Notes on Growth Theory, Ec750

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Abstract

A suite of models with an emphasis on core models and growth theory. This handbook is designed with the structure of Ec750 in mind. Distribution is permitted as long as this page accompanies all copies.

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0 Introduction

These notes serve to organize and distill the main topics, results, and lessons from Macro I. Section headings correspond with the major topics of the course. Division into section and subsection reflects my subjective view of topics' importance. This first section summarizes the stylized facts and charts the course of the notes.

I’m going to start by writing down the simplest economic model in the world:

\[ Y_t = A_t L_t \]  

where \( Y \) is gross domestic product, \( A \) is an augmenting factor loosely called “labor productivity” or sometimes “technology,” and \( L \) is labor input. Notice that variables are stated in term of aggregates and that time is an important element. Macroeconomists care about the determinants of \( Y \). We care about other things, like prices, interest rates, capital, labor, or whatever, only to the extent that they can tell us useful things about \( Y \). Macroeconomics can be subdivided into two broad areas of study: growth (\( A \)) and business cycles (\( L \)).

This is a handbook on economic growth: its causes and the models by which economists organize their thinking on the subject. Along the way the book will introduce you to and familiarize you with the workhorse models of macroeconomics. These are as fundamental to macroeconomics as the consumer choice and firm production models in microeconomics and they serve the same function. The first four chapters provide compact introductions to the Solow, Ramsey and Diamond models. These models serve as meta-frameworks, as architectures and archetypes, onto which we drape more interesting model features.

The Solow, Ramsey and Diamond models share one critical feature: they are “exogenous growth models” in that long-run growth in income per capita comes from features which lay outside the model. Growth here is spurred by technological change; however, they posit a simple relationship by which \( A \) evolves over time and it almost will seem as though technology falls like manna from heaven in these models. It does. Despite this, Solow, Ramsey and Diamond will have much useful to say about macroeconomics by ruling out where growth does not come from.

The second half of the book, chapters five through eight, investigate several models (or classes of models) which attempt to explain economic growth endogenously. We will walk thorough these models roughly in order of their appearance in the literature. First we study quasi-endogenous growth models, which explore growth as an externality, a byproduct of productive economic activity. Then we will turn to research-and-development theories of growth in which \( A \) changes over time through innovation in new products. Finally we will study a model of creative-destruction, reminiscent of Schumpeter, in which firms compete with each other to produce goods that are better than their opponents’. In all three of these archetypes, \( A \) changes over time as the result of the behavior of agents inside the model. Hence they are referred to as endogenous growth theories. Some are more pursuasive than others. The final chapter of the book provides an introduction to dynamic programming as a way to link growth with business cycles.
The first statement of “growth facts” is due to Kaldor. His stylized facts were

1. The shares of income going to capital and labor are roughly constant over time
2. The rate of growth of the capital stock is roughly constant over time
3. The rate of growth of output per worker is roughly constant over time
4. The capital/output ratio is roughly constant over time
5. The rate of return on investment is roughly constant over time
6. The real wage grows over time

First-generation growth models, like Solow’s and Ramsey’s, attempted to account for these facts in a standard neoclassical framework. They were not unsuccessful; however these are just the beginning of growth theory. Additional insights on growth served to undermine, not strengthen, the core neoclassical model. Easterly and Levine (2001) provides an updated statement of the stylized facts of growth, as they are understood after five decades of macroeconomic research. They document:

1. TFP, rather than factor accumulation, accounts for most of the cross-country differences in income and growth;
2. Income diverges over the long run in the entire sample of countries;
3. Factor accumulation is persistent while growth is not;
4. Economic activity is concentrated: all factors flow towards the richest areas;
5. National policies (institutions) are closely associated with long-run growth rates

It is the task of any good growth model to account for these facts. We will have the time to explore many models, which will emphasize one portion of this list or the other. None of the models studied in the first half of the book will conform to all of them. However, first we must familiarize ourselves with the workhorse models of the field. Just as one cannot study game theory, labor, social choice, or public finance without a good grounding in consumer and producer theory, so one cannot investigate the deep causes of growth without first being familiar with the benchmark models.

To summarize: macroeconomists care about output, $Y_t$, and its growth rate, $\gamma_Y$. This is a book about the latter. The former, equally important, is the subject of the companion volume on business cycles.

Section 1 begins our journey with the influential Solow model. The model focuses on the firm’s side of the economy, providing a model of capital accumulation while leaving the consumer side of the economy relatively untouched. The steady-state is derived and analyzed; an application is given to log-linearization. The key paper is of course Solow (1956). The section ends by explaining some
popular extensions of the model and confronting the model’s predictions with the cross-country postwar data.

Section 2 transitions into an econometric evaluation of the fundamental causes of growth. It moves away from factor accumulation and considers the role of institutions, geography, and culture in growth. The institutions literature is represented by Acemoglu, Johnson, and Robinson (2001) and responses by Albouy (2008) and Glaeser et al. (2004); the geographic literature by Sachs (2003); and the culture literature by Michalopoulos and Papaioannou (2011).

Section 3 leaves behind the empirical world and returns us to theory. The Ramsey model is essentially a Solow model which takes consumer behavior seriously. The supply-side of the economy is identical, with a representative firm which maximizes profits. The consumer side, however, consists of an infinitely-lived, intertemporally-optimizing dynastic household. The implications of such optimization on long-run growth and income are explored.

Section 4 introduces the overlapping generations model. Two cases are considered: the classical endowment-economy OLG model and the Diamond OLG model with production. In contrast with Ramsey, the OLG model consists of a world in which people are born and die. This seemingly minor modification will be shown to have interesting effects on the model’s equilibrium and dynamics.

Section 5 transitions between the exogenous-growth models seen in Solow, Ramsey and Diamond and the new endogenous-growth models by exploring the so-called “AK” framework. These models generate long-run growth without exogeneous technological change or endogenous R&D by firms: instead growth is caused by externalities in production. While maligned as something of a halfway-house, they will get our feet wet in examining the determinants of technological change.

Section 6 is the first of two sections dedicated explicitly to endogenous growth. This section focuses on different models of the technological process, with particular attention given to models in which the type and number of products expands endogenously. I survey models from the (relatively) recent literature, focusing on classic articles.

Section 7 is devoted entirely to the Schumpeterian growth framework, in which competition to beat out one’s competitors spurs technological improvement.

Finally, section 8 introduces stochastic economics and provides examples of the benchmark “investment” and “consumption” functions used in short-run macroeconomics.
1 The Solow Model

Solow’s basic contribution was to work out the growth path of an economy under a constant-returns-to-scale production function. The findings are all well-known at this point: the economy converges to a steady-state growth path regardless of the initial savings rate. Including technological progress allowed for indefinite growth; that is, all long-run growth was explained by the progress of an exogenously given technological process. Such a result, which effective in showing that the ultimate cause of growth was not factor accumulation, was nevertheless unsatisfactory.

1.1 Setup

The basic Solow model in its modern form consists of three aggregate equations and three laws of motion:

<table>
<thead>
<tr>
<th>Aggregate Equations</th>
<th>Dynamic Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = F(K, AL) )</td>
<td>( \dot{L} = nL )</td>
</tr>
<tr>
<td>( Y = C + I )</td>
<td>( \dot{A} = gA )</td>
</tr>
<tr>
<td>( I = sY )</td>
<td>( \dot{K} = sY - \delta K )</td>
</tr>
</tbody>
</table>

Start with the left-hand side. The first equation states that output is created by mixing aggregate capital \( K \) with augmented labor \( AL \). The second states the equilibrium condition of the output market: aggregate supply \( Y \) must equal aggregate demand \( C + I \). Finally, consumer behavior is pinned down in the third equation: \( C = (1 - s)Y \) and \( I = sY \). Intuitively, this states that our consumers save a constant fraction \( s \) of their income each period.

The critical assumptions on \( F \) are:

- CRS in all arguments: \( F(xK, xAL) = xF(K, AL) \),
- DMR in each argument: \( F_K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0 \), and
- the Inada conditions: \( F_K(0) = \infty, F_L(0) = \infty, F_K(\infty) = 0, F_L(\infty) = 0 \)

Together these are known as the neoclassical assumptions and \( F(\cdot, \cdot) \) is referred to as a neoclassical production function. The first conditions is typically considered innocuous: one can scale operations arbitrarily (if one factory produces \( Y \) units, two factories can produce \( 2Y \)). One popular relaxation of the CRS assumption introduces increasing returns to scale, which can generate endogenous growth. The second set of conditions ensure concavity of the production function in each argument. Output may increase in each argument, but piling on more and more of one factor, while keeping the other factor fixed, has diminishing value in production. Finally, the Inada conditions are technical assumptions about the shape of the first derivatives of \( F \); they will eventually ensure an equilibrium. This completes the setup. We can now derive the steady state.
1.2 Steady-State and Dynamics

It is useful now to reformulate the model in **intensive form**. The intensive form of a model is simply the model, divided through by (effective) labor. Perhaps more accurately, writing a model in intensive form is an attempt to recast all variables into a form that is stationary in the steady-state. Usually this involves knowing the model well enough to know what variables grow at the same rates in the steady-state, so that one knows the appropriate variables to divide by. In general, if any variable is growing exogenously, it is a candidate for dividing through. Since here $A$ and $L$ grow exogenously, we will divide through by them.

Letting $k = K/AL$ and $y = Y/AL$, the entire model may be recast in terms of $\{y, k\}$. The dynamics of $k$ are:

$$\dot{k} = sf(k) - (n + g + \delta)k \quad (1.1)$$

This is the key equation of the Solow model so it’s worth the time needed to write out a formal proof. How do we get from the table of aggregate dynamics to this per-effective-worker dynamic?

**Proof.** In gory detail. Start with $Y = F(K, AL)$; constant returns to scale implies that $Y/AL = F(k, 1) \equiv f(k)$.

The dynamics of $K/AL$ are:

$$K/AL = \frac{1}{(AL)^2} \left( AL\dot{K} - K\dot{AL} \right)$$

$$= \frac{1}{(AL)^2} \left( AL\dot{K} - KA\dot{L} - KL\dot{A} \right)$$

$$= \frac{1}{(AL)^2} \left( AL(sY - \delta K) - KAnL - KLgA \right)$$

$$= \frac{1}{AL} \left( sY - (n + \delta + g)K \right)$$

$$= sy - (n + \delta + g)k$$

$$= sf(k) - (n + \delta + g)k$$

which is the desired expression.

Intuition: capital per effective worker grows with saving and decreases with depreciation and whenever it is “thinned out” across a larger base of effective labor. Note that $f(k)$ inherits: diminishing marginal returns and the Inada conditions (prove?).

This is a first-order nonlinear differential equation in $k$. Given this differential equation we are interested in the existence and stability of a steady-state. A steady-state is simply a level of $k$ where the differential equation is at rest, i.e. $sf(k^*) = (n + g + \delta)k^*$; a steady-state is stable if the relevant second-order conditions are satisfied (intuitively, that if the endogenous variable starts “near” the steady-state, it will tend towards, and now away, from the steady-state).
1. *Existence.* $f(k)$ is globally increasing and concave; $(n+\delta+g)k$ is increasing and linear. Their difference is globally concave and has two zeros: the trivial case of $k = 0$ and one interior case $k = k^\ast$. The interior optimum is the one we are interested in.

2. *Stability.* It can be shown that the interior steady-state is stable for any strictly positive initial value of $k$.

Now we can look at steady-state dynamics. By definition in the steady-state $\dot{k} = 0$. Since $y = f(k)$, it is clear that $\dot{y} = 0$ as well. So $\dot{y}/y \equiv \gamma_y = 0$. What about the growth rate of $Y/L$, output per worker?

\[
\frac{Y}{L} = Ay \\
\gamma_{Y/L} = \gamma_A + \gamma_y \\
= g + 0 \\
= g
\]

So output per worker grows at the constant rate $g$ forever. Similar analysis yields that $\gamma_Y = n + g$; that is, total national output increases at the rate of population growth plus productivity growth whenever the economy is in the steady state. The results of the model are: long-run growth comes entirely from the exogenous technological growth process, and is unrelated to the savings rate.

The above analysis applies to any generic neoclassical production function and focused on long-run growth rates. We can add some additional comments about long-run levels of output if we have a Cobb-Douglas formulation. Suppose that $Y = K^\alpha(AL)^{1-\alpha}$. Then there is a unique steady-state in which:

\[
k^\ast = \left(\frac{s}{n+\delta+g}\right)^{1/1-\alpha} \tag{1.2}
\]
\[
y^\ast = \left(\frac{s}{n+\delta+g}\right)^{\alpha/1-\alpha} \tag{1.3}
\]

Both of the steady-state level equations are increasing in $s$; decreasing in $n,g$, and $\delta$; and increasing in $\alpha$. Suppose countries are identical except for their initial level of income and national levels of $s$ and $n$, but that $g$ is constant across countries. Then the model would predict that countries would converge to their own individual steady-states and that $s$ would be positively correlated with $y^\ast$, while $n$ would be negatively correlated with $y^\ast$. This is a quantitative prediction that can be brought to the data, and I review that literature in the next section on extensions to the model.

The Solow model brings out a key distinction between level effects and growth effects. Note that while changes in $g$ will affect both the steady-state level of output (per effective labor) and steady-state growth rate of output per effective labor, changes in $s$ and $n$ will only affect the steady-state level of output and will have no lasting effect on economic growth. For all of its faults, the Solow model retains the strength of providing clarity in our thoughts on level- versus growth-rate effects.
1.3 Convergence

This subsection moves from an examination at the steady-state to an examination of how countries behave as they approach the steady-state from an initial level of \( k \). These are in general known as \textit{out of steady-state} dynamics, and play an important role in business-cycle theory.

Begin by linearizing around the steady-state. This implies \( \dot{k} = 0 \) or \( sf(k^*) = (n + g + \delta)k^* \). From here, we have:

\[
\dot{k}_t - \dot{k}^* = \dot{k}_t
\]

\[
\approx sf'(k^*)dk_t - (n + \delta + g)dk_t
\] (totally differentiate)

\[
= \frac{f'(k^*)}{f(k^*)}sf(k^*)dk_t - (n + \delta + g)dk_t
\] (multiply by 1)

\[
= \frac{f'(k^*)}{f(k^*)}(n + \delta + g)k^*dk_t - (n + \delta + g)dk_t
\] (substitute \( sf = (n + g + d)k \))

\[
= \left( \frac{f'(k^*)k^*}{f(k^*)} - 1 \right)(n + \delta + g)dk_t
\] (rearrange)

\[
= [\varepsilon_k - 1](n + \delta + g)dk_t
\] (simplify)

That is, \( \dot{k} \approx [\varepsilon_k - 1](n + \delta + g)(k_t - k^*) \). This is a linear first-order differential equation and is quickly solvable.

Define \( x_t = k_t - k^* \); note that \( \dot{x} = \dot{k} \). Then \( \dot{x} = -\lambda x_t \) where \( \lambda = (1 - \varepsilon_k)(n + \delta + g) > 0 \). The solution to this differential equation is \( x_t = Be^{-\lambda t} \), where \( B \) is a constant. To pin down the value of the constant, we need an initial condition. Consider \( t = 0 \), then \( x_0 = B - \lambda x_0 \) or \( B = x_0 \). Perfect.

