ dy dx .

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90. Solution: C

Let T_1 be the time until the next Basic Policy claim, and let T_2 be the time until the next Deluxe policy claim. Then the joint pdf of T_1 and T_2 is

$f(t_1,t_2) = \left(\frac{1}{2}e^{-t_1/2}\right)\left(\frac{1}{3}e^{-t_2/3}\right) = \frac{1}{6}e^{-t_1/2}e^{-t_2/3}$, $0 < t_1 < \infty$, $0 < t_2 < \infty$ and we need to find

P[$T_2 < T_1$] = $\displaystyle\int_0^\infty\int_0^{t_1}\frac{1}{6}e^{-t_1/2}e^{-t_2/3}dt_2dt_1 = \int_0^\infty\left[-\frac{1}{2}e^{-t_1/2}e^{-t_2/3}\right]_0^{t_1}dt_1$

$= \displaystyle\int_0^\infty\left[\frac{1}{2}e^{-t_1/2} - \frac{1}{2}e^{-t_1/2}e^{-t_1/3}\right]dt_1 = \int_0^\infty\left[\frac{1}{2}e^{-t_1/2} - \frac{1}{2}e^{-5t_1/6}\right]dt_1 = \left[-e^{-t_1/2} + \frac{3}{5}e^{-5t_1/6}\right]_0^\infty = 1 - \frac{3}{5} = \frac{2}{5}$

= 0.4 .

----

91. Solution: D

We want to find P[X + Y > 1] . To this end, note that P[X + Y > 1]

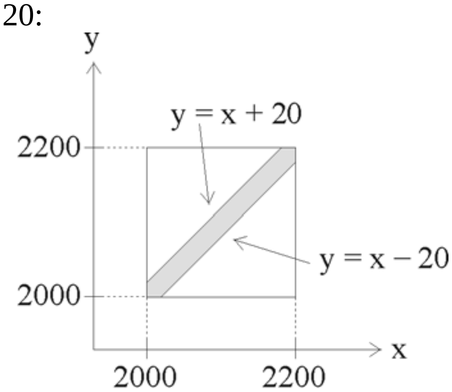
$= \displaystyle\int_0^1\int_{1-x}^2\left[\frac{2x+2-y}{4}\right]dydx = \int_0^1\left[\frac{1}{2}xy + \frac{1}{2}y - \frac{1}{8}y^2\right]_{1-x}^2 dx$

$= \displaystyle\int_0^1\left[x + 1 - \frac{1}{2} - \frac{1}{2}x(1-x) - \frac{1}{2}(1-x) + \frac{1}{8}(1-x)^2\right]dx = \int_0^1\left[x + \frac{1}{2}x^2 + \frac{1}{8} - \frac{1}{4}x + \frac{1}{8}x^2\right]dx$

$= \displaystyle\int_0^1\left[\frac{5}{8}x^2 + \frac{3}{4}x + \frac{1}{8}\right]dx = \left[\frac{5}{24}x^3 + \frac{3}{8}x^2 + \frac{1}{8}x\right]_0^1 = \frac{5}{24} + \frac{3}{8} + \frac{1}{8} = \frac{17}{24}$

92. Solution: B

Let X and Y denote the two bids. Then the graph below illustrates the region over which X and Y differ by less than 20:



Based on the graph and the uniform distribution:

$$\Pr\left[|X-Y|<20\right]=\frac{\text{Shaded Region Area}}{(2200-2000)^2}=\frac{200^2-2\cdot\frac{1}{2}(180)^2}{200^2}$$

$$=1-\frac{180^2}{200^2}=1-(0.9)^2=0.19$$

More formally (still using symmetry)

$$\Pr\left[|X-Y|<20\right]=1-\Pr\left[|X-Y|\geq 20\right]=1-2\Pr\left[X-Y\geq 20\right]$$

$$=1-2\int_{2020}^{2200}\int_{2000}^{x-20}\frac{1}{200^2}\,dydx=1-2\int_{2020}^{2200}\frac{1}{200^2}\,y\Big|_{2000}^{x-20}dx$$

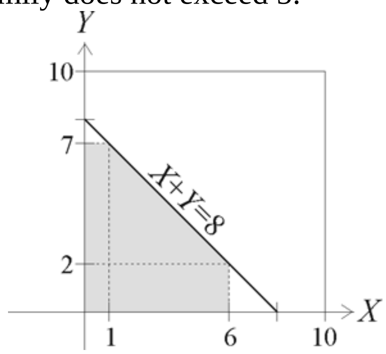
$$=1-\frac{2}{200^2}\int_{2020}^{2200}(x-20-2000)\,dx=1-\frac{1}{200^2}(x-2020)^2\Big|_{2020}^{2200}$$

$$=1-\left(\frac{180}{200}\right)^2=0.19$$

---

93. Solution: C

Define X and Y to be loss amounts covered by the policies having deductibles of 1 and 2, respectively. The shaded portion of the graph below shows the region over which the total benefit paid to the family does not exceed 5:



