

Baric Structures on Triangulated Categories and Coherent Sheaves

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We introduce the notion of a *baric structure* on a triangulated category, as an abstraction of S. Morel's weight truncation formalism for mixed ℓ -adic sheaves. We study these structures on the derived category $\mathcal{D}_G^b(X)$ of G -equivariant coherent sheaves on a G -scheme X . Our main result shows how to endow this derived category with a family of nontrivial baric structures when G acts on X with finitely many orbits.

We also describe a general construction for producing a new t -structure on a triangulated category equipped with given t - and baric structures, and we prove that the staggered t -structures on $\mathcal{D}_G^b(X)$ introduced by Achar arise in this way.

1 Introduction

Let Z be a variety over a finite field. The triangulated category of ℓ -adic sheaves on Z has a full subcategory $\mathcal{D}_m^b(Z)$ of "mixed sheaves," defined in terms of eigenvalues of the Frobenius morphism. The existence and good formal properties of this category are among the most important consequences of Deligne's proof of the Weil conjectures. It plays a major role in the theory of perverse sheaves and in their applications in

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representation theory. An important part of the formalism of mixed sheaves is a certain filtration of $D_m^b(Z)$ by full subcategories $\{D_m^b(Z)_{\leq w}\}_{w \in \mathbb{Z}}$, known as the *weight filtration*.

Let us now turn our attention to the world of equivariant coherent sheaves. Let X be a scheme (say, of finite type over a field) and let G be an affine group scheme acting on X with finitely many orbits. The first author [1] introduced a class of t -structures, called *staggered t -structures*, on the bounded derived category $\mathcal{D}_G^b(X)$ of G -equivariant coherent sheaves on X . These t -structures depend on the choice of a certain kind of filtration of the abelian category of equivariant coherent sheaves. These filtrations, known as *s-structures*, bear, at least, a superficial resemblance to the weight filtration of $D_m^b(Z)$.

The main goal of this paper is to try to make this resemblance into a precise statement and thereby to place these two kinds of structures in a unified setting. We do this by introducing the notion of a *baric structure* on a triangulated category. The usual weight filtration on $D_m^b(Z)$ is not a baric structure, but a modified version of it due to Morel [12] is. Indeed, the definition of a baric structure is largely motivated by Morel's results. (The authors thank the referee for pointing out that this notion has appeared much earlier under the name *f-structure* in [5]; see Section 3.4.) An s -structure is not a baric structure either: for one thing, it is a filtration of an abelian category, not of a triangulated category. We show in this paper how to construct baric structures on $\mathcal{D}_G^b(X)$ using an s -structure on X . We also exhibit several other examples of baric structures that have appeared in the literature.

The second goal of the paper is to recast the construction in [1] as an instance of an abstract operation that can be done on any triangulated category. Specifically, given a triangulated category with "compatible" t - and baric structures, we outline a procedure, which we call *staggering*, for producing a new t -structure. Note that in [1], "staggered" was simply a name assigned to certain specific t -structures by definition, whereas in this paper, "to stagger" is a verb. We prove that these two uses of the word are consistent: that is, that the t -structures of [1] arise by staggering the standard t -structure on $\mathcal{D}_G^b(X)$ with respect to a suitable baric structure. (The staggering operation can also be applied to the weight baric structure on $D_m^b(Z)$, as well as to other baric structures. This yields a new t -structure that has not previously been studied.)

An outline of the paper is as follows. We begin in Section 2 by giving the definition of a baric structure and of the staggering operation. In Section 3, we give examples of baric structures, including Morel's version of the weight filtration. Next, in Section 4, we begin the study of baric structures on derived categories of equivariant coherent sheaves, especially those that behave well with respect to the geometry of the underlying scheme.

The next three sections are devoted to the relationship between baric structures and s -structures. First, in Section 5, we review relevant definitions and results from [1]. Section 6 contains the main result of the paper, showing how s -structures on the abelian category of coherent sheaves give rise to baric structures on the derived category. In Section 7, we briefly consider the reverse problem, that of producing s -structures from baric structures.

Finally, in Section 8, we study staggered t -structures associated to the baric structures produced in Section 6. Specifically, we prove that their hearts are finite-length categories, and we give a description of their simple objects. This was done in some cases in [1], but remarkably, the machinery of baric structures allows us to remove the assumptions that were imposed in [1].

We conclude by mentioning an application of the machinery developed in this paper. The language of baric structures allows one to define the notion of “purity,” similar to the one for ℓ -adic mixed constructible sheaves. In a subsequent paper, the authors [3] prove that every simple staggered sheaf is pure and that every pure object in the derived category is a direct sum of shifts of simple staggered sheaves. These results are analogous to the well-known Purity and Decomposition Theorems for ℓ -adic mixed perverse sheaves.

2 Baric Structures

In this section, we introduce baric structures on triangulated categories (Definition 2.1) and the operation of *staggering* a t -structure with respect to a baric structure (Definition 2.8). Staggering produces, out of a t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on a triangulated category \mathcal{D} , a new pair of orthogonal subcategories $({}^s\mathcal{D}^{\leq 0}, {}^s\mathcal{D}^{\geq 0})$. Our main result is a criterion that guarantees that $({}^s\mathcal{D}^{\leq 0}, {}^s\mathcal{D}^{\geq 0})$ is itself a t -structure (Theorem 2.11).

2.1 Baric structures

Definition 2.1. Let \mathcal{D} be a triangulated category. A *baric structure* on \mathcal{D} is a pair of collections of thick subcategories $(\{\mathcal{D}_{\leq w}\}, \{\mathcal{D}_{\geq w}\})_{w \in \mathbb{Z}}$ satisfying the following axioms:

- (1) $\mathcal{D}_{\leq w} \subset \mathcal{D}_{\leq w+1}$ and $\mathcal{D}_{\geq w} \supset \mathcal{D}_{\geq w+1}$ for all w .
- (2) $\mathrm{Hom}(A, B) = 0$ whenever $A \in \mathcal{D}_{\leq w}$ and $B \in \mathcal{D}_{\geq w+1}$.
- (3) For any object $X \in \mathcal{D}$, there is a distinguished triangle $A \rightarrow X \rightarrow B \rightarrow$ with $A \in \mathcal{D}_{\leq w}$ and $B \in \mathcal{D}_{\geq w+1}$. □

This definition is at least superficially very similar to that of *t-structure*, and in fact arguments identical to those given in [4, Sections 1.3.3–1.3.5] yield the following basic properties of baric structures.

Proposition 2.2. Let \mathcal{D} be a triangulated category equipped with a baric structure $(\{\mathcal{D}_{\leq w}\}, \{\mathcal{D}_{\geq w}\})_{w \in \mathbb{Z}}$. The inclusion $\mathcal{D}_{\leq w} \hookrightarrow \mathcal{D}$ admits a right adjoint $\beta_{\leq w} : \mathcal{D} \rightarrow \mathcal{D}_{\leq w}$, and the inclusion $\mathcal{D}_{\geq w} \rightarrow \mathcal{D}$ admits a left adjoint $\beta_{\geq w} : \mathcal{D} \rightarrow \mathcal{D}_{\geq w}$. There is a distinguished triangle

$$\beta_{\leq w} X \longrightarrow X \longrightarrow \beta_{\geq w+1} X \longrightarrow,$$

and any distinguished triangle as in axiom (3) above is canonically isomorphic to this one. Furthermore, if $v \leq w$, then we have the following isomorphisms of functors:

$$\beta_{\leq v} \circ \beta_{\leq w} \cong \beta_{\leq v}, \quad \beta_{\geq w} \circ \beta_{\geq v} \cong \beta_{\geq w}, \quad \beta_{\geq v} \circ \beta_{\leq w} \cong \beta_{\leq w} \circ \beta_{\geq v}.$$

Moreover, if $v < w$, then $\beta_{\leq v} \circ \beta_{\geq w} \cong \beta_{\geq w} \circ \beta_{\leq v} = 0$. □

Note that in a baric structure, unlike in a *t-structure*, the subcategories $\mathcal{D}_{\leq w}$ and $\mathcal{D}_{\geq w}$ are required to be stable under shifts in both directions, and it is not assumed that there is an autoequivalence $\mathcal{D} \rightarrow \mathcal{D}$ taking $\mathcal{D}_{\leq w}$ to, say, $\mathcal{D}_{\leq w+1}$. Moreover, baric truncation functors enjoy the following important property.

Proposition 2.3. The baric truncation functors $\beta_{\leq w}$ and $\beta_{\geq w}$ take distinguished triangles to distinguished triangles. □

Proof. Let $X \rightarrow Y \rightarrow Z \rightarrow$ be a distinguished triangle in \mathcal{D} and consider the natural morphism $\beta_{\leq w} X \rightarrow X$. The composition of this morphism with $X \rightarrow Y$ factors through $\beta_{\leq w} Y \rightarrow Y$ (since $\text{Hom}(\beta_{\leq w} X, Y) \cong \text{Hom}(\beta_{\leq w} X, \beta_{\leq w} Y)$), so we obtain a commutative diagram:

$$\begin{array}{ccc} \beta_{\leq w} X & \rightarrow & \beta_{\leq w} Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Let us complete this diagram using the 9-lemma [4, Proposition 1.1.11]:

$$\begin{array}{ccccccc}
 \beta_{\leq w} X & \longrightarrow & \beta_{\leq w} Y & \longrightarrow & Z' & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \beta_{\geq w+1} X & \longrightarrow & \beta_{\geq w+1} Y & \longrightarrow & Z'' & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & &
 \end{array}$$

Since $\mathcal{D}_{\leq w}$ and $\mathcal{D}_{\geq w+1}$ are full triangulated subcategories of \mathcal{D} , we see that $Z' \in \mathcal{D}_{\leq w}$ and $Z'' \in \mathcal{D}_{\geq w+1}$. But then Proposition 2.2 tells us that $Z' \cong \beta_{\leq w} Z$ and $Z'' \cong \beta_{\geq w+1} Z$, so we obtain distinguished triangles

$$\beta_{\leq w} X \longrightarrow \beta_{\leq w} Y \longrightarrow \beta_{\leq w} Z \longrightarrow \quad \text{and} \quad \beta_{\geq w+1} X \longrightarrow \beta_{\geq w+1} Y \longrightarrow \beta_{\geq w+1} Z \longrightarrow,$$

as desired. ■

Definition 2.4. Let \mathcal{D} be a triangulated category equipped with a baric structure $(\{\mathcal{D}_{\leq w}\}, \{\mathcal{D}_{\geq w}\})_{w \in \mathbb{Z}}$. We will use the following terminology:

- (1) The adjoints $\beta_{\leq w}$ and $\beta_{\geq w}$ to the inclusions $\mathcal{D}_{\leq w} \hookrightarrow \mathcal{D}$ and $\mathcal{D}_{\geq w} \hookrightarrow \mathcal{D}$ are called *baric truncation functors*.
- (2) The baric structure is *bounded* if for each object $A \in \mathcal{D}$, there exist integers v and w such that $A \in \mathcal{D}_{\geq v} \cap \mathcal{D}_{\leq w}$.
- (3) It is *nondegenerate* if there is no nonzero object belonging to all $\mathcal{D}_{\leq w}$ or to all $\mathcal{D}_{\geq w}$. Note that a bounded baric structure is automatically nondegenerate.
- (4) Let \mathcal{D}' be another triangulated category and suppose it is equipped with a baric structure $(\{\mathcal{D}'_{\leq w}\}, \{\mathcal{D}'_{\geq w}\})$. A functor of triangulated categories $F : \mathcal{D} \rightarrow \mathcal{D}'$ is said to be *left baryexact* if $F(\mathcal{D}_{\geq w}) \subset \mathcal{D}'_{\geq w}$ for all $w \in \mathbb{Z}$, and *right baryexact* if $F(\mathcal{D}_{\leq w}) \subset \mathcal{D}'_{\leq w}$ for all $w \in \mathbb{Z}$. □

Let us also record the following definitions, although we will not use them until later in the paper.

Definition 2.5. Let \mathcal{D} be a triangulated category equipped with a baric structure $(\{\mathcal{D}_{\leq w}\}, \{\mathcal{D}_{\geq w}\})_{w \in \mathbb{Z}}$.

- (1) Suppose \mathcal{D} is equipped with an involutive antiequivalence $\mathbb{D}: \mathcal{D} \rightarrow \mathcal{D}$. The baric structure is *self-dual* if $\mathbb{D}(\mathcal{D}_{\leq w}) = \mathcal{D}_{\geq -w}$.
- (2) Suppose \mathcal{D} has the structure of a tensor category, with the tensor product \otimes . The baric structure is *multiplicative* with respect to \otimes if for any $A \in \mathcal{D}_{\leq v}$ and $B \in \mathcal{D}_{\leq w}$, we have $A \otimes B \in \mathcal{D}_{\leq v+w}$.
- (3) Suppose \mathcal{D} has an internal Hom functor $\mathcal{H}om$. The baric structure is *multiplicative* with respect to $\mathcal{H}om$ if for any $A \in \mathcal{D}_{\leq v}$ and $B \in \mathcal{D}_{\geq w}$, we have $\mathcal{H}om(A, B) \in \mathcal{D}_{\geq w-v}$.

Note that whenever we have an adjunction between \otimes and $\mathcal{H}om$, the multiplicativity conditions are equivalent. □

2.2 Staggering

Below, if \mathcal{D} is equipped with a t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, we write $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ for its heart, and we denote the associated truncation functors by $\tau^{\leq n}$ and $\tau^{\geq n}$. The n th cohomology functor associated to the t -structure is denoted by $h^n: \mathcal{D} \rightarrow \mathcal{C}$.

Definition 2.6. Let \mathcal{D} be a triangulated category equipped with both a t -structure and a baric structure. These structures are said to be *compatible* if $\tau^{\leq n}$ and $\tau^{\geq n}$ are right baryexact, and $\beta_{\leq w}$ and $\beta_{\geq w}$ are left t -exact. □

Remark 2.7. Of course there is a dual notion of compatibility, but it does not seem to arise as often. □

Definition 2.8. Let \mathcal{D} be a triangulated category equipped with compatible t - and baric structures. Define two full subcategories of \mathcal{D} as follows:

$${}^s\mathcal{D}^{\leq 0} = \{A \in \mathcal{D} \mid h^k(A) \in \mathcal{D}_{\leq -k} \text{ for all } k \in \mathbb{Z}\},$$

$${}^s\mathcal{D}^{\geq 0} = \{B \in \mathcal{D} \mid \beta_{\leq k} B \in \mathcal{D}_{\geq -k} \text{ for all } k \in \mathbb{Z}\}.$$

Assume that the pair $({}^s\mathcal{D}^{\leq 0}, {}^s\mathcal{D}^{\geq 0})$ constitutes a t -structure. It is called the *staggered t -structure*, or the t -structure obtained by *staggering* the original t -structure with respect to the given baric structure. □

As usual, we let ${}^s\mathcal{D}^{\leq n} = {}^s\mathcal{D}^{\leq 0}[-n]$ and ${}^s\mathcal{D}^{\geq n} = {}^s\mathcal{D}^{\geq 0}[-n]$.

Lemma 2.9. Let \mathcal{D} be a triangulated category equipped with compatible t - and baric structures. Assume that the t -structure is bounded.

- (1) $A \in \mathcal{D}_{\leq w}$ if and only if $h^k(A) \in \mathcal{D}_{\leq w}$ for all k .
- (2) $B \in \mathcal{D}_{\geq w}$ if and only if $\beta_{\leq w-1} \tau^{\leq k} B \in \mathcal{D}^{\geq k+2}$ for all k .
- (3) We have

$$\mathcal{D}_{\geq w} \cap \mathcal{C} = \{B \in \mathcal{C} \mid \text{Hom}^k(A, B) = 0 \text{ for all } A \in \mathcal{D}_{\leq w-1} \cap \mathcal{C} \text{ and all } k \geq 0\}.$$

- (4) $\mathcal{D}_{\leq w} \cap \mathcal{C}$ is a Serre subcategory of \mathcal{C} , and $\mathcal{D}_{\geq w} \cap \mathcal{C}$ is stable under extensions.
- (5) ${}^s\mathcal{D}^{\leq 0}$ and ${}^s\mathcal{D}^{\geq 0}$ are stable under extensions.
- (6) $\mathcal{D}^{\leq k} \cap \mathcal{D}_{\leq w} \subset {}^s\mathcal{D}^{\leq k+w}$ and $\mathcal{D}^{\geq k} \cap \mathcal{D}_{\geq w} \subset {}^s\mathcal{D}^{\geq k+w}$. □

Recall that a bounded t -structure is necessarily nondegenerate.

Proof. (1) Since $\mathcal{D}_{\leq w}$ is stable under $\tau^{\leq k}$ and $\tau^{\geq k}$, it is clear that $A \in \mathcal{D}_{\leq w}$ implies that $h^k(A) \in \mathcal{D}_{\leq w}$. Conversely, suppose $h^k(A) \in \mathcal{D}_{\leq w}$ for all k . Recall (e.g. [14, Proposition 4.4.6]) that we have a spectral sequence

$$E_2^{ab} = \text{Hom}(h^{-b}(A), B[a]) \Rightarrow \text{Hom}(A, B[a+b]). \tag{2.1}$$

Since $\text{Hom}(h^{-b}(A), B[a]) = 0$ for all $B \in \mathcal{D}_{\geq w+1}$ and all $a, b \in \mathbb{Z}$, we see that $\text{Hom}(A, B) = 0$ for all $B \in \mathcal{D}_{\geq w+1}$, and hence that $A \in \mathcal{D}_{\leq w}$.

(2) Consider the distinguished triangle

$$\beta_{\leq w-1} \tau^{\leq k} B \longrightarrow \beta_{\leq w-1} B \longrightarrow \beta_{\leq w-1} \tau^{\geq k+1} B \longrightarrow .$$

The last term is always in $\mathcal{D}^{\geq k+1}$ by the left t -exactness of $\beta_{\leq w-1}$. If $B \in \mathcal{D}_{\geq w}$, so that $\beta_{\leq w-1} B = 0$, then $\beta_{\leq w-1} \tau^{\leq k} B \cong (\beta_{\leq w-1} \tau^{\geq k+1} B)[-1] \in \mathcal{D}^{\geq k+2}$. Conversely, if the t -structure is nondegenerate, and if $\beta_{\leq w-1} \tau^{\leq k} B \in \mathcal{D}^{\geq k+2}$ for all k , the distinguished triangle above shows that $\beta_{\leq w-1} B \in \mathcal{D}^{\geq k+1}$ for all k , and hence that $\beta_{\leq w-1} B = 0$, so $B \in \mathcal{D}_{\geq w}$, as desired.

(3) If $B \in \mathcal{D}_{\geq w} \cap \mathcal{C}$, then clearly $\text{Hom}(A[-k], B) = 0$ for all $A \in \mathcal{D}_{\leq w-1} \cap \mathcal{C}$ and all $k \geq 0$, since $A[-k] \in \mathcal{D}_{\leq w-1}$ for all k . Conversely, if $\text{Hom}(A, B[k]) = 0$ for all $A \in \mathcal{D}_{\leq w-1} \cap \mathcal{C}$ and all $k \geq 0$, the spectral sequence (2.1) shows that $\text{Hom}(A, B) = 0$ for all $A \in \mathcal{D}_{\leq w-1}$, and hence that $B \in \mathcal{D}_{\geq w}$.

(4) Suppose we have a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{C} . If A and C are in $\mathcal{D}_{\leq w}$, then B must be as well, since $\mathcal{D}_{\leq w}$ is stable under extensions. Conversely, suppose $B \in \mathcal{D}_{\leq w}$. Assume that $C \notin \mathcal{D}_{\leq w}$, and consider the distinguished triangle

$$\beta_{\leq w} C \longrightarrow C \longrightarrow \beta_{\geq w+1} C \longrightarrow .$$

By left t -exactness of the baric truncation functors, we have an exact sequence

$$0 \longrightarrow h^0(\beta_{\leq w} C) \longrightarrow C \longrightarrow h^0(\beta_{\geq w+1} C).$$

We must have $h^0(\beta_{\geq w+1} C) \neq 0$: otherwise, we would have $C \cong h^0(\beta_{\leq w} C) \in \mathcal{D}_{\leq w}$. Next, from the distinguished triangle

$$\beta_{\geq w+1} A \longrightarrow 0 \longrightarrow \beta_{\geq w+1} C \longrightarrow ,$$

we see that $\beta_{\geq w+1} A \cong \beta_{\geq w+1} C[-1]$. In particular, $h^0(\beta_{\geq w+1} A) = 0$. But then the exact sequence

$$0 \longrightarrow h^0(\beta_{\leq w} A) \longrightarrow A \longrightarrow h^0(\beta_{\geq w+1} A) = 0$$

shows that $A \cong h^0(\beta_{\leq w} A) \in \mathcal{D}_{\leq w}$, and hence that $\beta_{\geq w+1} A = 0$ and $\beta_{\geq w+1} C = 0$. Thus, A and C are in $\mathcal{D}_{\leq w}$, as desired.

That $\mathcal{D}_{\geq w} \cap \mathcal{C}$ is stable under extensions follows immediately from the fact that $\mathcal{D}_{\geq w}$ is stable under extensions.

(5) Let $A \rightarrow B \rightarrow C \rightarrow$ be a distinguished triangle with $A \in {}^s\mathcal{D}^{\leq 0}$ and $C \in {}^s\mathcal{D}^{\leq 0}$, and consider the exact sequence

$$h^k(A) \xrightarrow{f} h^k(B) \xrightarrow{g} h^k(C).$$

Since $h^k(A) \in \mathcal{D}_{\leq -k}$, its quotient $\text{im } f$ is in $\mathcal{D}_{\leq -k}$ as well. Similarly, $\text{im } g \in \mathcal{D}_{\leq -k}$ because it is a subobject of $h^k(C)$. Now, from the short exact sequence $0 \rightarrow \text{im } f \rightarrow h^k(B) \rightarrow \text{im } g \rightarrow 0$, we deduce that $h^k(B) \in \mathcal{D}_{\leq -k}$. Thus, $B \in {}^s\mathcal{D}^{\leq 0}$.

On the other hand, if $A \rightarrow B \rightarrow C \rightarrow$ is a distinguished triangle with $A, C \in {}^s\mathcal{D}^{\geq 0}$, consider the distinguished triangle

$$\beta_{\leq k}A \longrightarrow \beta_{\leq k}B \longrightarrow \beta_{\leq k}C \longrightarrow .$$

Since $\beta_{\leq k}A$ and $\beta_{\leq k}C$ lie in $\mathcal{D}^{\geq -k}$, $\beta_{\leq k}B \in \mathcal{D}^{\geq -k}$ as well, so $B \in {}^s\mathcal{D}^{\geq 0}$.

(6) If $A \in \mathcal{D}^{\leq k} \cap \mathcal{D}_{\leq w}$, then $h^i(A[k+w]) = h^{i+k+w}(A) = 0$ if $i > -w$, and $h^i(A[k+w]) \in \mathcal{D}_{\leq w} \subset \mathcal{D}_{\leq -i}$ if $i \leq -w$. Thus, $A[k+w] \in {}^s\mathcal{D}^{\leq 0}$, or $A \in {}^s\mathcal{D}^{\leq k+w}$. Next, suppose $B \in \mathcal{D}^{\geq k} \cap \mathcal{D}_{\geq w}$. Then $\beta_{\leq i}B[k+w] = 0$ if $i < w$, and $\beta_{\leq i}B[k+w] \in \mathcal{D}^{\geq k}[k+w] = \mathcal{D}^{\geq -w} \subset \mathcal{D}^{\geq -i}$ if $i \geq w$. Hence, $B[k+w] \in {}^s\mathcal{D}^{\geq 0}$, or $B \in {}^s\mathcal{D}^{\geq k+w}$. ■

Proposition 2.10. Let \mathcal{D} be a triangulated category equipped with compatible t - and baric structures. Assume the t -structure is bounded.