Our final expression is:

\[
x_t = x_0e^{-\lambda t}
\]

\[
\lambda = (1 - \varepsilon_k)(n + \delta + g)
\]

\[
k_t - k^* = (k_0 - k^*)e^{-\lambda t}
\]

This pins down the transitional dynamics of \( k \). What about \( y \)? It’s a hop, skip, and a jump to

\[
y_t - y^* = (y_0 - y^*)e^{-(1-\varepsilon_k)(n+\delta+g)t}
\] (1.4)

Hence the gross growth rate increases as one’s initial condition is farther from the steady-state. Intuition: the decay rate is \( \lambda \), or one moves to the steady state at a rate of \( \lambda \) per year. Application: \( n = .01, g = .02, \delta = .03 \) and \( \varepsilon_k = .33 \) imply that \( \lambda = 4\% \).

The Solow model, calibrated to match average real-world data, suggests that countries ought to close 4% of the gap between their current-value output per capita and their steady-state output per capita each year. The implied convergence speed across countries (for countries with similar values for the underlying parameters, but different current levels of output) is implausibly high: half of the income gap should close each 35 years.
1.4 The Golden Rule savings rate

There’s no utility in this model, so it is difficult to do welfare analysis. One approach is by analyzing the Golden Rule (GR) savings: the savings rate which maximizes consumption over all time periods.

The problem to be solved is

$$\max_k (1 - s)f(k)$$
$$= \max_k f(k) - (n + g + \delta)k$$

or $$f'(k^g) = n + g + \delta$$

The final expression implicitly defines the golden-rule level of capital as a function of the population growth rate, depreciation rate, and growth rate of technology: $$k^g(n, g, \delta)$$. To obtain the golden-rule savings rate $$s^g$$, note that we simply need to choose $$s$$ to ensure that $$sf(k^g) = (n + g + \delta)k^g$$; this implicitly defines $$s$$ as a function of the exogenous variables.

Suppose we take the specific functional form $$f(k) = k^\alpha$$. Then $$k^g$$ becomes

$$\alpha k^{\alpha - 1} = n + g + \delta$$

$$k^g = \left(\frac{\alpha}{n + g + \delta}\right)^{1/(1-\alpha)}$$

and the savings rate which generates the golden-rule level is $$s = \alpha$$.

1.5 Log-linearization

We now log-linearize the steady-state equation near the steady state. Suppose that a unique interior optimum exists; call it $$k^*$$. We wish to analyze the behavior of $$Y/L$$ near $$(k^*, y^*)$$. Here is a step-by-step derivation of how we do that.

For this section only we make a small change in notation. Let $$y = Y/L$$, so that it is in terms of workers and not effective workers, and let $$k = K/AL$$ as before. Begin with:

$$y_t = A_t f(k_t)$$ (1.5)
$$\dot{k}_t = sf(k_t) - (n + g + \delta)k_t$$ (1.6)

Now there is a two-step procedure. First, we take logs of (1.5) and employ a trick of time derivatives that $$d\ln(x_t)/dt = \dot{x}_t/x_t$$. This will yield an equation for the growth rate of $$y$$ in terms of $$k$$. Second, we will log-linearize $$k$$ around the steady-state and plug that log-linearization back into the relationship derived in the first part.
Part 1: Logs and derivative trick:

\[
\ln(y_t) = \ln(A_t) + \ln(f(k_t))
\]

\[
\frac{d\ln(y_t)}{dt} = \frac{d\ln(A_t)}{dt} + \frac{d\ln(f(k_t))}{dt}
\]

\[
= g + \frac{f'(k_t)\dot{k}_t}{f(k_t)} \frac{k_t}{k_t}
\]

(see below for justification)

\[
= g + \varepsilon_k \dot{k}_t/k_t
\]

where \( \varepsilon = (df/dk)(k/f) \). Keep a hold of \( \dot{y}_t/y_t = g + \varepsilon_k(\dot{k}_t/k_t) \). We’ll need it later.

Part 2: We now take the total differential of (1.6) around the steady-state. This will give us the log-linearization of \( \dot{k} \).

\[
d\dot{k}_t = sf'(k_*)dk_* - (n + g + \delta)dk_*
\]

\[
= \left[ sf'(k_*) - (n + g + \delta) \right] dk_*
\]

\[
= \left[ \frac{f'(k_*)}{k_t} \right] \left[ sf(k_t) - (n + g + \delta) \right] dk_*
\]

(see first term)

\[
= \left[ \frac{f'(k_*)k_t}{k_t} \right] \left[ sf(k_t) - (n + g + \delta) \right] dk_*
\]

(see first term)

\[
= [\varepsilon_k(n + g + \delta)k_t] - (n + g + \delta)dk_*
\]

\[
= [\varepsilon_k(n + g + \delta)k_t - (n + g + \delta)] dk_*
\]

(1.7)

which is, finally, a linear form. So hold on to (1.7): \( \dot{k}_t = (\varepsilon_k - 1)(n + g + \delta)dk_* \). But it uses \( k \) and we want \( y \). First we’ll get rid of the clunky differential notation and second we’ll use what we know about the relationship between \( k \) and \( y \) to make substitutions.

Part 3: examine (1.7). Now \( \dot{k}_t = \dot{\hat{k}}_t - \dot{k}_t = \dot{\hat{k}}_t - 0 \). Make the substitution on the left-hand side and divide both sides through by \( k_t \):

\[
\frac{\dot{k}_t}{k_t} = (\varepsilon_k - 1) \frac{n + g + \delta}{k_t} \frac{dk_*}{k_t}
\]

\[
\approx (\varepsilon_k - 1) \frac{n + g + \delta}{k_t} \frac{\ln k_t - \ln k_*}{k_t}
\]

Now combine the previous equation with \( \dot{y}_t/y_t = g + \varepsilon_k(\dot{k}_t/k_t) \) to obtain:

\[
\frac{\dot{y}_t}{y_t} \approx g - \varepsilon_k(1 - \varepsilon_k)(n + g + \delta)(\ln k_t - \ln k_*')
\]

(1.8)

We’re almost home-free. We just need to get some \( y_t \) terms on the right-hand side and eliminate the \( k_t \) terms.

Part 4: back all the way up to \( y_t = Af(k) \). Totally differentiate this, with \( A \) constant (wtf, try
with A growing later):

\[ dy_t = Af'(k_t^*) dk_t \]
\[ dy_t/y_t = Af'(k_t^*) dk_t k_t^*/k_t^{**} \]
\[ = \varepsilon_k k_t^*/k_t^{**} \]
\[ = \varepsilon_k (\ln k_t - \ln k_t^*) \]

Substitute this last line into our friend (1.8) to get the log-linearization around the steady-state:

\[
\frac{\dot{y}_t}{y_t} \approx g - (1 - \varepsilon_k)(n + g + \delta)(\ln y_t - \ln y_t^*) \tag{1.9}
\]

And that’s the end of that. Try to clean up the derivation later.

Aside: some of the trickier transformations.

**Proof.**

\[
\frac{d \ln(k(t))}{dt} = \frac{f(k_t)}{f(k_t)} 
\]
(definition)
\[ = \frac{1}{f(k_t)} \frac{df(k_t)}{dt} \]
(pee apart the above)
\[ = \frac{1}{f(k_t)} \frac{df(k_t)}{dk_t} \frac{dk_t}{dt} \]
(chain rule)
\[ = \frac{f'(k)\dot{k}}{f(k)} \]

multiply the last line by \(k/k\), which is just 1, and we have the desired expression. ∎

### 1.6 Augmented Solow models

The models discussed in the following two subsections provide extensions and generalizations of Solow’s framework.

#### 1.6.1 Mankiw, Romer, Weil

MRW begin by testing the Solow model with a Cobb-Douglas specification.

The equilibrium equation is:

\[
\begin{bmatrix} Y \\ L \end{bmatrix} = A \left( \frac{s}{n + g + \delta} \right)^{\alpha / (1 - \alpha)}
\]
which may be made additive via a log transformation:

\[
\ln \left[ \frac{Y}{L} \right] = \ln A + \frac{\alpha}{1 - \alpha} \ln(s) - \frac{\alpha}{1 - \alpha} \ln(n + g + \delta)
\]

and, assuming that \( \ln A = a + \varepsilon \), we turn this into an econometric model:

\[
\ln \left[ \frac{Y}{L} \right] = a + \frac{\alpha}{1 - \alpha} \ln(s) - \frac{\alpha}{1 - \alpha} \ln(n + g + \delta) + \varepsilon \tag{1.10}
\]

Nothing of the above three equations should be new. They are straightforward algebraic manipulations of (1.3). The only difference is that they are putting some structure on \( A \), namely that it is distributed across countries with mean value \( a \) and dispersion measured by \( \varepsilon \).

This model can be taken to the data. The identifying assumption is that \( \varepsilon \) is uncorrelated with everything on the right-hand side; that is, “good shocks” to output are not correlated with national savings rates or population growth. This is perhaps a dubious claim; but let us take MRW seriously for now. Simply gather data on \( Y/L, s, \) and \( n \) while making some assumptions about \( g + \delta \) (MRW assume they sum to 5%). These three variables are available for a wide cross-section of countries through the Penn World Tables dataset. Specifically MRW use the average level of \( s \) and \( n \) for 1960-1985 for each country, to smooth out some of the noise that might have arisen from using one specific year. Equation (1.10) can be estimated through simple OLS.

The main results are that \( \alpha \) is around 0.60, far higher than the national accounts data suggest, and that variation in the right-hand side variables can explain about one-half of the variation of log GDP per capita. A growth-formulaic version yields similar results with a convergence coefficient around 0.01, much lower than the implied \( \lambda \) of 4%.

MRW’s substantive extension is to consider a new model, \( Y = F(K, H, AL) \) where \( H \) enters as a new form of capital, not embodied in labor. The model equations become

<table>
<thead>
<tr>
<th>Aggregate Equations</th>
<th>Dynamic Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = F(K, H, AL) )</td>
<td>( \dot{L} = nL )</td>
</tr>
<tr>
<td>( Y = C + I_H + I_K )</td>
<td>( \dot{A} = gA )</td>
</tr>
<tr>
<td>( I_K = s_K Y )</td>
<td>( \dot{K} = s_K Y - \delta K )</td>
</tr>
<tr>
<td>( I_H = s_H Y )</td>
<td>( \dot{H} = s_H Y - \delta H )</td>
</tr>
</tbody>
</table>

That is, output is now a function of effective labor and two forms of capital, “physical” and “human”. MRW make the assumption that both types of capital are accumulated in the same fashion: investment into either one is financed through sacrificing current consumption, the same production technology can produce all three of these goods, and both types of capital depreciate at the same rate. These assumptions keep the model small-scale and tractible. Savings rates continue to be exogenously imposed and constant.

The model can be transformed into per-effective-worker terms and solved. The new steady-state
levels of output, capital, and human capital are:

\[ y^* = (k^*)^\alpha (h^*)^\beta \]  

\[ k^* = \left( \frac{s_k^1 - \beta s_h^\beta}{s_n + \delta + g} \right)^{1/1-\alpha-\beta} \]  

\[ h^* = \left( \frac{s_k^\alpha s_h^{1-\alpha}}{s_n + \delta + g} \right)^{1/1-\alpha-\beta} \] 

We still have a balanced growth path, with a stable equilibrium. Output per worker continues to grow at rate \( g \). There are no truly fundamental changes to the model, only differences in detail. It nicely has the benefit of fitting the data somewhat better.

Substitute \( k^* \) and \( h^* \) into \( y^* \) and take logs of the above equilibrium:

\[
\log \left[ \frac{Y}{L} \right] = A + gt + \frac{\alpha}{1-\alpha-\beta} \log(s_k) + \frac{\beta}{1-\alpha-\beta} \log(s_h) - \frac{\alpha + \beta}{1-\alpha-\beta} \log(n + \delta + g) \tag{1.14}
\]

Finally, we must turn this into a regression equation that we can use.

The identifying assumption made by MRW is that levels of technology are uncorrelated with everything on the right-hand side; hence, one writes \( A + gt = a + \varepsilon \) so that

\[
\log \left[ \frac{Y}{L} \right] = a + \frac{\alpha}{1-\alpha-\beta} \log(s_k) + \frac{\beta}{1-\alpha-\beta} \log(s_h) - \frac{\alpha + \beta}{1-\alpha-\beta} \log(n + \delta + g) + \varepsilon_t \tag{1.15}
\]

at which point one can run straight OLS, as before. Results are: the estimate on \( \alpha \) is around one-third, as expected from national accounts data. The estimate on \( \beta \) is between one-fourth and one-third, which MRW argue is reasonable. The equation explains about three-fourths of cross-country differences in income and about one-half of cross-country differences in growth rates.

The identifying assumption is critical and may be an unsupported claim. In particular, it’s not unreasonable to think that countries with good shocks (high \( \varepsilon \)'s) might also save more, leading to upward bias in the estimated regression coefficients.

### 1.6.2 Hall and Jones

\( Y = F(K, AhL) \) where \( h \) is determined through a Mincerian wage regression. Here, human capital enters the production function via labor; we say this is “embodied” human capital. While MRW performed an econometric evaluation of their model, Hall and Jones instead perform a calibration exercise.

<table>
<thead>
<tr>
<th>Aggregate Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = F(K, AH) )</td>
</tr>
<tr>
<td>( H = e^{\phi(E)}L )</td>
</tr>
<tr>
<td>( \phi(E) ) measures the return to ( E ) years of schooling</td>
</tr>
<tr>
<td>( \phi'(E) ) estimated through Mincerian wage calculation</td>
</tr>
</tbody>
</table>
Now let $y = Y/L$, $k = K/L$ and $h = H/L$ be units per worker. Then the above set of equations simplifies to:

$$ y = k^{\alpha/(1-\alpha)} h A $$  \hspace{1cm} (1.16)

Additionally, to compare country $i$ with country $j$ it is only necessary to take the ratio of their incomes per capita:

$$ \frac{y_i}{y_j} = \left( \frac{k_i}{k_j} \right)^{\alpha/(1-\alpha)} \frac{h_i}{h_j} \frac{A_i}{A_j} $$  \hspace{1cm} (1.17)

Now $y$ is observable from the Penn World Tables; $k$ can be constructed using the perpetual-inventory method to derive a consistent $K$ series; and $h$ can be measured with average educational attainment. $A$ can then be calculated as a residual if one uses a value for $\alpha$; Hall and Jones use the standard $1/3$.

The calibration exercise thus uses gathered data on $y$, $k$, and $h$ along with an assumption about $\alpha$ to derive a series for the $A_i/A_j$ ratio; then, they decompose the $y_i/y_j$ ratio using those three multiplicative arguments. The base country is the United States, $j = US$.

The key findings are that (1) factor intensity per worker is significantly correlated with productivity and (2) the variance in factor intensities is not sufficient to explain the differences in cross-country income. The first finding casts doubt on the validity of MRW’s identifying assumption and suggests that growth regressions must be supplemented with instrumental variables techniques to obtain unbiased parameter estimates. The second finding highlights that understanding components to income other than factor accumulation is important; we now dive in to the attempts to capture three such components: institutions, geography, and culture.
2 Fundamental Causes of Growth

Hall and Jones do more than a calibration exercise with the Solow model: they propose a model wherein “social capital” (institutions) affects economic growth. This section explores that latter half of their paper and also addresses the substantial contributions of AJR, Albouy and Glaeser; Sachs; and Michalopoulos, representing institutions, geography, and culture, respectively. Each of these six papers will have a dedicated subsection, preceded by a primer on the instrumental variable regression technique which underlies all of their analyses.