We can also infer from the graph that the uniform random variables X and Y have joint density function $f(x,y)=\frac{1}{100}$, $0<x<10$, $0<y<10$

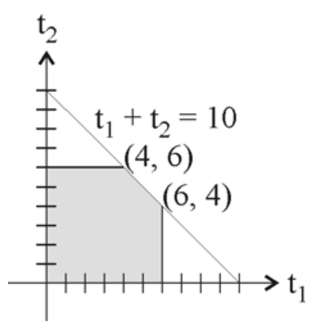
We could integrate f over the shaded region in order to determine the desired probability. However, since X and Y are uniform random variables, it is simpler to determine the portion of the 10 x 10 square that is shaded in the graph above. That is,

$$\Pr(\text{Total Benefit Paid Does not Exceed 5})$$
$$=\Pr(0<X<6,\ 0<Y<2)+\Pr(0<X<1,\ 2<Y<7)+\Pr(1<X<6,\ 2<Y<8-X)$$
$$=\frac{(6)(2)}{100}+\frac{(1)(5)}{100}+\frac{(1/2)(5)(5)}{100}=\frac{12}{100}+\frac{5}{100}+\frac{12.5}{100}=0.295$$

---

94. Solution: C

Let $f(t_1,t_2)$ denote the joint density function of T_1 and T_2 . The domain of f is pictured below:



Now the area of this domain is given by

$$A=6^2-\frac{1}{2}(6-4)^2=36-2=34$$

Consequently, $f(t_1, t_2) = \begin{cases} \dfrac{1}{34} & , \ 0 < t_1 < 6 \ , \ 0 < t_2 < 6 \ , \ t_1 + t_2 < 10 \\ 0 & \text{elsewhere} \end{cases}$

and

$$E[T_1 + T_2] = E[T_1] + E[T_2] = 2E[T_1] \qquad \text{(due to symmetry)}$$

$$= 2\left\{ \int_0^4 t_1 \int_0^6 \frac{1}{34} dt_2 dt_1 + \int_4^6 t_1 \int_0^{10-t_1} \frac{1}{34} dt_2 dt_1 \right\} = 2\left\{ \int_0^4 t_1 \left[\frac{t_2}{34} \Big|_0^6 \right] dt_1 + \int_4^6 t_1 \left[\frac{t_2}{34} \Big|_0^{10-t_1} \right] dt_1 \right\}$$

$$= 2\left\{ \int_0^4 \frac{3t_1}{17} dt_1 + \int_4^6 \frac{1}{34}\left(10t_1 - t_1^2\right) dt_1 \right\} = 2\left\{ \frac{3t_1^2}{34} \Big|_0^4 + \frac{1}{34}\left(5t_1^2 - \frac{1}{3}t_1^3\right) \Big|_4^6 \right\}$$

$$= 2\left\{ \frac{24}{17} + \frac{1}{34}\left[180 - 72 - 80 + \frac{64}{3} \right] \right\} = 5.7$$

---

95. Solution: E

$$M(t_1, t_2) = E\left[e^{t_1 W + t_2 Z} \right] = E\left[e^{t_1(X+Y) + t_2(Y-X)} \right] = E\left[e^{(t_1 - t_2)X} e^{(t_1 + t_2)Y} \right]$$

$$= E\left[e^{(t_1 - t_2)X} \right] E\left[e^{(t_1 + t_2)Y} \right] = e^{\frac{1}{2}(t_1 - t_2)^2} e^{\frac{1}{2}(t_1 + t_2)^2} = e^{\frac{1}{2}\left(t_1^2 - 2t_1 t_2 + t_2^2\right)} e^{\frac{1}{2}\left(t_1^2 + 2t_1 t_2 + t_2^2\right)} = e^{t_1^2 + t_2^2}$$

---

96. Solution: E

Observe that the bus driver collect 21x50 = 1050 for the 21 tickets he sells. However, he may be required to refund 100 to one passenger if all 21 ticket holders show up. Since passengers show up or do not show up independently of one another, the probability that all 21 passengers will show up is $(1 - 0.02)^{21} = (0.98)^{21} = 0.65$. Therefore, the tour operator's expected revenue is $1050 - (100)(0.65) = 985$.

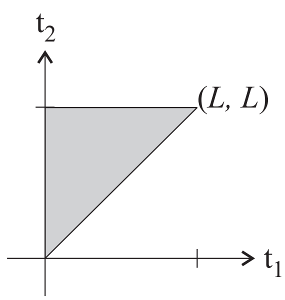
97. Solution: C

We are given $f(t_1, t_2) = 2/L^2$, $0 \le t_1 \le t_2 \le L$.

