Section 8.7 Improper Integrals—Unbounded Intervals

2. The improper integral diverges. We reach this conclusion as follows:

$$\int_9^\infty \left(\frac{1}{\sqrt{x}}\right) dx = \lim_{N\to\infty} \int_9^N \left(\frac{1}{\sqrt{x}}\right) dx = \lim_{N\to\infty} \left(\left(\frac{\sqrt{x}}{2}\right)\Bigg|_9^N \right) = \lim_{N\to\infty} \left(\frac{\sqrt{N}}{2} - \frac{3}{2} \right) = \infty .$$

3. The improper integral converges. We evaluate as follows:

$$\int_{-1}^\infty \left(\frac{1}{(3+x)^{3/2}}\right) dx = \lim_{N\to\infty} \int_{-1}^N \left(\frac{1}{(3+x)^{3/2}}\right) dx = \lim_{N\to\infty} \left(-2(3+x)^{-1/2} \Big|_{-1}^N \right)$$

$$= \lim_{N\to\infty} \left(-2(3+N)^{-1/2} + \frac{2}{\sqrt{2}} \right) = \sqrt{2}.$$

7. The improper integral converges. We evaluate as follows:

$$\int_0^\infty \left(\frac{x}{\left(1+x^2\right)^2}\right) dx = \lim_{N\to\infty} \int_0^N \left(\frac{x}{\left(1+x^2\right)^2}\right) dx = \lim_{N\to\infty} \left(\frac{-1}{2\left(1+x^2\right)} \Bigg|_0^N \right)$$

$$= \lim_{N\to\infty} \left(\frac{-1}{2\left(1+N^2\right)} + \frac{1}{2} \right) = \frac{1}{2}.$$

8. The improper integral converges. We evaluate as follows:

$$\int_2^\infty e^{-3x} dx = \lim_{N\to\infty} \int_2^N e^{-3x} dx = \lim_{N\to\infty} \left(\frac{-e^{-3x}}{3} \Bigg|_2^N \right) = \lim_{N\to\infty} \left(\frac{-e^{-3N}}{3} + \frac{e^{-6}}{3} \right) = \frac{e^{-6}}{3} .$$

13. The improper integral diverges. We reach this conclusion as follows:

$$\int_e^\infty \left(\frac{1}{x\ln(x)}\right) dx = \lim_{N\to\infty} \int_e^N \left(\frac{1}{x\ln(x)}\right) dx = \lim_{N\to\infty} \left(\ln\left|\ln(x)\right| \Big|_e^N \right) = \lim_{N\to\infty} \left(\ln(\ln(N)) \right) = \infty .$$

20. We evaluate as follows:

$$\int_{-\infty}^{0} \sin(x)\,dx = \lim_{M\to-\infty} \int_{M}^{0} \sin(x)\,dx = \lim_{M\to-\infty} \left(-\cos(x)\Big|_{M}^{0}\right) = \lim_{M\to-\infty} \left(-1+\cos(M)\right).$$

The last limit does not exist; hence, the improper integral diverges.

21. The improper integral converges. We evaluate as follows:

$$\int_{-\infty}^{4} e^{x/3}\,dx = \lim_{M\to-\infty} \int_{M}^{4} e^{x/3}\,dx = \lim_{M\to-\infty} \left(3e^{x/3}\Big|_{M}^{4}\right) = \lim_{M\to-\infty} \left(3e^{4/3} - 3e^{M/3}\right) = 3e^{4/3}.$$

27. The improper integral converges. We evaluate as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \left(\frac{1}{1+x^2}\right)dx &= \lim_{N\to\infty} \lim_{M\to-\infty} \int_{M}^{N} \left(\frac{1}{1+x^2}\right)dx = \lim_{N\to\infty} \lim_{M\to-\infty} \left(\arctan(x)\Big|_{M}^{N}\right) \\ &= \lim_{N\to\infty} \lim_{M\to-\infty} \left(\arctan(N) - \arctan(M)\right) \\ &= \lim_{N\to\infty} \arctan(N) - \lim_{M\to-\infty} \arctan(M) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.\end{aligned}$$

28. We evaluate as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \left(\frac{x}{1+x^2}\right)dx &= \lim_{N\to\infty} \lim_{M\to-\infty} \int_{M}^{N} \left(\frac{x}{1+x^2}\right)dx = \lim_{N\to\infty} \lim_{M\to-\infty} \left(\frac{\ln\left(1+x^2\right)}{2}\Bigg|_{M}^{N}\right) \\ &= \lim_{N\to\infty} \lim_{M\to-\infty} \left(\frac{\ln\left(1+N^2\right)}{2} - \frac{\ln\left(1+M^2\right)}{2}\right) = \lim_{N\to\infty} \lim_{M\to-\infty} \ln\left(\frac{1+N^2}{1+M^2}\right).\end{aligned}$$

The last limit does not exist; hence, the integral diverges.

44. We have for $p \neq 1$

$$\int_e^{\infty}\left(\frac{1}{x\ln^p(x)}\right)dx = \lim_{N\to\infty}\int_e^{N}\left(\frac{1}{x\ln^p(x)}\right)dx = \lim_{N\to\infty}\left(\left.\frac{\ln^{1-p}(x)}{1-p}\right|_e^N\right) = \lim_{N\to\infty}\left(\frac{\ln^{1-p}(N)}{1-p} - \frac{1}{1-p}\right).$$

For $p<1$ this limit is ∞ and the integral diverges. For $p>1$ this limit is $\frac{1}{p-1}$ and the integral converges. For $p=1$ we have

$$\begin{aligned}\int_e^{\infty}\left(\frac{1}{x\ln(x)}\right)dx &= \lim_{N\to\infty}\int_e^{N}\left(\frac{1}{x\ln(x)}\right)dx = \lim_{N\to\infty}\left(\ln(\ln(x))\Big|_e^N\right)\\ &= \lim_{N\to\infty}\left(\ln(\ln(N)) - \ln(\ln(e))\right) = \infty\end{aligned}$$

and the integral diverges. In summary, the given improper integral converges precisely when $p>1$.