- (1) $\text{Hom}(A, B) = 0$ for all $A \in {}^s\mathcal{D}^{\leq 0}$ and $B \in {}^s\mathcal{D}^{\geq 1}$.
- (2) If $\text{Hom}(A, B) = 0$ for all $B \in {}^s\mathcal{D}^{\geq 1}$, then $A \in {}^s\mathcal{D}^{\leq 0}$. If $\text{Hom}(A, B) = 0$ for all $A \in {}^s\mathcal{D}^{\leq 0}$, then $B \in {}^s\mathcal{D}^{\geq 1}$.
- (3) ${}^s\mathcal{D}^{\leq 0} \subset {}^s\mathcal{D}^{\leq 1}$ and ${}^s\mathcal{D}^{\geq 0} \supset {}^s\mathcal{D}^{\geq 1}$.
- (4) If the baric structure is nondegenerate, there is no nonzero object belonging to all ${}^s\mathcal{D}^{\leq n}$ or to all ${}^s\mathcal{D}^{\geq n}$.
- (5) If the baric structure is also bounded, then for any $A \in D$, there are integers n and m such that $A \in {}^s\mathcal{D}^{\geq n} \cap {}^s\mathcal{D}^{\leq m}$. □

Proof. (1) For any $k \in \mathbb{Z}$, $h^{-k}(A) \in \mathcal{D}_{\leq k}$, and therefore $\text{Hom}(h^{-k}(A), B[-k]) \cong \text{Hom}(h^{-k}(A), \beta_{\leq k}B[-k])$. But $\beta_{\leq k}B \in \mathcal{D}^{\geq -k+1}$, so $\text{Hom}(h^{-k}(A), \beta_{\leq k}B[-k]) = 0$ for all k . It follows from the spectral sequence (2.1) that $\text{Hom}(A, B) = 0$.

(2) Suppose $\text{Hom}(A, B) = 0$ for all $B \in {}^s\mathcal{D}^{\geq 1}$, and suppose for some k , $h^k(A) \notin \mathcal{D}_{\leq -k}$. That implies that $\tau^{\geq k}A \notin \mathcal{D}_{\leq -k}$, so $\beta_{\geq -k+1}\tau^{\geq k}A \neq 0$. In particular, the natural adjunction morphism $A \rightarrow \beta_{\geq -k+1}\tau^{\geq k}A$ is nonzero. However, $\beta_{\geq -k+1}\tau^{\geq k}A \in \mathcal{D}^{\geq k} \cap \mathcal{D}_{\geq -k+1} \subset {}^s\mathcal{D}^{\geq 1}$. This contradicts the assumption that $\text{Hom}(A, B) = 0$ for all $B \in {}^s\mathcal{D}^{\geq 1}$, so we must have $h^k(A) \in \mathcal{D}_{\leq -k}$ for all k , and hence $A \in {}^s\mathcal{D}^{\leq 0}$.

On the other hand, if $\text{Hom}(A, B) = 0$ for all $A \in {}^s\mathcal{D}^{\leq 0}$, a similar argument involving the morphism $\tau^{\leq -k}\beta_{\leq k}B \rightarrow B$ shows that $B \in {}^s\mathcal{D}^{\geq 1}$.

(3) If $A \in {}^s\mathcal{D}^{\leq 0}$, then $h^k(A[1]) = h^{k+1}(A) \in \mathcal{D}_{\leq -k-1} \subset \mathcal{D}_{\leq -k}$, so $A[1] \in {}^s\mathcal{D}^{\leq 0}$, and hence ${}^s\mathcal{D}^{\leq 0} \subset {}^s\mathcal{D}^{\leq 1}$. Similarly, if $B \in {}^s\mathcal{D}^{\geq 0}$, then $\beta_{\leq k}B[-1] \in \mathcal{D}^{\geq -k+1} \subset \mathcal{D}^{\geq -k}$, so $B[-1] \in {}^s\mathcal{D}^{\geq 0}$.

(4) Suppose $A \in {}^s\mathcal{D}^{\leq n}$ for all n . Then $h^k(A) \in \mathcal{D}_{\leq n-k}$ for all n and all k . The nondegeneracy of the baric structure implies that $h^k(A) = 0$; then, the nondegeneracy of

the t -structure implies that $A = 0$. Next, suppose $A \in {}^s\mathcal{D}^{\geq n}$ for all n , and assume $A \neq 0$. Choose some w such that $\beta_{\leq w} A \neq 0$, and then choose some k such that $\tau^{\leq k} \beta_{\leq w} A \neq 0$. By right baryexactness of $\tau^{\leq k}$, we know that $\tau^{\leq k} \beta_{\leq w} A \in \mathcal{D}_{\leq w}$, so we obtain a sequence of isomorphisms:

$$\text{Hom}(\tau^{\leq k} \beta_{\leq w} A, \tau^{\leq k} \beta_{\leq w} A) \cong \text{Hom}(\tau^{\leq k} \beta_{\leq w} A, \beta_{\leq w} A) \cong \text{Hom}(\tau^{\leq k} \beta_{\leq w} A, A).$$

In particular, the natural map $\tau^{\leq k} \beta_{\leq w} A \rightarrow A$ is nonzero. But clearly $\tau^{\leq k} \beta_{\leq w} A \in {}^s\mathcal{D}^{\leq k+w}$, so $A \notin {}^s\mathcal{D}^{\geq k+w+1}$, a contradiction.

(5) This follows from Lemma 2.9(6). ■

We will not prove in general that $({}^s\mathcal{D}^{\leq 0}, {}^s\mathcal{D}^{\geq 0})$ is a t -structure.

Theorem 2.11. Let \mathcal{D} be a triangulated category endowed with compatible bounded, nondegenerate t - and baric structures. Suppose that we have a function $\mu : \mathcal{D} \rightarrow \mathbb{N}$ with the following properties:

- (1) $\mu(X) = 0$ if and only if $X = 0$.
- (2) If $X \in \mathcal{D}^{\geq n}$ but $X \notin \mathcal{D}^{\geq n+1}$, then $\mu(\tau^{\geq n+1} \beta_{\leq -n} X) < \mu(X)$.

Then $({}^s\mathcal{D}^{\leq 0}, {}^s\mathcal{D}^{\geq 0})$ is a bounded, nondegenerate t -structure on \mathcal{D} . □

Proof. It will be convenient to use “ $*$ ” operation on triangulated categories (cf. [4, Section 1.3.9]): given two classes of objects $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$, we denote by $\mathcal{A} * \mathcal{B}$ the class of all objects $X \in \mathcal{D}$ such that there exists a distinguished triangle $A \rightarrow X \rightarrow B \rightarrow$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. In view of the preceding proposition, the present theorem will be proved once we show that every object of \mathcal{D} belongs to ${}^s\mathcal{D}^{\leq 0} * {}^s\mathcal{D}^{\geq 1}$. We proceed by induction on $\mu(X)$. If $\mu(X) = 0$, then $X = 0$, and there is nothing to prove. Otherwise, let n be the smallest integer such that $h^n(X) \neq 0$. Let $A_1 = \tau^{\leq n} \beta_{\leq -n} X$, $X' = \tau^{\geq n+1} \beta_{\leq -n} X$, and $B_1 = \beta_{\geq -n+1} X$. It follows from the right baryexactness of $\tau^{\leq n}$ that $A_1 \in {}^s\mathcal{D}^{\leq 0}$, and, similarly, it follows from the left t -exactness of $\beta_{\geq -n+1}$ that $B_1 \in {}^s\mathcal{D}^{\geq 1}$. Recall [4, Proposition 1.3.10] that the “ $*$ ” operation is associative. By construction, we have

$$X \in \{A_1\} * \{X'\} * \{B_1\} \subset {}^s\mathcal{D}^{\leq 0} * \{X'\} * {}^s\mathcal{D}^{\geq 1}.$$

Since $\mu(X') < \mu(X)$ by assumption, we know that $X' \in {}^s\mathcal{D}^{\leq 0} * {}^s\mathcal{D}^{\geq 1}$, and hence

$$X \in {}^s\mathcal{D}^{\leq 0} * {}^s\mathcal{D}^{\leq 0} * {}^s\mathcal{D}^{\geq 1} * {}^s\mathcal{D}^{\geq 1}.$$

Since ${}^s\mathcal{D}^{\leq 0}$ and ${}^s\mathcal{D}^{\geq 1}$ are stable under extensions, we have ${}^s\mathcal{D}^{\leq 0} * {}^s\mathcal{D}^{\leq 0} = {}^s\mathcal{D}^{\leq 0}$ and ${}^s\mathcal{D}^{\geq 1} * {}^s\mathcal{D}^{\geq 1} = {}^s\mathcal{D}^{\geq 1}$, so $X \in {}^s\mathcal{D}^{\leq 0} * {}^s\mathcal{D}^{\geq 1}$, as desired. ■

3 Examples

In this section, we exhibit several examples of baric structures occurring “in nature.” In the first one, the staggering operation of Definition 2.8 is a new approach to a known t -structure. In two others, this operation gives what appears to be a previously unknown t -structure. The main example of this paper—baric structures on derived categories of coherent sheaves—will be discussed in the next section.

3.1 Perverse sheaves

Let X be a topologically stratified space (as in [9]), with all strata of even real dimension. (This example can be easily modified to relax that condition, or to treat stratified varieties over a field instead.) Let $D = D_c^b(X)$ be the bounded derived category of sheaves of complex vector spaces that are constructible with respect to the given stratification. For any $w \in \mathbb{Z}$, let X_w be the union of all strata of dimension at most $2w$. (Thus, $X_w = \emptyset$ if $w < 0$.) This is a closed subspace of X . Let $i_w : X_w \rightarrow X$ be the inclusion map. Let $D_{\leq w}$ be the full subcategory consisting of complexes the support of which is contained in X_w , and let $D_{\geq w+1}$ be the full subcategory of complexes \mathcal{F} such that $i_w^! \mathcal{F} = 0$.

If $\mathcal{F} \in D_{\leq w}$ and $\mathcal{G} \in D_{\geq w+1}$, then $\mathcal{F} \cong i_{w*} i_w^{-1} \mathcal{F}$, and

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(i_{w*} i_w^{-1} \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(i_w^{-1} \mathcal{F}, i_w^! \mathcal{G}) = 0.$$

Next, let $j_{w+1} : (X \setminus X_w) \rightarrow X$ be the open inclusion of the complement of X_w . For any complex \mathcal{F} , the distinguished triangle

$$i_{w*} i_w^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow (j_{w+1})_* j_{w+1}^{-1} \mathcal{F} \longrightarrow$$

is one the first term of which lies in $D_{\leq w}$ and the last term of which lies in $D_{\geq w+1}$. Thus, we see that $(\{D_{\leq w}\}, \{D_{\geq w}\})_{w \in \mathbb{Z}}$ is a baric structure on $D_c^b(X)$, with baric truncation functors

$$\beta_{\leq w} = i_{w*} i_w^! \quad \text{and} \quad \beta_{\geq w} = j_{w*} j_w^{-1}.$$

It is easy to see that this baric structure is compatible with the standard t -structure on D . If \mathcal{F} is supported on X_w , it is obvious that any truncation of it is as well,

so $D_{\leq w}$ is stable under $\tau^{\leq n}$ and $\tau^{\geq n}$. On the other hand, it is clear from the formulas above that $\beta_{\leq w}$ and $\beta_{\geq w}$ are both left t -exact.

In the associated staggered t -structure $({}^sD^{\leq 0}, {}^sD^{\geq 0})$, we have $\mathcal{F} \in {}^sD^{\leq 0}$ if and only if $h^k(\mathcal{F}) \in D_{\leq -k}$, or, in other words,

$$\dim \operatorname{supp} h^k(\mathcal{F}) \leq -2k.$$

The staggered t -structure in this case is none other than the perverse t -structure of middle perversity.

3.2 Quasi-exceptional sets

Let \mathcal{D} be a triangulated category. A set of objects $\{\nabla^w\}_{w \in \mathbb{N}}$ in \mathcal{D} indexed by nonnegative integers is called a *quasi-exceptional set* if it generates \mathcal{D} as a triangulated category, and if the following conditions hold:

- (1) If $v < w$, then $\operatorname{Hom}(\nabla^v, \nabla^w[k]) = 0$ for all $k \in \mathbb{Z}$.
- (2) For any $w \in \mathbb{N}$, $\operatorname{Hom}(\nabla^w, \nabla^w[k]) = 0$ if $k < 0$, and $\operatorname{End}(\nabla^w)$ is a division ring.

For $w \in \mathbb{N}$, let $\mathcal{D}_{\leq w}$ be the full triangulated subcategory of \mathcal{D} generated by $\nabla^0, \dots, \nabla^w$, and for an integer $w < 0$, let $\mathcal{D}_{\leq w}$ be the full triangulated subcategory containing only zero objects. (Here, we are following the notation of [6], but this will turn out to be consistent with our notation for baric structures as well.) A quasi-exceptional set is *dualizable* if there is another collection of objects $\{\Delta_w\}_{w \in \mathbb{N}}$ such that

- (3) If $v > w$, then $\operatorname{Hom}(\Delta_v, \nabla^w[k]) = 0$ for all $k \in \mathbb{Z}$.
- (4) For any $w \in \mathbb{N}$, we have $\Delta_w \cong \nabla^w \operatorname{mod} \mathcal{D}_{\leq w-1}$.

The last condition means that Δ_w and ∇^w give rise to isomorphic objects in the quotient category $\mathcal{D}_{\leq w} / \mathcal{D}_{\leq w-1}$.

Next, let $\mathcal{D}_{\geq w}$ be the full triangulated subcategory generated by the objects $\{\nabla^k \mid k \geq w\}$. If $A \in \mathcal{D}_{\leq w}$ and $B \in \mathcal{D}_{\geq w+1}$, then Axiom (1) above implies that $\operatorname{Hom}(A, B) = 0$. In addition, by [6, Lemma 4(e)], each inclusion $\mathcal{D}_{\leq w} \rightarrow \mathcal{D}_{\leq w+1}$ admits a right adjoint ι_w . By a straightforward argument, these functors can be used to construct distinguished triangles as in Definition 2.1(3). Thus, $(\{\mathcal{D}_{\leq w}\}, \{\mathcal{D}_{\geq w}\})_{w \in \mathbb{Z}}$ is a baric structure on \mathcal{D} . It is nondegenerate and bounded by construction.

A key result of [6] is the construction of a bounded, nondegenerate t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ associated to a quasi-exceptional set. This t -structure is defined as follows

(see [6, Proposition 1]):

$$\mathfrak{D}^{\leq 0} = \langle \{\Delta_w[n] \mid n \geq 0\} \rangle,$$

$$\mathfrak{D}^{\geq 0} = \langle \{\nabla^w[n] \mid n \leq 0\} \rangle.$$

Here, the notation $\langle S \rangle$ stands for the smallest strictly full subcategory of \mathfrak{D} that is stable under extensions and contains all objects in the set S .

We claim that this t -structure and the baric structure defined above are compatible. It follows from Axiom (1) above that

$$\beta_{\leq w} \nabla^v = \begin{cases} 0 & \text{if } w < v, \\ \nabla^v & \text{if } w \geq v, \end{cases} \quad \text{and} \quad \beta_{\geq w} \nabla^v = \begin{cases} 0 & \text{if } w > v, \\ \nabla^v & \text{if } w \leq v. \end{cases}$$

This calculation shows that the baric truncation functors preserve $\mathfrak{D}^{\geq 0}$. On the other hand, Axiom (3) implies that $\tau^{\leq 0} \nabla^w$ is contained in the subcategory generated by $\Delta_0, \dots, \Delta_w$, and that subcategory coincides with $\mathfrak{D}_{\leq w}$ by Axiom (4). Thus, $\tau^{\leq 0}$ preserves $\mathfrak{D}_{\leq w}$, so $\tau^{\geq 0}$ does as well.

Finally, given a nonzero object $X \in \mathfrak{D}$, let $a(X)$ be the smallest integer n such that $X \in \mathfrak{D}^{\geq -n}$, and let $b(X)$ be the smallest integer w such that $X \in \mathfrak{D}_{\leq w}$. Note that $b(X) \geq 0$. Let

$$\mu(X) = \begin{cases} \max\{a(X) + 1, b(X)\} + 1 & \text{if } X \neq 0, \\ 0 & \text{if } X = 0. \end{cases}$$

Clearly, μ takes nonnegative integer values, and $\mu(X) = 0$ if and only if $X = 0$. Moreover, if $a(X) = -n$ (which implies $\mu(X) \geq -n + 2$), then $a(\tau^{\geq n+1} \beta_{\leq -n} X) \leq -n - 1$ and $b(\tau^{\geq n+1} \beta_{\leq -n} X) \leq -n$, so $\mu(\tau^{\geq n+1} \beta_{\leq -n} X) \leq -n + 1$. Thus, the conditions of Theorem 2.11 are satisfied, and there is a staggered t -structure $({}^s\mathfrak{D}^{\leq 0}, {}^s\mathfrak{D}^{\geq 0})$ on \mathfrak{D} .

3.3 Weight truncation for ℓ -adic mixed constructible sheaves

Let X be a scheme of finite type over a finite field \mathbb{F}_q , and let ℓ be a fixed prime number distinct from the characteristic of \mathbb{F}_q . Let $D = D_m^b(X, \mathbb{Q}_\ell)$ be the bounded derived category of mixed constructible \mathbb{Q}_ℓ -sheaves on X . Let ${}^p\mathcal{H}^n$ denote the n th cohomology functor with respect to the perverse t -structure on D with respect to the middle perversity. Let $D_{\leq w}$ (resp. $D_{\geq w}$) be the full subcategory of $D_m^b(X, \mathbb{Q}_\ell)$ consisting of objects \mathcal{F} such that ${}^p\mathcal{H}^n(\mathcal{F})$ is of weight $\leq w$ (resp. $\geq w$) for all $n \in \mathbb{Z}$. Morel has shown [12, Proposition 4.1.1] that $(\{D_{\leq w}\}, \{D_{\geq w}\})_{w \in \mathbb{Z}}$ is a baric structure on $D_m^b(X, \mathbb{Q}_\ell)$.

Since all objects in the heart of this t -structure have finite length, we may attach a nonnegative integer $\mu(\mathcal{F})$ to each complex \mathcal{F} by the formula

$$\mu(\mathcal{F}) = \sum_{n \in \mathbb{Z}} (\text{length of } {}^p\mathcal{H}^n(\mathcal{F})).$$

Moreover, by [12, Proposition 4.1.3], the baric truncation functors are t -exact for the perverse t -structure. This implies that μ satisfies the assumptions of Theorem 2.11, so the perverse t -structure on $D_m^b(X, \mathbb{Q}_\ell)$ can be staggered with respect to Morel’s baric structure to obtain a new t -structure. The authors are not aware of any previous appearance of this “staggered-perverse” t -structure on ℓ -adic mixed constructible sheaves.

3.4 f -structures

Some instances of the staggering construction have appeared much earlier, in the work Beilinson et al. [5] on f -structures. (The authors are grateful to the referee for bringing this paper to their attention.) An f -structure is identical to a baric structure, and the basic notions and results of Section 2.1 all appear in [5], with slightly different terminology. That paper also introduces a notion of *compatible* f - and t -structures that is narrower than ours: they require the truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$ to be baryexact, and the baric truncation functors $\beta_{\leq w}$ and $\beta_{\geq w}$ to be t -exact.

Under the additional assumption (called “condition (F)” in [5]) that the baric truncation functors admit adjoints on *both* sides, they describe a construction for producing new t -structures on a triangulated category with compatible f - and t -structures. This construction requires as additional input a function $p: \mathbb{Z} \rightarrow \mathbb{Z}$. For the function $p(n) = -n$, it produces the staggered t -structure of the present paper, but many other t -structures can be produced in this way as well. However, the most important examples of staggered t -structures in the present paper do not arise by the construction of [5], since they come from t - and baric structures that are not compatible in the stronger sense of that paper.

3.5 Diagonal Complexes and Torsion Pairs

We conclude with two examples of t -structures that closely resemble staggered t -structures, although they do not in general arise by staggering with respect to a baric structure. (The authors thank Ramachandran and the referee for pointing out these works to them.) First, let \mathcal{D} be a triangulated category with a bounded, nondegenerate t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, and as usual, let $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. Suppose

$\{\mathcal{C}_{\leq w}\}_{w \in \mathbb{Z}}$ is an increasing collection of Serre subcategories of \mathcal{C} , and let $\mathcal{C}_{\geq w} = \{B \in \mathcal{C} \mid \text{Hom}(A, B) = 0 \text{ for all } A \in \mathcal{C}_{\leq w-1}\}$. Following Ekedahl [8], the collection $\{\mathcal{C}_{\leq w}\}$ is called a *radical filtration* of the pair $(\mathcal{D}, \mathcal{C})$ if the following axioms hold:

- (1) For each object $A \in \mathcal{C}$, there exist integers v, w such that $A \in \mathcal{C}_{\geq v} \cap \mathcal{C}_{\leq w}$.
- (2) If $A \in \mathcal{C}_{\leq w}$ and $B \in \mathcal{C}_{\geq v}$, then $\text{Hom}^{v-w-1}(A, B) = 0$ in \mathcal{D} .

If $(\mathcal{D}, \mathcal{C})$ is equipped with a radical filtration, Ekedahl shows that the categories

$$\begin{aligned} \tilde{\mathcal{D}}^{\leq 0} &= \{A \in \mathcal{D} \mid h^k(A) \in \mathcal{C}_{\leq -k} \text{ for all } k \in \mathbb{Z}\}, \\ \tilde{\mathcal{D}}^{\geq 0} &= \{B \in \mathcal{D} \mid h^k(B) \in \mathcal{C}_{\geq -k} \text{ for all } k \in \mathbb{Z}\} \end{aligned} \tag{3.1}$$

constitute a bounded, nondegenerate t -structure on \mathcal{D} . This is called the *diagonal t -structure*, and the objects in its heart are called *diagonal complexes*.

A related notion appears in the work of Happel–Reiten–Smalø [10] on tilting theory. Recall that a *torsion pair* in \mathcal{C} is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that the following conditions hold:

- (1) $\text{Hom}(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
- (2) For any object $X \in \mathcal{C}$, there is a short exact sequence $0 \rightarrow t(X) \rightarrow X \rightarrow X/t(X) \rightarrow 0$ with $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$.

Assume now that $\mathcal{D} = D^b(\mathcal{C})$. A key construction of [10] associates to the torsion pair $(\mathcal{T}, \mathcal{F})$ a new t -structure on \mathcal{D} . This t -structure can also be described by the formulas (3.1) if we set $\mathcal{C}_{\leq -1} = \mathcal{C}_{\geq 2} = 0$, $\mathcal{C}_{\leq 0} = \mathcal{T}$, and $\mathcal{C}_{\geq 1} = \mathcal{F}$. (Since \mathcal{T} and \mathcal{F} are not required to be Serre subcategories, this is not quite an instance of Ekedahl's construction.)

The formulas (3.1) are, of course, strongly reminiscent of those in Definition 2.8. Let us comment briefly on the relationship between the two constructions. Given a radical filtration or a torsion pair, one could hope to define a baric structure by setting $\mathcal{D}_{\leq w} = \{A \in \mathcal{D} \mid h^k(A) \in \mathcal{C}_{\leq w} \text{ for all } k \in \mathbb{Z}\}$. However, the construction of a baric truncation functor turns out to require a stronger Hom-vanishing condition between $\mathcal{C}_{\leq w}$ and $\mathcal{C}_{\geq w+1}$ than that stated above: one needs something like Lemma 2.9(3).