Therefore, $E[T_1^2 + T_2^2] = \int_0^L \int_0^{t_2} (t_1^2 + t_2^2) \frac{2}{L^2} dt_1 dt_2 =$

$$\frac{2}{L^2}\left\{\int_0^L \left[\frac{t_1^3}{3} + t_2^2 t_1\right]_0^{t_2} dt_1\right\} = \frac{2}{L^2}\left\{\int_0^L \left(\frac{t_2^3}{3} + t_2^3\right) dt_2\right\}$$

$$= \frac{2}{L^2}\int_0^L \frac{4}{3} t_2^3 dt_2 = \frac{2}{L^2}\left[\frac{t_2^4}{3}\right]_0^L = \frac{2}{3}L^2$$



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98. Solution: A

Let g(y) be the probability function for $Y = X_1X_2X_3$. Note that Y = 1 if and only if $X_1 = X_2 = X_3 = 1$. Otherwise, Y = 0 . Since $P[Y = 1] = P[X_1 = 1 \cap X_2 = 1 \cap X_3 = 1]$ $= P[X_1 = 1]\, P[X_2 = 1]\, P[X_3 = 1] = (2/3)^3 = 8/27$.

We conclude that $g(y) = \begin{cases} \frac{19}{27} & \text{for } y = 0 \\ \frac{8}{27} & \text{for } y = 1 \\ 0 & \text{otherwise} \end{cases}$

and M(t) = $E\left[e^{y_t}\right] = \frac{19}{27} + \frac{8}{27}e^t$

99. Solution: C

We use the relationships $\text{Var}(aX+b) = a^2 \text{ Var}(X)$, $\text{Cov}(aX, bY) = ab \text{ Cov}(X,Y)$, and $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{ Cov}(X,Y)$. First we observe $17{,}000 = \text{Var}(X+Y) = 5000 + 10{,}000 + 2 \text{ Cov}(X,Y)$, and so $\text{Cov}(X,Y) = 1000$.

We want to find $\text{Var}\left[(X+100)+1.1Y\right] = \text{ Var}\left[(X+1.1Y)+100\right]$

$= \text{Var}\left[X+1.1Y\right] = \text{Var } X + \text{Var}\left[(1.1)Y\right] + 2 \text{ Cov}(X, 1.1Y)$

$= \text{Var } X + (1.1)^2 \text{ Var} Y + 2(1.1)\text{Cov}(X,Y) = 5000 + 12{,}100 + 2200 = 19{,}300.$

----

100. Solution: B

Note

P(X = 0) = 1/6

P(X = 1) = 1/12 + 1/6 = 3/12

P(X = 2) = 1/12 + 1/3 + 1/6 = 7/12 .

E[X] = (0)(1/6) + (1)(3/12) + (2)(7/12) = 17/12

E[X^2] = $(0)^2$(1/6) + $(1)^2$(3/12) + $(2)^2$(7/12) = 31/12

Var[X] = 31/12 – $(17/12)^2$ = 0.58 .

----

101. Solution: D

Note that due to the independence of X and Y

Var(Z) = Var(3X – Y – 5) = Var(3X) + Var(Y) = 3^2 Var(X) + Var(Y) = 9(1) + 2 = 11 .

----

102. Solution: E

Let *X* and *Y* denote the times that the two backup generators can operate. Now the variance of an exponential random variable with mean β is β^2. Therefore,

$$\text{Var}\left[X\right] = \text{Var}\left[Y\right] = 10^2 = 100$$

Then assuming that *X* and *Y* are independent, we see

$$\text{Var}\left[\text{X+Y}\right] = \text{Var}\left[\text{X}\right] + \text{Var}\left[\text{Y}\right] = 100 + 100 = 200$$

103. Solution: E

Let $X_1, X_2,$ and X_3 denote annual loss due to storm, fire, and theft, respectively. In addition, let $Y = Max\left(X_1, X_2, X_3\right)$.

Then

$$\Pr\left[Y > 3\right] = 1 - \Pr\left[Y \le 3\right] = 1 - \Pr\left[X_1 \le 3\right]\Pr\left[X_2 \le 3\right]\Pr\left[X_3 \le 3\right]$$

$$= 1 - \left(1 - e^{-3}\right)\left(1 - e^{-3/1.5}\right)\left(1 - e^{-3/2.4}\right) \qquad *$$

$$= 1 - \left(1 - e^{-3}\right)\left(1 - e^{-2}\right)\left(1 - e^{-5/4}\right)$$

$$= 0.414$$

* Uses that if X has an exponential distribution with mean μ

$$\Pr\left(X \le x\right) = 1 - \Pr\left(X \ge x\right) = 1 - \int_x^\infty \frac{1}{\mu} e^{-t/\mu} dt = 1 - \left(-e^{-t/\mu}\right)\Big|_x^\infty = 1 - e^{-x/\mu}$$

