Section 10.2 Operations on Power Series

2. Using geometric series expansion we obtain $\frac{x}{1+x} = \frac{x}{1-(-x)} = x\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^{n+1}$.

Given that $a_n = -(-1)^n$, the radius of convergence is $R = 1$.

4. Using geometric series expansion we obtain $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.

Given that $a_n = (-1)^n$, the radius of convergence is $R = 1$.

6. Using geometric series expansion we obtain $\frac{1}{9-x^2} = \left(\frac{1}{9}\right)\cdot\frac{1}{1-(x/3)^2} = \left(\frac{1}{9}\right)\cdot\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{2n}$. Given that $\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{2n} = \sum_{n=0}^{\infty}\left(\frac{u}{9}\right)^n$ for $u = x^2$, we have $a_n = \left(\frac{1}{9}\right)^n$ so the series converges for $|u| < 9$, or $|x| < 3$. Hence, $R = 3$.

8. Using geometric series expansion we obtain $\frac{x}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}$. Given that $a_n = (-1)^n$, the radius of convergence is $R = 1$ (See Exercise 4).

10. Using geometric series expansion we obtain $\frac{1}{16-x^4} = \left(\frac{1}{16}\right)\cdot\frac{1}{1-(x/2)^4} = \left(\frac{1}{16}\right)\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{4n}$. We have $\frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{4n} = \frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{u}{16}\right)^n$ for $u = x^4$. So $a_n = \left(\frac{1}{16}\right)^n$, and the series converges for $|u| < 16$, hence $|x| < 2$. So $R = 2$.

11. Using the series expansion for the logarithmic function we obtain

$$\ln(1+4x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(4x)^m}{m} = \sum_{m=1}^{\infty} (-1)^{m-1} 4^m \frac{x^m}{m}.$$

Given that $|a_m| = 4^m \frac{1}{m}$, we find $\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty} \left(\frac{4m}{m+1}\right) = 4$. The radius of convergence is thus $R = \frac{1}{4}$.

12. Using the series expansion for the logarithmic function we obtain

$$\ln(5+2x) = \ln\left(5\left(1+\frac{2x}{5}\right)\right) = \ln(5) + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(2x/5)^m}{m} = \ln(5) + \sum_{m=1}^{\infty} (-1)^{m-1} \left(\frac{2}{5}\right)^m \frac{x^m}{m}.$$

We use the Ratio Test. Given that $|a_m| = \left(\frac{2}{5}\right)^m \cdot \frac{1}{m}$, we find

$$\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty} \left(\frac{(2/5)m}{m+1}\right) = \frac{2}{5}.$$

The radius of convergence is thus $R = \frac{5}{2}$.

17. Using the series expansion for the trigonometric function we obtain

$$\arctan\left(x^2\right)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(x^2\right)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}.$$

Given that $|a_m|=\frac{1}{2m+1}$, we find $\lim_{m\to\infty}|a_m|^{1/m}=\lim_{m\to\infty}\left(\frac{1}{2m+1}\right)^{1/m}=1$. The radius of convergence is $R=1$.

18. Using the series expansion for the trigonometric function we obtain

$$\int_0^x \arctan(t)dt=\int_0^x\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}t^{2n+1}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^x t^{2n+1}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)(2n+1)}x^{2n+2}.$$

Given that $|a_m|=\frac{1}{(2m+1)(2m+2)}$, we find

$$\lim_{m\to\infty}|a_m|^{1/m}=\lim_{m\to\infty}\left(\frac{1}{(2m+1)(2m+2)}\right)^{1/m}=1.$$

The radius of convergence is $R=1$.

27. To apply geometric series expansion we first express the given function as a derivative: $\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)$. Next, using the geometric series expansion we find

$$\frac{1}{(1-x)^2}=\frac{d}{dx}\sum_{n=0}^{\infty}x^n=\sum_{n=1}^{\infty}nx^{n-1}=\sum_{m=0}^{\infty}(m+1)x^m.$$

28. Using the result of Exercise 10.2.27 and replacing *x* with 2*x* in this result we find

$$\frac{1}{(1-2x)^2}=\sum_{m=0}^{\infty}(m+1)(2x)^m=\sum_{m=0}^{\infty}(m+1)2^m\cdot x^m.$$