Conversely, given a baric structure, one could hope to define a radical filtration by setting $\mathcal{C}_{\leq w} = \mathcal{D}_{\leq w} \cap \mathcal{C}$. This fails, because a baric structure imposes no higher Hom-vanishing conditions on the right-orthogonal of $\mathcal{C}_{\leq w}$. However, in the special case where $\mathcal{D}_{\leq -1} = \mathcal{D}_{\geq 2} = 0$, the pair $(\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 1})$ is indeed a torsion pair in the sense of [10], and the staggered t -structure on \mathcal{D} coincides with the t -structure of [10].

4 Baric Structures on Coherent Sheaves

In this section, we will investigate baric structures on derived categories of coherent sheaves. Let X be a scheme of finite type over a Noetherian base scheme, and let G be an affine group scheme over the same base, acting on X . We adopt the convention that all statements about subschemes are to be understood in the G -invariant sense. Thus, “open subscheme” will always mean “ G -stable open subscheme,” and “irreducible” will mean “not a union of two proper G -stable closed subschemes.” This convention will remain in effect for the remainder of the paper.

Let $\mathcal{C}_G(X)$ and $\mathcal{Q}_G(X)$ denote the categories of G -equivariant coherent and quasi-coherent sheaves, respectively, on X . One of the headaches of the subject is the need to work with three closely related triangulated categories, which we denote as follows:

- (1) $\mathcal{D}_G^b(X)$ is the bounded derived category of $\mathcal{C}_G(X)$.
- (2) $\mathcal{D}_G^-(X)$ is the bounded-above derived category of $\mathcal{C}_G(X)$.
- (3) $\mathcal{D}_G^+(X)$ is the full subcategory of the bounded-below derived category of $\mathcal{Q}_G(X)$ consisting of objects with coherent cohomology sheaves.

$\mathcal{D}_G^b(X)$ will be the focus of our attention, but it will be necessary to work with $\mathcal{D}_G^-(X)$ and $\mathcal{D}_G^+(X)$ as well, simply because most operations on sheaves take values in one of those categories, even when acting on bounded complexes.

Definition 4.1. A *baric structure* on X is a baric structure on $\mathcal{D}_G^b(X)$ which is compatible with the standard t -structure. □

Remark 4.2. Implicit in this definition are some finiteness conditions; for example, it is conceivable that there are interesting baric structures on $\mathcal{D}_G^+(X)$ that take advantage of the fact that the functors $\beta_{\leq w}$ can take bounded complexes to unbounded complexes. Nevertheless, this is the definition we will work with. □

Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a baric structure on X . Inspired by parts (1) and (2) of Lemma 2.9, we define the following subcategories of $\mathcal{D}_G^-(X)$ and $\mathcal{D}_G^+(X)$:

$$\mathcal{D}_G^-(X)_{\leq w} = \{\mathcal{F} \in \mathcal{D}_G^-(X) \mid h^k(\mathcal{F}) \in \mathcal{D}_G^b(X)_{\leq w} \text{ for all } k\},$$

$$\mathcal{D}_G^+(X)_{\geq w} = \{\mathcal{F} \in \mathcal{D}_G^+(X) \mid \beta_{\leq w-1} \tau^{\leq k} \mathcal{F} \in \mathcal{D}_G^b(X)^{\geq k+2} \text{ for all } k\}.$$

It is unknown whether these categories constitute parts of baric structures on $\mathcal{D}_G^-(X)$ or on $\mathcal{D}_G^+(X)$. Nevertheless, they will be useful in the sequel, in part because they admit the alternate characterization given in the lemma below. If Y is another scheme endowed with a baric structure, we will, by a minor abuse of terminology, call a functor $\mathcal{D}_G^-(X) \rightarrow \mathcal{D}_G^-(Y)$ *right baryexact* if it takes objects of $\mathcal{D}_G^-(X)_{\leq w}$ to objects of $\mathcal{D}_G^-(Y)_{\leq w}$. Similarly, we call a functor $\mathcal{D}_G^+(X) \rightarrow \mathcal{D}_G^+(Y)$ *left baryexact* if it takes objects of $\mathcal{D}_G^+(X)_{\geq w}$ to $\mathcal{D}_G^+(Y)_{\geq w}$.

Lemma 4.3. Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a baric structure on X .

- (1) For $\mathcal{F} \in \mathcal{D}_G^-(X)$, we have $\mathcal{F} \in \mathcal{D}_G^-(X)_{\leq w}$ if and only if $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathcal{D}_G^b(X)_{\geq w+1}$.
- (2) For $\mathcal{F} \in \mathcal{D}_G^+(X)$, we have $\mathcal{F} \in \mathcal{D}_G^+(X)_{\geq w}$ if and only if $\text{Hom}(\mathcal{G}, \mathcal{F}) = 0$ for all $\mathcal{G} \in \mathcal{D}_G^b(X)_{\leq w-1}$. □

In particular, we see from this lemma that

$$\begin{aligned} \mathcal{D}_G^-(X)_{\leq w} \cap \mathcal{D}_G^b(X) &= \mathcal{D}_G^b(X)_{\leq w}, \\ \mathcal{D}_G^+(X)_{\geq w} \cap \mathcal{D}_G^b(X) &= \mathcal{D}_G^b(X)_{\geq w}. \end{aligned} \tag{4.1}$$

Proof. (1) Suppose $\mathcal{F} \in \mathcal{D}_G^-(X)_{\leq w}$. By Lemma 2.9(1), $\tau^{\geq k}\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$ for all k . In particular, given $\mathcal{G} \in \mathcal{D}_G^b(X)_{\geq w+1}$, let k be such that $\mathcal{G} \in \mathcal{D}_G^b(X)^{\geq k}$. Then $\text{Hom}(\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\tau^{\geq k}\mathcal{F}, \mathcal{G}) = 0$. Conversely, suppose $\mathcal{F} \in \mathcal{D}_G^-(X)$ but $\mathcal{F} \notin \mathcal{D}_G^-(X)_{\leq w}$, so that for some k , $h^k(\mathcal{F}) \notin \mathcal{D}_G^b(X)_{\leq w}$. Then $\tau^{\geq k}\mathcal{F} \notin \mathcal{D}_G^b(X)_{\leq w}$. Let $\mathcal{G} = \beta_{\geq w+1}\tau^{\geq k}\mathcal{F}$. We then have a nonzero morphism $\tau^{\geq k}\mathcal{F} \rightarrow \mathcal{G}$. Moreover, since the baric structure on $\mathcal{D}_G^b(X)$ is compatible with the standard t -structure, we have that $\mathcal{G} \in \mathcal{D}_G^b(X)^{\geq k}$, so there is a natural isomorphism $\text{Hom}(\tau^{\geq k}\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\mathcal{F}, \mathcal{G})$. Thus, $\text{Hom}(\mathcal{F}, \mathcal{G}) \neq 0$.

(2) Suppose $\mathcal{F} \in \mathcal{D}_G^+(X)_{\geq w}$. Given $\mathcal{G} \in \mathcal{D}_G^b(X)_{\leq w-1}$, let k be such that $\mathcal{G} \in \mathcal{D}_G^b(X)^{\leq k}$. Then $\text{Hom}(\mathcal{G}, \mathcal{F}) \cong \text{Hom}(\mathcal{G}, \tau^{\leq k}\mathcal{F}) \cong \text{Hom}(\mathcal{G}, \beta_{\leq w-1}\tau^{\leq k}\mathcal{F}) = 0$. Conversely, if $\mathcal{F} \in \mathcal{D}_G^+(X)$ but $\mathcal{F} \notin \mathcal{D}_G^+(X)_{\geq w}$, then for some k , $\beta_{\leq w-1}\tau^{\leq k}\mathcal{F} \notin \mathcal{D}_G^b(X)^{\geq k+2}$. Let $\mathcal{G} = \tau^{\leq k+1}\beta_{\leq w-1}\tau^{\leq k}\mathcal{F}$. Then, clearly $\mathcal{G} \in \mathcal{D}_G^b(X)^{\leq k+1}$ and $\mathcal{G} \in \mathcal{D}_G^b(X)_{\leq w-1}$, and there is a nonzero morphism $\mathcal{G} \rightarrow \beta_{\leq w-1}\tau^{\leq k}\mathcal{F}$. In particular, the group $\text{Hom}(\mathcal{G}, \beta_{\leq w-1}\tau^{\leq k}\mathcal{F}) \cong \text{Hom}(\mathcal{G}, \tau^{\leq k}\mathcal{F})$ is nonzero. Now, consider the exact sequence

$$\text{Hom}(\mathcal{G}, (\tau^{\geq k+1}\mathcal{F})[-1]) \longrightarrow \text{Hom}(\mathcal{G}, \tau^{\leq k}\mathcal{F}) \longrightarrow \text{Hom}(\mathcal{G}, \mathcal{F}).$$

The first term vanishes because $(\tau^{\geq k+1}\mathcal{F})[-1] \in \mathcal{D}_G^+(X)^{\geq k+2}$, so the natural map $\text{Hom}(\mathcal{G}, \tau^{\leq k}\mathcal{F}) \rightarrow \text{Hom}(\mathcal{G}, \mathcal{F})$ is injective. It follows that $\text{Hom}(\mathcal{G}, \mathcal{F}) \neq 0$. ■

4.1 Hereditary, local, and right baric structures

We do not wish to work with arbitrary baric structures on $\mathcal{D}_G^b(X)$; rather, we want them to be well behaved in relation to the scheme structure on X . We have already imposed the condition that the baric structure be compatible with the standard t -structure. We may also ask that it give rise to baric structures on subschemes, in the following sense.

Definition 4.4. Suppose X is equipped with a baric structure, and let $\kappa : Y \hookrightarrow X$ be a locally closed subscheme. A baric structure on Y is said to be *induced* by the one on X if $L\kappa^*$ is right baryexact and $R\kappa^!$ is left baryexact. □

The class of “HLR (hereditary, local, and rigid) baric structures,” defined below, is particularly well behaved. For instance, every locally closed subscheme of a scheme with an HLR baric structure admits a unique induced baric structure. (See Theorem 4.10.) The remainder of Section 4 is devoted to establishing various properties of HLR baric structures, and the main result of the paper, Theorem 6.4, is a statement about a class of nontrivial HLR baric structures.

Definition 4.5. A baric structure on X is said to be *hereditary* if every closed subscheme admits an induced baric structure. A hereditary baric structure on X is said to be *local* if every open subscheme admits an induced baric structure that is also hereditary.

Next, a hereditary baric structure on X is *rigid* if for every sequence of closed subschemes $Z \xrightarrow{t} Z_1 \hookrightarrow X$ where Z_1 is a nilpotent thickening of Z (i.e., Z_1 has the same underlying topological space as Z), the induced baric structures on Z and Z_1 are related as follows:

$$\begin{aligned} \mathcal{D}_G^b(Z_1)_{\leq w} &= \text{the thick closure of } t_*(\mathcal{D}_G^b(Z)_{\leq w}), \\ \mathcal{D}_G^b(Z_1)_{\geq w} &= \text{the thick closure of } t_*(\mathcal{D}_G^b(Z)_{\geq w}). \end{aligned} \tag{4.2}$$

Finally, a baric structure that is hereditary, local, and rigid is called an *HLR baric structure*. □

It turns out that the “local” and “rigid” conditions on an HLR baric structure are redundant.

Theorem 4.6. Every hereditary baric structure is HLR. □

This theorem will be proved in Section 4.3. We first require a couple of preliminary lemmas about induced baric structures, proved below. Following that, in Section 4.2, we will establish a number of useful properties of HLR baric structures.

Lemma 4.7. Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a baric structure on X , and let $i : Z \hookrightarrow X$ be a closed subscheme. If Z admits an induced baric structure, it is given by

$$\begin{aligned} \mathcal{D}_G^b(Z)_{\leq w} &= \{\mathcal{F} \in \mathcal{D}_G^b(Z) \mid i_*\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}\}, \\ \mathcal{D}_G^b(Z)_{\geq w} &= \{\mathcal{F} \in \mathcal{D}_G^b(Z) \mid i_*\mathcal{F} \in \mathcal{D}_G^b(X)_{\geq w}\}. \end{aligned} \tag{4.3}$$

Conversely, if the categories (4.3) constitute a baric structure on Z , then that baric structure is induced from the one on X .

If an open subscheme $j : U \hookrightarrow X$ admits an induced baric structure, it is given by

$$\begin{aligned} \mathcal{D}_G^b(U)_{\leq w} &= \{\mathcal{F} \in \mathcal{D}_G^b(U) \mid \mathcal{F} \cong j^*\mathcal{F}_1 \text{ for some } \mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\leq w}\}, \\ \mathcal{D}_G^b(U)_{\geq w} &= \{\mathcal{F} \in \mathcal{D}_G^b(U) \mid \mathcal{F} \cong j^*\mathcal{F}_1 \text{ for some } \mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\geq w}\}. \end{aligned} \tag{4.4}$$

Conversely, if the categories (4.4) constitute a baric structure on U , then that baric structure is induced from the one on X . □

Proof. Let $(\{\mathcal{D}_G^b(Z)_{\leq w}\}, \{\mathcal{D}_G^b(Z)_{\geq w}\})_{w \in \mathbb{Z}}$ be an induced baric structure on a closed subscheme $i : Z \hookrightarrow X$. If $\mathcal{F} \in \mathcal{D}_G^b(Z)_{\leq w}$, then for all $\mathcal{G} \in \mathcal{D}_G^+(X)_{\geq w+1}$, we have (by Lemma 4.3) that $\text{Hom}(\mathcal{F}, Ri^!\mathcal{G}) = 0$, and therefore $\text{Hom}(i_*\mathcal{F}, \mathcal{G}) = 0$. The latter implies that $i_*\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$. Similarly, if $\mathcal{F} \in \mathcal{D}_G^b(Z)_{\geq w}$, then $\text{Hom}(Li^*\mathcal{G}, \mathcal{F}) = \text{Hom}(\mathcal{G}, i_*\mathcal{F}) = 0$ for all $\mathcal{G} \in \mathcal{D}_G^-(X)_{\leq w-1}$, so $i_*\mathcal{F} \in \mathcal{D}_G^b(X)_{\geq w}$. For the opposite inclusion, given an object $\mathcal{F} \in \mathcal{D}_G^b(Z)$, form the distinguished triangle

$$i_*\beta_{\leq w}\mathcal{F} \longrightarrow i_*\mathcal{F} \longrightarrow i_*\beta_{\geq w+1}\mathcal{F} \longrightarrow$$

in $\mathcal{D}_G^b(X)$. By the reasoning above, we have $i_*\beta_{\leq w}\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$ and $i_*\beta_{\geq w+1}\mathcal{F} \in \mathcal{D}_G^b(X)_{\geq w+1}$, so the first and last terms above must be the baric truncations of $i_*\mathcal{F}$:

$$i_*\beta_{\leq w}\mathcal{F} \cong \beta_{\leq w}i_*\mathcal{F} \quad \text{and} \quad i_*\beta_{\geq w+1}\mathcal{F} \cong \beta_{\geq w+1}i_*\mathcal{F}.$$

Thus, if $i_*\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$, then $\beta_{\geq w+1}i_*\mathcal{F} = i_*\beta_{\geq w+1}\mathcal{F} = 0$. Since i_* is faithful, this implies that $\beta_{\geq w+1}\mathcal{F} = 0$, so that $\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$. The same argument shows that $i_*\mathcal{F} \in \mathcal{D}_G^b(X)_{\geq w}$ implies that $\mathcal{F} \in \mathcal{D}_G^b(X)_{\geq w}$.

Next, assume the categories (4.3) constitute a baric structure on Z . We will show that this baric structure is induced from the one on X . If $\mathcal{F} \in \mathcal{D}_G^-(X)_{\leq w}$, then $\text{Hom}(\mathcal{F}, i_*\mathcal{G}) = 0$ for all $\mathcal{G} \in \mathcal{D}_G^b(Z)_{\geq w+1}$ by Lemma 4.3, so $\text{Hom}(Li^*\mathcal{F}, \mathcal{G}) = 0$, and hence $Li^*\mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w}$. Similarly, if $\mathcal{F} \in \mathcal{D}_G^+(X)_{\geq w}$, then $\text{Hom}(i_*\mathcal{G}, \mathcal{F}) = \text{Hom}(\mathcal{G}, Ri^!\mathcal{F}) = 0$ for all $\mathcal{G} \in \mathcal{D}_G^b(Z)_{\leq w-1}$, so $Ri^!\mathcal{F} \in \mathcal{D}_G^+(Z)_{\geq w}$. Thus, Li^* is right baryexact, and $Ri^!$ is left baryexact, as desired.

We turn now to open subschemes. Suppose $(\{\mathcal{D}_G^b(U)_{\leq w}\}, \{\mathcal{D}_G^b(U)_{\geq w}\})_{w \in \mathbb{Z}}$ is an induced baric structure on an open subscheme $j : U \hookrightarrow X$. In view of the equalities (4.1), the definition of "induced" implies that $j^* : \mathcal{D}_G^b(X) \rightarrow \mathcal{D}_G^b(U)$ is baryexact. In other words, if $\mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\leq w}$, then $j^*\mathcal{F}_1 \in \mathcal{D}_G^b(U)_{\leq w}$, and if $\mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\geq w}$, then $j^*\mathcal{F}_1 \in \mathcal{D}_G^b(U)_{\geq w}$. Conversely, if $\mathcal{F} \in \mathcal{D}_G^b(U)_{\leq w}$, then there exists some object $\mathcal{F}' \in \mathcal{D}_G^b(X)$ such that $j^*\mathcal{F}' \cong \mathcal{F}$. Form the distinguished triangle $\beta_{\leq w}\mathcal{F}' \rightarrow \mathcal{F}' \rightarrow \beta_{\geq w+1}\mathcal{F}' \rightarrow$, and apply j^* to it. We know that $j^*\beta_{\leq w}\mathcal{F}' \in \mathcal{D}_G^b(U)_{\leq w}$ and that $j^*\beta_{\geq w+1}\mathcal{F}' \in \mathcal{D}_G^b(U)_{\geq w+1}$. Since $j^*\mathcal{F}' \cong \mathcal{F}$, we see from the triangle

$$j^*\beta_{\leq w}\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow j^*\beta_{\geq w+1}\mathcal{F}' \longrightarrow$$

that $j^*\beta_{\geq w+1}\mathcal{F}' \cong \beta_{\geq w+1}\mathcal{F} = 0$, and hence that $\mathcal{F} \cong j^*\beta_{\leq w}\mathcal{F}'$. Thus, setting $\mathcal{F}_1 = \beta_{\leq w}\mathcal{F}'$, we have found an $\mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\leq w}$ such that $j^*\mathcal{F}_1 \cong \mathcal{F}$. The argument for $\mathcal{D}_G^b(U)_{\geq w}$ is similar.

Finally, assume the categories (4.4) constitute a baric structure on U . We must show that this baric structure is induced. Clearly, j^* is baryexact as a functor of bounded derived categories $\mathcal{D}_G^b(X) \rightarrow \mathcal{D}_G^b(U)$. Since j^* is also exact, it commutes with truncation and cohomology functors, and it takes $\mathcal{D}_G^b(X)^{\geq w}$ to $\mathcal{D}_G^b(U)^{\geq w}$. It follows from these observations that it takes $\mathcal{D}_G^-(X)_{\leq w}$ to $\mathcal{D}_G^-(U)_{\leq w}$ and $\mathcal{D}_G^+(X)_{\geq w}$ to $\mathcal{D}_G^+(U)_{\geq w}$. ■

Lemma 4.8. Let $j : U \hookrightarrow X$ be the inclusion of an open subscheme, and let $i : Z \hookrightarrow X$ be the inclusion of a closed subscheme. Assume that U and Z are equipped with baric

structures induced from one on X . Then:

- (1) j^* takes $\mathcal{D}_G^-(X)_{\leq w}$ to $\mathcal{D}_G^-(U)_{\leq w}$ and $\mathcal{D}_G^+(X)_{\geq w}$ to $\mathcal{D}_G^+(U)_{\geq w}$.
- (2) Li^* takes $\mathcal{D}_G^-(X)_{\leq w}$ to $\mathcal{D}_G^-(Z)_{\leq w}$.
- (3) $Ri^!$ takes $\mathcal{D}_G^+(X)_{\geq w}$ to $\mathcal{D}_G^+(Z)_{\geq w}$.
- (4) i_* takes $\mathcal{D}_G^-(Z)_{\leq w}$ to $\mathcal{D}_G^-(X)_{\leq w}$ and $\mathcal{D}_G^+(Z)_{\geq w}$ to $\mathcal{D}_G^+(X)_{\geq w}$. □

Proof. Parts (1)–(3) hold by definition.

(4) We saw in the proof of Lemma 4.7 that as a functor of bounded derived categories $\mathcal{D}_G^b(Z) \rightarrow \mathcal{D}_G^b(X)$, i_* is baryexact. Since i_* is also an exact functor, we have $h^k(i_*\mathcal{F}) \cong i_*h^k(\mathcal{F})$ for any $\mathcal{F} \in \mathcal{D}_G^-(Z)$. Thus, if $\mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w}$, we have $h^k(i_*\mathcal{F}) \in \mathcal{D}_G^b(X)_{\leq w}$ for all k ; in other words, $i_*\mathcal{F} \in \mathcal{D}_G^-(X)_{\leq w}$. On the other hand, suppose $\mathcal{F} \in \mathcal{D}_G^+(Z)_{\geq w}$. Since i_* is exact and baryexact on $\mathcal{D}_G^b(Z)$, we have $i_*\beta_{\leq w-1}\tau^{\leq k}\mathcal{F} \cong \beta_{\leq w-1}\tau^{\leq k}i_*\mathcal{F}$. Moreover, the fact that $\beta_{\leq w-1}\tau^{\leq k}\mathcal{F} \in \mathcal{D}_G^b(Z)^{\geq k+1}$ for all k implies that $i_*\beta_{\leq w-1}\tau^{\leq k}\mathcal{F} \in \mathcal{D}_G^b(X)^{\geq k+1}$ for all k . Thus, $i_*\mathcal{F} \in \mathcal{D}_G^+(X)_{\geq w}$. ■

Lemma 4.9. Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a hereditary baric structure on X , and let $i : Z \hookrightarrow X$ be the inclusion of a closed subscheme. The induced baric structure on Z is also hereditary. □

Proof. Let $\kappa : Y \hookrightarrow Z$ be a closed subscheme of Z . We must show that Y admits a baric structure induced from the one on Z . In fact, we claim that the baric structure on Y induced from the one on X (via $i \circ \kappa : Y \hookrightarrow X$) has the desired property. Suppose $\mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w}$. If $L\kappa^*\mathcal{F} \notin \mathcal{D}_G^-(Y)_{\leq w}$, then there is some $\mathcal{G} \in \mathcal{D}_G^b(Y)_{\geq w+1}$ such that $\text{Hom}(L\kappa^*\mathcal{F}, \mathcal{G}) \neq 0$. Then $\text{Hom}(\mathcal{F}, \kappa_*\mathcal{G}) \neq 0$ and, because i_* is faithful, $\text{Hom}(i_*\mathcal{F}, i_*\kappa_*\mathcal{G}) \neq 0$. But this is impossible, because according to Lemma 4.8, $i_*\mathcal{F} \in \mathcal{D}_G^-(X)_{\leq w}$ and $(i \circ \kappa)_*\mathcal{G} \in \mathcal{D}_G^b(X)_{\geq w+1}$. Thus, $L\kappa^*\mathcal{F} \in \mathcal{D}_G^-(Y)_{\leq w}$. Similarly, if $\mathcal{F} \in \mathcal{D}_G^+(Z)_{\geq w}$, then a consideration of $\text{Hom}(\mathcal{G}, R\kappa^!\mathcal{F})$ and $\text{Hom}(i_*\kappa_*\mathcal{G}, i_*\mathcal{F})$ for $\mathcal{G} \in \mathcal{D}_G^b(Y)_{\leq w-1}$ shows that $R\kappa^!\mathcal{F} \in \mathcal{D}_G^+(Y)_{\geq w}$. Thus, $L\kappa^*$ is right baryexact and $R\kappa^!$ is left baryexact, so the baric structure on Y induced from the one on X is also induced from the one on Z . The induced baric structure on Z is therefore hereditary. ■

4.2 Properties of HLR baric structures

In this section, we prove three useful results about HLR baric structures. First, we prove that the HLR property is inherited by induced baric structures on subschemes. Next, we prove an additional rigidity property for nilpotent thickenings of closed subschemes.