----

104. Solution: B

Let us first determine k:

$$1 = \int_0^1 \int_0^1 kxdxdy = \int_0^1 \frac{1}{2} kx^2 \Big|_0^1 \, dy = \int_0^1 \frac{k}{2} dy = \frac{k}{2}$$

$$k = 2$$

Then

$$E\left[X\right] = \int_0^1 \int_0^1 2x^2 dydx = \int_0^1 2x^2 dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$E\left[Y\right] = \int_0^1 \int_0^1 y \;\; 2x \;\; dxdy = \int_0^1 ydy = \frac{1}{2} y^2 \Big|_0^1 = \frac{1}{2}$$

$$E\left[XY\right] = \int_0^1 \int_0^1 2x^2 ydxdy = \int_0^1 \frac{2}{3} x^3 y \Big|_0^1 \, dy = \int_0^1 \frac{2}{3} ydy$$

$$= \frac{2}{6} y^2 \Big|_0^1 = \frac{2}{6} = \frac{1}{3}$$

$$\text{Cov}\left[X, Y\right] = E\left[XY\right] - E\left[X\right]E\left[Y\right] = \frac{1}{3} - \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3} - \frac{1}{3} = 0$$

(Alternative Solution)

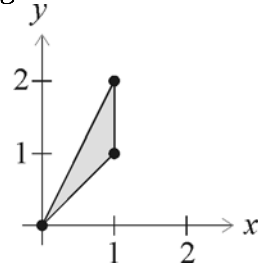
Define $g(x) = kx$ and $h(y) = 1$. Then

$f(x,y) = g(x)h(x)$

In other words, $f(x,y)$ can be written as the product of a function of x alone and a function of y alone. It follows that X and Y are independent. Therefore, Cov[X, Y] = 0 .

105. Solution: A

The calculation requires integrating over the indicated region.



$$E(X)=\int_0^1\int_x^{2x}\frac{8}{3}x^2y\,dy\,dx=\int_0^1\frac{4}{3}x^2y^2\Big|_x^{2x}dx=\int_0^1\frac{4}{3}x^2\left(4x^2-x^2\right)dx=\int_0^1 4x^4dx=\frac{4}{5}x^5\Big|_0^1=\frac{4}{5}$$

$$E(Y)=\int_0^1\int_x^{2x}\frac{8}{3}xy^2dy\,dx=\int_0^1\frac{8}{9}xy^3\Big|_x^{2x}dy\,dx=\int_0^1\frac{8}{9}x\left(8x^3-x^3\right)dx=\int_0^1\frac{56}{9}x^4dx=\frac{56}{45}x^5\Big|_0^1=\frac{56}{45}$$

$$E(XY)=\int_0^1\int_x^{2x}\frac{8}{3}x^2y^2dy\,dx=\int_0^1\frac{8}{9}x^2y^3\Big|_x^{2x}dx=\int_0^1\frac{8}{9}x^2\left(8x^3-x^3\right)dx=\int_0^1\frac{56}{9}x^5dx=\frac{56}{54}=\frac{28}{27}$$

$$\text{Cov}(X,Y)=E(XY)-E(X)E(Y)=\frac{28}{27}-\left(\frac{56}{45}\right)\left(\frac{4}{5}\right)=0.04$$

---

106. Solution: C

The joint pdf of X and Y is $f(x,y) = f_2(y|x)\ f_1(x)$
$= (1/x)(1/12),\ 0 < y < x,\ 0 < x < 12$.
Therefore,

E[X] = $\int_0^{12}\int_0^{x} x\cdot\frac{1}{12x}dydx=\int_0^{12}\frac{y}{12}\Big|_0^{x}\ dx=\int_0^{12}\frac{x}{12}dx=\frac{x^2}{24}\Big|_0^{12}\ = 6$

E[Y] = $\int_0^{12}\int_0^{x}\frac{y}{12x}dydx=\int_0^{12}\left[\frac{y^2}{24x}\right]_0^{x}\ dx=\int_0^{12}\frac{x}{24}dx=\frac{x^2}{48}\Big|_0^{12}\ =\frac{144}{48}\ = 3$

E[XY] = $\int_0^{12}\int_0^{x}\frac{y}{12}dydx=\int_0^{12}\left[\frac{y^2}{24}\right]_0^{x}\ dx=\int_0^{12}\frac{x^2}{24}dx=\frac{x^3}{72}\Big|_0^{12}\ =\frac{(12)^3}{72}\ = 24$

Cov(X,Y) = E[XY] – E[X]E[Y] = 24 – (3)(6) = 24 – 18 = 6 .

107. Solution: A

$$\begin{aligned}\text{Cov}(C_1, C_2) &= \text{Cov}(X+Y,\ \ X+1.2Y)\\ &= \text{Cov}(X,X)+\text{Cov}(Y,X)+\text{Cov}(X,1.2Y)+\text{Cov}(\text{Y},1.2\text{Y})\\ &= \text{Var}\,X+\text{Cov}(X,Y)+1.2\text{Cov}(X,Y)+1.2\text{Var}Y\\ &= \text{Var}\,X+2.2\,\text{Cov}(X,Y)+1.2\text{Var}Y\end{aligned}$$

$$\text{Var}\,X = E\left(X^2\right)-\left(E(X)\right)^2 = 27.4-5^2 = 2.4$$

$$\text{Var}\,Y = E\left(Y^2\right)-\left(E(Y)\right)^2 = 51.4-7^2 = 2.4$$

$$\text{Var}(X+Y) = \text{Var}\,X+\text{Var}\,Y+2\,\text{Cov}(X,Y)$$

$$\text{Cov}(X,Y) = \frac{1}{2}\left(\text{Var}(X+Y)-\text{Var}\,X-\text{Var}\,Y\right) = \frac{1}{2}(8-2.4-2.4) = 1.6$$