2.1 Preliminary Note: Instrumental Variables Regression

Forget general equilibrium for the moment and focus on the partial effect of institutions on growth. We have in mind a model of the form:

\[
\log \left( \frac{Y}{L} \right)_i = \alpha_y + \beta R_i + \varepsilon_i
\]  

(2.1)

where \( R_i \) is institutions in country \( i \). Now there might be feedback effects of the level of economic development on institutions, so write down the equation

\[
R_i = \alpha_R + \delta \log \left( \frac{Y}{L} \right)_i + X_i \theta + \eta_i
\]  

(2.2)

where \( X \) is a vector of other covariates, \( \theta \) is a vector of coefficients and \( \eta_i \) is an error term which is uncorrelated with the right-hand side. That is, while institutions might have an effect on income it’s also true that institutions are themselves affected by economic development, summarized by income.

Now, there is clearly reason for concern here. Can we use OLS to estimate \( \beta \)? To do that, \( R_i \) must be uncorrelated with \( \varepsilon_i \). Let’s combine the two equations above to see if there might be a problem:

\[
R_i = (1 - \delta \beta)^{-1}(\delta \varepsilon_i + \eta_i + X_i \theta + \alpha_r + \delta \alpha_Y)
\]  

Now re-arrange this equation to obtain:

\[
R_i = (1 - \delta \beta)^{-1}(X_i \theta + \alpha_r + \delta \alpha_Y) + (1 - \delta \beta)^{-1}(\delta \varepsilon_i + \eta_i)
\]  

(2.3)

It is clear that, given \( \delta > 0 \) and \( \delta \beta < 1 \), \( R_i \) and \( \varepsilon_i \) are correlated: higher values of \( \varepsilon_i \) lead to higher values of \( R_i \). Hence, one of the identifying restrictions of OLS is violated; using OLS here with positive \( \text{Cov}(R_i, \varepsilon_i) \) will bias upwards the estimate of \( \beta \). One cannot simply OLS into (2.1).

Finally, we introduce one more wrinkle. Keep the same two equations as before, but now we don’t observe \( R \), we observe \( \tilde{R} \) (a noisy measure of \( R \)). Hence we add a third equation:

\[
\tilde{R}_i = R_i + \mu_i
\]  

(2.4)
where \( \mu \) is a classical white-noise measurement error. Now what? Substitute \( \tilde{R} \) into our first equation:

\[
\log \left( \frac{Y_i}{L_i} \right) = \alpha_y + \beta (\tilde{R}_i - \mu_i) + \varepsilon_i
\]

or,

\[
\log \left( \frac{Y_i}{L_i} \right) = \alpha_y + \beta \tilde{R}_i + (\varepsilon_i - \beta \mu_i)
\]

(2.5)

so that the error term is now \( \varepsilon_i - \beta \mu \). The covariance of \( R_i \) with \( \varepsilon_i \) is positive and that with \( -\beta \mu_i \) negative (called “attenuation bias”), so the overall effect is uncertain but almost surely different from zero.

Here is a formal proof that the OLS estimator is inconsistent in the face of endogeneity. Let \( y = \log(Y/L) \).

Proof.

\[
\hat{\beta} = \frac{\text{cov}(y, R)}{\text{cov}(R, R)}
\]

\[
= \frac{\text{cov}(\alpha_y + \beta R + (\varepsilon - \beta \mu), R)}{\text{cov}(R, R)}
\]

\[
= \beta \frac{\text{cov}(R, R)}{\text{cov}(R, R)} + \frac{\text{cov}(R, \varepsilon)}{\text{cov}(R, R)} - \beta \frac{\text{cov}(\mu, R)}{\text{cov}(R, R)}
\]

\[
= \beta + \frac{\text{cov}(R, \varepsilon)}{\text{cov}(R, R)} - \beta \frac{\text{cov}(\mu, R)}{\text{cov}(R, R)}
\]

\[
E[\hat{\beta}] \neq \beta
\] since the other covariance terms are nonzero

\( \square \)

The solution is to find an instrumental variable, \( Z_i \), which is correlated with \( R_i \) but uncorrelated with \( \varepsilon - \beta \mu_i \). Then one may use IV regression techniques to obtain unbiased estimates for \( \beta \). The critical task, then is finding a suitable instrument.

### 2.2 Hall and Jones, European Language

Hall and Jones (1999) (HJ) find that productivity \( A \) is significantly correlated with factor intensity, which casts serious doubt on MRW’s identifying assumption. So Hall and Jones suggest that there is an underlying factor, “social capital”, which drives factor accumulation (and hence income). Their operational measure of social capital is derived by combining two indices: the International Country Risk Guide and a standard measure of openness.

As indicated above, the measure of institutions itself is likely positively correlated with the error term in a standard OLS regression, and since the risk index is likely only observed with measurement
error, attenuation bias is also a concern. Hence, an instrumental approach is necessary. Hall and Jones instrument for institutions using measures of European influence in the sixteenth through nineteenth centuries. Countries which were heavily influenced by Western European ideas tended to have stronger initial institutions, and it is plausible that European influence is uncorrelated with current levels of income. While some areas (the US, Canada, Australia and New Zealand in particular) are among the wealthiest countries in the world and were heavily influenced by Europe, it is equally true that African nations, also influenced by Europe, are among the poorest of nations today. The exclusion restriction seems plausible.

The exact instrument chosen is the proportion of the population speaking a European language today. This does correlate strongly with measured social infrastructure and is uncorrelated with $\varepsilon$ by the argument above. Hall-Jones’ result is that social infrastructure correlates significantly and positively with income per capita in the instrumental variables framework. They conclude that social infrastructure drives both factor accumulation and levels of income.

### 2.3 AJR, Settler Mortality

Acemoglu, Johnson, and Robinson (2001) (AJR) take another tack. They also try to estimate the effect of current institutions on current levels of income, but use a different instrument: settler mortality rates among colonizers of modern-day America, Asian, and African nations. Their sample is restricted to those “non-Europe” nations; nevertheless they collect a reasonably large sample of over 60 countries. The validity of the instrument remains to be argued.

First, they argue that settler mortality rates influenced early institutions. The logic is as follows: those areas with high settler mortality were unlikely to develop lasting institutions, as settlers would not wish to invest the time and effort required to bring up strong institutions. By contrast, in low-mortality areas early settlers would be more likely to build up strong institutions. More formally, initial settlement conditions influenced the type of settlement: extractive or permanent. The type of settlement determined early local institutions: extractive states had poor property rights and dictatorial policies, while permanent settlements were more democratic and had stronger property rights protection. Finally, these early institutions persist to today, so that: settlement conditions correlate with early institutions, and early institutions correlate with modern institutions, and modern institutions correlate with income levels. Indeed first-stage regressions report that settler mortality during the nineteenth century is (negatively) associated with current levels of expropriation risk; a remarkable finding in itself.

Second, the identification assumption is that settler mortality has an effect on current income only through its effect on institutions. The main cause for concern is that early settler mortality may be correlated with current disease environment, which is a predictor of current income levels. However, AJR argue that the identification holds through the following reasoning. Most of the settlers who died from disease, died from malaria or yellow fever; these diseases had little effect on the indigenous population, which had developed immunity. Hence these diseases are unlikely to be the reason that Africa and parts of Asia are undeveloped today, and the effect of early settler
mortality on growth flows solely through institutions.

Finally, we must consider the specification of institutions. AJR use the average risk of property expropriation, as measured by the Political Research Service. The claim is that this single metric may be used as a proxy for the whole cluster of “good-governance” institutions. With the usual measurement errors in mind, it is likely that expropriation risk does correlate with the actual quality of institutions; given the lack of alternatives there is no reason to criticize their choice.

With these two arguments and one specification in hand, we may proceed to the results. AJR find that current institutional arrangements (instrumented by settler mortality) is a significant driver of current income levels. This result holds robustly across a variety of specifications: regardless of whether controls are introduced, or if the “neo-Europes” are excluded, or if Africa is excluded, or with continent dummy variables. Importantly, the IV estimate of $\beta$ is larger than the corresponding OLS estimate, indicating that attenuation bias is stronger than endogeneity bias for the measure of expropriation risk used by AJR. The results look robust and substantively convincing: institutions matter for current levels of income.

### 2.4 Albouy, Response to AJR

Albouy (2008) levies serious criticism of AJR’s methodology. Given that their causal link is reasonably well-established, the main criticism comes from the empirical evidence AJR use to support their argument. Albouy finds that much of the original settler mortality data was either imputed or incorrectly specified, often in ways that favored AJR’s argument. His two criticisms (data imputation and mismeasurement) and their effects on the empirical analysis (on the first- and second-stage regressions) are explored in turn.

The actual settler mortality data used in AJR comes from Phillip Curtin’s work on the mortality of European soldiers from disease while on campaign in Africa from 1800-1850. This provides raw data for 28 countries out of the 64-country sample, leaving 38 countries to be imputed. Imputations are performed based on countries with “similar disease environments” as countries for which raw data exist. Albouy claims that these imputations are virtually indefensible under AJR’s own argument: since the disease environment can vary dramatically even across neighboring countries, there is virtually no way to non-arbitrarily assign mortality rates to countries with no raw data. Indeed, data from Mali are used to impute seven countries on a seemingly random basis. Further, AJR assign mortality rates to the Latin American countries in their sample based on bishop mortality with adjustments on the basis of Mexican mortality data; the bishop data is extremely thin and suffers from small-sample bias. In short, 22 countries are given data using thin evidence from Mali and Mexico.

Mismeasurement in the form of inconsistent data merging also plague the AJR dataset. While some of the data used are for soldiers in barracks, other data are for soldiers on campaign; no correction is made for the higher (disease-related) mortality of campaigning soldiers. Indeed for three countries in which multiple sources of data exist (Sudan, Madagascar and Egypt) AJR choose not the earliest data (peacetime in barracks) but the data which has the higher mortality rate
(soldiers on campaign). These choices drive a considerable part of the empirical relationship between settler mortality and institutions. Additional measurement error comes from the use of data on mortality for African laborers, whose mortality rates were far higher than the norm due to their harsh working conditions.

Once corrected for imputation and mismeasurement, Albouy’s reconstructed settler mortality data shows no significant link between mortality and institutions, critically weakening the validity of the first-stage regression of the IV procedure. His revised first-stage regressions correct AJR by (1) considering only the countries for which raw data exist; (2) correcting the Latin American and Mali-based data; and (3) adding dummy variables to control for the data source. All three of these reduce the significance of the first-stage coefficient, and in many cases the point estimate of the coefficient falls as well.

Finally, the insignificance of the first-stage coefficient leads to a weak instrument problem. Using consistent standard errors generated from the AR statistic leads to explosive and in some cases infinite confidence intervals for the second-stage regression. Subsampling without- and within-Africa leads to similarly disappointing results: the without-Africa data are driven entirely by the Anglican offshoots (US, Canada, New Zealand) and the within-Africa results show an insignificant effect of institutions on growth. This finding is particularly important, as it is within Africa that all of AJR’s identifying assumptions are satisfied: the Europeans there imported their institutions but not much else, whereas outside Africa settlers tended to import not just institutions but entire cultures.

To summarize: Albouy finds that AJR’s data methodology is often questionable and rests on extremely thin primary source data. Even small and plausible alterations to their dataset produce unstable empirical results, casting serious doubt on the robustness of their results. Whether all of Albouy’s adjustments are correct, they show that AJR’s empirical work is incredibly fragile and sensitive to exact data specification.

2.5 Glaeser, Critique of Institutions

Glaeser et al. (2004) argue that economists’ focus on institutions is misled and misleading. In particular, they highlight the possibility of reverse causation: not that institutions cause higher standards of living, but that high standards of living and high levels of human capital make good institutional environments possible. On an empirical level, Glaeser levels the charge that the IV procedure used in AJR and similar studies fundamentally fails to properly identify the exclusion restriction.

Glaeser et al identify two basic approaches to practical economic development: factor accumulation and institutional development. Glaeser identifies “checks on the executive” as the foremost institutional criterion explored in that strand of the literature; as we have seen, some form of executive checks and balances were indeed used in Hall-Jones (implicitly, through their average of several indicators) and AJR (explicitly, through looking at expropriation risk). Glaeser’s critique of the second formulation is that a certain level of human capital is necessary to realize the benefits of
improved institutions; hence, estimating the partial effect of institutions on income without regards to the level of human capital is misleading.

Glaeser’s first substantive subsection deals with the three most common measurement of institutions in the literature. First, we have the International Country Risk Guide, a subset of which underlies both HJ and AJR. Second is an assortment of governance indicators collected by World Bank researchers. Third is the Polity IV data which attempts to directly measure constraints on the executive branch. Glaeser documents that all three of these indicators vary directly with per-capita income and that they are highly volatile, reflecting recent country experience rather than “deep” parameters of governance. Glaeser hence argues that the commonly-used institutions metrics do not, in fact, perform satisfactorily in measuring the underlying institutions which economists are actually interested in. In addition, he claims (p.282) that if one is to use the Polity IV metrics, their constitutional variables are more stable and hence likely more reflective of underlying institutions, than checks on the executive.

His second subsection reviews the empirical correlations among income, human capital, and institutions. His growth regressions include a lagged income term as well as an initial human capital term; his results show the standard conditional convergence patterns. His findings on institutions are: checks on the executive, broadly defined, have a positive effect on growth; judicial and legislative measures of institutions tend to be insignificant. However, recall his argument: the causality is not running from institutional measures to growth, but vice-versa. Countries are scored high on institutional metrics when they have good growth years, and poorly when they have bad growth years. The more stable constitutional, judicial and legislative indicators, by contrast, show no correlation with growth.

2.6 Sachs, Geography

Contrary to AJR, Sachs (2003) argues that the disease environment is of primary importance in determining current income levels. The claim made in the institutions literature is that the effect of geography on income is through choice of institutions, which depended on geographic climate; and once institutions are controlled for, geography has little direct effect on income. Sachs addresses these claims through his own empirical analysis using malaria transmission.

Sachs’ paper is short and his claim is modest: that there is at least one instance in which geography and climate affect growth in a way apart from their effect on institutions. His data are on malaria risk relative to population size, using 1994 data from the World Health Organization. The data suffers from underreported cases in Africa, leading to nonuniform measurement error, particularly in the downward direction for the most diseased areas.

Sachs then regresses income on both institutions and malaria risk, to determine whether malaria risk affects income outside of its effect on institutions. He instruments both of these variables using (1) the proportion of the population in temperate climate zones, (2) settler mortality data from AJR, and (3) a measure of “malaria ecology” which is a uniquely constructed indicator of malaria risk designed solely from climatological factors. In larger samples, Sachs drops the settler
mortality instrument which fortunately rids his analysis of the concerns raised by Albouy. He finds that malaria risk is strongly and consistently related to levels of income; the null hypothesis that “geography doesn’t matter” (explicitly advanced in AJR) is soundly rejected.