Finally, we prove a “gluing theorem” that states that an HLR baric structure is determined by the baric structures it induces on a closed subscheme and the complementary open subscheme. It should be noted that the proofs of these results depend on Theorem 4.6.

Theorem 4.10. Suppose X is endowed with an HLR baric structure. Every locally closed subscheme $\kappa : Y \hookrightarrow X$ admits a unique induced baric structure. Moreover, this baric structure is also HLR. □

Proof. We have already seen the uniqueness of the induced baric structure in the case of open or closed subschemes, in Lemma 4.7. For a general locally closed subscheme, let us factor the inclusion map $\kappa : Y \rightarrow X$ as a closed imbedding $i : Y \hookrightarrow U$ followed by an open imbedding $j : U \hookrightarrow X$. Then U acquires a unique induced hereditary baric structure from the baric structure on X , and it in turn induces a unique baric structure on its closed subscheme Y . This baric structure is also induced from the one on X : clearly, $L\kappa^* = Li^* \circ j^*$ is right baryexact, and $R\kappa^! = Ri^! \circ j^*$ is left baryexact.

To show that this is the unique baric structure on Y induced from the one on X , we must show that the baryexactness assumptions on $L\kappa^*$ and $R\kappa^!$ imply the same conditions on Li^* and $Ri^!$. (It then follows that any baric structure induced from the one on X is actually induced from the one on U .) Suppose $\mathcal{F} \in \mathcal{D}_G^-(U)_{\leq w}$, and consider a distinguished triangle of the form

$$Li^* \tau^{\leq k-1} \mathcal{F} \longrightarrow Li^* \mathcal{F} \longrightarrow Li^* \tau^{\geq k} \mathcal{F} \longrightarrow .$$

Since $Li^* \tau^{\leq k-1} \mathcal{F} \in \mathcal{D}_G^-(Y)^{\leq k-1}$, we see that $h^k(Li^* \mathcal{F}) \cong h^k(Li^* \tau^{\geq k} \mathcal{F})$. Now, $\tau^{\geq k} \mathcal{F}$ is an object in $\mathcal{D}_G^b(U)_{\leq w}$, so there exists an object $\mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\leq w}$ such that $j^* \mathcal{F}_1 \cong \tau^{\geq k} \mathcal{F}$. By assumption, $L\kappa^* \mathcal{F}_1 \in \mathcal{D}_G^-(Y)_{\leq w}$. But $L\kappa^* \mathcal{F}_1 \cong Li^* \tau^{\geq k} \mathcal{F}$, so we conclude that $h^k(Li^* \tau^{\geq k} \mathcal{F}) \cong h^k(Li^* \mathcal{F}) \in \mathcal{D}_G^b(Y)_{\leq w}$. Thus, $Li^* \mathcal{F} \in \mathcal{D}_G^-(Y)_{\leq w}$.

On the other hand, suppose that $\mathcal{F} \in \mathcal{D}_G^+(U)_{\geq w}$, and consider a distinguished triangle of the form

$$Ri^! \tau^{\leq k} \mathcal{F} \longrightarrow Ri^! \mathcal{F} \longrightarrow Ri^! \tau^{\geq k+1} \mathcal{F} \longrightarrow .$$

Since $Ri^! \tau^{\geq k+1} \mathcal{F} \in \mathcal{D}_G^+(Y)^{\geq k+1}$, we see that $\tau^{\leq k} Ri^! \mathcal{F} \cong \tau^{\leq k} Ri^! \tau^{\leq k} \mathcal{F}$. Next, consider the distinguished triangle

$$Ri^! \beta_{\leq w-1} \tau^{\leq k} \mathcal{F} \longrightarrow Ri^! \tau^{\leq k} \mathcal{F} \longrightarrow Ri^! \beta_{\geq w} \tau^{\leq k} \mathcal{F} \longrightarrow .$$

By assumption, $\beta_{\leq w-1} \tau^{\leq k} \mathcal{F} \in \mathcal{D}_G^b(U)^{\geq k+2}$, so $Ri^! \beta_{\leq w-1} \tau^{\leq k} \mathcal{F} \in \mathcal{D}_G^+(Y)^{\geq k+2}$. It follows that $\tau^{\leq k} Ri^! \tau^{\leq k} \mathcal{F} \cong \tau^{\leq k} Ri^! \beta_{\geq w} \tau^{\leq k} \mathcal{F}$. Now, $\beta_{\geq w} \tau^{\leq k} \mathcal{F} \in \mathcal{D}_G^b(U)_{\geq w}$, so there is some $\mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\geq w}$ such that $j^* \mathcal{F}_1 \cong \beta_{\geq w} \tau^{\leq k} \mathcal{F}$. Since $R\kappa^! \mathcal{F}_1$ belongs to $\mathcal{D}_G^+(Y)_{\geq w}$ by assumption, we have $\beta_{\leq w-1} \tau^{\leq k} R\kappa^! \mathcal{F}_1 \in \mathcal{D}_G^b(Y)^{\geq k+2}$. But we also have $R\kappa^! \mathcal{F}_1 \cong Ri^! \beta_{\geq w} \tau^{\leq k} \mathcal{F}$, and from the chain of isomorphisms

$$\tau^{\leq k} Ri^! \mathcal{F} \cong \tau^{\leq k} Ri^! \tau^{\leq k} \mathcal{F} \cong \tau^{\leq k} Ri^! \beta_{\geq w} \tau^{\leq k} \mathcal{F} \cong \tau^{\leq k} R\kappa^! \mathcal{F}_1,$$

we see that $\beta_{\leq w-1} \tau^{\leq k} Ri^! \mathcal{F} \in \mathcal{D}_G^b(Y)^{\geq k+2}$. Thus, $Ri^! \mathcal{F} \in \mathcal{D}_G^+(Y)_{\geq w}$. We now conclude that any baric structure on Y induced from the one on X is also induced from the one on U , and is therefore uniquely determined.

To show that the induced baric structure on a locally closed subscheme is HLR, it suffices, by Theorem 4.6, to show that it is hereditary. In the case of a closed subscheme, this was done in Lemma 4.9, and in the case of an open subscheme, there is nothing to prove: this property is part of the definition of “local.” The assertion then follows for a general locally closed subscheme, since, by construction, the induced baric structure on such a subscheme is obtained by first passing to an open subscheme, and then to a closed subscheme of that. ■

Next, we turn to nilpotent thickenings of a closed subscheme.

Proposition 4.11. Suppose X is endowed with an HLR baric structure, and let $Z \xrightarrow{t} Z_1 \hookrightarrow X$ be a sequence of closed subschemes of X with the same underlying topological space. Then:

- (1) For $\mathcal{F} \in \mathcal{D}_G^-(Z_1)$, $\mathcal{F} \in \mathcal{D}_G^-(Z_1)_{\leq w}$ if and only if $Lt^* \mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w}$.
- (2) For $\mathcal{F} \in \mathcal{D}_G^+(Z_1)$, $\mathcal{F} \in \mathcal{D}_G^+(Z_1)_{\geq w}$ if and only if $Rt^! \mathcal{F} \in \mathcal{D}_G^+(Z)_{\geq w}$. □

Proof. If $\mathcal{F} \in \mathcal{D}_G^-(Z_1)_{\leq w}$, it is obvious that $Lt^* \mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w}$, since the baric structure on Z is induced from that on Z_1 . Conversely, suppose $\mathcal{F} \in \mathcal{D}_G^-(Z_1)$ and $Lt^* \mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w}$. Then $\text{Hom}(\mathcal{F}, t_* \mathcal{G}) \cong \text{Hom}(Lt^* \mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathcal{D}_G^b(Z)_{\geq w+1}$. But by the definition of “rigid,” $\mathcal{D}_G^b(Z_1)_{\geq w+1}$ is generated as a triangulated category by objects of the form $t_* \mathcal{G}$ with $\mathcal{G} \in \mathcal{D}_G^b(Z)_{\geq w+1}$, so it follows that $\text{Hom}(\mathcal{F}, \mathcal{G}') = 0$ for all $\mathcal{G}' \in \mathcal{D}_G^b(Z_1)_{\geq w+1}$, and hence that $\mathcal{F} \in \mathcal{D}_G^-(Z_1)_{\leq w}$. The proof of part (2) is entirely analogous and will be omitted. ■

Finally, we prove a “gluing theorem” for HLR baric structures.

Theorem 4.12. Suppose X is endowed with an HLR baric structure. Let $i : Z \hookrightarrow X$ be a closed subscheme of X , and let $j : U \hookrightarrow X$ be its open complement. Endow U and Z with the baric structures induced from that on X . Then we have

$$\begin{aligned} \mathcal{D}_G^b(X)_{\leq w} &= \{ \mathcal{F} \in \mathcal{D}_G^b(X) \mid j^* \mathcal{F} \in \mathcal{D}_G^b(U)_{\leq w} \text{ and } Li^* \mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w} \}, \\ \mathcal{D}_G^b(X)_{\geq w} &= \{ \mathcal{F} \in \mathcal{D}_G^b(X) \mid j^* \mathcal{F} \in \mathcal{D}_G^b(U)_{\geq w} \text{ and } Ri^! \mathcal{F} \in \mathcal{D}_G^+(Z)_{\geq w} \}. \end{aligned}$$

In particular, there is a unique HLR baric structure on X which induces the baric structures $(\{\mathcal{D}_G^b(U)_{\leq w}\}, \{\mathcal{D}_G^b(U)_{\geq w}\})_{w \in \mathbb{Z}}$ and $(\{\mathcal{D}_G^b(Z)_{\leq w}\}, \{\mathcal{D}_G^b(Z)_{\geq w}\})_{w \in \mathbb{Z}}$ on U and Z . \square

Proof. If $\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$, then $j^* \mathcal{F} \in \mathcal{D}_G^b(U)_{\leq w}$ and $Li^* \mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w}$ by the definition of the induced baric structure. For the other direction, suppose that $j^* \mathcal{F} \in \mathcal{D}_G^b(U)_{\leq w}$ and $Li^* \mathcal{F} \in \mathcal{D}_G^-(Z)_{\leq w}$. We will prove that $\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$ by showing $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathcal{D}_G^b(X)_{\geq w+1}$.

Fix $\mathcal{G} \in \mathcal{D}_G^b(X)_{\geq w+1}$. We have an exact sequence

$$\lim_{\substack{\longrightarrow \\ Z_1}} \text{Hom}(i_{Z_1*} Li_{Z_1}^* \mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}(j^* \mathcal{F}, j^* \mathcal{G}),$$

where the limit runs over nilpotent thickenings of Z . (See, for instance, [7, Proposition 2 and Lemma 3(a)] for an explanation of this exact sequence.) We have $j^* \mathcal{F} \in \mathcal{D}_G^b(U)_{\leq w}$ and $j^* \mathcal{G} \in \mathcal{D}_G^b(U)_{\geq w+1}$, and by Lemma 4.8, we have $i_{Z_1*} Li_{Z_1}^* \mathcal{F} \in \mathcal{D}_G^-(X)_{\leq w}$ so the first and third terms vanish. We conclude that $\text{Hom}(\mathcal{F}, \mathcal{G})$ also vanishes. The argument for $\mathcal{D}_G^b(X)_{\geq w}$ is similar. \blacksquare

4.3 Proof of Theorem 4.6

In this section, we will prove that the hereditary baric structures are automatically also local and rigid. We begin with a result about baric truncation functors with respect to a hereditary baric structure. If X is endowed with a hereditary baric structure, and $\mathcal{F} \in \mathcal{D}_G^b(X)$ is actually supported on some closed subscheme $i : Z \hookrightarrow X$, then the baric truncations of \mathcal{F} are obtained by taking baric truncations in the induced baric structure on Z , and then pushing them forward by i_* . In other words, hereditary baric structures have the property that baric truncation functors preserve supports. More precisely:

Proposition 4.13. Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a hereditary baric structure on X . Then

- (1) If $\mathcal{F} \in \mathcal{D}_G^b(X)$ has set-theoretic support on a closed set $Z \subset X$, then so do $\beta_{\leq w}\mathcal{F}$ and $\beta_{\geq w}\mathcal{F}$.
- (2) If a morphism $u: \mathcal{F} \rightarrow \mathcal{G}$ in $\mathcal{D}_G^b(X)$ has set-theoretic support on Z , in the sense that $u|_{X \setminus Z} = 0$, then so do $\beta_{\leq w}(u)$ and $\beta_{\geq w}(u)$. □

Proof. If \mathcal{F} is set-theoretically supported on Z then there is a subscheme $i: Z_1 \hookrightarrow X$ of X , the underlying closed set of which is Z , such that $\mathcal{F} = i_*\mathcal{F}'$ for some $\mathcal{F}' \in \mathcal{D}_G^b(Z_1)$. Form the distinguished triangle

$$\beta_{\leq w}\mathcal{F}' \longrightarrow \mathcal{F}' \longrightarrow \beta_{\geq w+1}\mathcal{F}' \longrightarrow .$$

By Lemma 4.7, we have that $i_*\beta_{\leq w}\mathcal{F}' \in \mathcal{D}_G^b(X)_{\leq w}$ and $i_*\beta_{\geq w+1}\mathcal{F}' \in \mathcal{D}_G^b(X)_{\geq w+1}$. Since we have a distinguished triangle

$$i_*\beta_{\leq w}\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow i_*\beta_{\geq w+1}\mathcal{F}' \longrightarrow ,$$

we must have $i_*\beta_{\leq w}\mathcal{F}' \cong \beta_{\leq w}\mathcal{F}$ and $i_*\beta_{\geq w+1}\mathcal{F}' \cong \beta_{\geq w+1}\mathcal{F}$. In particular, these objects are set-theoretically supported on Z , proving the first assertion.

To prove the second assertion, consider the exact sequence

$$\varinjlim_{Z'} \text{Hom}(\mathcal{F}, i_{Z'*}Ri_{Z'}^!\mathcal{G}) \longrightarrow \text{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}(\mathcal{F}|_{X \setminus Z}, \mathcal{G}|_{X \setminus Z}),$$

where $i_{Z'}: Z' \hookrightarrow X$ ranges over all closed subscheme structures on Z . By assumption, $u \in \text{Hom}(\mathcal{F}, \mathcal{G})$ vanishes upon restriction to $X \setminus Z$, so we see from the exact sequence above that it must factor through $i_{Z'*}Ri_{Z'}^!\mathcal{G} \rightarrow \mathcal{G}$ for some closed subscheme structure $i_{Z'}: Z' \hookrightarrow X$ on Z . Now, $i_{Z'*}Ri_{Z'}^!\mathcal{G}$ is in general an object of $\mathcal{D}_G^+(X)$, but since \mathcal{F} lies in $\mathcal{D}_G^b(X)$, any morphism $\mathcal{F} \rightarrow i_{Z'*}Ri_{Z'}^!\mathcal{G}$ factors through $\tau^{\leq n}i_{Z'*}Ri_{Z'}^!\mathcal{G}$ for sufficiently large n . It follows that $\beta_{\leq w}(u)$ and $\beta_{\geq w}(u)$ factor through $\beta_{\leq w}\tau^{\leq n}i_{Z'*}Ri_{Z'}^!\mathcal{G}$ and $\beta_{\geq w}\tau^{\leq n}i_{Z'*}Ri_{Z'}^!\mathcal{G}$, respectively. These objects have set-theoretic support on Z by the first part of the proposition, so $\beta_{\leq w}(u)$ and $\beta_{\geq w}(u)$ have set-theoretic support on Z as well, as desired. ■

We may use this fact to prove the following:

Theorem 4.14. Every hereditary baric structure is local. □

We will prove this theorem over the course of the following three propositions. Recall from Lemma 4.7 that in a local baric structure, the induced baric structures on open subschemes necessarily have the form given in the following proposition.

Proposition 4.15. Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a hereditary baric structure on X , and let U be an open subscheme of X . For any $w \in \mathbb{Z}$, define full subcategories of $\mathcal{D}_G^b(U)$ as follows:

$$\begin{aligned} \mathcal{D}_G^b(U)_{\leq w} &= \{\mathcal{F} \in \mathcal{D}_G^b(U) \mid \mathcal{F} \cong j^* \mathcal{F}_1 \text{ for some } \mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\leq w}\}, \\ \mathcal{D}_G^b(U)_{\geq w} &= \{\mathcal{F} \in \mathcal{D}_G^b(U) \mid \mathcal{F} \cong j^* \mathcal{F}_1 \text{ for some } \mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\geq w}\}. \end{aligned}$$

Then $\mathcal{D}_G^b(U)_{\leq w}$ and $\mathcal{D}_G^b(U)_{\geq w}$ are thick subcategories of $\mathcal{D}_G^b(U)$. □

Proof. Suppose that \mathcal{F} and \mathcal{G} belong to $\mathcal{D}_G^b(U)_{\leq w}$, so that there exist \mathcal{F}_1 and \mathcal{G}_1 in $\mathcal{D}_G^b(X)_{\leq w}$ with $\mathcal{F}_1|_U \cong \mathcal{F}$ and $\mathcal{G}_1|_U \cong \mathcal{G}$. Since $\mathcal{D}_G^b(U)$ is a localization of $\mathcal{D}_G^b(X)$, we may find for every morphism $u: \mathcal{F} \rightarrow \mathcal{G}$ an object $\mathcal{G}_2 \in \mathcal{D}_G^b(X)$ and a diagram $\mathcal{F}_1 \rightarrow \mathcal{G}_2 \leftarrow \mathcal{G}_1$ such that $(\mathcal{G}_2 \leftarrow \mathcal{G}_1)|_U$ is an isomorphism, and the composition

$$\mathcal{F} \cong \mathcal{F}_1|_U \longrightarrow \mathcal{G}_2|_U \cong \mathcal{G}_1|_U \cong \mathcal{G}$$

coincides with u . We claim that the diagram

$$\beta_{\leq w} \mathcal{F}_1 \longrightarrow \beta_{\leq w} \mathcal{G}_2 \longleftarrow \beta_{\leq w} \mathcal{G}_1$$

has the same property. In that case, the cone on the composition $\mathcal{F}_1 \cong \beta_{\leq w} \mathcal{F}_1 \rightarrow \beta_{\leq w} \mathcal{G}_2$ belongs to $\mathcal{D}_G^b(X)_{\leq w}$, which shows that the cone on $u: \mathcal{F} \rightarrow \mathcal{G}$ belongs to $\mathcal{D}_G^b(U)_{\leq w}$. To prove the claim, note that the cone on the map $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ is set-theoretically supported on the closed set $X \setminus U$, and since the baric structure $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is hereditary, the same must be true for the cone on $\beta_{\leq w} \mathcal{G}_1 \rightarrow \beta_{\leq w} \mathcal{G}_2$; in particular, the restriction of the latter map to U is an isomorphism.

We have shown that $\mathcal{D}_G^b(U)_{\leq w} \subset \mathcal{D}_G^b(U)$ is a triangulated subcategory. To show that it is thick we have to show that it is also closed under summands—that is, that if $\mathcal{F} \oplus \mathcal{G} \in \mathcal{D}_G^b(U)_{\leq w}$ then \mathcal{F} and \mathcal{G} also belong to $\mathcal{D}_G^b(U)_{\leq w}$. Thus, suppose that $\mathcal{F} \oplus \mathcal{G}$ belongs

to $\mathcal{D}_G^b(U)_{\leq w}$, and choose an object $\mathcal{H} \in \mathcal{D}_G^b(X)_{\leq w}$ such that $\mathcal{H}|_U \cong \mathcal{F} \oplus \mathcal{G}$. Since $\mathcal{D}_G^b(U)$ is a localization of $\mathcal{D}_G^b(X)$, we may find an object $\mathcal{G}_1 \in \mathcal{D}_G^b(X)$ and a morphism $\mathcal{H} \rightarrow \mathcal{G}_1$ the restriction to U of which is isomorphic to the natural map $\mathcal{F} \oplus \mathcal{G} \rightarrow \mathcal{G}$. We may then complete this to a triangle

$$\mathcal{F}_1 \longrightarrow \mathcal{H} \longrightarrow \mathcal{G}_1 \longrightarrow$$

the restriction to U of which is isomorphic to the triangle

$$\mathcal{F} \longrightarrow \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow .$$

In particular, the map $\mathcal{G}_1 \rightarrow \mathcal{F}_1[1]$ is set-theoretically supported on $X \setminus U$, so by Proposition 4.13 the same must be true of $\beta_{\leq w}\mathcal{G}_1 \rightarrow \beta_{\leq w}\mathcal{F}_1[1]$. From the diagram

$$\begin{array}{ccccccc} \beta_{\leq w}\mathcal{F}_1 & \longrightarrow & \mathcal{H} & \longrightarrow & \beta_{\leq w}\mathcal{G}_1 & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{F}_1 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ \beta_{\geq w+1}\mathcal{F}_1 & \longrightarrow & 0 & \longrightarrow & \beta_{\geq w+1}\mathcal{G}_1 & \longrightarrow & \end{array}$$

the rows and columns of which are distinguished triangles, we see that $\beta_{\geq w+1}\mathcal{G}_1 \rightarrow \beta_{\geq w+1}\mathcal{F}_1[1]$ is an isomorphism. But since this morphism has set-theoretic support on $X - U$ the objects $\beta_{\geq w+1}\mathcal{F}_1$ and $\beta_{\geq w+1}\mathcal{G}_1$ must have set-theoretic support on $X - U$ which implies that there are isomorphisms $\beta_{\leq w}\mathcal{F}_1|_U \cong \mathcal{F}$ and $\beta_{\leq w}\mathcal{G}_1|_U \cong \mathcal{G}$. Thus, \mathcal{F} and \mathcal{G} belong to $\mathcal{D}_G^b(U)_{\leq w}$.