$$\text{Cov}(C_1,C_2) = 2.4+2.2(1.6)+1.2(2.4) = 8.8$$

----

107. Alternate solution:

We are given the following information:

$$C_1 = X+Y$$

$$C_2 = X+1.2Y$$

$$E[X]=5$$

$$E\left[X^2\right]=27.4$$

$$E[Y]=7$$

$$E\left[Y^2\right]=51.4$$

$$\text{Var}[X+Y]=8$$

Now we want to calculate

$$\begin{aligned}\text{Cov}(C_1,C_2) &= \text{Cov}(X+Y, X+1.2Y)\\ &= E\left[(X+Y)(X+1.2Y)\right]-E[X+Y]\bullet E[X+1.2Y]\\ &= E\left[X^2+2.2XY+1.2Y^2\right]-\left(E[X]+E[\text{Y}]\right)\left(E[X]+1.2E[\text{Y}]\right)\\ &= E\left[X^2\right]+2.2E[XY]+1.2E\left[Y^2\right]-(5+7)\left(5+(1.2)7\right)\\ &= 27.4+2.2E[XY]+1.2(51.4)-(12)(13.4)\\ &= 2.2E[XY]-71.72\end{aligned}$$

Therefore, we need to calculate $E[XY]$ first. To this end, observe

$$8 = \mathrm{Var}[X+Y] = E\left[(X+Y)^2\right] - \left(E[X+Y]\right)^2$$
$$= E\left[X^2 + 2XY + Y^2\right] - \left(E[X] + E[Y]\right)^2$$
$$= E\left[X^2\right] + 2E[XY] + E\left[Y^2\right] - (5+7)^2$$
$$= 27.4 + 2E[XY] + 51.4 - 144$$
$$= 2E[XY] - 65.2$$
$$E[XY] = (8 + 65.2)/2 = 36.6$$

Finally, $\mathrm{Cov}(C_{1,}C_2) = 2.2(36.6) - 71.72 = 8.8$

----

108. Solution: A

The joint density of T_1 and T_2 is given by

$$f(t_1, t_2) = e^{-t_1} e^{-t_2} \quad , \quad t_1 > 0 \quad , \quad t_2 > 0$$

Therefore,

$$\Pr[X \le x] = \Pr[2T_1 + T_2 \le x]$$
$$= \int_0^x \int_0^{\frac{1}{2}(x-t_2)} e^{-t_1} e^{-t_2} dt_1 dt_2 = \int_0^x e^{-t_2} \left[-e^{-t_1} \Big|_0^{\frac{1}{2}(x-t_2)} \right] dt_2$$
$$= \int_0^x e^{-t_2} \left[1 - e^{-\frac{1}{2}x + \frac{1}{2}t_2} \right] dt_2 = \int_0^x \left(e^{-t_2} - e^{-\frac{1}{2}x} e^{-\frac{1}{2}t_2} \right) dt_2$$
$$= \left[-e^{-t_2} + 2e^{-\frac{1}{2}x} e^{-\frac{1}{2}t_2} \right] \Big|_0^x = -e^{-x} + 2e^{-\frac{1}{2}x} e^{-\frac{1}{2}x} + 1 - 2e^{-\frac{1}{2}x}$$
$$= 1 - e^{-x} + 2e^{-x} - 2e^{-\frac{1}{2}x} = 1 - 2e^{-\frac{1}{2}x} + e^{-x} \quad , \quad x > 0$$

It follows that the density of X is given by

$$g(x) = \frac{d}{dx}\left[1 - 2e^{-\frac{1}{2}x} + e^{-x} \right] = e^{-\frac{1}{2}x} - e^{-x} \quad , \quad x > 0$$

109. Solution: B

Let

u be annual claims,
v be annual premiums,
$g(u, v)$ be the joint density function of U and V,
$f(x)$ be the density function of X, and
$F(x)$ be the distribution function of X.

Then since U and V are independent,

$$g(u,v) = \left(e^{-u}\right)\left(\frac{1}{2}e^{-v/2}\right) = \frac{1}{2}e^{-u}e^{-v/2} \quad , \quad 0 < u < \infty \quad , \quad 0 < v < \infty$$

and

$$F(x) = \Pr[X \le x] = \Pr\left[\frac{u}{v} \le x\right] = \Pr[U \le Vx]$$

$$= \int_0^\infty \int_0^{vx} g(u,v)\,du\,dv = \int_0^\infty \int_0^{vx} \frac{1}{2}e^{-u}e^{-v/2}\,du\,dv$$

$$= \int_0^\infty -\frac{1}{2}e^{-u}e^{-v/2}\Big|_0^{vx}\,dv = \int_0^\infty \left(-\frac{1}{2}e^{-vx}e^{-v/2} + \frac{1}{2}e^{-v/2}\right)dv$$

$$= \int_0^\infty \left(-\frac{1}{2}e^{-v(x+1/2)} + \frac{1}{2}e^{-v/2}\right)dv$$

$$= \left[\frac{1}{2x+1}e^{-v(x+1/2)} - e^{-v/2}\right]_0^\infty = -\frac{1}{2x+1} + 1$$