Sachs’ results show that empirical results depend crucially on the choice of proxy variable; the substantive message is that the empirical literature must be backed by strong theoretical support to ensure that omitted variables are included and that appropriate proxies are used. Despite his strong evidence that malarial risk drives down income, he does not advance the stronger hypothesis that geography is a primary driver of growth. His results show that both institutions and geography matter, and that the contribution of either one cannot be simply subsumed in the other.

2.7 Michalopoulos, Culture

Michalopoulos and Papaioannou (2011) exploits the arbitrary country divisions in Africa to explore the relative importance of nationa-level rules (“institutions”) and local-level rules (“culture”). Instead of an IV procedure, Michalopoulos’ paper follows a regression discontinuity / difference-in-differences design. In the 1800s, much of Africa was partitioned by European colonial interests with little regard to the local mix of ethnic groups. This leads to a variety of ethnic groups which are “cut” by a border; this gives rise to an excellent natural experiment when the two countries on either side of the border have different institutional structures.

Consistently throughout his empirical analysis, Michalopoulos finds that institutions are positively correlated with economic growth (proxied by light emissions at the 1km x 1km level) in an unconditional sense; but that after conditioning on ethnic group, the country-level institutional effects vanish and local-level fixed effects dominate.

One way to interpret these data is that for Africa, national-level institutions are simply unimportant as one moves further from the national capitol and that as one moves away from the capitol, the dominant institutional feature is local rules and customs. However, these results directly challenge the prevailing view in the institutions literature, that the main driver of economic development is rules and norms at the national level.

2.8 General critique

I have one general critique of the AJR methodology, which may not be solvable given data limitations. Mankiw, Romer, and Weil (1992) shows the importance of including savings and population growth rates in any growth or level of income regression; hence it is possible that AJR have upwardly biased estimates of their key point coefficients. At minimum, it would be illuminating to re-run their analysis with such variables included.

The Albouy and Sachs papers show that the choice of instrument, choice of control, and construction of data are critical determinants of the empirical results, highlighting the fragility of much of the empirical literature. Albouy levies serious and likely justified criticism of the methods AJR use to construct their settler mortality series; highlighting that in small samples the construction of the data is critical.
Sachs shows that the choice of proxy also leads to widely varying results. In his case, the institutions literature argues that the disease environment is unimportant except for its effect in determining institutions; Sachs shows that by construction of an appropriate variable it is possible to obtain results in which geography and disease climate enter the analysis significantly apart from their effect on institutions. Hence, claims that one has correctly proxied and controlled for a particular confounder must be inspected carefully.

Finally, Glaeser shows that the common metrics used as proxies for underlying, unobservable “institutions” are fraught with measurement error which is correlated with GDP and GDP growth rates. Hence these common measures will show spurious correlation with GDP growth; if this measurement error is not accounted for, such metrics will lead to biased estimates of the effects of institutions on income and growth. His preferred metrics, which tend to change more slowly and are less directly related to the yearly ups and downs of an economy, show little systematic correlation with growth. In some sense this is an artifact of the data, since he deliberately chooses measures of institutions which show little variation (and without variation in a covariate, it is impossible to establish correlation between the covariate and the dependent variable).

Sachs’, Glaeser’s and Albouy’s critiques highlight that even careful empirical work can be fragile under slight data modifications. However, in general, the weight of evidence is that institutions do matter and that one cannot explain levels of income simply with factor accumulation (a la MRW). Clearly research in this area remains unfinished; avenues going forward must focus on finding robust, appropriate instruments and—given the uncertainty of the empirical work—may benefit from more theorizing to guide the choice of empirical design, rather than yet more reduced-form regressions.

2.9 Returning to Klenow and Easterly

What, if anything, have we learned from this exercise? Recall the fundamental equation from MRW’s and HJ’s model:

\[
\log(Y/L) = \log(A) + \log(f(k^*, h^*))
\]

So differences in income per capita come from either differences in \(A\) or differences in factor intensity. We’ve learned from MRW that factor intensity can explain between one-half and two-thirds of variation in income per capita across countries. We learned from HJ that these estimates are probably on the high side, given the endogeneity of MRW’s regressors, so that a significant amount of variation in cross-country income can be attributed to \(A\). This conforms with the stylized facts mentioned in the introduction.

So what goes into \(A\)? Hall-Jones and AJR claim that “institutions” is a key component of \(A\) and set out to prove it with instrumental-variable regression techniques. These regressions were shown to have significant explanatory power, but were undermined by critiques from Sachs, Glaeser, and Albouy. The other main factors that go into \(A\) is what we loosely term “technology” and “efficiency”. We will later explore models which try to capture endogenous changes in technology and efficiency, generating cross-country differences in income driven by different efficiencies of labor.
3 The Ramsey Model

The Ramsey-Cass-Koopmans model is a generalization of the Solow model with optimizing agents: a household which maximizes utility and a firm which maximizes profits. The model was developed in Ramsey (1928), Cass (1965), and Koopmans (1965). This is a fundamental, benchmark model; variants of it are used across the business-cycle and growth literatures. It can be easily modified to account for government, distortionary taxation, open-economy macroeconomics, and more.

The core assumptions of the model are:

1. Perfect competition (price-taking behavior)
2. Exogenous growth in population (at rate $n$) and technology (at rate $g$)
3. Full employment
4. Representative household (either Gorman form utility or total homogeneity in households)
5. A single homogenous good produced at each instant of time
6. No market imperfections

Time is continuous and continues forever. I choose to suppress the time subscript for notational convenience.

3.1 Firms

The representative firms creates output $Y$ by mixing capital $K$ with effective labor $AL$ in a neo-classical production function

$$Y = F(K, AL)$$ (3.1)

The production function satisfies constant returns to scale in both arguments, positive marginal product in each argument, and decreasing marginal returns in each argument.

The firm’s aim is to maximize period-by-period profits:

$$\max \pi \equiv pY - RK - wAL$$

where wages $w$ are paid per effective worker and $R = r + \delta$ is the user cost of capital: the real interest rate plus depreciation. First-order conditions for the firm are

$$F_K - R = 0 \quad \text{and} \quad F_{AL} - w = 0$$

which may be rearranged to yield

$$f'(k) - R = 0 \quad \text{(3.2)}$$
$$f(k) - kf'(k) - w = 0 \quad \text{(3.3)}$$
where \( f(k) = F(K/AL, 1) \) is the production function in intensive form; these two equations implicitly define the firm’s factor demand functions in intensive form.

Proof. For the first proposition:

\[
F(K, AL) = ALf(k)
\]

\[
\frac{\partial F}{\partial K} = \frac{\partial (ALf(k))}{\partial K} = ALf'(k) \frac{1}{AL} = f'(k)
\]

so that \( F_K = f'(k) \), as desired

For the second proposition,

\[
\frac{\partial}{\partial AL} [F(K, AL)] = \frac{\partial}{\partial AL} [ALf(k)]
\]

\[
\frac{\partial}{\partial AL} [ALf(k)] = ALf'(k) \frac{K}{(AL)^2} + f(k) = kf'(k) + f(k)
\]

which is the desired expression.

\[\square\]

3.2 Households

Households are characterized by the following system of equations:

\[
u(C) = \frac{C^{1-\sigma}}{1-\sigma}
\]

\[
U = \int_0^\infty e^{-\rho t} u(C) \frac{L(t)}{H} dt \tag{3.4}
\]

where \( L \) is persons, \( H \) is households, and \( C \) is consumption per person of a homogenous good. Hence \( L/H \) is the size of the average household and \( C \cdot L/H \) is total consumption per household. The exponential term \( e^{-\rho t} \) denotes the continuous-time discount factor at the household level. The first equation is instantaneous felicity function at time \( t \), given by a CRRA utility function. Low values of \( \sigma \) reflect a willingness to accept a rough consumption pattern; high values of \( \sigma \) reflect a stronger preference for smooth consumption patterns across time. The second equation is the household’s lifetime utility, obtained by summing up the felicity function through time. Note that household instantaneous utility depends on per-capita consumption within the household, but lifetime utility for the household depends in part on the size of the household.
The lifetime utility function can be rewritten in intensive form as

\[
U = B \int_0^\infty e^{-\beta t} \frac{c^{1-\sigma}}{1-\sigma} dt
\]

(3.5)

where \( \beta = \rho - n - (1 - \sigma)g \) is the intensive form discount factor, \( B = L(0)A(0)^{1-\sigma}(1/H) \) and \( c = C/A \) is consumption per effective person.

Proof. Deriving (3.5) from the aggregate equations is simple algebra:

\[
U = \int_0^\infty e^{-\rho t} u(C) \frac{L(t)}{H} dt
\]

\[
= \int_0^\infty e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} \frac{L(t)}{H} dt
\]

\[
= \int_0^\infty e^{-\rho t} \frac{(Ac)^{1-\sigma}}{1-\sigma} \frac{L(t)}{H} dt
\]

\[
= \int_0^\infty e^{-\rho t} \frac{(A_0 e^{\sigma t}c)^{1-\sigma}}{1-\sigma} \frac{L(t)}{H} dt
\]

\[
= \frac{A_0^{1-\sigma} L_0}{H} \int_0^\infty e^{-\rho t} e^{nt}(1-\sigma)g t \frac{c^{1-\sigma}}{1-\sigma} dt
\]

\[
= \frac{A_0^{1-\sigma} L_0}{H} \int_0^\infty e^{-(\rho - n - (1 - \sigma)g)t} \frac{c^{1-\sigma}}{1-\sigma} dt
\]

\[
= B \int_0^\infty e^{-\beta t} \frac{c^{1-\sigma}}{1-\sigma} dt
\]

which is the desired expression. \( \Box \)

Finally, the model is closed by a resource constraint:

\[
Y = C + I
\]

\[
Y = C + (\frac{dK}{dt} - \delta K)
\]

\[
y = c + \frac{dk}{dt} + (n + g + \delta)k
\]

(3.6)

which rearranges to

\[
\dot{k} = f(k) - c - (n + g + \delta)k
\]

(3.6')

Equation (3.6) is society’s flow budget constraint stated in terms of effective labor. The consumer’s problem is to maximize (3.5) subject to (3.6). This is an optimization problem in continuous time and we employ the Hamiltonian approach to solve it. Consumption per effective worker \( c \) is our control variable, capital per effective worker \( k \) is the state variable. Note that (3.6') is identical to the savings-investment identity in the Solow model, with “sy” replaced by “\( f(k) - c \)”. 

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3.3 The Planner’s Solution

There are two ways to solve the consumer’s maximization problem. First is to take the approach of a benevolent central planner who maximizes consumer utility subject to the aggregate resource constraint. Second is to use the agent’s own budget constraint. I will solve this model both ways and show that they are identical.

The mathematical technique used to solve the planner’s problem is called a (current-value) Hamiltonian. Think of it as a dynamic version of the Lagrangian.

\[
H_c := [\text{objective function}] - \lambda [\text{flow budget constraint}]
\]

or, for our problem,

\[
H_c := \left[ \frac{c^{1-\sigma}}{1-\sigma} \right] + \lambda [f(k) - c - (n + g + \delta)k]
\]

The first-order conditions are:

\[
\frac{\partial H_c}{\partial c} = c^{-\sigma} - \lambda = 0 \quad (3.7)
\]

\[
\frac{\partial H_c}{\partial k} = \lambda (f'(k) - (n + g + \delta)) + \frac{1}{\sigma} \dot{\lambda} = \beta \lambda - \dot{\lambda} \quad (3.8)
\]

which, along with a transversality condition

\[
\lim_{t \to \infty} k(t)\lambda(t)e^{-\beta t} = 0 \quad (3.9)
\]

gives three equations in three unknowns, \(c, k,\) and \(\lambda\). So solve them:

\[
c^{-\sigma} = \lambda \quad (3.10)
\]

\[
\lambda (f(k) - (n + g + \delta)) = \beta \lambda - \dot{\lambda} \quad (3.11)
\]

\[
-\sigma c^{-\sigma-1} \dot{c} = \dot{\lambda} \quad \text{(differentiate w.r.t. time)} \quad (3.12)
\]

This has given us three equations without having to deal with the TVC. Plug (3.10) and (3.11) into (3.12) to obtain:

\[
\dot{c} \quad \frac{1}{c} (f'(k) - \delta - \rho - \sigma g)
\]

(recall that \(\beta = \rho - n - (1 - \sigma)g\)). Equation (3.13) is the fundamental equation of motion for consumption; it is the consumer’s Euler equation. Equations (3.13) and (3.6) define two differential equations in \(c\) and \(k\).

The steady-state in the model will come from exploring the behavior of these two functions when both are at rest. I will explore this in more detail soon; first, I walk through the decentralized version of the problem and address a few technical notes.
3.4 The Decentralized Solution

We now set up the same problem, but look at it from the point of view of the representative consumer. In particular, this means that the budget constraint is no longer society’s aggregate resource constraint but instead is the representative agent’s flow budget constraint.

Denote by $F_A$ the entire household’s net financial assets, without being too specific about what goes into $F_A$ just yet. (We’ll get there.) Now note that:

$$
\dot{F}_A = F_A \cdot r + wAL - C
$$

that is, the change in financial assets is the return on previous assets $F_A \cdot r$, wage income $wAL$, less consumption expenditure $C$. First we rewrite in per-effective-worker terms:

$$
\dot{a} = a \cdot r + w - c - (n + g)a
$$

(3.14)

where the last term falls out of the algebra and calculus of the time derivative. (3.14) is the representative household’s flow budget constraint expressed in per-effective-worker terms. We let $a$ be $F_A/(AL)$.

Second, we set up the usual no-ponzi-games (NPG) condition. This states that:

$$
\lim_{t \to \infty} a(t)e^{-\int_0^t r(\tau) - n - g)d\tau} \geq 0
$$

(3.15)

in English: the net present value of assets cannot rise faster than the interest rate. This ensures that our household isn’t borrowing indefinitely.

We now set up the present-value Hamiltonian:

$$
H := e^{-\beta t} \left[ \frac{c^{1-\sigma}}{1-\sigma} \right] + \nu [a(r - n - g) + w - c]
$$

Take first-order conditions:

$$
\frac{\partial H}{\partial c} \equiv 0
$$

$$
\frac{\partial H}{\partial a} \equiv -\dot{\nu}
$$

$$
\lim_{t \to \infty} a \cdot \nu = 0 \quad \text{(TVC)}
$$

and again we have three equations in $c$, $a$, and $\nu$.

Calculating the first two partial derivatives, and differentiating the partial derivative of con-
sumption with respect to time, yields operational conditions

\[ e^{-\beta t}c^{-\sigma} = \nu \] (3.16)

\[ \nu(r - n - g) = -\dot{\nu} \] (3.17)

\[ \sigma e^{-\beta t}c^{-\sigma-1}\dot{c} + c^{-\sigma}\beta e^{-\beta t} = -\dot{\nu} \] (3.18)

Combining these three equations yields

\[ \frac{\dot{c}}{c} = \frac{1}{\sigma}(r - \rho - \sigma g) \] (3.19)

... identical to what we had previously given that in equilibrium \( r + \delta = f'(k) \). That is, the decentralized solution [3.19] is identical to the solution that would be obtained by a benevolent central planner [3.13]; the market solution is Pareto optimal. The equality of the solutions comes directly from the perfect competition assumption in factor markets, because the key to linking the two expressions is \( r + \delta = f'(k) \).

### 3.5 Two Technical Notes

(I do not guarantee the minute accuracy of this subsection! See Barro for a better treatment.)