A similar proof shows that the subcategories $\mathcal{D}_G^b(U)_{\geq w}$ are thick. ■

Proposition 4.16. Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a hereditary baric structure on X , let U be an open subscheme of X , and let $(\{\mathcal{D}_G^b(U)_{\leq w}\}, \{\mathcal{D}_G^b(U)_{\geq w}\})_{w \in \mathbb{Z}}$ be as in Proposition 4.15. Then $(\{\mathcal{D}_G^b(U)_{\leq w}\}, \{\mathcal{D}_G^b(U)_{\geq w}\})_{w \in \mathbb{Z}}$ is a baric structure on U . □

Proof. It is clear that $\mathcal{D}_G^b(U)_{\leq w} \subset \mathcal{D}_G^b(U)_{\leq w+1}$ and $\mathcal{D}_G^b(U)_{\geq w} \supset \mathcal{D}_G^b(U)_{\geq w+1}$, and that $\mathcal{D}_G^b(U) = \mathcal{D}_G^b(U)_{\leq w} * \mathcal{D}_G^b(U)_{\geq w+1}$. If $\mathcal{F} \in \mathcal{D}_G^b(U)_{\leq w}$ and $\mathcal{G} \in \mathcal{D}_G^b(U)_{\geq w+1}$, then we have an exact

sequence

$$\mathrm{Hom}(\mathcal{F}_1, \mathcal{G}_1) \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \varinjlim_{i: Z \hookrightarrow X} \mathrm{Hom}(i_* Li^* \mathcal{F}_1, \mathcal{G}_1[1]) \longrightarrow,$$

where \mathcal{F}_1 is an extension of \mathcal{F} to $\mathcal{D}_G^b(X)_{\leq w}$, \mathcal{G}_1 is an extension of \mathcal{G} to $\mathcal{D}_G^b(X)_{\geq w+1}$, and $i: Z \hookrightarrow X$ runs over all subscheme structures on $X \setminus U$. The first term above vanishes automatically, and each of the terms $\mathrm{Hom}(i_* Li^* \mathcal{F}_1, \mathcal{G}_1[1])$ vanishes because, by Lemma 4.8, $i_* Li^* \mathcal{F}_1 \in \mathcal{D}_G^-(X)_{\leq w}$. Thus, $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$ and $(\{\mathcal{D}_G^b(U)_{\leq w}\}, \{\mathcal{D}_G^b(U)_{\geq w}\})_{w \in \mathbb{Z}}$ is a baric structure on $\mathcal{D}_G^b(U)$.

By assumption the baric structure $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is compatible with the standard t -structure on $\mathcal{D}_G^b(X)$. Thus, if \mathcal{F}_1 belongs to $\mathcal{D}_G^b(X)_{\leq w}$ then so do $\tau^{\leq n} \mathcal{F}_1$ and $\tau^{\geq n} \mathcal{F}_1$. The objects $\mathcal{F}_1|_U, (\tau^{\leq n} \mathcal{F}_1)|_U \cong \tau^{\leq n}(\mathcal{F}_1|_U)$ and $(\tau^{\geq n} \mathcal{F}_1)|_U \cong \tau^{\geq n}(\mathcal{F}_1|_U)$ therefore all belong to $\mathcal{D}_G^b(U)_{\leq w}$. Similarly, we have $(\beta_{\leq w} \mathcal{F}_1)|_U \cong \beta_{\leq w}(\mathcal{F}_1|_U)$ and $(\beta_{\geq w} \mathcal{F}_1)|_U \cong \beta_{\geq w}(\mathcal{F}_1|_U)$ so that the baric truncation functors preserve $\mathcal{D}_G^b(U)^{\geq 0}$. Thus, the baric structure $(\{\mathcal{D}_G^b(U)_{\leq w}\}, \{\mathcal{D}_G^b(U)_{\geq w}\})_{w \in \mathbb{Z}}$ is compatible with the standard t -structure on $\mathcal{D}_G^b(U)$. ■

Proposition 4.17. Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a hereditary baric structure on X , and let U be an open subscheme of X . Then the collection of categories $(\{\mathcal{D}_G^b(U)_{\leq w}\}, \{\mathcal{D}_G^b(U)_{\geq w}\})_{w \in \mathbb{Z}}$ defined in Proposition 4.15 constitute a hereditary baric structure on U . □

Proof. Using Lemma 4.7 and the previous proposition, we know that the baric structure $(\{\mathcal{D}_G^b(U)_{\leq w}\}, \{\mathcal{D}_G^b(U)_{\geq w}\})_{w \in \mathbb{Z}}$ is induced from the one on X . It remains only to show that this baric structure is hereditary. Let $i: Y \hookrightarrow U$ be a closed subscheme of U . By Lemma 4.7, we must prove that the following categories constitute a baric structure on Y :

$$\mathcal{D}_G^b(Y)_{\leq w} = \{\mathcal{F} \in \mathcal{D}_G^b(Y) \mid i_* \mathcal{F} \cong \mathcal{F}_1|_U \text{ for some } \mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\leq w}\},$$

$$\mathcal{D}_G^b(Y)_{\geq w} = \{\mathcal{F} \in \mathcal{D}_G^b(Y) \mid i_* \mathcal{F} \cong \mathcal{F}_1|_U \text{ for some } \mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\geq w}\}.$$

Let \bar{Y} be the closure of Y in X , and let $i_1 : \bar{Y} \hookrightarrow X$ be the inclusion map, so that we have a commutative square of inclusions

$$\begin{array}{ccc} Y & \xrightarrow{i} & U \\ \downarrow & & \downarrow \\ \bar{Y} & \xrightarrow{i_1} & X \end{array}$$

By definition, the hereditary baric structure on X induces a baric structure on \bar{Y} . This baric structure is itself hereditary, by Lemma 4.9. Thus, by the previous proposition, the baric structure on \bar{Y} induces one on its open subscheme Y . This is given by

$$\begin{aligned} (\mathcal{D}_G^b(Y)_{\leq w})' &= \{ \mathcal{F} \mid \mathcal{F} \cong \mathcal{F}_2|_Y \text{ for some } \mathcal{F}_2 \in \mathcal{D}_G^b(\bar{Y}) \text{ with } i_{1*}\mathcal{F}_2 \in \mathcal{D}_G^b(X)_{\leq w} \}, \\ (\mathcal{D}_G^b(Y)_{\geq w})' &= \{ \mathcal{F} \mid \mathcal{F} \cong \mathcal{F}_2|_Y \text{ for some } \mathcal{F}_2 \in \mathcal{D}_G^b(\bar{Y}) \text{ with } i_{1*}\mathcal{F}_2 \in \mathcal{D}_G^b(X)_{\geq w} \}. \end{aligned}$$

It suffices now to show that $\mathcal{D}_G^b(Y)_{\leq w} = (\mathcal{D}_G^b(Y)_{\leq w})'$ and $\mathcal{D}_G^b(Y)_{\geq w} = (\mathcal{D}_G^b(Y)_{\geq w})'$. If $\mathcal{F} \in \mathcal{D}_G^b(Y)$ is such that we may find $\mathcal{F}_2 \in \mathcal{D}_G^b(\bar{Y})$ with $\mathcal{F}_2|_Y \cong \mathcal{F}$ and $i_{1*}\mathcal{F}_2 \in \mathcal{D}_G^b(X)_{\leq w}$ then $\mathcal{F}_1 := i_{1*}\mathcal{F}_2$ has the property that $\mathcal{F}_1|_U \cong i_*\mathcal{F}$. Thus, $(\mathcal{D}_G^b(Y)_{\leq w})' \subset \mathcal{D}_G^b(Y)_{\leq w}$. To show the reverse inclusion, let $\mathcal{F} \in \mathcal{D}_G^b(Y)$ and $\mathcal{F}_1 \in \mathcal{D}_G^b(X)_{\leq w}$ be such that $\mathcal{F}_1|_U \cong i_*\mathcal{F}$. Let \mathcal{F}'_2 be any object of $\mathcal{D}_G^b(\bar{Y})$ such that $\mathcal{F}'_2|_Y \cong \mathcal{F}$. Thus, $(i_{1*}\mathcal{F}'_2)|_U \cong \mathcal{F}_1|_U$. By [7, Corollary 2], there is a third object $\mathcal{G} \in \mathcal{D}_G^b(X)$ together with maps

$$i_{1*}\mathcal{F}'_2 \longleftarrow \mathcal{G} \longrightarrow \mathcal{F}_1,$$

where both maps are isomorphisms over U . Apply the functor $\beta_{\leq w}$ to this diagram. It follows from Proposition 4.16 that $\beta_{\leq w}$ commutes with restriction to U , so the maps in the diagram

$$i_{1*}\beta_{\leq w}\mathcal{F}'_2 \cong \beta_{\leq w}i_{1*}\mathcal{F}'_2 \longleftarrow \beta_{\leq w}\mathcal{G} \longrightarrow \beta_{\leq w}\mathcal{F}_1 \cong \mathcal{F}_1,$$

are still isomorphisms over U . Therefore, the object $\mathcal{F}_2 := \beta_{\leq w}\mathcal{F}'_2$ has the property that $\mathcal{F}_2|_Y \cong \mathcal{F}$ and $i_{1*}\mathcal{F}_2 \in \mathcal{D}_G^b(X)_{\leq w}$. Thus, $(\mathcal{D}_G^b(Y)_{\leq w})' = \mathcal{D}_G^b(Y)_{\leq w}$. A similar argument shows that $(\mathcal{D}_G^b(Y)_{\geq w})' = \mathcal{D}_G^b(Y)_{\geq w}$. ■

Let us finally show that hereditary baric structures are rigid.

Proposition 4.18. Let $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ be a hereditary baric structure on X . Then $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is rigid. \square

Proof. Let Z be a subscheme of X and let Z_1 be a nilpotent thickening of Z in X , and write t for inclusion of Z into Z_1 . If \mathcal{F} is a bounded chain complex of coherent sheaves on Z_1 , then we may find a filtration of \mathcal{F} by subcomplexes \mathcal{F}_k the subquotients of which are scheme-theoretically supported on Z . Thus, in $\mathcal{D}_G^b(Z_1)$ we may find a sequence of objects and maps

$$0 = \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \cdots \longrightarrow \mathcal{F}_n = \mathcal{F}$$

such that the cone on $\mathcal{F}_{k-1} \rightarrow \mathcal{F}_k$ is of the form $t_* \mathcal{G}_k$. Now suppose \mathcal{F} belongs to $\mathcal{D}_G^b(Z_1)_{\leq w}$. Then we may apply $\beta_{\leq w}$ to the sequence to obtain

$$0 = \beta_{\leq w} \mathcal{F}_0 \longrightarrow \beta_{\leq w} \mathcal{F}_1 \longrightarrow \cdots \longrightarrow \beta_{\leq w} \mathcal{F}_n = \mathcal{F}$$

and distinguished triangles

$$\beta_{\leq w} \mathcal{F}_{k-1} \longrightarrow \beta_{\leq w} \mathcal{F}_k \longrightarrow \beta_{\leq w} t_* \mathcal{G}_k \longrightarrow .$$

It follows from Lemma 4.7 that the object $\beta_{\leq w} t_* \mathcal{G}_k$ is isomorphic to $t_* \beta_{\leq w} \mathcal{G}_k$. Thus, \mathcal{F} is in the thick closure of the image of $\mathcal{D}_G^b(Z)_{\leq w}$ under t_* . A similar proof gives the same result for $\mathcal{D}_G^b(Z_1)_{\geq w}$. \blacksquare

This completes the proof of Theorem 4.6.

5 Background on s -structures and Staggered Sheaves

In this section, we review the t -structures on derived categories of equivariant coherent sheaves that were introduced in [1]. (They were called “staggered t -structures” in [1]; in Section 8, we will prove that they usually arise by the staggering construction of Definition 2.8.) These t -structures depend on two auxiliary data: an s -structure, and a *perversity function*. After fixing notation, we briefly recall some facts about these objects, and we then describe the t -structures themselves. We will also prove a few useful lemmas about these objects.

As before, let X be a scheme of finite type over a Noetherian base scheme, acted on by an affine group scheme G over the same base. We adopt the additional assumptions that the base scheme admits a dualizing complex in the sense of

[11, Chapter V], and that the category $\mathcal{C}_G(X)$ has enough locally free objects. It follows (see [7, Proposition 1]) that X admits an equivariant dualizing complex. Fix one, and denote it $\omega_X \in \mathcal{D}_G^b(X)$. Next, let $\mathbb{D} = R\mathcal{H}om(\cdot, \omega_X)$ denote the equivariant Serre–Grothendieck duality functor. Let X^{gen} denote the set of generic points of G -invariant subschemes of X , and for any $x \in X^{\text{gen}}$, we denote by \overline{Gx} the smallest G -stable closed subset of X containing x . (We do not usually regard \overline{Gx} as having a fixed subscheme structure.)

For any point $x \in X^{\text{gen}}$ and any closed subscheme structure $i: Z \hookrightarrow X$ on \overline{Gx} , there is an open subscheme $V \subset Z$ such that $Ri^!\omega_X|_V$ is concentrated in a single degree in $\mathcal{D}_G^b(V)$. Let $\text{cod } \overline{Gx}$ be the unique integer such that $h^{\text{cod } \overline{Gx}}(Ri^!\omega_X|_V) \neq 0$. This number is independent of the choice of closed subscheme structure $i: Z \hookrightarrow X$ and of open subscheme $V \subset Z$. If X is, say, an equidimensional scheme of finite type over a field, ω_X may be normalized so that $\text{cod } \overline{Gx}$ is the ordinary (Krull) codimension of \overline{Gx} .

An *s-structure* on the scheme X is a pair of collections of full subcategories $(\{\mathcal{C}_G(X)_{\leq w}\}, \{\mathcal{C}_G(X)_{\geq w}\})_{w \in \mathbb{Z}}$ of $\mathcal{C}_G(X)$ satisfying a list of nine axioms, called (S1)–(S9) in [1]. We will not review all the axioms here, but we do recall some of the key properties of *s-structures*:

- Each $\mathcal{C}_G(X)_{\leq w}$ is a Serre subcategory, and each $\mathcal{C}_G(X)_{\geq w}$ is closed under extensions and subobjects.
- $\mathcal{C}_G(X)_{\geq w}$ is the right orthogonal to $\mathcal{C}_G(X)_{\leq w-1}$.
- Each sheaf \mathcal{F} contains a unique maximal subsheaf in $\mathcal{C}_G(X)_{\leq w}$, denoted $\sigma_{\leq w}\mathcal{F}$. The quotient $\sigma_{\geq w+1}\mathcal{F} \cong \mathcal{F}/\sigma_{\leq w}\mathcal{F}$ is the largest quotient of \mathcal{F} in $\mathcal{C}_G(X)_{\geq w+1}$.
- An *s-structure* on X induces *s-structures* on all locally closed subschemes of X .

Assume henceforth that X is equipped with a fixed *s-structure*. Given a point $x \in X^{\text{gen}}$ and a closed subscheme structure $i: Z \hookrightarrow X$ on \overline{Gx} , choose an open subscheme $V \subset Z$ such that $Ri^!\omega_X|_V$ is concentrated in degree $\text{cod } \overline{Gx}$. There is a unique integer, called the *altitude* of \overline{Gx} and denoted $\text{alt } \overline{Gx}$, such that

$$Ri^!\omega_X|_V[\text{cod } \overline{Gx}] \in \mathcal{C}_G(V)_{\leq \text{alt } \overline{Gx}} \cap \mathcal{C}_G(V)_{\geq \text{alt } \overline{Gx}}.$$

Again, $\text{alt } \overline{Gx}$ is independent of the choice of i and V .

The *staggered codimension* of \overline{Gx} is defined by

$$\text{scod } \overline{Gx} = \text{alt } \overline{Gx} + \text{cod } \overline{Gx}.$$

A (staggered) *perversity function* is a function $p: X^{\text{gen}} \rightarrow \mathbb{Z}$ such that

$$0 \leq p(x) - p(y) \leq \text{scod } \overline{Gx} - \text{scod } \overline{Gy} \quad \text{if } x \in \overline{Gy}.$$

Given a perversity $p: X^{\text{gen}} \rightarrow \mathbb{Z}$, the function $\bar{p}: X^{\text{gen}} \rightarrow \mathbb{Z}$ given by

$$\bar{p}(x) = \text{scod } \overline{Gx} - p(x)$$

is also a perversity function, known as the *dual perversity*. Given a staggered perversity function p , we define a full subcategory of $\mathcal{D}_G^-(X)$ by

$$\begin{aligned} {}^p\mathcal{D}_G^-(X)^{\leq 0} &= \{ \mathcal{F} \mid \text{for any } x \in X^{\text{gen}}, \text{ any closed subscheme structure } i: Z \hookrightarrow X \text{ on } \overline{Gx}, \\ &\text{and any } k \in \mathbb{Z}, \text{ there is a dense open subscheme } V \subset Z \text{ such that} \\ &h^k(Li^*\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq p(x)-k} \}, \end{aligned}$$

and a full subcategory of $\mathcal{D}_G^+(X)$ by

$${}^p\mathcal{D}_G^+(X)^{\geq 0} = \mathbb{D}({}^p\mathcal{D}_G^-(X)^{\leq 0}).$$

The t -structure associated in [1] to the given s -structure and to a perversity p is the pair $({}^p\mathcal{D}_G^b(X)^{\leq 0}, {}^p\mathcal{D}_G^b(X)^{\geq 0})$, where

$${}^p\mathcal{D}_G^b(X)^{\leq 0} = {}^p\mathcal{D}_G^-(X)^{\leq 0} \cap \mathcal{D}_G^b(X) \quad \text{and} \quad {}^p\mathcal{D}_G^b(X)^{\geq 0} = {}^p\mathcal{D}_G^+(X)^{\geq 0} \cap \mathcal{D}_G^b(X).$$

The remainder of the section will be spent establishing a number of useful lemmas about these objects. Let $q: X^{\text{gen}} \rightarrow \mathbb{Z}$ be a function such that

$$q(x) = q(y) \quad \text{whenever} \quad \overline{Gx} = \overline{Gy}. \tag{5.1}$$

Given such a function, let

$$\begin{aligned} {}_q\mathcal{C}_G(X)_{\leq w} &= \{ \mathcal{F} \in \mathcal{C}_G(X) \mid \text{for any closed subscheme } i: \overline{Gx} \hookrightarrow X \text{ with } x \in X^{\text{gen}}, \\ &\text{there is a dense open subscheme } V \subset \overline{Gx} \text{ such that } i^*\mathcal{F}|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}. \} \end{aligned}$$

One may either regard this definition as a condition only on *reduced* closed subschemes of the form \overline{GX} , or as a condition on all possible closed subscheme structures on the various closed sets \overline{GX} . These two interpretations are equivalent by [1, Proposition 4.1], however, so there is no ambiguity in the definition. The first viewpoint is more convenient for checking explicit examples, but the second is sometimes more useful in proofs.

Lemma 5.1. Let $x \in X^{\text{gen}}$, and let $i : Z \hookrightarrow X$ be a closed subscheme structure on \overline{GX} . For any sheaf $\mathcal{F} \in {}_q\mathcal{C}_G(X)_{\leq w}$ and any $r \geq 0$, there is a dense open subscheme $V \subset Z$ such that $h^{-r}(Li^*\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}$. □

Proof. The proof of this lemma follows that of [1, Lemma 3.10] nearly verbatim. By the definition of ${}_q\mathcal{C}_G(X)_{\leq w}$, we know that there is a dense open subset $Z' \subset Z$ such that $i^*\mathcal{F}|_{Z'} \in \mathcal{C}_G(Z')_{\leq w+q(x)}$. Let $X' = X \setminus (Z \setminus Z')$. Then X' is a dense open subset of X , and $Z' \subset X$ is a closed subscheme of X' . It clearly suffices to prove the lemma in the case where X and Z are replaced by X' and Z' . We therefore henceforth assume, without loss of generality, that $i^*\mathcal{F} \in \mathcal{C}_G(Z)_{\leq w+q(x)}$.

We now proceed by induction on r . For $r = 0$, the lemma is trivial: we have $i^*\mathcal{F} \in \mathcal{C}_G(Z)_{\leq w+q(x)}$ by assumption. Now, suppose $r > 0$. According to Axiom (S9) in the definition of an s -structure [1], there is an open subscheme $V' \subset Z$ such that for any open set $U \subset X$ with $U \cap Z \subset V'$, we have $\text{Ext}^r(\mathcal{F}|_U, i_*\mathcal{G}|_U) = 0$ for all $\mathcal{G} \in \mathcal{C}_G(Z)_{\geq w+q(x)+1}$. (In fact, Axiom (S9) guarantees this vanishing for all \mathcal{G} in a slightly larger category, denoted $\tilde{\mathcal{C}}_G(Z)_{\geq w+q(x)+1}$, but we will not require that additional information.) Equivalently, for any open $V \subset V'$, we have $\text{Hom}(Li^*\mathcal{F}|_V, \mathcal{G}[r]|_V) = 0$ for all $\mathcal{G} \in \mathcal{C}_G(Z)_{\geq w+q(x)+1}$. We also have $\text{Hom}(Li^*\mathcal{F}|_V, \mathcal{G}[r]|_V) \cong \text{Hom}(\tau^{\geq -r}Li^*\mathcal{F}|_V, \mathcal{G}[r]|_V)$, and then, using the distinguished triangle

$$\tau^{\leq -r}\tau^{\geq -r}Li^*\mathcal{F} \longrightarrow \tau^{\geq -r}Li^*\mathcal{F} \longrightarrow \tau^{\geq -r+1}Li^*\mathcal{F} \longrightarrow,$$

we obtain the exact sequence

$$\begin{aligned} \cdots &\longrightarrow \text{Hom}(\tau^{\geq -r}Li^*\mathcal{F}|_V, \mathcal{G}[r]|_V) \longrightarrow \text{Hom}(\tau^{\leq -r}\tau^{\geq -r}Li^*\mathcal{F}|_V, \mathcal{G}[r]|_V) \\ &\longrightarrow \text{Hom}(\tau^{\geq -r+1}Li^*\mathcal{F}[-1]|_V, \mathcal{G}[r]|_V) \longrightarrow \cdots \end{aligned}$$

Since $\tau^{\leq -r} \tau^{\geq -r} Li^* \mathcal{F} \cong h^{-r}(Li^* \mathcal{F})[r]$, the sequence above can be rewritten as

$$\begin{aligned} \cdots &\longrightarrow \text{Hom}(Li^* \mathcal{F}|_V, \mathcal{G}[r]|_V) \longrightarrow \text{Hom}(h^{-r}(Li^* \mathcal{F})|_V, \mathcal{G}|_V) \\ &\longrightarrow \text{Hom}(\tau^{\geq -(r-1)} Li^* \mathcal{F}|_V, \mathcal{G}[r+1]|_V) \longrightarrow \cdots \end{aligned}$$

The first term above vanishes. Note that

$$h^k(\tau^{\geq -(r-1)} Li^* \mathcal{F}) \cong \begin{cases} h^k(Li^* \mathcal{F}) & \text{if } -(r-1) \leq k \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by the inductive assumption, the cohomology sheaves of $\tau^{\geq -(r-1)} Li^* \mathcal{F}$ have the property that for each k , there is a dense open subscheme $V_k \subset Z$ such that $h^k(\tau^{\geq -(r-1)} Li^* \mathcal{F})|_{V_k} \in \mathcal{C}_G(V_k)_{\leq w+q(x)}$. This property is precisely the hypothesis of [1, Lemma 3.9], which then tells us that there is a dense open subscheme $V'' \subset Z$ such that the last term in the exact sequence above vanishes whenever $V \subset V''$. In particular, let us take $V = V' \cap V''$. The middle term above then clearly vanishes. Since $\text{Hom}(h^{-r}(Li^* \mathcal{F})|_V, \mathcal{G}_1) = 0$ for all $\mathcal{G}_1 \in \mathcal{C}_G(V)_{\leq w+q(x)+1}$, we have $h^{-r}(Li^* \mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}$, as desired. ■

Lemma 5.2. ${}_q\mathcal{C}_G(X)_{\leq w}$ is a Serre subcategory of $\mathcal{C}_G(X)$. □

Proof. Suppose we have a short exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ in $\mathcal{C}_G(X)$. Given $x \in X^{\text{gen}}$ and a closed subscheme structure $i : Z \hookrightarrow X$ on \overline{Gx} , consider the exact sequence

$$h^{-1}(Li^* \mathcal{F}'') \longrightarrow i^* \mathcal{F}' \longrightarrow i^* \mathcal{F} \longrightarrow i^* \mathcal{F}'' \longrightarrow 0.$$

Suppose \mathcal{F}' and \mathcal{F}'' are in ${}_q\mathcal{C}_G(X)_{\leq w}$. Then there are dense open subschemes $V', V'' \subset Z$ such that $i^* \mathcal{F}'|_{V'} \in \mathcal{C}_G(V')_{\leq w+q(x)}$ and $i^* \mathcal{F}''|_{V''} \in \mathcal{C}_G(V'')_{\leq w+q(x)}$. Let $V = V' \cap V''$. Then, since $\mathcal{C}_G(V)_{\leq w+q(x)}$ is a Serre subcategory of $\mathcal{C}_G(V)$, we see that $i^* \mathcal{F}|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}$, so $\mathcal{F} \in {}_q\mathcal{C}_G(X)_{\leq w}$.