Finally, $f(x) = F'(x) = \dfrac{2}{(2x+1)^2}$

---

110. Solution: C

Note that the conditional density function

$$f\left(y \middle| x = \frac{1}{3}\right) = \frac{f(1/3, y)}{f_x(1/3)}, \quad 0 < y < \frac{2}{3},$$

$$f_x\left(\frac{1}{3}\right) = \int_0^{2/3} 24(1/3)\,y\,dy = \int_0^{2/3} 8y\,dy = 4y^2\Big|_0^{2/3} = \frac{16}{9}$$

It follows that $f\left(y \middle| x = \frac{1}{3}\right) = \frac{9}{16}f(1/3, y) = \frac{9}{2}y \quad , \quad 0 < y < \frac{2}{3}$

Consequently, $\Pr\left[Y < X \middle| X = 1/3\right] = \int_0^{1/3} \frac{9}{2}y\,dy = \frac{9}{4}y^2\Big|_0^{1/3} = \frac{1}{4}$

111. Solution: E

$$\Pr\left[1<Y<3 \middle| X=2\right]=\int_1^3 \frac{f(2,y)}{f_x(2)}dy$$

$$f(2,y)=\frac{2}{4(2-1)}y^{-(4-1)/2-1}=\frac{1}{2}y^{-3}$$

$$f_x(2)=\int_1^\infty \frac{1}{2}y^{-3}dy=-\frac{1}{4}y^{-2}\Big|_1^\infty=\frac{1}{4}$$

Finally, $\Pr\left[1<Y<3 \middle| X=2\right]=\dfrac{\int_1^3 \frac{1}{2}y^{-3}dy}{\frac{1}{4}}=-y^{-2}\Big|_1^3=1-\dfrac{1}{9}=\dfrac{8}{9}$

----

112. Solution: D

We are given that the joint pdf of X and Y is f(x,y) = 2(x+y), 0 < y < x < 1 .

Now $f_x(x) = \int_0^x (2x+2y)dy=\left[2xy+y^2\right]_0^x$ $= 2x^2 + x^2 = 3x^2$, 0 < x < 1

so f(y|x) = $\dfrac{f(x,y)}{f_x(x)}=\dfrac{2(x+y)}{3x^2}=\dfrac{2}{3}\left(\dfrac{1}{x}+\dfrac{y}{x^2}\right)$, 0 < y < x

f(y|x = 0.10) = $\dfrac{2}{3}\left[\dfrac{1}{0.1}+\dfrac{y}{0.01}\right]=\dfrac{2}{3}\left[10+100y\right]$, 0 < y < 0.10

P[Y < 0.05|X = 0.10] = $\int_0^{0.05}\dfrac{2}{3}\left[10+100y\right]dy=\left[\dfrac{20}{3}y+\dfrac{100}{3}y^2\right]_0^{0.05}=\dfrac{1}{3}+\dfrac{1}{12}=\dfrac{5}{12}$ = 0.4167 .

----

113. Solution: E

Let

W = event that wife survives at least 10 years
H = event that husband survives at least 10 years
B = benefit paid
P = profit from selling policies

Then $\Pr\left[H\right]=P\left[H\cap W\right]+\Pr\left[H\cap W^c\right]=0.96+0.01=0.97$

and

$$\Pr\left[W \mid H\right]=\frac{\Pr\left[W\cap H\right]}{\Pr\left[H\right]}=\frac{0.96}{0.97}=0.9897$$

$$\Pr\left[W^c \mid H\right]=\frac{\Pr\left[H\cap W^c\right]}{\Pr\left[H\right]}=\frac{0.01}{0.97}=0.0103$$

It follows that

$$E[P] = E[1000 - B] = 1000 - E[B] = 1000 - \{(0)\Pr[W \mid H] + (10{,}000)\Pr[W^c \mid H]\}$$
$$= 1000 - 10{,}000(0.0103) = 1000 - 103 = 897$$

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114. Solution: C

Note that

$$P(Y = 0 \mid X = 1) = \frac{P(X=1, Y=0)}{P(X=1)} = \frac{P(X=1, Y=0)}{P(X=1, Y=0) + P(X=1, Y=1)} = \frac{0.05}{0.05 + 0.125}$$

= 0.286

$P(Y = 1 \mid X=1) = 1 – P(Y = 0 \mid X = 1) = 1 – 0.286 = 0.714$

Therefore, $E(Y \mid X = 1) = (0)\ P(Y = 0 \mid X = 1) + (1)\ P(Y = 1 \mid X = 1) = (1)(0.714) = 0.714$