**The Transversality Condition**

Consider (3.17). We integrate it to solve:

\[-\dot{\nu} = (r(t) - n - g)\nu\]

\[\nu(t) = \nu(0)exp\{-\int_0^t (r(\tau) - n - g)d\tau\}\]

Now combine this with (TVC) above to write:

\[\lim_{t \to \infty} \nu(0)exp\{-\int_0^t (r(\tau) - n - g)d\tau\}a(t) = 0\]

Now \( \nu(0) \) is simply a a positive constant and can be eliminated, yielding

\[\lim_{t \to \infty} exp\{-\int_0^t (r(\tau) - n - g)d\tau\}a(t) = 0\]

so that net assets cannot increase faster than the rate of interest; TVC and optimization imply the no-Ponzi condition.
The Consumption Function

It is sometimes useful to be able to formulate consumption in terms of income. I do this now for the Ramsey model.

First, we formally state the intertemporal budget constraint facing the consumer:

$$\int_0^\infty e^{-R(t)}c(t)A(t)L(t)dt \leq a(0)A(0)L(0) + \int_0^\infty w(t)A(t)L(t)dt$$

where $R(t) = \int_0^t r(\tau)d\tau$. English: The entire discounted stream of aggregate household consumption must not exceed starting assets $a(0)A(0)L(0)$ plus discounted wage income.

Recall that $A(t) = A(0)e^{gt}$, $L(t) = L(0)e^{nt}$. Use this and divide through by $A(t)L(t)$ to obtain

$$\int_0^\infty e^{-(\tilde{r}(t) - n - g)t}c(t)dt \leq a(0) + \int_0^\infty e^{-(\tilde{r}(t) - n - g)t}w(t)dt$$

$$c(0) = a(0) + \tilde{w}(0)$$

where $\tilde{r}(t) = R(t)/t$ is the average interest rate up until time $t$. Again, this simply states that per-effective-person discounted consumption cannot exceed the discounted present value of assets plus labor income.

Continue by integrating the Euler condition:

$$\int \dot{c}dt = \int \frac{1}{\sigma}(r - \rho - \sigma g)cdt$$

$$c(t) = c(0)e^{\frac{1}{\sigma}(\tilde{r}(t) - \rho - \sigma g)t}$$

and replace $c(t)$ in the last expression with that above:

$$c(0) = \mu(0) \cdot [a(0) + \tilde{w}(0)]$$

where $\mu(0)$ is given by $1/\mu(0) = \int_0^\infty \exp[(1-\sigma)\tilde{r}(t) - n - \rho/\sigma]dt$. The punchline is that current consumption depends on the entire expected future path of real interest rates, not just on the current interest rate; but this lesson is obscured violently by an impenetrable algebraic soup. For what might be a better explication, see Barro (2003), p.93.

3.6 Steady State

We are now prepared to analyze the steady-state of the Ramsey economy. Recall the two equations of motion:

$$\dot{k} = f(k) - c - (n + g + \delta)k$$

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(f'(k) - \delta - \rho - \sigma g)$$
The first is the equation of motion for capital, identical to what we had in the Solow model previously. Note by way of proof that \( y = c + i \) so that \( i = f(k) - c \); then the \( \dot{k} \) equation is identical to that in Solow. The second is the consumption Euler equation, derived from solving the consumer’s maximization problem. In equilibrium, both of these differential equations is at rest. So we now look at the loci in \( k, c \) space for which they vanish:

\[
\begin{align*}
c &= f(k) - (n + g + \delta)k \\
f'(k) &= \delta + \rho + \sigma g
\end{align*}
\]  

Which produces the qualitative graph:

Figure 1: Equilibrium in the Ramsey model

Equation (3.20) defines a parabola in \( c, k \) space; equation (3.21) implicitly defines a level of capital \( k^* \). These are depicted on the graph above.

The two lines split the coordinate space into four pieces. Dynamics are such that the steady-state point is saddle-path stable. The substantive punchline is: there exists a steady-state in which \( \dot{c} = \dot{k} = 0 \). Therefore:

1. \( k \) is constant in steady-state, hence \( y = f(k) \) is constant in steady-state: \( \gamma_y = 0 \);
2. \( Y/L = Af(k) \) grows at rate \( g \) in steady-state: \( \gamma_{Y/L} = g \);
3. \( Y = ALf(k) \) grows at rate \( n + g \) in steady-state: \( \gamma_Y = n + g \)

that is, including consumer optimization yields the same substantive results as the Solow model. There is a bit more going on, but the results are nearly identical. Long-run economic growth in \( Y/L \) comes entirely from the exogenous technological process.

### 3.7 Adding a Government

I now expand the model to incorporate government spending. There are several ways to generally integrate a government into economic models. Governments can act as firms that produce public
goods which enter the consumer’s utility function; alternatively, governments can transfer income or endowments and act as insurance vehicles. Government transfers can enter in the utility function as perfect substitutes for private consumption, or as imperfect substitutes (say, as public goods) or even not enter at all (wasteful government spending). Government can be financed by taxation (lump-sum or distortionary) or by issuing bonds.

Here we will add government expenditures as purely wasteful. This is often a useful benchmark case. Hence the consumer’s Euler equation will not change but the asset accumulation equation will change.

Let \( G(t) \) denote government expenditures. The new centralized asset equation is

\[
\dot{k} = f(k) - c - G - (n + g + \delta)k
\]  

(3.22)

Government expenditure is to be financed by lump-sum taxes. Hence the decentralized budget constraint is:

\[
\int_0^\infty e^{-R(t)} c(t)e^{(n+g)t} dt = a(0) + \int_0^\infty e^{-R(t)} [w(t) - G(t)]e^{(n+g)t} dt
\]  

(3.23)

This is the lifetime discounted budget constraint, as opposed to the flow budget constraint. The left-hand side is the discounted value of consumption; the right-hand side is the discounted value of wages less taxes. Again at each instant \( R(t) \) denotes the average interest rate from time 0 to time \( t \).

Rewriting this in terms of the familiar flow budget constraint yields:

\[
\dot{a} = a(\rho - n - g) + (w - G) - c
\]  

(3.24)

identical to what we had before, except now wages are reduced by \( G \). This is a vertical parallel downward shift of the \( \dot{a} = 0 \) locus.

The new equilibrium conditions are:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} (f'(k) - \delta - \rho - \sigma g)
\]  

(3.25)

\[
\dot{a} = a(\rho - n - g) + (w - G) - c
\]  

(3.26)

Again, this is identical to the old set of equilibrium conditions, but one locus has been shifted down by a vertical distance of \( G \).

3.8 Government Debt and Ricardian Equivalence

We now allow for the possibility of debt. For simplicity normalize \( A(0)L(0) = 1 \).
The new government intertemporal budget constraint (ITBC) is:

\[
b^g(0) + \int_0^\infty e^{-R(t)[G(t) - T(t)]} e^{(n+g)t} dt = 0 \tag{3.27}
\]

that is, the present value of government debt must equal zero, or the government cannot run debt into perpetuity.

Next we rewrite the household’s budget constraint:

\[
\int_0^\infty e^{-R(t)c(t)e^{(n+g)t}} dt = k(0) + b(0) + \int_0^\infty e^{-R(t+(n+g)t}[w(t) - T(t)]} dt \tag{3.28}
\]

now solve for \(b(0)\) and substitute into the previous equation to obtain the household’s budget constraint in slightly different form:

\[
\int_0^\infty e^{-R(t)c(t)e^{(n+g)t}} dt = a(0) + \int_0^\infty e^{-R(t)[w(t) - G(t)]} e^{(n+g)t} dt \tag{3.29}
\]

... identical to what we had in the no-bonds case. Punchline: for a given government spending stream \(\{G(t)\}_{t=0}^\infty\), the manner of financing has no effect on interest rates, wages, consumption, or output.
4 Overlapping Generations (work in progress)

The overlapping generations model makes one simple change to the Ramsey model. Instead of analyzing a single infinitely-lived household, now the economy is populated by agents which are born, live for a few periods, then die. At any given point in time there are agents of different age alive, hence the term “overlapping generations”. We have introduced, in a crude but effective way, some heterogeneity into the model. OLG models provide the minimum necessary heterogeneity needed to analyze intergenerational conflict and debt dynamics in a meaningful fashion.

In particular the OLG model may be usefully employed to study

1. savings
2. the determinants of the capital stock and its evolution
3. the effect of government policy on capital accumulation and welfare
4. pension systems

OLG models are the first model we will consider in which the decentralized market solution is not Pareto optimal.

4.1 An Endowment Economy

Consider an endowment economy in which agents live for a finite number of periods. Agents are endowed with the consumption good in non-negative quantities each period and act to maximize utility, defined over consumption in each period. This is a quite general setup; but for now, I focus on a specific case:

1. Agents live for two periods, “young” and “old”
2. Agents only receive positive endowment in the first period of life
3. All agents in a given generation have identical endowments and preferences

It is easily shown that there is no possibility for trade in such a setup. (The young will not trade with the old, as there is nothing for the old to offer them; and they will not trade with other young, since preferences and endowments are identical.) Hence agents are stuck with consuming their endowment. If they have any sort of convex preferences, this is a suboptimal outcome.

Now let’s put a bit more structure on things. Time is discrete and continues forever. The present is period 1; the immediate past is period 0 and the immediate future is period 2. There are \( N_t \) people in any given period and population growth follows \( N_t = (1 + n)N_{t-1} \), so that \( n \) is the net rate of population growth. Choose units so that \( N_0 = 1 \). Choose units so that each agent is given an endowment of 1 unit when young; they receive no endowment when old. Recall that agents within a generation are identical. Treat the good as nondurable consumption; treat the time periods as “long,” say 40 years or so. Youth is spent working and old age is spent in retirement.
First we start with the usual definitions. The economy’s resource constraint is

\[ N_t c_{1,t} + N_{t-1} c_{2,t} \leq N_t \]

or, doing some rearrangement,

\[ c_{1,t} + \frac{1}{1+n} c_{2,t} \leq 1 \quad (4.1) \]

Note carefully that the constraint binds per time period and not per life cycle.

The economy’s efficient feasible stationary allocation (EFSA) is

\[ c_1 + \frac{1}{1+n} c_2 = 1 \quad (4.2) \]

note that the constraint now holds (feasible) with equality (efficient) and we have dropped the time subscripts (stationary). Note that the stationarity allows us to play fast and loose with the distinction between “a single consumer’s two-period consumption bundle” and “the bundle at time \( t \), consumed by the time-\( t \) old and time-\( t \) young”. By stationarity, these two concepts are in some sense equivalent from the perspective of a benevolent social planner who wishes to maximize utility.

Note carefully the argument we stated before: there is no trade in this economy. Each agent is stuck with her endowment despite having utility which has positive MU in both periods’ consumption. There is scope for Pareto improvements; the market solution is not optimal.

Formally, endowments are \([1, 0]\) for each agent in each generation; and the consumption path is \([1, 0]\) for each agent in each generation regardless of preferences. This is in general suboptimal.

### 4.2 Government in the Endowment Economy

Now add a government which can tax and transfer the consumption good. Let utility be of the form \( u_i = u(c_1, c_2) \). The social planner maximizes \( u(c_1, c_2) \) subject to \( c_1 + c_2/(1+n) = 1 \). Call the ideal allocation \( c_1^*, c_2^* \). Since everyone’s identical and the population growth rate is constant, we can talk about an allocation within a period as “equivalent” to an allocation across periods. Again, the stationary notation hides many of these subtleties.

Now suppose the government taxes away \( 1 - c_1^* \) from each young agent and gives \((1 - c_1^*)(1+n)\) to each old individual, period-by-period. This is feasible given population growth. It can easily be shown that this reallocation will yield the Pareto efficient outcome. Think of this as social security; the government taxes the young to pay for the pension of the old.

### 4.3 A Production Economy

The basic result from the endowment economy is that a private allocation might not be Pareto efficient. Now let’s add production to see if that changes anything.

The supply side of the economy is carried over from Solow and Ramsey, and should be familiar
Recall in the Ramsey model we presumed all agents had constant-relative-risk aversion (CRRA) preferences. This was done to ensure the existence and uniqueness of equilibrium in the infinitely-lived dynamic optimization problem. Such a restriction is not necessary here; however, we will continue to make the CRRA assumption for analytical convenience. Hence utility for any agent is given by

\[ u(c_1, c_2) = \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{1}{1 + \rho} \frac{c_2^{1-\sigma}}{1 - \sigma} \] (4.7)

where \( \sigma \), as in Ramsey, is the coefficient of relative risk aversion and \( \rho > -1 \) is the discount factor. \( \rho > 0 \) implies discounting of the future; \( \rho < 0 \) implies the agent values the future more than the present. In general we assume \( \rho \) is non-negative.

Each individual is endowed with one unit of labor in the first period of life, which she inelastically supplies to the labor market. The budget constraint is:

\[ c_{1,t} + \frac{1}{1 + r_t} c_{2,t} \leq w_t \] (4.8)

that is, the discounted sum of consumption expenditures cannot exceed the real wage. Implicitly we assume a bond market in which the consumer can borrow and lend at rate \( r_t \). We have enough now to solve the consumer side of the problem and derive labor supply and output demand functions. Labor supply is simple, one unit supplied inelastically per household. As for output demand, set up the Lagrangian:

\[ L(c_1, c_2, \lambda) = \left[ \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{1}{1 + \rho} \frac{c_2^{1-\sigma}}{1 - \sigma} \right] - \lambda \left[ c_{1,t} + \frac{1}{1 + r_t} c_{2,t} - w_t \right] \]

which yields the current consumption demand function, the future consumption demand function, and the current saving supply function.

Let’s run through the calculus and algebra to obtain a closed-form solution. Assume an interior optimum, which is guaranteed by the utility function. Then the Lagrangian has three first-order
conditions:

\[
\begin{align*}
    c_1^{-\sigma} &= \lambda \\
    \frac{1}{1 + \rho} c_2^{-\sigma} &= \frac{1}{1 + r_t} \lambda \\
    c_1 + \frac{1}{1 + r_t} c_2 &= w_t
\end{align*}
\]

Combine the first two expressions to eliminate \( \lambda \) and obtain the discrete-time Euler equation under certainty:

\[
\frac{c_2}{c_1} = \left( \frac{1 + r_t}{1 + \rho} \right)^{1/\sigma}
\]  

(4.9)

A higher interest rate raises the growth rate of consumption; similarly lower \( \rho \). Hence low \( \rho \) indicates more “patience” on the part of the consumer; all of this conditional on \( \sigma > 0 \). Substitute this into the budget constraint and rearrange to obtain the first-period consumption function as a function of \( w \) and \( r \):

[Derive \( c_1, c_2 \), and the savings rate in the general case. Lots and lots of algebra to do yet. Somewhat tedious]

4.4 A Special Case

We now consider a special case: log utility and Cobb-Douglas production. The model setup is:

\[
\begin{align*}
    u(c_1, c_2) &= \ln c_1 + \frac{1}{1 + \rho} \ln c_2 \\
    A_t w_t &= C_1 + \frac{1}{1 + r_t} C_2 \\
    f(k) &= k^\alpha \\
    K_{t+1} &= A_t L_t s_t w_t; \quad L_{t+1} = (1 + n) L_t; \quad A_{t+1} = (1 + g) A_t
\end{align*}
\]

First we solve for the consumption function in terms of the interest rate \( r_t \) and the effective wage \( A_t w_t \), both of which are taken as given by the consumer. We begin by setting up a Lagrangian:

\[
L(c_1, c_2, \lambda) = \left[ \log c_1 + \frac{1}{1 + \rho} \log c_2 \right] + \lambda \left[ A_t w_t - c_t - \frac{1}{1 + r_t} c_2 \right]
\]

\[
\begin{align*}
    \frac{1}{c_1} &= \lambda \\
    \frac{1}{1 + \rho} &= \frac{1}{1 + r_t} \lambda \\
    A_t w_t &= c_t + \frac{1}{1 + r_t} c_2
\end{align*}
\]

One can rearrange the second FOC to obtain \( c_2(\lambda) \), then plug in to the first foc to obtain \( c_1(c_2) \),

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and finally use that plus the third FOC to get $c_1$ as a function solely of exogenous parameters. We obtain

$$c_1 = A_t w_t \frac{1 + \rho}{2 + \rho}$$

that is, first-period consumption is independent of the interest rate. This result is a feature of the logarithmic utility specification, in which income and substitution effects just cancel each other out. We use this result to find the savings rate:

$$s_t \equiv 1 - \frac{C}{Y} = 1 - \frac{1 + \rho}{2 + \rho} = \frac{1}{2 + \rho}$$

and the savings rate also does not depend on any time-varying parameters. Hence this model reduces to the Solow model, just like our main specification of the Ramsey model. The assumptions we have made rob the model of much of its richness, but provide analytical tractability.

