Section 10.5 Taylor Series

5. We substitute $u = x^2$ in the Maclaurin series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ of $\cos(x)$ to obtain

$$\cos\left(x^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}.$$

Thus, $x^3 + \cos\left(x^2\right) = 1 + x^3 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$.

6. We multiply the Maclaurin series $\sum_{n=0}^{\infty} x^n$ of $\frac{1}{1-x}$ by x^2 to obtain $\frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2} = \sum_{m=2}^{\infty} x^m$ for $-1 < x < 1$.

7. We observe that $\frac{1}{1+x^2} = \frac{1}{1-\left(-x^2\right)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$. We then multiply this Maclaurin series by x to obtain $\frac{x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$ for $-1 < x < 1$.

10. We divide each term of the Maclaurin series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1}$ of $\sin(x)$ to obtain

$$\frac{\sin(x)}{x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2(n-1)} \overset{m=n-1}{=} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}.$$