$E(Y^2 \mid X = 1) = (0)^2\ P(Y = 0 \mid X = 1) + (1)^2\ P(Y = 1 \mid X = 1) = 0.714$

$\text{Var}(Y \mid X = 1) = E(Y^2 \mid X = 1) – [E(Y \mid X = 1)]^2 = 0.714 – (0.714)^2 = 0.20$

----

115. Solution: A

Let $f_1(x)$ denote the marginal density function of X. Then

$$f_1(x) = \int_x^{x+1} 2x\,dy = 2xy\big|_x^{x+1} = 2x(x+1-x) = 2x \quad , \quad 0 < x < 1$$

Consequently,

$$f(y \mid x) = \frac{f(x,y)}{f_1(x)} = \begin{cases} 1 & \text{if: } \quad x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y \mid X] = \int_x^{x+1} y\,dy = \frac{1}{2}y^2\big|_x^{x+1} = \frac{1}{2}(x+1)^2 - \frac{1}{2}x^2 = \frac{1}{2}x^2 + x + \frac{1}{2} - \frac{1}{2}x^2 = x + \frac{1}{2}$$

$$E[Y^2 \mid X] = \int_x^{x+1} y^2\,dy = \frac{1}{3}y^3\big|_x^{x+1} = \frac{1}{3}(x+1)^3 - \frac{1}{3}x^3$$
$$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} - \frac{1}{3}x^3 = x^2 + x + \frac{1}{3}$$

$$\text{Var}[Y \mid X] = E[Y^2 \mid X] - \{E[Y \mid X]\}^2 = x^2 + x + \frac{1}{3} - \left(x + \frac{1}{2}\right)^2$$
$$= x^2 + x + \frac{1}{3} - x^2 - x - \frac{1}{4} = \frac{1}{12}$$

116. Solution: D

Denote the number of tornadoes in counties P and Q by N_P and N_Q, respectively. Then

$E[N_Q|N_P = 0]$

$= [(0)(0.12) + (1)(0.06) + (2)(0.05) + 3(0.02)] / [0.12 + 0.06 + 0.05 + 0.02] = 0.88$

$E[N_Q^2|N_P = 0]$

$= [(0)^2(0.12) + (1)^2(0.06) + (2)^2(0.05) + (3)^2(0.02)] / [0.12 + 0.06 + 0.05 + 0.02]$

$= 1.76$ and $Var[N_Q|N_P = 0] = E[N_Q^2|N_P = 0] – \{E[N_Q|N_P = 0]\}^2 = 1.76 – (0.88)^2$

$= 0.9856$.

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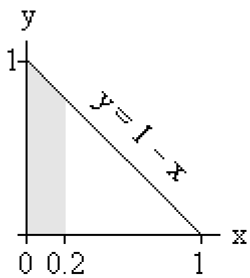
117. Solution: C

The domain of X and Y is pictured below. The shaded region is the portion of the domain over which $X<0.2$.



Now observe

$$\Pr\left[X < 0.2\right] = \int_0^{0.2}\int_0^{1-x} 6\left[1-\left(x+y\right)\right]dydx = 6\int_0^{0.2}\left[y - xy - \frac{1}{2}y^2\right]_0^{1-x} dx$$

$$= 6\int_0^{0.2}\left[1-x-x\left(1-x\right)-\frac{1}{2}\left(1-x\right)^2\right]dx = 6\int_0^{0.2}\left[\left(1-x\right)^2-\frac{1}{2}\left(1-x\right)^2\right]dx$$

$$= 6\int_0^{0.2}\frac{1}{2}\left(1-x\right)^2 dx = -\left(1-x\right)^3 \Big|_0^{0.2} = -\left(0.8\right)^3 + 1 = 0.488$$

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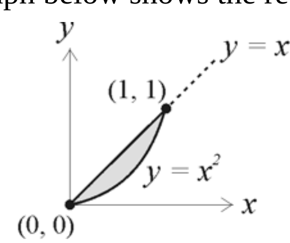
118. Solution: E

The shaded portion of the graph below shows the region over which $f\left(x,y\right)$ is nonzero:



We can infer from the graph that the marginal density function of Y is given by

$$g\left(y\right) = \int_y^{\sqrt{y}} 15y\,dx = 15xy\Big|_y^{\sqrt{y}} = 15y\left(\sqrt{y}-y\right) = 15y^{3/2}\left(1-y^{1/2}\right),\ 0<y<1$$

or more precisely, $g(y) = \begin{cases} 15y^{3/2}(1-y)^{1/2}, & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

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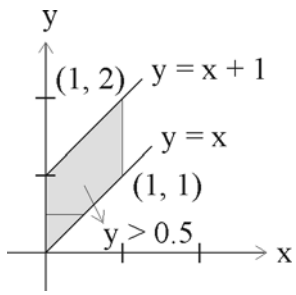
119. Solution: D

The diagram below illustrates the domain of the joint density $f(x, y)$ of X and Y.