The second step is to find the equation of motion of $k$, capital per effective worker. Given that the savings rate is time-invariant, this isn’t too hard. Note, however, that since we’re in discrete time (rather than continuous time, as in Solow and Ramsey) the equation of motion will look slightly different: it will be a function $k_{t+1}(k_t)$. We find the steady-state by locating a fixed point of this function. Start with the definition of $k_{t+1}$:

$$k_{t+1} = \frac{K_{t+1}}{A_{t+1} L_{t+1}}$$

$$= \frac{s_t A_t L_t w_t}{A_t L_t (1 + n)(1 + g)}$$

$$= \frac{s_t}{(1 + n)(1 + g)} w_t$$

$$= \frac{1}{(2 + \rho)(1 + n)(1 + g)} (f(k_t) - kf'(k_t))$$

$$= \frac{1}{(2 + \rho)(1 + n)(1 + g)} (Bk_t^\alpha - B\alpha k_t^{\alpha-1})$$

$$= \frac{B(1 - \alpha)}{(2 + \rho)(1 + n)(1 + g)} k_t^\alpha$$

which is globally concave in $(k_t, k_{t+1})$ space. It satisfies the Inada conditions and yields a unique, globally stable steady-state.

We can next derive the steady-state level of $k$.

$$k^* = \frac{B(1 - \alpha)}{(2 + \rho)(1 + n)(1 + g)} (k^*)^\alpha$$

$$\implies k^* = \left[ \frac{B(1 - \alpha)}{(2 + \rho)(1 + n)(1 + g)} \right]^{1/(1-\alpha)}$$

It may be worthwhile to compare this to (1.2), the analogous expression under the Solow model.
As in Solow, $n$ and $g$ appear in the denominator; $s = 1/(2 + \rho)$ appears in the numerator of both expressions. Indeed they differ only by the multiplicative constant $B(1 - \alpha)$. 
5 AK Models

AK models were born with the goal of generating endogenous growth while maintaining the assumption of perfect competition in factor and output markets. The basic feature common to all of the models studied in this chapter is that they end up having constant returns to scale in physical capital. This will generate long-run endogenous growth in the key variable, income per capita.

AK models are built upon a Ramsey foundation and involve only minor tweaks to obtain endogenous growth. Usually they employ some form of constant returns to scale to capital inputs (as opposed to total inputs) or otherwise exploit simple externalities. Hence the work done in analyzing the Ramsey model continues to be useful here.

Recall that in the Solow, Ramsey and Diamond models, growth in income per capita was fixed at $g$ in the long run, and that $g$ was an exogenous parameter that “fell from heaven” in the form of technological progress. In each model in this chapter, there will be no explicit technological progress; $g = 0$. Nevertheless, the models will feature growth in consumption per worker.

5.1 Baseline case

Consider first the following model. Consumers wish to maximize utility:

$$ U = \int_0^\infty Be^{-(\rho-n-\sigma g)t} \frac{c^{1-\sigma}}{1-\sigma} dt $$

(5.1)

with $n = g = 0$ to draw out the most stark results. Recall that in the Ramsey and Solow models, $g = 0$ would imply zero steady-state growth in output per capita. Assets evolve in the usual way:

$$ \dot{a} = ra + w - c $$

(5.2)

Firms have the production function

$$ y = Ak $$

(5.3)

where $y = Y/L$ and $k = K/L$ are quantities per worker, which is simply a special case of $Y = K^\alpha(AL)^{1-\alpha}$ with $\alpha = 1$ and $y = Y/L$, $k = K/L$. Note that the production function is CRS with respect to $K$ and does not exhibit DMR.

Let us set up the equilibrium conditions under this setup. Begin by normalizing everything.

Firms maximize profit $\pi = AK - RK$ and have the FOC:

$$ A = r + \delta $$

(5.4)

that is, the usual $f'(k) = r + \delta$ condition. There is no FOC for labor, as labor is not used in production.
Consumers are characterized by the Hamiltonian:

\[ H := \left[ \frac{c^{1-\sigma}}{1-\sigma} \right] + \mu(rk + w - c) \quad (5.5) \]

Take FOCs to obtain:

\[
\begin{align*}
    c^{-\sigma} &= \mu \\
    \mu r &= \rho \mu - \dot{\mu} \\
    -\sigma c^{-\sigma-1} \dot{c} &= \dot{\mu}
\end{align*}
\]

Rearrange the FOCs to obtain the usual Euler equation:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} (r - \rho)
\]

Hence equilibrium is characterized by:

\[
\gamma_c = \frac{\dot{c}}{c} = \frac{1}{\sigma} (A - \delta - \rho)
\quad (5.6)
\]

Note that these are all constants! So consumption grows at a constant rate for all time; there is no steady-state at which consumption per worker, despite the fact that \( g = 0 \).

However, this model is unsatisfactory for several reasons. First, the capital share of income \( rK/Y \) approaches unity as time goes on; but in the data the capital share is roughly constant over time. Hence there are some dynamics that the model is missing.

### 5.2 Physical and human capital

This is essentially a “patch” on the previous setup to keep the capital share from exploding.

Consider the following:

\[ Y = F(K, H) \quad (5.7) \]

that is, output is produced entirely through human and physical capital. Normalize through by capital:

\[ y = Y/K = F(1, h) = f(h) \quad (5.8) \]

The firm’s FOCs are:

\[
\begin{align*}
    F_K(K, H) &= f(h) - hf'(h) = r + \delta_K \\
    F_H(K, H) &= f'(h) = r + \delta_H
\end{align*}
\quad (5.9)
\]
Now we perform a trick. Note that $\delta_K$ and $\delta_H$ are constants. So in equilibrium, the firm chooses optimally and the no-arbitrage condition implies that $r + \delta_K = r + \delta_H$. Thus:

$$\delta_K - \delta_H = f(h) - hf(h) - f'(h)$$  \hspace{1cm} (5.11)

but the left-hand side is a constant; so the above equation implicitly defines an equilibrium level of $h$. Hence, in turn, $f(h)$ is a constant.

Go back to the production function and rewrite:

$$Y = f(h)K$$

now define $A \equiv f(h)$ and we have an AK model as before. The consumption growth equation is

$$\gamma_c = \frac{\dot{c}}{c} = \frac{1}{\sigma}(f'(h) - \delta_H - \rho)$$  \hspace{1cm} (5.12)

The advantage of this model is that $K/H$ is stationary in equilibrium, so the share of income going to capital $K$ (as a fraction of income going to all capital, $K$ and $H$) is constant and less than unity. As before, the fraction of income going to both $K$ and $H$ does approach unity as time goes on. The share of income going to unskilled labor declines over time.

5.3 Knowledge spillovers

Now the AK model gets interesting. Abandon the assumption that there is one firm. Now the economy is populated by a continuum of firms on the unit interval. Each firm has the production function:

$$Y_i = F(K_i, AL_i).$$  \hspace{1cm} (5.13)

Note that labor and capital are specific to a firm, but technology is shared across firms. As usual $g = n = 0$.

Now suppose that one can summarize the state of technology with the aggregate level of capital, $K$. Divide through by $L_i$ to obtain:

$$\frac{y_i}{L_i} = \frac{Y_i}{L_i} = F(k_i, K)$$  \hspace{1cm} (5.14)

Firm FOCs are as usual:

$$F_1(k_i, K) = r + \delta$$  \hspace{1cm} (5.15)

$$F(k_i, K) - kF_1k_i, K) = w$$  \hspace{1cm} (5.16)
All firms are identical so that in any equilibrium we have \( k_i = k_j = k \). Hence,

\[
y = F(k, K) \tag{5.17}
\]

Now, can we turn the above equation into an AK equation? The answer, it turns out, is yes.

\[
\frac{y}{k} = F(1, K/k) = F\left( k, \frac{K}{K/L} \right) = F(1, L) \equiv f(L) \tag{5.18}
\]

note that \( f(L) \) is identically the average product of capital. Now simply rearrange:

\[
y = f(L)k \tag{5.19}
\]

an AK equation.

Now we solve further for the competitive equilibrium.

The consumer solves his Hamiltonian as before. Skipping some steps:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma}(r - \rho)
\]

Now \( r + \delta = f(L) - Lf'(L) \) so the decentralized Euler equation is:

\[
\gamma_c = \frac{\dot{c}}{c} = \frac{1}{\sigma}(f(L) - Lf'(L) - \delta - \rho) \tag{5.20}
\]

Proof. That \( r + \delta = f(L) - Lf'(L) \). It is not immediately obvious that this is the case. So let’s do the calculus and algebra in gory detail:

\[
r + \delta = F_1(k, K) \quad \text{Firm FOC}
\]

\[
= \frac{\partial}{\partial k}(kf(L)) \quad \text{since } y = F(k, K) = kf(L)
\]

\[
= \frac{\partial}{\partial k}[kf(K/k)] \quad K/k = K/(K/L) = L
\]

\[
= f(L) - kf'(L)\frac{K}{k^2} \quad \text{product and chain rules}
\]

\[
= f(L) - f'(L)\frac{K}{k} \quad \text{simplify}
\]

\[
= f(L) - Lf'(L) \quad K/k = L
\]

which is, finally, the desired expression.

Now let’s solve the problem from the planner’s side. The knowledge spillover creates an externality; let us see how the planner deals with the externality.
Set up the Hamiltonian:

\[ H : = \left[ \frac{c^{1-\sigma}}{1-\sigma} \right] + \mu[kf(L) - c - \delta k] \]

The necessary conditions from the Maximum Principle are:

\[
\begin{align*}
&c^{-\sigma} = \mu \\
&\frac{\dot{c}}{c} = \frac{1}{\sigma} \frac{\dot{\mu}}{\mu} \\
&\mu f(L) - \delta = \rho \mu - \dot{\mu}
\end{align*}
\]

Rearranging, we obtain the planner’s Euler equation:

\[
\gamma_c = \frac{\dot{c}}{c} = \frac{1}{\sigma}(f(L) - \delta - \rho) \tag{5.21}
\]

The planner’s Euler equation is always higher than the competitive Euler equation; hence, the growth rate of consumption is higher under the planner’s solution than under the competitive solution. In the private solution, consumption grows with the marginal product of capital; under the planner, consumption grows with the average product of capital. The intuition for the competitive case is just as in all other models; firms invest until the MPK equals the user cost of capital. In this model, due to the knowledge spillover, the social planner invests until the APK equals the user cost of capital; this maximizes the growth rate of consumption.

The natural policy implication is that a government should subsidize capital formation through an investment subsidy so that MPK = APK.

This model also has an implication that will be drawn out in later models: scale effects. Note that both the market and planner solutions yield growth rates of consumption that are increasing in \(L\). That is, larger economies should grow more quickly than small economies. A cursory glance at the historical record shows that this is not true, at least when we consider “size” to be the population of a country. An alternate explanation, explored later, considers “size” to be the labor force of the trading partners of a country, reasoning that knowledge spillovers are not restricted to a country’s internal population but are also affected by the populations with which it regularly interacts.

5.4 Public goods spillovers

This is the first of two models where the government explicitly enters in the production function. Consider a world with a continuum of firms on the unit interval, producing according to:

\[ Y_i = AL_i^{1-\alpha}K_i^{\alpha}G^{1-\alpha}. \tag{5.22} \]
The production function has CRS in \((K, L)\) and in \((K, G)\) and DMR in all three arguments individually. Technology is constant; \(g = 0\). The government is a constant share \(\tau\) of the economy:

\[ G = \tau Y. \]  
(5.23)

and finances its purchases through either capital taxation or lump-sum taxation. We can interpret the government’s appearance in the production function as government services (defense, law, courts) which in some way affect production. Note that aggregate government spending appears in the individual firm’s production function: government spending is nonrival and nonexcludable.

**AK formulation**

We need only transform this into an \(AK\) format. Suppose that \(n = 0\) so that labor, as well as technology, is constant. Plug in the government’s budget constraint into the (aggregate) production function to obtain:

\[ Y = AL^{1-\alpha}K^\alpha (\tau Y)^{1-\alpha} \]  
(5.24)

and rearrange:

\[ Y^\alpha = A(\tau L)^{1-\alpha}K^\alpha \]

\[ Y = \left[A^{1/\alpha}(\tau L)^{(1-\alpha)/\alpha}\right] K \]  
(5.25)

which is the AK-type equation for this model. Output has constant returns to scale in capital alone, once the endogeneity of government spending is acknowledged.

**Firm FOC**

As usual the demand side of the economy is characterized by the Ramsey-type formulation

\[ \frac{\dot{c}}{c} = \frac{1}{\sigma}(r - \rho) \]  
(5.26)

and \(r = f'(k) - \delta\) so we must find \(f'(k)\). Typically I’d just take the derivative using (5.25), but that’s not what we should do here: since \(\tau\) is a function of \(Y\). Instead take firm FOC with respect to the original production function:

\[ F_K = \alpha AL^{1-\alpha}K^{\alpha-1}G^{1-\alpha} \]  
(5.27)

Drop the FOC into the consumer’s Euler equation to obtain the growth rate of consumption:

\[ \gamma_c = \frac{\dot{c}}{c} = \frac{1}{\sigma} \left( A^{1/\alpha}(\tau L)^{(1-\alpha)/\alpha} - \delta - \rho \right) \]  
(5.28)
Now suppose the government finances itself via a capital tax. New solution:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left( \left[ (1 - \tau)A^{1/\alpha}(\tau L)^{(1-\alpha)/\alpha} \right] - \delta - \rho \right)
\]  

(5.29)

There exists an optimal \(\tau\) to maximize consumption growth; turns out to be \(1 - \alpha\).

### 5.5 Public goods with congestion (work in progress)

do it later

### 5.6 Lessons and Review

Now let’s back away from the results and try to figure out what, if anything, we’ve learned in the past few chapters.