Conversely, if $\mathcal{F} \in {}_q\mathcal{C}_G(X)_{\leq w}$, then there is a dense open subscheme $V \subset Z$ such that $i^* \mathcal{F}|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}$. It follows that $i^* \mathcal{F}''|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}$ as well, so $\mathcal{F}'' \in {}_q\mathcal{C}_G(X)_{\leq w}$. Next, by Lemma 5.1, there is some dense open subscheme $V' \subset Z$ such that $h^{-1}(Li^* \mathcal{F}'')|_{V'} \in \mathcal{C}_G(V')_{\leq w+q(x)}$, and it follows that $i^* \mathcal{F}'|_{V \cap V'} \in \mathcal{C}_G(V \cap V')_{\leq w+q(x)}$. Thus, $\mathcal{F}' \in {}_q\mathcal{C}_G(X)_{\leq w}$ as well. ■

Next, let p be a staggered perversity function. The following alternate characterization of ${}^p\mathcal{D}_G^-(X)^{\leq 0}$ will be useful.

Lemma 5.3. We have

$${}^p\mathcal{D}_G^-(X)^{\leq 0} = \{\mathcal{F} \in \mathcal{D}_G^-(X) \mid h^k(\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -k} \text{ for all } k \in \mathbb{Z}\}. \quad \square$$

Remark 5.4. Note the similarity between the right-hand side of this equation and the definition of ${}^s\mathcal{D}^{\leq 0}$ of Definition 2.8. □

Remark 5.5. Although the category ${}^p\mathcal{D}_G^-(X)^{\leq 0}$ was considered in [1] only under the assumption that p is a staggered perversity function, its definition actually makes sense for any function p satisfying (5.1). The proof below applies in this more general setting. □

Proof. Throughout the proof, x will denote a point of X^{gen} , and $i : Z \hookrightarrow X$ will denote a closed subscheme structure on \overline{Gx} .

First, suppose \mathcal{F} is concentrated in a single degree with respect to the standard t -structure, say in degree n , and that $h^n(\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -n}$. If $k > n$, then of course $h^k(Li^*\mathcal{F}) = 0$. If $k \leq n$, then by Lemma 5.1, there is a dense open subscheme $V \subset Z$ such that $h^k(Li^*\mathcal{F})|_V \in \mathcal{C}_G(X)_{\leq p(x)-n} \subset \mathcal{C}_G(X)_{\leq p(x)-k}$, so $\mathcal{F} \in {}^p\mathcal{D}_G^-(X)^{\leq 0}$.

Next, if $\mathcal{F} \in \mathcal{D}_G^b(X)$ and $h^k(\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -k}$ for all k , it follows that $\mathcal{F} \in {}^p\mathcal{D}_G^-(X)^{\leq 0}$ by the preceding paragraph and a standard induction argument on the number of nonzero cohomology sheaves of \mathcal{F} . Finally, suppose that $\mathcal{F} \in \mathcal{D}_G^-(X)$ and that $h^k(\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -k}$ for all k . For any $k \in \mathbb{Z}$, $\tau^{\geq k}\mathcal{F}$ is in $\mathcal{D}_G^b(X)$, so we already know that $\tau^{\geq k}\mathcal{F} \in {}^p\mathcal{D}_G^-(X)^{\leq 0}$. But consideration of the distinguished triangle

$$Li^*\tau^{\leq k-1}\mathcal{F} \longrightarrow Li^*\mathcal{F} \longrightarrow Li^*\tau^{\geq k}\mathcal{F} \longrightarrow$$

shows that $h^k(Li^*\mathcal{F}) \cong h^k(Li^*\tau^{\geq k}\mathcal{F})$, so in particular, there is a dense open subscheme $V \subset Z$ with $h^k(Li^*\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq p(x)-k}$, so $\mathcal{F} \in {}^p\mathcal{D}_G^-(X)^{\leq 0}$, as desired.

Conversely, suppose $\mathcal{F} \in {}^p\mathcal{D}_G^-(X)^{\leq 0}$. Let a be the largest integer such that $h^a(\mathcal{F}) \neq 0$. Then of course $h^a(Li^*\mathcal{F}) \cong h^a(Li^*\tau^{\geq a}\mathcal{F}) \cong i^*h^a(\mathcal{F})$, and we know that there is a dense open subscheme $V \subset Z$ such that $i^*h^a(\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq p(x)-a}$, so $h^a(\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -a}$.

Now, we will prove by downward induction on k that $h^k(\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -k}$ and that $\tau^{\leq k-1}\mathcal{F} \in {}^p\mathcal{D}_G^-(X)^{\leq 0}$ for all k . These statements hold trivially if $k > a$. Suppose we

know that $h^{k+1}(\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -k-1}$ and $\tau^{\leq k}\mathcal{F} \in {}_p\mathcal{D}_G^-(X)^{\leq 0}$. By the preceding paragraph, we know that $h^k(\mathcal{F}) = h^k(\tau^{\leq k}\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -k}$. Next, from the distinguished triangle $\tau^{\leq k-1}\mathcal{F} \rightarrow \tau^{\leq k}\mathcal{F} \rightarrow \tau^{[k,k]}\mathcal{F} \rightarrow$, we obtain the exact sequence

$$h^{r-1}(Li^*\tau^{[k,k]}\mathcal{F}) \longrightarrow h^r(Li^*\tau^{\leq k-1}\mathcal{F}) \longrightarrow h^r(Li^*\tau^{\leq k}\mathcal{F}).$$

Assume $r \leq k - 1$ (otherwise, the middle term above vanishes). By Lemma 5.1, for some dense open $V \subset Z$, $h^{r-1}(Li^*\tau^{[k,k]}\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq p(x)-k} \subset \mathcal{C}_G(V)_{\leq p(x)-r}$. Replacing V by a smaller open subscheme if necessary, we may also assume that $h^r(Li^*\tau^{\leq k}\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq p(x)-r}$. It follows that $h^r(Li^*\tau^{\leq k-1}\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq p(x)-r}$, so $\tau^{\leq k-1}\mathcal{F} \in {}_p\mathcal{D}_G^-(X)^{\leq 0}$.

In particular, $h^k(\mathcal{F}) \in {}_p\mathcal{C}_G(X)_{\leq -k}$ for all k , as desired. ■

In the course of the preceding proof, we have also established the following statement.

Corollary 5.6. The category ${}_p\mathcal{D}_G^-(X)^{\leq 0}$ is stable under all standard truncation functions $\tau^{\leq k}$ and $\tau^{\geq k}$. □

6 Baric Structures Associated to s -structures

In this section, we achieve the main goal of the paper: the construction of a class of baric structures on derived categories of equivariant coherent sheaves. These baric structures depend on a function on X^{gen} that plays a role analogous to that played by a staggered perversity in Section 5.

We henceforth assume that the base scheme over which we are working is $\text{Spec } \mathbb{k}$ for some field \mathbb{k} , and that G is a linear algebraic group over \mathbb{k} . (These are also the assumptions of [1, Sections 9 and 10].) In particular, G is itself a reduced variety. An orbit of G on X is a reduced, locally closed subscheme isomorphic to a homogeneous space for G .

Definition 6.1. Suppose G acts on X with finitely many orbits. For each orbit $C \subset X$, let $\mathcal{I}_C \subset \mathcal{O}_X$ denote the ideal sheaf corresponding to the reduced closed subscheme structure on $\bar{C} \subset X$. An s -structure on X is said to be *recessed* if for each C , $\mathcal{I}_C/\mathcal{I}_C^2 \in \mathcal{C}_G(X)_{\leq -1}$. □

For the remainder of the paper, we assume that G acts on X with finitely many orbits, and that X is endowed with a recessed s -structure. (See Remarks 6.11 and 8.3,

however.) The assumption that the s -structure is recessed is a mild one: “most” of the s -structures appearing in [13] are recessed, as is the one used in [2].

Note that $\mathcal{I}_C/\mathcal{I}_C^2$ is always at least in $\mathcal{C}_G(X)_{\leq 0}$, since it is a subquotient of $\mathcal{O}_X \in \mathcal{C}_G(X)_{\leq 0}$. In addition, since the coherent pullback functor to a locally closed subscheme is right s -exact, it follows that the restriction of a recessed s -structure to any locally closed subscheme is also recessed.

Remark 6.2. It is certainly possible to define the notion of “recessed s -structure” in a way that does not assume finiteness of the number of orbits. (One simply imposes a condition on the ideal sheaf of \overline{Gx} for every $x \in X^{\text{gen}}$, not just for every orbit closure.) However, it seems likely that when there are infinitely many orbits, there are no recessed s -structures. \square

Given a function $q: X^{\text{gen}} \rightarrow \mathbb{Z}$ satisfying (5.1), define a new function $\hat{q}: X^{\text{gen}} \rightarrow \mathbb{Z}$ given by

$$\hat{q}(x) = \text{alt } \overline{Gx} - q(x).$$

Note that when G acts on X with finitely many orbits, a function $q: X^{\text{gen}} \rightarrow \mathbb{Z}$ satisfying (5.1) may be regarded as a \mathbb{Z} -valued function on the set of orbits. It will sometimes be convenient to adopt this point of view, and, given an orbit $C \subset X$, we sometimes write as follows:

$$q(C) = q(x_C) \quad \text{where } x_C \in X^{\text{gen}} \text{ is any generic point of } C.$$

Lemma 6.3. Let $\mathcal{G} \in \mathcal{C}_G(X)$, and let $j: U \hookrightarrow X$ be an open subscheme. Suppose $\mathcal{F}_1 \subset \mathcal{G}|_U$ is such that $\mathcal{F}_1 \in {}_q\mathcal{C}_G(U)_{\leq w}$. Then there exists a subsheaf $\mathcal{F} \subset \mathcal{G}$ such that $\mathcal{F}|_U \cong \mathcal{F}_1$ and $\mathcal{F} \in {}_q\mathcal{C}_G(X)_{\leq w}$. \square

Proof. If U is closed (i.e., if U is a connected component of X), then $j_*\mathcal{F}_1$ is naturally a subsheaf of \mathcal{G} , so we simply take $\mathcal{F} \cong j_*\mathcal{F}_1$. Otherwise, let C be an open orbit in $\bar{U} \setminus U$, and let V be the open subscheme $U \cup C$. By induction on the number of orbits in $\bar{U} \setminus U$, it suffices to find $\mathcal{F}' \subset \mathcal{G}|_V$ such that $\mathcal{F}' \in {}_q\mathcal{C}_G(V)_{\leq w}$ and $\mathcal{F}'|_U \cong \mathcal{F}_1$. Let $\kappa: C \hookrightarrow V$ be the inclusion map, and let \mathcal{I}_C be the ideal sheaf of C in V . Finally, let \mathcal{F}' be some subsheaf of $\mathcal{G}|_V$ such that $\mathcal{F}'|_U \cong \mathcal{F}_1$. Suppose $\kappa^*\mathcal{F}' \in \mathcal{C}_G(C)_{\leq v}$. If $v \leq w + q(C)$, then we may take $\mathcal{F} = \mathcal{F}'$, and we are finished. On the other hand, if $v > w + q(C)$, let $\mathcal{F} = \mathcal{I}_C^{v-w-q(C)}\mathcal{F}'$. Since $\mathcal{I}_C|_U \cong \mathcal{O}_U$, we clearly still have $\mathcal{F}|_U \cong \mathcal{F}_1$. The fact that the s -structure is recessed means that $\kappa^*\mathcal{I}_C \in \mathcal{C}_G(C)_{\leq -1}$, so $\kappa^*\mathcal{I}_C^{\otimes v-w-q(C)} \in \mathcal{C}_G(C)_{\leq -v+w+q(C)}$, and therefore

$\kappa^* \mathcal{I}_C^{\otimes v-w-q(C)} \otimes \kappa^* \mathcal{F}' \in \mathcal{C}_G(C)_{\leq w+q(C)}$. Now, $\kappa^* \mathcal{F}$ is a quotient of $\kappa^* \mathcal{I}_C^{\otimes v-w-q(C)} \otimes \kappa^* \mathcal{F}'$, so $\kappa^* \mathcal{F} \in \mathcal{C}_G(C)_{\leq w+q(C)}$, as desired. ■

Given a function $q : X^{\text{gen}} \rightarrow \mathbb{Z}$, we define a full subcategory of $\mathcal{D}_G^-(X)$ by

$${}_q \mathcal{D}_G^-(X)_{\leq w} = \{ \mathcal{F} \in \mathcal{D}_G^-(X) \mid h^k(\mathcal{F}) \in {}_q \mathcal{C}_G(X)_{\leq w} \text{ for all } k \in \mathbb{Z} \}.$$

We also define a full subcategory of $\mathcal{D}_G^+(X)$ by

$${}_q \mathcal{D}_G^+(X)_{\geq w} = \mathbb{D}({}_q \mathcal{D}_G^-(X)_{\leq -w}).$$

Finally, we put

$${}_q \mathcal{D}_G^b(X)_{\leq w} = {}_q \mathcal{D}_G^-(X)_{\leq w} \cap \mathcal{D}_G^b(X) \quad \text{and} \quad {}_q \mathcal{D}_G^b(X)_{\geq w} = {}_q \mathcal{D}_G^+(X)_{\geq w} \cap \mathcal{D}_G^b(X).$$

The main result of the paper is the following.

Theorem 6.4. The collection of subcategories $(\{{}_q \mathcal{D}_G^b(X)_{\leq w}\}, \{{}_q \mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is a bounded, nondegenerate HLR baric structure on X . □

The proof of this theorem will occupy the rest of this section. Note that the definition of ${}_q \mathcal{D}_G^-(X)_{\leq w}$ is consistent with the notation used in Section 4. We will see in Corollary 6.10 that the same holds for ${}_q \mathcal{D}_G^+(X)_{\geq w}$.

Lemma 6.5. ${}_q \mathcal{D}_G^b(X)_{\leq w}$ and ${}_q \mathcal{D}_G^b(X)_{\geq w}$ are thick subcategories of $\mathcal{D}_G^b(X)$. Moreover, ${}_q \mathcal{D}_G^b(X)_{\leq w} \subset {}_q \mathcal{D}_G^b(X)_{\leq w+1}$ and ${}_q \mathcal{D}_G^b(X)_{\geq w} \supset {}_q \mathcal{D}_G^b(X)_{\geq w+1}$. □

Proof. It is obvious that ${}_q \mathcal{D}_G^b(X)_{\leq w}$ is stable under shift. Since it is defined by the requirement that cohomology sheaves belong to a Serre subcategory of $\mathcal{C}_G(X)$ (see Lemma 5.2), it is stable under extensions as well, so it is indeed a thick subcategory of $\mathcal{D}_G^b(X)$. It follows that ${}_q \mathcal{D}_G^b(X)_{\geq w}$ is as well. It is obvious that ${}_q \mathcal{D}_G^b(X)_{\leq w} \subset {}_q \mathcal{D}_G^b(X)_{\leq w+1}$, and hence that ${}_q \mathcal{D}_G^b(X)_{\geq w} \supset {}_q \mathcal{D}_G^b(X)_{\geq w+1}$. ■

Lemma 6.6. Let $j : U \hookrightarrow X$ be the inclusion of an open subscheme, and $i : Z \hookrightarrow X$ the inclusion of a closed subscheme. Then:

- (1) j^* takes ${}_q \mathcal{D}_G^-(X)_{\leq w}$ to ${}_q \mathcal{D}_G^-(U)_{\leq w}$ and ${}_q \mathcal{D}_G^+(X)_{\geq w}$ to ${}_q \mathcal{D}_G^+(U)_{\geq w}$.

- (2) Li^* takes ${}_q\mathcal{D}_G^-(X)_{\leq w}$ to ${}_q\mathcal{D}_G^-(Z)_{\leq w}$.
- (3) $Ri^!$ takes ${}_q\mathcal{D}_G^+(X)_{\geq w}$ to ${}_q\mathcal{D}_G^+(Z)_{\geq w}$. □

Remark 6.7. This statement closely resembles the first three parts of Lemma 4.8, but it requires an independent proof because we have not yet established Theorem 6.4. The “missing” assertion

- (3) i_* takes ${}_q\mathcal{D}_G^-(Z)_{\leq w}$ to ${}_q\mathcal{D}_G^-(X)_{\leq w}$ and ${}_q\mathcal{D}_G^+(Z)_{\geq w}$ to ${}_q\mathcal{D}_G^+(X)_{\geq w}$

is also true, of course; it will follow from Lemma 4.8 after the proof of Theorem 6.4 is complete. □

Proof. (1) It is immediate from the definition of ${}_q\mathcal{C}_G(X)_{\leq w}$ that j^* takes ${}_q\mathcal{C}_G(X)_{\leq w}$ to ${}_q\mathcal{C}_G(U)_{\leq w}$. Since j^* is an exact functor, it follows that it takes ${}_q\mathcal{D}_G^-(X)_{\leq w}$ to ${}_q\mathcal{D}_G^-(U)_{\leq w}$. Since j^* commutes with \mathbb{D} , we also see that it takes ${}_q\mathcal{D}_G^+(X)_{\geq w}$ to ${}_q\mathcal{D}_G^+(U)_{\geq w}$.

(2) We proceed by the Noetherian induction: assume that the statement is known if X is replaced by a proper closed subscheme, or if X is retained and Z is replaced by a proper closed subscheme. Suppose $\mathcal{F} \in {}_q\mathcal{D}_G^-(X)_{\leq w}$. We show by downward induction on k that $h^k(Li^*\mathcal{F}) \in {}_q\mathcal{C}_G(Z)_{\leq w}$. For large k , $h^k(Li^*\mathcal{F}) = 0$, so this holds trivially. Now, assume that $h^r(Li^*\mathcal{F}) \in {}_q\mathcal{C}_G(Z)_{\leq w}$ for all $r > k$, and consider the distinguished triangle $\tau^{\leq k}Li^*\mathcal{F} \rightarrow Li^*\mathcal{F} \rightarrow \tau^{\geq k+1}Li^*\mathcal{F} \rightarrow$. Then $\tau^{\geq k+1}Li^*\mathcal{F}$ is an object of ${}_q\mathcal{D}_G^b(Z)_{\leq w}$, so for any $x \in Z^{\text{gen}}$ and any closed subscheme structure $\kappa : Y \hookrightarrow Z$ on \overline{Gx} , we know that $L\kappa^*\tau^{\geq k+1}Li^*\mathcal{F} \in {}_q\mathcal{D}_G^-(Y)_{\leq w}$. Consider the exact sequence

$$h^{k-1}(L\kappa^*\tau^{\geq k+1}Li^*\mathcal{F}) \longrightarrow h^k(L\kappa^*\tau^{\leq k}Li^*\mathcal{F}) \longrightarrow h^k(L\kappa^*Li^*\mathcal{F}).$$

The first term above belongs to ${}_q\mathcal{C}_G(Y)_{\leq w}$. Observe that $h^k(L\kappa^*\tau^{\leq k}Li^*\mathcal{F}) \cong \kappa^*h^k(Li^*\mathcal{F})$. Thus, to prove that $h^k(Li^*\mathcal{F}) \in {}_q\mathcal{C}_G(Z)_{\leq w}$, we must show that there is a dense open subscheme $V \subset Y$ such that $h^k(L\kappa^*\tau^{\leq k}Li^*\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}$.

If Y is a proper closed subscheme of Z , then we have assumed inductively that $L(\kappa \circ i)^*\mathcal{F} \in {}_q\mathcal{D}_G^-(Y)_{\leq w}$, and in that case, the last term in the sequence above belongs to ${}_q\mathcal{C}_G(Y)_{\leq w}$ as well. By Lemma 5.2, the middle term as well, and the existence of the desired open subscheme $V \subset Y$ follows.

On the other hand, if $Y = Z$, and κ is the identity map, then Lemma 5.1 gives us a dense open subscheme $V \subset Z$ such that $h^k(Li^*\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}$. Since $h^{k-1}(\tau^{\geq k+1}Li^*\mathcal{F}) = 0$, we see from the exact sequence above that $h^k(\tau^{\leq k}Li^*\mathcal{F})|_V \in \mathcal{C}_G(V)_{\leq w+q(x)}$, as desired.

(3) If $\mathcal{F} \in {}_q\mathcal{D}_G^+(X)_{\geq w}$, let $\mathcal{F}' \in {}_q\mathcal{D}_G^-(X)_{\leq -w}$ be such that $\mathbb{D}\mathcal{F}' \cong \mathcal{F}$. Then $Ri^!\mathcal{F} \cong \mathbb{D}(Li^*\mathcal{F}') \in {}_q\mathcal{D}_G^+(Z)_{\geq w}$ since $Li^*\mathcal{F}' \in {}_q\mathcal{D}_G^-(Z)_{\leq w}$. ■

The next two propositions both rely on results from [1, Section 6] concerning the interaction between s -structures and Serre–Grothendieck duality. The reader who wishes to refer to those results is reminded that in [1], as in the present paper, the term “irreducible” is always used in the G -invariant sense.

Proposition 6.8. If $\mathcal{F} \in {}_q\mathcal{D}_G^-(X)_{\leq w}$ and $\mathcal{G} \in {}_q\mathcal{D}_G^+(X)_{\geq w+1}$, then $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$. □

Proof. We proceed by the Noetherian induction: assume that the theorem is known for all proper closed subschemes of X . Let a and b be such that $\mathcal{G} \in \mathcal{D}_G^+(X)^{\geq a}$ and $\mathcal{F} \in \mathcal{D}_G^-(X)^{\leq b}$. Since $\text{Hom}(\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\tau^{\geq a}\mathcal{F}, \mathcal{G})$, we may replace \mathcal{F} by $\tau^{\geq a}\mathcal{F}$ and assume that $\mathcal{F} \in {}_q\mathcal{D}_G^b(X)_{\leq w}$. Next, let $\mathcal{G}' \in \mathcal{D}_G^-(X)_{\leq -w-1}$ be such that $\mathbb{D}\mathcal{G}' \cong \mathcal{G}$. For a sufficiently small integer c , we will have $\mathbb{D}(\tau^{\leq c}\mathcal{G}') \in \mathcal{D}_G^+(X)^{\geq b+1}$. From this, it follows that $\text{Hom}(\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\mathcal{F}, \mathbb{D}(\tau^{\geq c+1}\mathcal{G}'))$. Replacing \mathcal{G} by $\mathbb{D}(\tau^{\geq c+1}\mathcal{G}')$, we may assume that $\mathcal{G} \in {}_q\mathcal{D}_G^b(X)_{\geq w+1}$.