We are told that the marginal density function of X is $f_X(x) = 1$, $0 < x < 1$

while $f_{y|x}(y|x) = 1$, $x < y < x+1$

It follows that $f(x, y) = f_X(x)\, f_{y|x}(y|x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1,\ x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$

Therefore,

$$\Pr[Y > 0.5] = 1 - \Pr[Y \le 0.5] = 1 - \int_0^{1/2} \int_x^{1/2} dy dx$$

$$= 1 - \int_0^{1/2} y \Big|_x^{1/2} dx = 1 - \int_0^{1/2} \left(\frac{1}{2} - x\right) dx = 1 - \left[\frac{1}{2}x - \frac{1}{2}x^2\right]\Bigg|_0^{1/2} = 1 - \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

[Note since the density is constant over the shaded parallelogram in the figure the solution is also obtained as the ratio of the area of the portion of the parallelogram above $y = 0.5$ to the entire shaded area.]

120. Solution: A

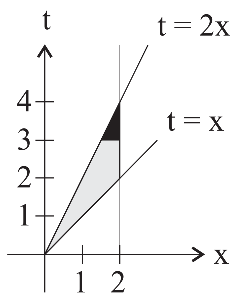
We are given that X denotes loss. In addition, denote the time required to process a claim by T.

Then the joint pdf of X and T is $f(x,t)=\begin{cases}\frac{3}{8}x^2\cdot\frac{1}{x}=\frac{3}{8}x, & x<t<2x, 0\le x\le 2\\ 0, & \text{otherwise.}\end{cases}$

Now we can find P[T ≥ 3] =

$$\int_3^4\int_{t/2}^2 \frac{3}{8}x\,dx\,dt=\int_3^4\left[\frac{3}{16}x^2\right]_{t/2}^2 dt=\int_3^4\left(\frac{12}{16}-\frac{3}{64}t^2\right)dt=\left[\frac{12}{16}t-\frac{1}{64}t^3\right]_3^4=\frac{12}{4}-1-\left(\frac{36}{16}-\frac{27}{64}\right)$$

= 11/64 = 0.17 .



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121. Solution: C

The marginal density of *X* is given by

$$f_x(x)=\int_0^1\frac{1}{64}\left(10-xy^2\right)dy=\frac{1}{64}\left(10y-\frac{xy^3}{3}\right)\Big|_0^1=\frac{1}{64}\left(10-\frac{x}{3}\right)$$

Then $E(X)=\int_2^{10}x\,f_x(x)dx=\int_2^{10}\ \frac{1}{64}\left(10x-\frac{x^2}{3}\right)dx\ =\ \frac{1}{64}\left(5x^2-\frac{x^3}{9}\right)\Big|_2^{10}$

$$=\ \frac{1}{64}\left[\left(500-\frac{1000}{9}\right)-\left(20-\frac{8}{9}\right)\right]=5.778$$

122. Solution: D

The marginal distribution of Y is given by $f_2(y) = \int_0^y 6\, e^{-x}\, e^{-2y}\, dx = 6\, e^{-2y} \int_0^y e^{-x}\, dx$

$= -6\, e^{-2y}\, e^{-y} + 6e^{-2y} = 6\, e^{-2y} - 6\, e^{-3y}$, $0 < y < \infty$

Therefore, $E(Y) = \int_0^\infty y\ f_2(y)\, dy = \int_0^\infty (6ye^{-2y} - 6ye^{-3y})\, dy = 6 \int_0^\infty ye^{-2y}\ dy - 6 \int_0^\infty y\ e^{-3y}\, dy =$

$\frac{6}{2}\int_0^\infty 2\ ye^{-2y}\, dy - \frac{6}{3}\int_0^\infty 3y\, e^{-3y}\, dy$

But $\int_0^\infty 2\, y\, e^{-2y}\, dy$ and $\int_0^\infty 3y\, e^{-3y}\, dy$ are equivalent to the means of exponential random variables with parameters 1/2 and 1/3, respectively. In other words, $\int_0^\infty 2\, y\, e^{-2y}\, dy = 1/2$ and $\int_0^\infty 3y\, e^{-3y}\, dy = 1/3$. We conclude that E(Y) = (6/2) (1/2) – (6/3) (1/3) = 3/2 – 2/3 = 9/6 – 4/6 = 5/6 = 0.83 .

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123. Solution: C

Observe

$$\begin{aligned}\Pr[4<S<8] &= \Pr[4<S<8 \mid N=1]\Pr[N=1] + \Pr[4<S<8 \mid N>1]\Pr[N>1] \\ &= \frac{1}{3}\left(e^{-4/5} - e^{-8/5}\right) + \frac{1}{6}\left(e^{-1/2} - e^{-1}\right) \ * \\ &= 0.122\end{aligned}$$

*Uses that if X has an exponential distribution with mean μ

$$\Pr(a \le X \le b) = \Pr(X \ge a) - \Pr(X \ge b) = \int_a^\infty \frac{1}{\mu} e^{-t/\mu} dt - \int_b^\infty \frac{1}{\mu} e^{-t/\mu} dt = e^{-\frac{a}{\mu}} - e^{-\frac{b}{\mu}}$$