The Solow model contains five exogenous parameters: \(\alpha\), the capital share of income; \(n\), the rate of population growth; \(\delta\), the depreciation rate; \(s\), the national savings rate; and \(g\), the rate of technological progress. In the final analysis, all five are important in determining \(y\), the level of output per worker. But only one, \(g\), is important in determining the growth rate of output per worker. \(\gamma_y = g\). Nothing related to the capital side of the economy (\(\delta\) and \(s\)) had anything to do with growth in the long run. In sum:

1. Solow: \(\gamma_y = g\)
2. Ramsey: \(\gamma_y = g\)
3. Diamond: \(\gamma_y = g\)

Identical results flow from the fully-specified Ramsey and Diamond model. In the end, the rate of growth of output depended only on the rate of growth of labor productivity/technical progress.

Now what about the AK models? There we had:

1. Baseline: \(\gamma_y = \frac{1}{\sigma}(A - \delta - \rho)\)
2. Capital variety: \(\gamma_y = \frac{1}{\sigma}(f'(h) - \delta_H - \rho)\)
3. Knowledge spillovers: \(\gamma_y = \frac{1}{\sigma}(f(L) - LF'(L) - \delta - \rho)\) or \(\gamma_y = \frac{1}{\sigma}(f(L) - \delta - \rho)\)
4. Public good spillovers: \(\gamma_y = \frac{1}{\sigma} \left( \left[ A^{1/\alpha}(\tau L)^{(1-\alpha)/\alpha} \right] - \delta - \rho \right)\)

Basically, that long-run growth depends on a number of factors. Let’s first look at the baseline case. There, \(\gamma_y\) depends on the level of the (constant) technological parameter, as opposed to its rate of change; an alternate interpretation is that growth depends on the marginal product of capital. In the second case, the growth rate depends on the marginal product of human capital. In the third, it depends on either the average product of capital or marginal product of capital, depending on whether the solution is planned or decentralized. Finally, in the public-goods model
the growth rate is critically affected by the tax rate as well as a host of other constants. The AK model generally predicts that long-run growth flows from the constant-returns portion of the model, whatever that may be.
6 Endogenous Technological Change

While the Solow/Ramsey/Diamond models engender growth through exogenous technological change and AK models generated growth through constant returns to capital, the expanding-product-variety model produces growth endogenously through the purposeful actions of monopolistically competitive firms.

6.1 Baseline Expanding Product Variety (work in progress)

Demographics first. There is total factor productivity $A$ which is kept constant and a constant $L$ of labor. $n = g = 0$ to draw out the growth implications most clearly. The world consists of three kinds of economic actors: final goods producers, intermediate goods producers, and consumers.

There is a unit mass of final-goods producing firms. Each has production function

$$Y_i = A_t L_i^{1-\alpha} \left( \sum_{j=1}^{N} X_{ij} \right)^{\alpha}$$

where $Y_i$ is output of the final good for firm $i$; $L_i$ is the labor input for the firm; $X_{ij}$ denotes the use of intermediate type $j$ by firm $i$; and $A$ is common total factor productivity. The final good is identical across firms and over time. Innovation consists of expanding the variety of intermediates, $N$. Each final good firm acts competitively, taking prices as given, in both its output market (selling to consumers) and in its input market (buying from intermediate good producers). The solution to the final goods producer’s problem yields final output supply, labor demand, and intermediate good demand.

Second, there is a mass of intermediate-goods firms. This mass grows endogenously and we will be vague about its size for now. Intermediate firms are indexed by $j$ and each individual intermediate firm wishes to maximize profits. The solution to the intermediate firm’s problem yields intermediate good supply.

Finally, there is a unit mass of consumers which derive utility from consuming the homogeneous final good.

6.2 Romer’s Model (work in progress)

Todo

6.3 Kremer’s Model

Kremer (1993) attempts to create a model which not only explains the postwar data but also the available economic data for the past several thousand years. He provides a model with scale effects, but argues that such effects are reasonable once the “scale” has been appropriately defined: not the size of an economy, but the size of an economy plus its trading partners. If technology flows
across borders, this is a reasonable assumption. He also builds a model with endogenous population
growth, another innovation over all other models presented in these notes.

Suppose output and technological change are characterized by:

\[ Y = AL^\alpha T^{1-\alpha} \]  \hspace{1cm} (6.2)
\[ \dot{A} \over A = gL \]  \hspace{1cm} (6.3)

where time is continuous; \( Y \) is output, \( A \) labor productivity, \( L \) labor and \( T \) land. Since land is
fixed for a given country (we don’t model country expansion here), choose units so that \( T = 1 \), and
\( Y = AL^\alpha \).

Then we have

\[ \frac{Y}{L} = AL^{\alpha-1} \]  \hspace{1cm} (6.4)

if we make the Malthusian assumption that output per capita is fixed at an upper bound \( \bar{y} \). Hence,

\[ \bar{y} = AL^{\alpha-1} \]

\[ \implies L = \bar{y}^{1/(\alpha-1)}A^{1/(1-\alpha)} \]

Hence,

\[ \gamma_L = \frac{1}{1 - \alpha} \gamma_A. \]

Plugging in the relation \( \gamma_A = gL \),

\[ \gamma_L = \frac{g}{1 - \alpha} L \]  \hspace{1cm} (6.5)

hence, the growth rate of the population is proportional to its level. Now, this result flows entirely
from the assumption on technology. Is it reasonable? It’s a classic scale effect, one which we will
explore in much greater detail in the next section. For now, Kremer argues that his specification
is reasonable, if we interpret \( L \) broadly enough.

Now walk backwards: what is \( \gamma_Y \)? Well:

\[ Y = \bar{y}L \]
\[ \gamma_Y = \gamma_L = \frac{g}{1 - \alpha} L \]  \hspace{1cm} (6.6)

so that the growth rate of GDP is proportional to population. Large countries grow quickly; small
countries slowly. Read the paper to see the empirical tests he brings to bear on the model. It’s a
fun little diversion, not to be taken too seriously.
6.4 Endogenous Growth and the Scale Effect

We now turn to the analysis of scale effects in a somewhat stylized setting. I follow Jones (1999); his article provides a compact summary of a variety of scale-effect models. Scale effects refer to the phenomenon in expanding-product variety and other endogenous growth models that economic growth depends in some way on the "size" of the economy. We now investigate these models and evaluate the empirical plausibility of scale effects.

6.4.1 Model 1

Consider first a model which may be summarized by:

\[ Y = A^\sigma L_y \]  
\[ \dot{A} = \delta AL_A \]

where \( Y \) is aggregate output, \( L \) is total labor; \( L_y \) is labor used in producing output and \( L_A \) is labor used in research. \( A \) summarizes technological development and the growth rate of such technology is proportional to the (absolute) size of the research sector. Note that output is CRS in labor and may be IRS, CRS, or DRS in technology depending on the size of \( \sigma \); however with \( \sigma > 0 \) output is more than CRS in labor and technology. There is no capital. In the steady-state, the fraction of labor used in research \( L_A/L = s \), a constant. Then we may rewrite the model’s equations as

\[ Y = A^\sigma (1 - s)L \]
\[ \dot{A} = \delta sAL \]

so that the growth rate of technology is proportional to the size of the overall population.

\[ \gamma_Y = \frac{\dot{Y}}{Y} = \sigma \frac{\dot{A}}{A} + \frac{\dot{L}}{L} \]

letting \( n \) as usual denote population growth,

\[ \gamma_Y = \frac{\dot{Y}}{Y} = \sigma \delta sL + n \]  
\[ \gamma_y = \frac{\dot{y}}{y} = \sigma \delta sL \]  

where \( y = Y/L \) is output per capita. This says that the growth rate of output (per capita) is proportional to the size of the economy, summarized by \( L \). Large countries have large research centers; large research centers imply higher growth of technology; higher growth in technology spurs higher growth in income. Note that this model shows endogenous growth even when \( n = 0 \). In particular, the model shows explosive growth when \( n \neq 0 \); only for \( n = 0 \) is there a steady-state in which \( \gamma_A \) is a constant.
6.4.2 Model 2

So the prior model implies scale effects. Let’s try to modify things slightly. Consider now a model summarized by

\[ Y = A^\sigma (1 - s)L \]
\[ \dot{A} = \delta s A^\phi L \]

(6.11)

The only difference is that now we let the exponent on \( A \) vary from unity. We impose decreasing returns to scale so that \( \phi < 1 \) Let’s solve this version of the model. Suppose the population grows exogenously at rate \( n \).

Now rewrite the second equation so that

\[ \frac{\dot{A}}{A} = \delta s A^{\phi - 1} L \]

In the steady state, \( \frac{\dot{A}}{A} \) is a constant. Label that constant \( g \) (reminiscent of Solow). This implies that the entire right-hand side is constant at \( g \). This in turn implies that \( A^{1-\phi} \) and \( L \) grow at the same rate, so that:

\[ 0 = (\phi - 1)g + n \]
\[ (1 - \phi)g = n \]
\[ g = \frac{n}{1 - \phi} \]

We have found the steady-state growth rate of \( A \). From here, the growth rate of \( Y \) is easily determined:

\[ \gamma_Y = \sigma \gamma_A + \gamma_L \]
\[ \implies \gamma_Y = \frac{\sigma n}{1 - \phi} + n \]

and \( \gamma_y = \frac{\sigma n}{1 - \phi} \)

Notice that we have eliminated the scale effect in both the aggregate and in per-worker terms: \( L \) appears nowhere in \( \gamma_Y \). We have done this for all cases except the degenerate case of \( \phi = 1 \). Notice also that economic growth here depends critically on \( n > 0 \); without population growth there is no aggregate economic growth. Further note that there is a steady-state for any population growth rate, unlike model 1.

It is useful to contrast this result with Solow/Ramsey. In Solow, the exogenous parameter \( g \) was independent of the population growth rate; in the current model, the growth rate of technology is proportional to the growth rate of the population with the constant of proportionality defined by the (inverse) returns to scale in the production of new technology.
6.4.3 Model 3

Continue to assume that

\[ Y = A^\sigma L \]

but now suppose that consumption is a CES aggregate of \( B \) varieties. New key equations are

\[
\begin{align*}
C &= B^\theta Y \\
B &= L^\beta
\end{align*}
\] (6.12) (6.13)

The first condition simply says that consumption is a function of output, adjusted for the number of types of commodity. The second condition implies that \( \gamma_B = \beta n \). Further modify the technology growth equation:

\[
\frac{\dot{A}}{A} = \delta s \frac{L}{B}
\] (6.14)

so that research growth depends negatively on the number of varieties (fishing-out effect). Solve the model:

\[
c = B^\theta y
\]

\[
\Rightarrow \gamma_c = \theta \gamma_B + \gamma_y
\]

now \( \gamma_B = \beta n \) as asserted previously. So we need to figure out \( \gamma_y \). That isn’t too hard: \( y = A^\sigma \) so that \( \gamma_y = \sigma \gamma_A = \sigma \delta s L/B \). Given that \( B = L^\beta \) we can simplify further, \( \gamma_y = \sigma \delta s L^{1-\beta} \). Put everything together:

\[
g_c = \beta n + \sigma \delta s L^{1-\beta}
\] (6.15)

which shows no scale effects under the condition that \( \beta = 1 \). Note that the growth rate of consumption (=output) per capita depends on the population growth rate, contrary to the usual Solow result. “Solow’s \( g \)” is related to the degree of returns-to-scale to technology in production \( \sigma \), the conversion rate \( \delta \), and the share of the population working in research \( s \).

6.4.4 Summary

We have examined the scale effect inherent in basic endogenous-growth models and two attempts to remove it. Let’s briefly discuss the general solution strategy used across the models. The production function is specified and the growth rate of output is decomposed into the growth rates of technology and labor. Then the growth rates of labor and technology are determined and that solution is substituted in to recover the growth rate of output. If either the growth rate of labor
or technology is not already constant, propose a steady-state in which it is constant and find the relevant steady-state growth rate.

Now look at the form of the models. Each summarized by just two equations, a production function and a technology accumulation equation. The difference across models is the technology accumulation equation; at first glance, these are all somewhat ad-hoc and it can be seen that they are engineered ex post to generate the desired growth rate for output. While they are nice general equilibrium models, I don’t find any of them particularly illuminating or convincing. Their main purpose is to explore the role of scale effects and bring out which modelling choices inexorably bring out scale effects, not provide a thorough description of actual economic growth.

6.5 Lucas’ Model

In this subsection I provide a simplified version of the model in [Lucas (1988)].

Suppose output, capital accumulation, and human capital accumulation are characterized by

\[ Y = K^\alpha[(1 - u)H]^\beta \]  
\[ \dot{K} = sY \]  
\[ \dot{H} = BuH \]

with \( 0 < \alpha < 1, 0 < \beta < 1, \) and \( \alpha + \beta > 1. \) \( 0 < u < 1 \) is the share of human capital devoted to learning; \( 1 - u \) is the share devoted to production.

We solve the model for its constant-growth steady-state. Note that

\[ \gamma_K = s \frac{Y}{K} \]
\[ \gamma_H = Bu \]

taking logarithms of the first expression,

\[ \log \gamma_K = \log Y - \log K \]
\[ \implies \frac{\dot{\gamma}_K}{\gamma_K} = \gamma_Y - \gamma_K \]
\[ = \alpha \gamma_K + \beta \gamma_H - \gamma_K \]

In the steady-state, \( \dot{\gamma}_K = 0 \) so that

\[ \gamma_K = \frac{\beta}{1 - \alpha} \gamma_H \]
\[ = \frac{\beta}{1 - \alpha} Bu \]  

In addition, \( \gamma_Y = \alpha \gamma_K + \beta \gamma_H = \gamma_K \) in the steady-state.

Output and capital grow at rates \( \beta/(1 - \alpha)Bu \), while human capital grows at rate \( Bu \); since
\( \beta > 1 - \alpha \), capital and output faster than human capital in the steady-state.

If we instead assume \( \alpha + \beta < 1 \), the model continues to generate endogenous growth. If \( \alpha + \beta = 1 \), all three quantities \( Y \), \( K \), and \( H \), grow at the same rate.
7 The Quality Ladder

The final “big model” we consider is the Schumpeterian quality ladder. In this setup, innovation comes from producing a superior intermediate good than one’s rivals, which one then has monopoly rents over and may sell to the final good producer for a profit. A continuum of firms engage in production of the intermediate good and in research towards a new intermediate. Introduction of a new intermediate renders the old intermediate obsolete; hence the model has a flavor of creative destruction from whence it gets its name.

7.1 Final goods sector

We wish to simplify brutally both the consumer and the final good producer, as all of the growth action will come from the intermediate goods sector. The consumer has linear discounted utility in the final good:

\[ U = \int_0^\infty e^{-rt}yd\tau \]  \hspace{1cm} (7.1)

where \( \tau \) is continuous “real time” as distinguished from discrete “product time” to be introduced later. There is no exogeneous technological progress or population growth to worry about. We have already imposed the equilibrium conditions that \( \rho = r \) and \( y = c \); final output is used only for consumption.

The final goods producer generates final output competitively according to

\[ y = Ax^\alpha \hspace{1cm} 0 < \alpha < 1 \]  \hspace{1cm} (7.2)

where \( A \) is the constant parameter of exogenous technology and \( x \) is the intermediate good.