With \mathcal{F} and \mathcal{G} both in $\mathcal{D}_G^b(X)$, induction on the number of cohomology sheaves allows us to reduce to the case where both \mathcal{F} and $\mathcal{G}' := \mathbb{D}\mathcal{G}$ are concentrated in a single degree. By shifting both objects simultaneously, we may assume without loss of generality that $\mathcal{F} \in \mathcal{C}_G(X)$, and let d' be such that $\mathcal{G}'[d'] \in \mathcal{C}_G(X)$. Let x be a generic point of X . There is an irreducible open subscheme $U \subset X$ containing x such that $\mathcal{G}'[d']|_U \in \mathcal{C}_G(U)_{\leq \text{alt } \overline{Gx} - q(x) - w - 1}$. By [1, Lemma 6.1 and Proposition 6.8], we may replace U by a smaller open subscheme containing x such that $\mathcal{G}|_U$ is also concentrated in a single degree, say d , and such that $\mathcal{G}[d]|_U \in \mathcal{C}_G(U)_{\geq q(x) + w + 1}$. If $d > 0$, then clearly $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) = 0$. Otherwise, we invoke [1, Axiom (S9)] to replace U by a smaller open subscheme such that $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) = 0$. Let Z be the complementary closed subspace to U , and consider the exact sequence

$$\lim_{\substack{\longrightarrow \\ Z'}} \text{Hom}(Li_{Z'}^*\mathcal{F}, Ri_{Z'}^!\mathcal{G}) \longrightarrow \text{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $i_{Z'} : Z' \hookrightarrow X$ ranges over all closed subscheme structures on Z . We have just seen that the last term vanishes. Since $Li_{Z'}^*\mathcal{F} \in {}_q\mathcal{D}_G^-(Z')_{\leq w}$ and $Ri_{Z'}^!\mathcal{G} \in {}_q\mathcal{D}_G^+(Z')_{\geq w+1}$, the first term vanishes by induction. So $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$, as desired. ■

Proposition 6.9. For any $\mathcal{F} \in \mathcal{D}_G^b(X)$, there is a distinguished triangle $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow$ with $\mathcal{F}' \in {}_q\mathcal{D}_G^b(X)_{\leq w}$ and $\mathcal{F}'' \in {}_q\mathcal{D}_G^b(X)_{\geq w+1}$. Moreover, if $\mathcal{F} \in \mathcal{D}_G^b(X)^{\geq 0}$, then \mathcal{F}' and \mathcal{F}'' lie in $\mathcal{D}_G^b(X)^{\geq 0}$ as well. \square

Proof. Once again, we proceed by the Noetherian induction, and assume that the result is known for all proper closed subschemes of X . Now, assume first that \mathcal{F} is a sheaf. Let $C \subset X$ be an open (and possibly nonreduced) orbit, and let $i : \bar{C} \hookrightarrow X$ be the inclusion of its closure. By Lemma 6.3, there exists a subsheaf $\mathcal{F}_1 \subset \mathcal{F}$ such that $\mathcal{F}_1 \in {}_q\mathcal{C}_G(X)_{\leq w}$ and $\mathcal{F}_1|_C \cong \sigma_{\leq w+q(C)}(\mathcal{F}|_C)$. Next, form a short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Let $b = \text{cod } \bar{C}$. Then $i_* Ri^! \mathbb{D}\mathcal{G} \in \mathcal{D}_G^+(X)^{\geq b}$, and, by [1, Lemma 6.1], we know that $i_* Ri^! \mathbb{D}\mathcal{G}|_C \cong \mathbb{D}\mathcal{G}|_C$ is concentrated in degree b . Furthermore, [1, Proposition 6.8] tells us that $\mathbb{D}\mathcal{G}[b]|_C \in \mathcal{C}_G(C)_{\leq \text{alt } \bar{C} - q(C) - w - 1}$. (If C is reduced, these assertions about $\mathbb{D}\mathcal{G}|_C$ are immediate from the fact that \mathbb{D} is an exact functor, but in general, we must invoke [1, Lemma 6.1 and Proposition 6.8].) Now, we use Lemma 6.3 again to find a subsheaf $\mathcal{G}_1 \subset h^b(i_* Ri^! \mathbb{D}\mathcal{G})$ such that $\mathcal{G}_1 \in {}_q\mathcal{C}_G(X)_{\leq -w-1}$ and $\mathcal{G}_1|_C \cong \mathbb{D}\mathcal{G}[b]|_C$. Form the composition

$$\mathcal{G}_1[-b] \longrightarrow h^b(i_* Ri^! \mathbb{D}\mathcal{G})[-b] \cong \tau^{\leq b} i_* Ri^! \mathbb{D}\mathcal{G} \longrightarrow i_* Ri^! \mathbb{D}\mathcal{G} \longrightarrow \mathbb{D}\mathcal{G},$$

and then complete it to a distinguished triangle

$$\mathcal{G}_1[-b] \longrightarrow \mathbb{D}\mathcal{G} \longrightarrow \mathcal{G}' \longrightarrow .$$

Here, \mathcal{G}' is necessarily supported on the complement of C . Let $\mathcal{F}_2 = \mathbb{D}(\mathcal{G}_1[-b])$, and let $\mathcal{H} = \mathbb{D}\mathcal{G}'$, so we have a distinguished triangle

$$\mathcal{H} \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}_2 \longrightarrow .$$

Since $\text{cod } \bar{C} = b$, we see that $\mathcal{F}_2 \in \mathcal{D}_G^b(X)^{\geq 0}$. This distinguished triangle then implies that $\mathcal{H} \in \mathcal{D}_G^b(X)^{\geq 0}$ as well. Note also that $\mathcal{F}_2 \in {}_q\mathcal{D}_G^b(X)_{\geq w+1}$, and that

$$\mathcal{F} \in \{\mathcal{F}_1\} * \{\mathcal{H}\} * \{\mathcal{F}_2\}.$$

Since $\mathcal{F}_1 \in {}_q\mathcal{D}_G^b(X)_{\leq w}$, $\mathcal{F}_2 \in {}_q\mathcal{D}_G^b(X)_{\geq w+1}$, and \mathcal{H} is supported on a proper closed subscheme, we conclude that $\mathcal{F} \in {}_q\mathcal{D}_G^b(X)_{\leq w} * {}_q\mathcal{D}_G^b(X)_{\geq w+1}$, as desired. The last statement of

the proposition holds by the Noetherian induction as well, since $\mathcal{F}_1, \mathcal{H}$, and \mathcal{F}_2 all lie in $\mathcal{D}_G^b(X)^{\geq 0}$ by construction.

The result also follows for any object of $\mathcal{D}_G^b(X)$ that is concentrated in a single degree. Finally, for general objects $\mathcal{F} \in \mathcal{D}_G^b(X)$, we proceed by induction on the number of nonzero cohomology sheaves. Let $a \in \mathbb{Z}$ be such that $\tau^{\leq a}\mathcal{F}$ and $\tau^{\geq a+1}\mathcal{F}$ are both nonzero. Then, they both have fewer nonzero cohomology sheaves than \mathcal{F} , and we assume inductively that there exist distinguished triangles

$$\begin{aligned} \mathcal{F}'_1 &\longrightarrow \tau^{\leq a}\mathcal{F} \longrightarrow \mathcal{F}''_1 \longrightarrow, \\ \mathcal{F}'_2 &\longrightarrow \tau^{\geq a+1}\mathcal{F} \longrightarrow \mathcal{F}''_2 \longrightarrow \end{aligned}$$

with $\mathcal{F}'_1, \mathcal{F}'_2 \in {}_q\mathcal{D}_G^b(X)_{\leq w}$ and $\mathcal{F}''_1, \mathcal{F}''_2 \in {}_q\mathcal{D}_G^b(X)_{\geq w+1}$. In other words, we have $\tau^{\leq a}\mathcal{F} \in \{\mathcal{F}'_1\} * \{\mathcal{F}''_1\}$ and $\tau^{\geq a+1}\mathcal{F} \in \{\mathcal{F}'_2\} * \{\mathcal{F}''_2\}$, so

$$\mathcal{F} \in \{\mathcal{F}'_1\} * \{\mathcal{F}''_1\} * \{\mathcal{F}'_2\} * \{\mathcal{F}''_2\}.$$

(See [4, Section 1.3.9] or the proof of Theorem 2.11 for this notation.) We clearly have $\text{Hom}(\mathcal{F}'_2, \mathcal{F}''_1[1]) = 0$, so every triangle $\mathcal{F}'_1 \rightarrow \mathcal{G} \rightarrow \mathcal{F}'_2 \rightarrow$ splits. In other words, $\{\mathcal{F}''_1\} * \{\mathcal{F}'_2\} = \{\mathcal{F}'_1 \oplus \mathcal{F}'_2\} \subset \{\mathcal{F}'_2\} * \{\mathcal{F}''_1\}$. Therefore, we have

$$\mathcal{F} \in \{\mathcal{F}'_1\} * \{\mathcal{F}'_2\} * \{\mathcal{F}''_1\} * \{\mathcal{F}''_2\}.$$

In other words, there are objects $\mathcal{F}' \in \{\mathcal{F}'_1\} * \{\mathcal{F}'_2\}$ and $\mathcal{F}'' \in \{\mathcal{F}''_1\} * \{\mathcal{F}''_2\}$ together with a distinguished triangle $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow$. Moreover, since ${}_q\mathcal{D}_G^b(X)_{\leq w}$ and ${}_q\mathcal{D}_G^b(X)_{\geq w+1}$ are stable under extensions, we see that $\mathcal{F}' \in {}_q\mathcal{D}_G^b(X)_{\leq w}$ and $\mathcal{F}'' \in {}_q\mathcal{D}_G^b(X)_{\geq w+1}$, as desired.

Lastly, if \mathcal{F} lies in $\mathcal{D}_G^b(X)^{\geq 0}$, then so do $\tau^{\leq a}\mathcal{F}$ and $\tau^{\geq a+1}\mathcal{F}$, and hence, by induction, the objects $\mathcal{F}'_1, \mathcal{F}''_1, \mathcal{F}'_2$, and \mathcal{F}''_2 all lie in $\mathcal{D}_G^b(X)^{\geq 0}$ as well. It then follows that \mathcal{F}' and \mathcal{F}'' are in $\mathcal{D}_G^b(X)^{\geq 0}$, as desired. ■

Proof of Theorem 6.4. Lemma 6.5 and Propositions 6.8 and 6.9 together state that all the axioms for a baric structure hold. Moreover, the last part of Proposition 6.9 tells us that the baric truncation functors are left t -exact (with respect to the standard t -structure), and it is obvious from the definition of ${}_q\mathcal{D}_G^b(X)_{\leq w}$ that it is preserved by the truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$. Thus, the baric structure $(\{{}_q\mathcal{D}_G^b(X)_{\leq w}\}, \{{}_q\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is compatible with the standard t -structure. Next, for any closed subscheme $i : Z \hookrightarrow X$,

Lemma 6.6 tells us that Li^* is right baryexact and that $Ri^!$ is left baryexact. Thus, this baric structure is hereditary, and hence HLR by Theorem 4.6.

It remains to prove that the baric structure is bounded (and therefore nondegenerate). Every sheaf in $\mathcal{C}_G(X)$ belongs to some $\mathcal{C}_G(X)_{\leq n}$, and hence to some ${}_q\mathcal{C}_G(X)_{\leq w}$ (simply take w to be the maximum value of $n - q(x)$). Since an object $\mathcal{F} \in \mathcal{D}_G^b(X)$ has finitely many nonzero cohomology sheaves, we can clearly find a w such that all its cohomology sheaves belong to ${}_q\mathcal{C}_G(X)_{\leq w}$, so that $\mathcal{F} \in {}_q\mathcal{D}_G^b(X)_{\leq w}$. The same reasoning yields an integer v such that $\mathbb{D}\mathcal{F} \in {}_q\mathcal{D}_G^b(X)_{\leq -v}$, and hence $\mathcal{F} \in {}_q\mathcal{D}_G^b(X)_{\geq v}$. Thus, the baric structure is bounded and nondegenerate. ■

We can now verify that the notation ${}_q\mathcal{D}_G^+(X)_{\geq w}$ is consistent with the notation of Section 4.

Corollary 6.10. We have

$${}_q\mathcal{D}_G^+(X)_{\geq w} = \{\mathcal{F} \in \mathcal{D}_G^+(X) \mid {}_q\beta_{\leq w-1}\tau^{\leq k}\mathcal{F} \in \mathcal{D}_G^b(X)^{\geq k+2} \text{ for all } k\}. \quad \square$$

Proof. We have already observed that the definition of ${}_q\mathcal{D}_G^-(X)_{\leq -w}$ is consistent with the notation of Section 4, so by Lemma 4.3, for $\mathcal{F} \in \mathcal{D}_G^-(X)$, we have $\mathcal{F} \in {}_q\mathcal{D}_G^-(X)_{\leq -w}$ if and only if $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in {}_q\mathcal{D}_G^b(X)_{\geq -w+1}$. Applying \mathbb{D} , we have $\mathcal{F} \in {}_q\mathcal{D}_G^+(X)_{\geq w}$ if and only if $\text{Hom}(\mathbb{D}\mathcal{F}, \mathbb{D}\mathcal{G}) = 0$ for all $\mathcal{G} \in {}_q\mathcal{D}_G^b(X)_{\leq w-1}$, or, equivalently, if $\text{Hom}(\mathcal{G}, \mathcal{F}) = 0$ for all $\mathcal{G} \in {}_q\mathcal{D}_G^b(X)_{\leq w-1}$. The corollary follows by another application of Lemma 4.3. ■

Remark 6.11. The proof of Lemma 6.3 depends in an essential way on the assumption of finitely many orbits and a recessed s -structure, but no other arguments given in this section do. (The role of the orbit closure \bar{C} in the proof of Proposition 6.9 could instead have been played by \overline{Gx} for some generic point x .) By imposing additional conditions that permit us to evade Lemma 6.3, we can find a version of Theorem 6.4 that holds in much greater generality.

Specifically, assume that the function $q: X^{\text{gen}} \rightarrow \mathbb{Z}$ is *monotone*: that is, if $x \in \overline{Gy}$, then $q(x) \geq q(y)$. Suppose we have a coherent sheaf $\mathcal{G} \in \mathcal{C}_G(X)$, an open subscheme $j: U \hookrightarrow X$, and a subsheaf $\mathcal{F}_1 \subset \mathcal{G}|_U$ with $\mathcal{F}_1 \in {}_q\mathcal{C}_G(U)_{\leq w}$. By replacing U by a smaller open subscheme, we may assume that $\mathcal{F}_1 \in \mathcal{C}_G(U)_{\leq q(x)+w}$, where x is a generic point of U . Then \mathcal{F}_1 is a subsheaf of $\sigma_{\leq q(x)+w}\mathcal{G}|_U$, and standard arguments show that there is a subsheaf $\mathcal{F} \subset \sigma_{\leq q(x)+w}\mathcal{G}$ supported on \bar{U} such that $\mathcal{F}|_U \cong \mathcal{F}_1$. The monotonicity assumption then implies that $\mathcal{F} \in {}_q\mathcal{C}_G(X)_{\leq w}$. This reasoning can be substituted for invocations of

Lemma 6.3 for ${}_q\mathcal{C}_G(X)_{\leq w}$. Similarly, if q is *comonotone*, meaning that \hat{q} is monotone, then the reasoning above can replace invocations of Lemma 6.3 for the category ${}_{\hat{q}}\mathcal{C}_G(X)_{\leq w}$. The proof of Theorem 6.4 uses Lemma 6.3 in both these ways.

We thus obtain the following result: suppose X is a scheme satisfying the assumptions of Section 5, equipped with an s -structure. In particular, we do not assume that G acts with finitely many orbits, or that the s -structure is recessed. If $q : X^{\text{gen}} \rightarrow \mathbb{Z}$ is both monotone and comonotone, then the collection of subcategories $(\{{}_q\mathcal{D}_G^b(X)_{\leq w}\}, \{{}_q\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is a bounded, nondegenerate HLR baric structure on X . □

7 Multiplicative Baric Structures and s -structures

In this section, we study the relationship between multiplicative baric structures on the triangulated category $\mathcal{D}_G^b(X)$ and s -structures on the abelian category $\mathcal{C}_G(X)$. The authors had originally hoped that under appropriate conditions the two notions would be equivalent, and that the developments in Sections 5 and 6 could be simplified by replacing the latter concept with the former. In other words, the hope was that there would be a one-to-one correspondence between multiplicative HLR baric structures and s -structures on a G -scheme X .

This turns out to be not quite correct. Rather, we prove here that there is a one-to-one correspondence between multiplicative baric structures and a certain class of *pre- s -structures*, including all s -structures. (A pre- s -structure is a collection of subcategories of $\mathcal{C}_G(X)$ satisfying the first six of the 10 axioms for an s -structure in [1].) It would be interesting to look for an additional axiom on multiplicative baric structures that is satisfied precisely by those baric structures corresponding to s -structures, but we have not pursued this here.

We say that a baric structure $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is *multiplicative* if either of the following two equivalent conditions holds:

- (1) If $\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$ and $\mathcal{G} \in \mathcal{D}_G^b(X)_{\leq v}$, then $\mathcal{F} \otimes^L \mathcal{G} \in \mathcal{D}_G^-(X)_{\leq w+v}$.
- (2) If $\mathcal{F} \in \mathcal{D}_G^b(X)_{\leq w}$ and $\mathcal{G} \in \mathcal{D}_G^b(X)_{\geq v}$, then $R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathcal{D}_G^+(X)_{\geq v-w}$.

Theorem 7.1. Suppose $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is a multiplicative HLR baric structure on X . Then the categories

$$\mathcal{C}_G(X)_{\leq w} = \mathcal{C}_G(X) \cap \mathcal{D}_G^b(X)_{\leq w},$$

$$\mathcal{C}_G(X)_{\geq w} = \{\mathcal{F} \in \mathcal{C}_G(X) \mid \text{Hom}(\mathcal{G}|_U, \mathcal{F}|_U) = 0 \text{ for all } \mathcal{G} \in \mathcal{C}_G(X)_{\leq w-1}$$

and all open subschemes $U \subset X\}$

constitute a pre-s-structure on X .

Conversely, given an s-structure $(\{\mathcal{C}_G(X)_{\leq w}\}, \{\mathcal{C}_G(X)_{\geq w}\})_{w \in \mathbb{Z}}$ on a scheme X with finitely many G -orbits, the categories

$$\mathcal{D}_G^b(X)_{\leq w} = \{\mathcal{F} \in \mathcal{D}_G^b(X) \mid h^k(\mathcal{F}) \in \mathcal{C}_G(X)_{\leq w} \text{ for all } k \in \mathbb{Z}\},$$

$$\mathcal{D}_G^b(X)_{\geq w} = \{\mathcal{F} \in \mathcal{D}_G^b(X) \mid \text{Hom}(\mathcal{G}, \mathcal{F}) = 0 \text{ for all } \mathcal{G} \in \mathcal{D}_G^b(X)_{\leq w-1}\}$$

constitute a multiplicative baric structure on X . □

Proof. Suppose first that $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is a multiplicative baric structure on X . To show that the categories above constitute a pre-s-structure, we must verify axioms (S1)–(S6) from [1]. (The reader is referred to [1] for the statements of these axioms.)

Axioms (S2) and (S3) are clear from the definitions, and axiom (S1) follows by Lemma 2.9(4) from the fact that $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ is compatible with the standard t -structure.

Let us prove axiom (S4). Let \mathcal{F} be an object of $\mathcal{C}_G(X)$. Since \mathcal{F} is Noetherian, and $\mathcal{C}_G(X)_{\leq w}$ is a Serre subcategory, there is a largest subobject $\mathcal{F}' \subset \mathcal{F}$ belonging to $\mathcal{C}_G(X)_{\leq w}$. Then \mathcal{F}/\mathcal{F}' must belong to $\mathcal{C}_G(X)_{\geq w+1}$: otherwise, there is a nonzero map $\mathcal{G} \rightarrow \mathcal{F}/\mathcal{F}'$ the image of which $\mathcal{I} \neq 0$ belongs to $\mathcal{C}_G(X)_{\leq w}$, but the inverse image of \mathcal{I} in \mathcal{F} contains the maximal \mathcal{F}' .

Axiom (S5) follows from the fact that the baric structure on $\mathcal{D}_G^b(X)$ is bounded, and Axiom (S6) follows from the multiplicativity of the baric structure and the fact that for $\mathcal{F}, \mathcal{G} \in \mathcal{C}_G(X)$, we have $\mathcal{F} \otimes \mathcal{G} \cong h^0(\mathcal{F} \otimes^L \mathcal{G})$.

Now, suppose we are given an s-structure $(\{\mathcal{C}_G(X)_{\leq w}\}, \{\mathcal{C}_G(X)_{\geq w}\})_{w \in \mathbb{Z}}$. Let $\mathbf{0}$ denote the constant function $X^{\text{gen}} \rightarrow \mathbb{Z}$ of value 0. We claim that ${}_0\mathcal{C}_G(X)_{\leq w} = \mathcal{C}_G(X)_{\leq w}$. It is clear from the definition that $\mathcal{C}_G(X)_{\leq w} \subset {}_0\mathcal{C}_G(X)_{\leq w}$. We will prove the opposite inclusion by Noetherian induction. Suppose $\mathcal{F} \in {}_0\mathcal{C}_G(X)_{\leq w}$, and let x be a generic point of X . By definition, there is an open subscheme $V \subset X$ such that $\mathcal{F}|_V \in \mathcal{C}_G(V)_{\leq w}$. Let $i: Z \hookrightarrow X$ denote the reduced closed subscheme complementary to V . It is clear from the definition of ${}_0\mathcal{C}_G(Z)_{\leq w}$ that $i^*\mathcal{F} \in {}_0\mathcal{C}_G(Z)_{\leq w}$, and so, by induction, we have $i^*\mathcal{F} \in \mathcal{C}_G(Z)_{\leq w}$. We now invoke the gluing theorem for s-structures [2, Theorem 1.1]. This theorem asserts that

a recessed s -structure on X is determined by its restrictions to V and Z . An explicit description is given in [1, Proposition 5.2]: an object belongs to $\mathcal{C}_G(X)_{\leq w}$ if and only if its restrictions to V and Z belong to $\mathcal{C}_G(V)_{\leq w}$ and $\mathcal{C}_G(Z)_{\leq w}$, respectively. In particular, we have $\mathcal{F} \in \mathcal{C}_G(X)_{\leq w}$, as desired.

Since ${}_0\mathcal{C}_G(X)_{\leq w} = \mathcal{C}_G(X)_{\leq w}$, the categories $(\{\mathcal{D}_G^b(X)_{\leq w}\}, \{\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ defined in the statement of the theorem coincide with the baric structure constructed in Theorem 6.4 by taking $q = 0$. The fact that this baric structure is multiplicative is a consequence of Proposition 7.2. ■

Proposition 7.2. Let X be a scheme with finitely many G -orbits, and let $p, q : X^{\text{gen}} \rightarrow \mathbb{Z}$ be functions satisfying (5.1). Suppose $\mathcal{F} \in {}_p\mathcal{D}_G^-(X)_{\leq w}$.

- (1) If $\mathcal{G} \in {}_q\mathcal{D}_G^-(X)_{\leq v}$, then $\mathcal{F} \otimes^L \mathcal{G} \in {}_{p+q}\mathcal{D}_G^-(X)_{\leq w+v}$.
- (2) If $\mathcal{G} \in {}_q\mathcal{D}_G^+(X)_{\geq v}$, then $R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in {}_{\hat{p}+q}\mathcal{D}_G^+(X)_{\geq v-w}$. □

Proof. (1) We will show by the Noetherian induction that $\text{Hom}(\mathcal{F} \otimes^L \mathcal{G}, \mathcal{H}) = 0$ for all $\mathcal{H} \in {}_{p+q}\mathcal{D}_G^b(X)_{\geq w+v+1}$. Assume the result is known for all proper closed subschemes of X , and let $C \subset X$ be an open orbit. Let Z denote the closed subset $X \setminus C$, and consider the exact sequence

$$\lim_{\substack{\longrightarrow \\ Z'}} \text{Hom}(Li^*(\mathcal{F} \otimes^L \mathcal{G}), Ri^!\mathcal{H}) \longrightarrow \text{Hom}(\mathcal{F} \otimes^L \mathcal{G}, \mathcal{H}) \longrightarrow \text{Hom}((\mathcal{F} \otimes^L \mathcal{G})|_C, \mathcal{H}|_C),$$

where $i' : Z' \hookrightarrow X$ ranges over all closed subscheme structures on Z . Now, $Li^*(\mathcal{F} \otimes^L \mathcal{G}) \cong Li^*\mathcal{F} \otimes^L Li^*\mathcal{G}$. We have $Li^*\mathcal{F} \in {}_p\mathcal{D}_G^-(Z')_{\leq w}$ and $Li^*\mathcal{G} \in {}_q\mathcal{D}_G^-(Z')_{\leq v}$ by Lemma 6.6, and then $Li^*\mathcal{F} \otimes^L Li^*\mathcal{G} \in {}_{p+q}\mathcal{D}_G^-(Z')_{\leq w+v}$ by assumption. We also have $Ri^!\mathcal{H} \in {}_{p+q}\mathcal{D}_G^+(Z')_{\geq w+v+1}$, so the first term above clearly vanishes.