Innovations consist of creating new intermediate goods which raise the level of technology by a factor \( \gamma > 1 \). That is,

\[ A_{t+1} = \gamma A_t \hspace{1cm} \gamma > 1 \]  \hspace{1cm} (7.3)

where \( t \) is in product time: \( t \) indexes the current intermediate good, \( t = 0, 1, 2, \ldots \)

7.2 Intermediate goods sector

The representative individual splits his labor \( L \) into two uses, production of intermediates and innovation in new intermediates:

\[ L = L_x + L_n = x + n \]  \hspace{1cm} (7.4)

so that production is linear in the intermediate and linear in technology formation. With \( L \) large we can think of \( n \) as being “the number of innovators.” When labor \( n \) is used in research, new
intermediates arrive at Poisson rate $n\lambda$; $\lambda$ is i.i.d. across individual innovators. This is a common shortcut; it leaves out some interesting aspects of differences in research firms but is analytically tractable.

Innovation in new technology has three effects:

1. Monopoly rents go to the innovator
2. Other researchers may now begin working on the next innovation
3. The current innovation destroys the monopoly rents going to the prior innovator

The labor market is pinned down by a simple arbitrage condition in equilibrium:

$$w = \lambda V_{t+1}$$  

(7.5)

where $w$ is the wage rate to manufacturing and $V_{t+1}$ is the monopoly rents accruing to the production of the $(t+1)$th good. The right-hand side is simply the monopoly rent scaled by the chance that an individual innovator will hit on the next innovation.

### 7.3 Equilibrium

The nature of the value function remains to be determined. So let’s do that. The value of producing the $(t+1)$th innovation must also be equal to the value of holding a license in the innovation:

$$rV_{t+1} = \pi_{t+1} - \lambda n_{t+1} V_{t+1}$$  

(7.6)

The left-hand side is the income generated by a license to the $(t+1)$th innovation in a unit time interval; the right-hand side is the flow profit $\pi_{t+1}$ minus the expected loss from the next innovation.

Of course this merely pushes the problem back one step; what is $\pi_{t+1}$? The innovator has exclusive access to the final goods producer so her problem is

$$\max_x xp(x) - wx$$

he sells the intermediate to the final goods producer and must pay wages to the other individuals working for him.

The demand for the intermediate good is determined by the final good producer. Since he acts competitively, demand for the intermediate good satisfies

$$p = \alpha Ax^{\alpha - 1}$$  

(7.7)

So plug in to the maximization problem:

$$x := \arg \max_x [Ax^\alpha - wx]$$
which yields
\[ x = \left( \frac{1}{\alpha} \frac{w}{A} \right)^{1/(1-\alpha)}. \] (7.8)

Let the productivity-adjusted wage be \( \omega \equiv w/A \). Plug in the solution for \( x \) to find the maximized profit function:

\[ \pi_t = \left( \frac{1}{\alpha} - 1 \right) w_t x_t \equiv A_t \tilde{\pi}_t(\omega_t) \] (7.9)

Hence we now have a full characterization of the model in the following two equations:

\[ \omega_t = \lambda \left( \frac{\gamma \tilde{\pi}(\omega_{t+1})}{r + \lambda n_{t+1}} \right) \]
\[ L_t = n_t + \tilde{x}_t(\omega_t) \] (7.10) (7.11)

Our endogenous variables are \( n_t \) and \( \omega_{t+1} \), the quantity of laborers in research and the equilibrium productivity-adjusted wage.

### 7.4 The Steady State

So we have two equations in two unknowns, but they are time-variant. We must fix this. Consider now the steady-state, so that \( n_t = \hat{n} \) and \( \omega_t = \hat{\omega} \). That got rid of the pesky time problem and now we truly have two equations in two unknowns:

\[ \hat{\omega} = \lambda \left( \frac{\gamma \tilde{\pi}(\hat{\omega})}{r + \lambda \hat{n}} \right) \] (7.10)
\[ L_t = \hat{n} + \tilde{x}_t(\hat{\omega}) \] (7.11)

Think of these as supply and demand curves for \( \hat{n} \) where the wage is \( \hat{\omega} \).

We can now start to answer some questions about how different variables impact research and growth.

From the above maximized-profit function, we have

\[ \pi = \frac{1 - \alpha}{\alpha} wx = \frac{1 - \alpha}{\alpha} w(L - \hat{n}) \]

so that

\[ 1 = \lambda \frac{\gamma \frac{1 - \alpha}{\alpha} (L - \hat{n})}{r + \lambda \hat{n}} \]

Now this equation implies that \( \hat{n} \) is strictly decreasing in \( \alpha \). We can put some economic interpretation on this result. \( \alpha \) is the elasticity of the demand curve faced by the intermediate firm; in essence it measures the level of competition in the output sector.
Consider the position of the innovator. If he faces a downward-sloping demand curve, he can extract monopoly profits; by contrast if he faces a flat demand curve, his monopoly profits vanish. Hence as the final output sector becomes “more price taking” the monopoly profits to the innovator, and the incentive to innovate in the first place, decline.

7.5 Growth in real time

So far we have focused on “product time”, discrete and indexed by \( t = 1, 2, \ldots \). We now return to real time, continuously denoted in \( \tau \). How does innovation in product time translate to growth in real time?

Note that
\[
y_t = A_t \hat{x}^\alpha = A_t (L - \hat{n})^\alpha
\]
which implies that
\[
y_{t+1} = A_{t+1} (L - \hat{n})^\alpha = \gamma A_t (L - \hat{n})^\alpha = \gamma y_t
\]
since the only thing that is growing is \( A \). We’re still in product time. Begin by taking logs of the prior equation:
\[
\ln(y_{t+1}) = \ln(\gamma) + \ln(y_t)
\]
(7.12)

Now instead of taking a unit product step, take a unit time interval:
\[
\ln(y(\tau + 1)) = \ln(\gamma)\varepsilon(\tau) + \ln(y(\tau))
\]
(7.13)
where the \( \varepsilon(\tau) \) term is the number of innovations in the time interval. One can then rearrange and take expectations:
\[
E[\ln(y(\tau + 1)) - \ln(y(\tau))] = E[\ln(\gamma)\varepsilon(\tau)]
\]
\[
\implies g = \hat{n}\lambda \ln(\gamma)
\]
(7.14)
The left-hand side is simply the expected growth rate; the RHS shows that the expected growth rate is equal to the expected number of innovations in the time period, scaled by \( \ln(\gamma) \), the size of each innovation. Solow’s \( g \) becomes a function of the expected number of innovations per time interval.

We can now do comparative statics on the effects of the size of the labor force, interest rate, and size of innovation and productivity of R&D sector on growth. Briefly, larger \( L \) and lower \( r \) increase \( g \) through affecting \( \hat{n} \); increases in \( \lambda \) and \( \gamma \) will increase the growth rate both directly and indirectly, through \( \hat{n} \).

A quick application: how does openness to trade affect growth? It increases the “effective
L" by expanding the available pool of researchers (assuming innovations flow across borders), but also increases the size of the output market and may increase competition, reducing $\alpha$ and also monopoly profits. The final effect is ambiguous.
8 Stochastic Dynamic Programming

In short-run macroeconomics it is necessary to solve for the consumption and investment functions. This section provides an overview of one popular specification for each of these.

8.1 Tobin’s Q Model of Investment

The Euler equation for capital

Consider first a model of investment with quadratic adjustment costs. The firm wishes to maximize the present discounted value of dividends. The maximization problem, stated in Bellman formulation, is:

\[ V_t(K_t, \varepsilon_t) = \max_{I_t} D_t + \beta E_t[V_{t+1}(K_{t+1}, \varepsilon_{t+1})] \]

\[ K_{t+1} = I_t - (1 - \delta)K_t \]

\[ D_t = \pi(\varepsilon_t)K_t - \frac{b}{2} \left( \frac{I_t}{K_t} \right)^2 - P^I I_t \]

The first equation is the Bellman, relating the value function \( V_t \) to the maximization of current-period dividends plus the value function one step ahead. The stock/state variable is capital \( K \); the flow/control variable is investment \( I \). The second line is the capital evolution equation; it is of standard form that we have seen since the Solow model in the first chapter. The final equation determines the flow of dividends and demands some explication. The right-hand side is cash-flow profit. The right-hand side consists of three terms. First, \( \pi(\varepsilon_t)K_t \) is a compact notation for \( Y - wL \), or revenue in excess of variable costs. It is linear in the capital stock. The second term is the quadratic adjustment cost for investment. The third is the direct cost of investment; each unit of investment costs \( P^I \).

So much for setup. Begin solving by taking the FOC and envelope conditions:

\[ \frac{\partial D_t}{\partial I_t} + \beta E_t \left[ \frac{\partial V_{t+1}}{\partial K_{t+1}} \right] = 0 \]

(8.1)

\[ \frac{\partial V_t}{\partial K_t} = \frac{\partial D_t}{\partial K_t} + (1 - \delta)\beta E_t \left[ \frac{\partial V_{t+1}}{\partial K_{t+1}} \right] \]

(8.2)

Iterate (8.2) forward by one period to obtain:

\[ \frac{\partial V_{t+1}}{\partial K_{t+1}} = \frac{\partial D_{t+1}}{\partial K_{t+1}} + (1 - \delta)\beta E_{t+1} \left[ \frac{\partial V_{t+2}}{\partial K_{t+2}} \right] \]

(8.3)

Now take expectations relative to time \( t \):

\[ E_t \left[ \frac{\partial V_{t+1}}{\partial K_{t+1}} \right] = E_t \left[ \frac{\partial D_{t+1}}{\partial K_{t+1}} \right] + (1 - \delta)\beta E_t \left[ \frac{\partial V_{t+2}}{\partial K_{t+2}} \right] \]

(8.4)
Now we have a $t+2$ term. We could continue iterating forward, but it’s easier to simply move (8.1) forward one period:

$$\frac{\partial D_{t+1}}{\partial I_{t+1}} + \beta E_{t+1} \left[ \frac{\partial V_{t+2}}{\partial K_{t+2}} \right] = 0$$  (8.5)

Take expectations relative to $t$:

$$\frac{\partial D_{t+1}}{\partial I_{t+1}} + \beta E_t \left[ \frac{\partial V_{t+2}}{\partial K_{t+2}} \right] = 0$$  (8.6)

and rearrange:

$$\frac{\partial D_{t+1}}{\partial I_{t+1}} = -\beta E_t \left[ \frac{\partial V_{t+2}}{\partial K_{t+2}} \right]$$  (8.7)

Now start making substitutions. (8.1) can be placed in the left-hand side of (8.4) and (8.7) provides a time-$(t+2)$ term to put on the right-hand side of (8.4) . . .

$$-\frac{\partial D_t}{\partial I_t} = \beta E_t \left[ \frac{\partial D_{t+1}}{K_{t+1}} - (1 - \delta) \frac{\partial D_{t+1}}{I_{t+1}} \right]$$  (8.8)

This is the Euler equation for capital. However, it’s not yet in a form that “looks” like a Euler equation. So we can do some rearranging.

First, we know $D_t(K,I)$ so we can take the relevant partial derivatives:

$$D_I = -b \frac{I_t}{K_t} - P^I_t$$

$$D_K = \pi(\varepsilon_t) + \frac{b}{2} \left( \frac{I_t}{K_t} \right)^2$$

Further, normalizing $P^I = 1$ (so that the price of the investment good is equal to that of the output good, and both can be used as numeraire) and denoting $V_k \equiv q_t$, we can rewrite (8.1) as:

$$-b \frac{I_t}{K_t} - 1 + \beta E_t[q_{t+1}] = 0$$

$$\frac{I_t}{K_t} = \frac{1}{b} E_t[\beta q_{t+1} - 1]$$  (8.9)

Expression (8.9) is a key equation going forward.

Finally, substituting these three derivations into (8.8) we obtain:

$$\frac{I_t}{K_t} + \frac{1}{b} = \beta E_t \left[ \frac{1}{b} \pi_{t+1} + \frac{1}{2} \left( \frac{I_{t+1}}{K_{t+1}} \right)^2 + (1 - \delta) \left( \frac{I_{t+1}}{K_{t+1}} + \frac{1}{b} \right) \right]$$  (8.10)

This is the Euler equation for investment/capital in standard form, relating current investment to expected future investment one period ahead.
The Q Model

The next step is to relate investment to the $q$ defined earlier. Start by multiplying (8.5) by $I_{t+1}$ and (8.4) by $K_{t+1}$ to obtain:

$$
\frac{\partial D_{t+1}}{\partial K_{t+1}} I_{t+1} + \beta I_{t+1} E_t[q_{t+2}] + \frac{\partial D_{t+1}}{\partial K_{t+1}} K_{t+1} - K_{t+1} q_{t+1} + (1 - \delta) \beta K_{t+1} E_t[q_{t+2}] = 0
$$

(8.11)

This neatly simplifies to

$$
K_{t+1} q_{t+1} = D_{t+1} + \beta K_{t+2} E_t[q_{t+2}]
$$

(8.12)

We can iterate this forward to obtain:

$$
K_{t+1} q_{t+1} = E_{t+1} \sum_{s=1}^{\infty} \beta^s D_{t+1+s} \equiv V_{t+1}
$$

where the last equality stems from the fact that the current value of the firm $V_{t+1}$ is simply the sum of discounted dividends. Hence, $q_{t+1} = V_{t+1}/K_{t+1}$. Plug this into (8.9) to obtain the compact expression:

$$
\frac{I_t}{K_t} = \frac{1}{b} E_t \left[ \beta \frac{V_{t+1}}{K_{t+1}} - 1 \right]
$$

(8.13)

This is the $q$ model of investment: given $q$ no other information is needed to determine the level of investment.

8.2 The Stochastic Consumption Euler Equation

The Euler equation for consumption

The consumer wishes to maximize

$$
\max_c E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)
$$

(8.14)

s.t. $c_t + q_t w_{t+1} = (q_t + q_t^*) w_t + y_t$

(8.15)

where $c_t$ is a stream of consumption services, $q_t$ is the price of bonds, $w_t$ is the level of bond holdings, and $y_t$ is exogenous income. Define the control variable

$$
u_t = q_t w_{t+1} = (q_t + q_t^*) w_t + y_t - c_t
$$

(8.16)

as the level of investment a consumer chooses in any period. Hence

$$
c_t = (q_t + q_t^*) w_t + y_t - u_t
$$

(8.17)
and we can plug this into the maximization problem

\[
\max_u E_0 \sum_{t=0}^{\infty} \beta^t u(q_t + q^*_t)w_t + y_t - u_t \tag{8.18}
\]

The first-order necessary condition is

\[
E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} r_{t+1} \right] = 1 \tag{8.19}
\]

where \( r_{t+1} = (q_{t+1} + q^*_{t+1})/q_t \) is the gross rate of interest. (8.19) is the consumer’s Euler equation.

**Quadratic utility and constant \( r \)**

With \( u(c) = c_t - \frac{a}{2}c_t^2 \), the Euler equation becomes

\[
E_t[\beta (1 - ac_{t+1}) r] = 1 - ac_t \tag{8.20}
\]

or, dropping constants,

\[
E_t c_{t+1} = c_t \tag{8.21}
\]

as \( \beta r = 1 \) in steady-state. Consumption is a random walk.

**CRRA**

\[
E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha} r_{t+1} \right] = 1 \tag{8.22}
\]

[todo: log-linearization]
References