It now suffices to show that $(\mathcal{F} \otimes^L \mathcal{G})|_C \in {}_{p+q}\mathcal{D}_G^-(C)_{\leq w+v}$: that implies the vanishing of the last term in the exact sequence above, and hence of the middle term as well. Recall that on a single G -orbit, the tensor product functor is exact (because all objects of $\mathcal{C}_G(C)$ are locally free), so there is a natural isomorphism

$$h^r((\mathcal{F} \otimes^L \mathcal{G})|_C) \cong \bigoplus_{i+j=r} h^i(\mathcal{F}|_C) \otimes h^j(\mathcal{G}|_C).$$

We know that $h^i(\mathcal{F}|_C) \in \mathcal{C}_G(C)_{\leq w+p(C)}$ for all i , and that $h^j(\mathcal{G}|_C) \in \mathcal{C}_G(C)_{\leq v+q(C)}$ for all j . It follows that $h^r((\mathcal{F} \otimes^L \mathcal{G})|_C) \in \mathcal{C}_G(C)_{\leq w+v+p(C)+q(C)}$ for all r , and hence that $(\mathcal{F} \otimes^L \mathcal{G})|_C \in {}_{p+q}\mathcal{D}_G^-(C)_{\leq w+v}$, as desired.

(2) Consider $\mathbb{D}\mathcal{G} \in {}^q\mathcal{D}_G^-(X)_{\leq -v}$. By part (1), $\mathcal{F} \otimes^L \mathbb{D}\mathcal{G} \in {}_{p+\hat{q}}\mathcal{D}_G^-(X)_{\leq w-v}$. Since $R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong \mathbb{D}(\mathcal{F} \otimes^L \mathbb{D}\mathcal{G})$, the result follows. ■

8 Staggered Sheaves

In this section, we retain the assumptions that G acts on X with finitely many orbits, and that X is equipped with a recessed s -structure.

Given a function $q : X^{\text{gen}} \rightarrow \mathbb{Z}$, let us define full subcategories of $\mathcal{D}_G^-(X)$ and $\mathcal{D}_G^+(X)$ as follows:

$${}^q\mathcal{D}_G^-(X)^{\leq 0} = \{\mathcal{F} \in \mathcal{D}_G^-(X) \mid h^k(\mathcal{F}) \in {}^q\mathcal{D}_G^b(X)_{\leq -k} \text{ for all } k \in \mathbb{Z}\},$$

$${}^q\mathcal{D}_G^+(X)^{\geq 0} = \{\mathcal{G} \in \mathcal{D}_G^+(X) \mid \text{Hom}(\mathcal{F}[1], \mathcal{G}) = 0 \text{ for all } \mathcal{F} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}\}.$$

We also define bounded versions of these categories:

$${}^q\mathcal{D}_G^b(X)^{\leq 0} = {}^q\mathcal{D}_G^-(X)^{\leq 0} \cap \mathcal{D}_G^b(X) \quad \text{and} \quad {}^q\mathcal{D}_G^b(X)^{\geq 0} = {}^q\mathcal{D}_G^+(X)^{\geq 0} \cap \mathcal{D}_G^b(X).$$

Let $\mathcal{G} \in {}^q\mathcal{D}_G^+(X)^{\geq 0}$. There is some integer n such that $\mathcal{G} \in \mathcal{D}_G^+(X)^{\geq n}$, and then for any $\mathcal{F} \in \mathcal{D}_G^-(X)$, we have $\text{Hom}(\mathcal{F}[1], \mathcal{G}) \cong \text{Hom}(\tau^{\geq n}(\mathcal{F}[1]), \mathcal{G})$, with $\tau^{\geq n}(\mathcal{F}[1]) \in \mathcal{D}_G^b(X)$. Thus, the definition of ${}^q\mathcal{D}_G^+(X)^{\geq 0}$ could be changed to require $\text{Hom}(\mathcal{F}[1], \mathcal{G})$ to vanish only when $\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$. By Proposition 2.10(2), it follows that

$${}^q\mathcal{D}_G^b(X)^{\geq 0} = \{\mathcal{G} \in \mathcal{D}_G^b(X) \mid {}_q\beta_{\leq k}\mathcal{G} \in \mathcal{D}_G^b(X)^{\geq -k} \text{ for all } k \in \mathbb{Z}\}.$$

The categories ${}^q\mathcal{D}_G^b(X)^{\leq 0}$ and ${}^q\mathcal{D}_G^b(X)^{\geq 0}$ are none other than the categories associated in Definition 2.8 to the standard t -structure on $\mathcal{D}_G^b(X)$ with the baric structure $(\{{}_q\mathcal{D}_G^b(X)_{\leq w}\}, \{{}_q\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$.

Theorem 8.1. The categories $({}^q\mathcal{D}_G^b(X)^{\leq 0}, {}^q\mathcal{D}_G^b(X)^{\geq 0})$ constitute a bounded, nondegenerate t -structure on $\mathcal{D}_G^b(X)$. □

Definition 8.2. The t -structure $({}^q\mathcal{D}_G^b(X)^{\leq 0}, {}^q\mathcal{D}_G^b(X)^{\geq 0})$ is called the *staggered t -structure of perversity q* . Its heart, denoted ${}^q\mathcal{M}(X)$, is the category of *staggered sheaves*. □

This terminology and notation is consistent with that of [1] by Lemma 5.3. That is, if q happens to be a perversity function in the sense of [1], then the t -structure constructed here coincides with the t -structure associated to q in [1]. However, neither this theorem nor the main result of [1] encompasses the other: in [1], no assumptions were made on the number of orbits or the s -structure; here, no restrictions are imposed on the function $q : X^{\text{gen}} \rightarrow \mathbb{Z}$.

Note that if q happens to be a perversity function in the sense of [1], then Theorem 8.1 follows immediately from Lemma 5.3, but in general, Theorem 8.1 produces t -structures that are not given by the construction of [1].

Proof of Theorem 8.1. We will prove this theorem by invoking Theorem 2.11. To that end, we must define an invariant $\mu(\mathcal{F})$ for any object $\mathcal{F} \in \mathcal{D}_G^b(X)$ satisfying the hypotheses of that theorem. For any nonzero object $\mathcal{F} \in \mathcal{D}_G^b(X)$, let

$$m(\mathcal{F}) = \min\{k \in \mathbb{Z} \mid h^k(\mathcal{F}) \neq 0\}.$$

Let C be the maximum value of $\text{cod } Z$ as Z ranges over all closed subschemes of X . (Of course, $\text{cod } Z$ takes only finitely many distinct values, since X has finite Krull dimension.) Note that $\mathbb{D}(\mathcal{D}_G^b(X)^{\geq 0}) \subset \mathcal{D}_G^b(X)^{\leq C}$, and, more generally, $\mathbb{D}(\mathcal{D}_G^b(X)^{\geq n}) \subset \mathcal{D}_G^b(X)^{\leq C-n}$. Let

$$\mu(\mathcal{F}) = \begin{cases} C + 1 - m(\mathcal{F}) - m(\mathbb{D}\mathcal{F}) & \text{if } \mathcal{F} \neq 0, \\ 0 & \text{if } \mathcal{F} = 0. \end{cases}$$

We first prove that $\mu(\mathcal{F}) > 0$ whenever $\mathcal{F} \neq 0$. If $m(\mathcal{F}) = n$, then $\mathcal{F} \in \mathcal{D}_G^b(X)^{\geq n}$, so $\mathbb{D}\mathcal{F} \in \mathcal{D}_G^b(X)^{\leq C-n}$, and in particular, $m(\mathbb{D}\mathcal{F}) \leq C - n$. It follows that

$$\mu(\mathcal{F}) = C + 1 - m(\mathcal{F}) - m(\mathbb{D}\mathcal{F}) \geq C + 1 - n - (C - n) = 1,$$

as desired.

Next, the left t -exactness of ${}_q\beta_{\leq -n}$ implies that if $m(\mathcal{F}) = n$, then ${}_q\beta_{\leq -n}\mathcal{F} \in \mathcal{D}_G^b(X)^{\geq n}$, so $m({}_q\beta_{\leq -n}\mathcal{F}) \geq n$. Now, consider the distinguished triangle

$$\mathbb{D}_q\beta_{\geq -n+1}\mathcal{F} \longrightarrow \mathbb{D}\mathcal{F} \longrightarrow \mathbb{D}_q\beta_{\leq -n}\mathcal{F} \longrightarrow .$$

Since $\mathbb{D}_q\beta_{\geq -n+1}\mathcal{F} \in \hat{q}\mathcal{D}_G^b(X)_{\leq n-1}$ and $\mathbb{D}_q\beta_{\leq -n}\mathcal{F} \in \hat{q}\mathcal{D}_G^b(X)_{\geq n}$, we have canonical isomorphisms

$$\mathbb{D}_q\beta_{\geq -n+1}\mathcal{F} \cong \hat{q}\beta_{\leq n-1}\mathbb{D}\mathcal{F} \quad \text{and} \quad \mathbb{D}_q\beta_{\leq -n}\mathcal{F} \cong \hat{q}\beta_{\geq n}\mathbb{D}\mathcal{F}.$$

Now, the left t -exactness of $\hat{q}\beta_{\geq n}$ shows that $m(\mathbb{D}_q\beta_{\leq -n}\mathcal{F}) \geq m(\mathbb{D}\mathcal{F})$.

Finally, consider $\tau^{\geq n+1}{}_q\beta_{\leq -n}\mathcal{F}$. Clearly,

$$m(\tau^{\geq n+1}{}_q\beta_{\leq -n}\mathcal{F}) \geq n + 1 > m(\mathcal{F}).$$

Now, let $k = m(\mathbb{D}_q\beta_{\leq -n}\mathcal{F})$. By definition, ${}_q\beta_{\leq -n}\mathcal{F} \in \mathbb{D}(\mathcal{D}_G^b(X)^{\geq k})$. The t -structure $(\mathbb{D}(\mathcal{D}_G^b(X)^{\geq k}), \mathbb{D}(\mathcal{D}_G^b(X)^{\leq k}))$, which is dual to (a shift of) the standard t -structure, is an example of a perverse coherent t -structure [7], and therefore of a staggered t -structure in the sense of [1], so Corollary 5.6 tells us that $\mathbb{D}(\mathcal{D}_G^b(X)^{\geq k})$ is stable under $\tau^{\geq n+1}$. In particular, $\tau^{\geq n+1}{}_q\beta_{\leq -n}\mathcal{F} \in \mathbb{D}(\mathcal{D}_G^b(X)^{\geq k})$, so

$$m(\mathbb{D}\tau^{\geq n+1}{}_q\beta_{\leq -n}\mathcal{F}) \geq k = m(\mathbb{D}_q\beta_{\leq -n}\mathcal{F}) \geq m(\mathbb{D}\mathcal{F}).$$

We conclude that if $m(\mathcal{F}) = n$, then $\mu(\tau^{\geq n+1}{}_q\beta_{\leq -n}\mathcal{F}) < \mu(\mathcal{F})$. Thus, the hypotheses of Theorem 2.11 are satisfied. ■

Remark 8.3. If G does not act with finitely many orbits, or if the s -structure is not recessed, Remark 6.11 tells us that $(\{{}_q\mathcal{D}_G^b(X)_{\leq w}\}, \{{}_q\mathcal{D}_G^b(X)_{\geq w}\})_{w \in \mathbb{Z}}$ still constitutes a baric structure if we require q to be monotone and comonotone. The proof of Theorem 8.1 goes through in this setting. However, the conditions imposed on q are more restrictive than the conditions imposed on perversity functions in [1], so in this case, the theorem we obtain is actually just a special case of [1, Theorem 7.4]. Similar remarks apply to Theorems 8.6 and 8.11; cf. [1, Theorems 9.7 and 9.9]. □

Next, we study how the duality functor \mathbb{D} interacts with the staggered t -structure. Let $j : U \hookrightarrow X$ be an open subscheme. The following functions defined in terms of $q : X^{\text{gen}} \rightarrow \mathbb{Z}$ will be useful in the sequel.

$${}^bq(x) = \begin{cases} q(x) & \text{if } x \in U^{\text{gen}}, \\ q(x) - 1 & \text{if } x \notin U^{\text{gen}}, \end{cases} \quad \sharp q(x) = \begin{cases} q(x) & \text{if } x \in U^{\text{gen}}, \\ q(x) + 1 & \text{if } x \notin U^{\text{gen}}. \end{cases} \tag{8.1}$$

Lemma 8.4. Let $j: U \hookrightarrow X$ be the inclusion of an open subscheme, and $i: Z \hookrightarrow X$ the inclusion of a closed subscheme. Then:

- (1) j^* takes ${}^q\mathcal{D}_G^-(X)^{\leq 0}$ to ${}^q\mathcal{D}_G^-(U)^{\leq 0}$ and ${}^q\mathcal{D}_G^+(X)^{\geq 0}$ to ${}^q\mathcal{D}_G^+(U)^{\geq 0}$.
- (2) Li^* takes ${}^q\mathcal{D}_G^-(X)^{\leq 0}$ to ${}^q\mathcal{D}_G^-(Z)^{\leq 0}$.
- (3) $Ri^!$ takes ${}^q\mathcal{D}_G^+(X)^{\geq 0}$ to ${}^q\mathcal{D}_G^+(Z)^{\geq 0}$.
- (4) i_* takes ${}^q\mathcal{D}_G^-(Z)^{\leq 0}$ to ${}^q\mathcal{D}_G^-(X)^{\leq 0}$ and ${}^q\mathcal{D}_G^+(Z)^{\geq 0}$ to ${}^q\mathcal{D}_G^+(X)^{\geq 0}$. □

Proof. We will prove the parts of this lemma in the order (2), (4), (3), and (1).

(2) First, suppose $\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$ is concentrated in a single degree, say $\mathcal{F} \cong h^k(\mathcal{F})[-k]$. Then $\mathcal{F} \in {}^q\mathcal{D}_G^b(X)_{\leq -k}$, so by Lemma 6.6, $Li^*\mathcal{F} \in {}^q\mathcal{D}_G^-(Z)_{\leq -k}$. We also clearly have $Li^*\mathcal{F} \in \mathcal{D}_G^-(Z)^{\leq k}$, so it follows that $Li^*\mathcal{F} \in {}^q\mathcal{D}_G^-(Z)^{\leq 0}$. Next, for general $\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$, induction on the number of nonzero cohomology sheaves (together with the fact that ${}^q\mathcal{D}_G^-(Z)^{\leq 0}$ is stable under extensions) allows us to reduce to the case already considered, and we conclude that Li^* takes ${}^q\mathcal{D}_G^b(X)^{\leq 0}$ to ${}^q\mathcal{D}_G^-(Z)^{\leq 0}$. Finally, if $\mathcal{F} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}$, consider the distinguished triangle

$$Li^*\tau^{\leq k-1}\mathcal{F} \longrightarrow Li^*\mathcal{F} \longrightarrow Li^*\tau^{\geq k}\mathcal{F} \longrightarrow .$$

Since $\tau^{\geq k}\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$, we know that $Li^*\tau^{\geq k}\mathcal{F} \in {}^q\mathcal{D}_G^-(Z)^{\leq 0}$. Moreover, we see from the long exact cohomology sequence associated to this distinguished triangle $h^k(Li^*\mathcal{F}) \cong h^k(Li^*\tau^{\geq k}\mathcal{F}) \in {}^q\mathcal{D}_G^b(Z)_{\leq -k}$. Thus, $Li^*\mathcal{F} \in {}^q\mathcal{D}_G^-(Z)^{\leq 0}$, as desired. (4) Because i_* is t -exact (with respect to the standard t -structure), and because ${}^q\mathcal{D}_G^-(Z)^{\leq 0}$ and ${}^q\mathcal{D}_G^-(X)^{\leq 0}$ are defined by conditions on cohomology sheaves, it follows from Remark 6.7 that i_* takes ${}^q\mathcal{D}_G^-(Z)^{\leq 0}$ to ${}^q\mathcal{D}_G^-(X)^{\leq 0}$. (A similar argument shows that j^* takes ${}^q\mathcal{D}_G^-(X)^{\leq 0}$ to ${}^q\mathcal{D}_G^-(U)^{\leq 0}$.) On the other hand, if $\mathcal{F} \in {}^q\mathcal{D}_G^+(Z)^{\geq 0}$, then for any $\mathcal{G} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}$, we have

$$\mathrm{Hom}(\mathcal{G}[1], i_*\mathcal{F}) \cong \mathrm{Hom}(Li^*\mathcal{G}[1], \mathcal{F}) = 0,$$

where, in the last step, we have used the fact that $Li^*\mathcal{G} \in {}^q\mathcal{D}_G^-(Z)^{\leq 0}$. Thus, $i_*\mathcal{F} \in {}^q\mathcal{D}_G^+(X)^{\geq 0}$.

- (3) Let $\mathcal{F} \in {}^q\mathcal{D}_G^+(X)^{\geq 0}$. For any $\mathcal{G} \in {}^q\mathcal{D}_G^-(Z)^{\leq 0}$, we have

$$\mathrm{Hom}(\mathcal{G}[1], Ri^!\mathcal{F}) \cong \mathrm{Hom}(i_*\mathcal{G}[1], \mathcal{F}) = 0.$$

Here, we have used the fact that $i_*\mathcal{G} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}$. Thus, $Ri^!\mathcal{F} \in {}^q\mathcal{D}_G^+(Z)^{\geq 0}$.

(1) It was observed in the proof of part (4) that j^* takes ${}^q\mathcal{D}_G^-(X)^{\leq 0}$ to ${}^q\mathcal{D}_G^-(U)^{\leq 0}$. Next, suppose $\mathcal{F} \in {}^q\mathcal{D}_G^+(X)^{\geq 0}$. To show that $j^*\mathcal{F} \in {}^q\mathcal{D}_G^+(U)^{\geq 0}$, it suffices, as noted at the beginning of this section, to show that $\text{Hom}(\mathcal{G}[1], \mathcal{F}) = 0$ for all $\mathcal{G} \in {}^q\mathcal{D}_G^b(U)^{\leq 0}$. But since ${}^q\mathcal{D}_G^b(U)^{\leq 0}$ is stable under $\tau^{\geq n}$ and $\tau^{\leq n}$, we can further reduce to showing that $\text{Hom}(\mathcal{G}[1], \mathcal{F}) = 0$ whenever \mathcal{G} is an object of ${}^q\mathcal{D}_G^b(U)^{\leq 0}$ that is concentrated in a single degree. Suppose $\mathcal{G} \cong h^k(\mathcal{G})[-k]$. We then have $h^k(\mathcal{G}) \in {}_q\mathcal{C}_G(U)_{\leq -k}$. Let ${}^bq: X^{\text{gen}} \rightarrow \mathbb{Z}$ be as in (8.1). Then of course ${}_q\mathcal{C}_G(U)_{\leq -k} = {}_bq\mathcal{C}_G(U)_{\leq -k}$, and by Lemma 6.3, $\mathcal{G}[k]$ may be extended to a sheaf $\mathcal{G}' \in {}_bq\mathcal{C}_G(X)_{\leq -k}$. Consider the exact sequence

$$\text{Hom}(\mathcal{G}'[-k+1], \mathcal{F}) \longrightarrow \text{Hom}(j^*\mathcal{G}'[-k+1], j^*\mathcal{F}) \longrightarrow \lim_{\substack{\longrightarrow \\ Z'}} \text{Hom}(L\kappa_{Z'}^*\mathcal{G}'[-k], R\kappa_{Z'}^1\mathcal{F}),$$

where $\kappa_{Z'}: Z' \hookrightarrow X$ ranges over all closed subscheme structures on the complement of U . We clearly have that $\mathcal{G}'[-k] \in {}_bq\mathcal{D}_G^b(X)^{\leq 0} \subset {}^q\mathcal{D}_G^b(X)^{\leq 0}$, so $\text{Hom}(\mathcal{G}'[-k+1], \mathcal{F}) = 0$. We already know that $R\kappa_{Z'}^1\mathcal{F} \in {}^q\mathcal{D}_G^+(Z')^{\geq 0}$, and that $L\kappa_{Z'}^*\mathcal{G}'[-k] \in {}_bq\mathcal{D}_G^-(Z')^{\leq 0} = {}^q\mathcal{D}_G^-(Z')^{\leq -1}$, so the last term above vanishes as well. Therefore, $\text{Hom}(j^*\mathcal{G}'[-k+1], j^*\mathcal{F}) \cong \text{Hom}(\mathcal{G}[1], j^*\mathcal{F}) = 0$, so $j^*\mathcal{F} \in {}^q\mathcal{D}_G^+(U)^{\geq 0}$, as desired. ■

Proposition 8.5. We have $\mathbb{D}({}^q\mathcal{D}_G^-(X)^{\leq 0}) = \bar{q}\mathcal{D}_G^+(X)^{\geq 0}$. □

Proof. We proceed by the Noetherian induction, and assume that the result is known for all proper closed subschemes of X . To show that $\mathbb{D}({}^q\mathcal{D}_G^-(X)^{\leq 0}) \subset \bar{q}\mathcal{D}_G^+(X)^{\geq 0}$, let us begin by considering the special case where $\mathcal{F} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}$ is concentrated in a single degree, say $\mathcal{F} \cong h^k(\mathcal{F})[-k]$. Then $\mathcal{F} \in {}_q\mathcal{D}_G^b(X)_{\leq -k}$, so $\mathbb{D}\mathcal{F} \in \hat{q}\mathcal{D}_G^b(X)_{\geq k}$. Choose an open orbit $C \subset X$. By [1, Lemma 6.6], $\mathbb{D}\mathcal{F}|_C$ is concentrated in a single degree, viz., in degree $\text{cod } \bar{C} - k$. We claim that $\mathbb{D}\mathcal{F}|_C \in \bar{q}\mathcal{D}_G^+(C)^{\geq 0}$. To prove this, it suffices to show that if $\mathcal{G} \in \hat{q}\mathcal{D}_G^b(C)^{\leq 0}$, then $\text{Hom}(\mathcal{G}[1], \mathbb{D}\mathcal{F}|_C) = 0$. Consider the exact sequence

$$\begin{aligned} \text{Hom}((\tau^{\geq \text{cod } \bar{C} - k + 1}\mathcal{G})[1], \mathbb{D}\mathcal{F}|_C) &\longrightarrow \text{Hom}(\mathcal{G}[1], \mathbb{D}\mathcal{F}|_C) \\ &\longrightarrow \text{Hom}((\tau^{\leq \text{cod } \bar{C} - k}\mathcal{G})[1], \mathbb{D}\mathcal{F}|_C). \end{aligned}$$

The last term clearly vanishes because $\mathbb{D}\mathcal{F}|_C \in \mathcal{D}_G^b(C)^{\geq \text{cod } X - k}$. On the other hand, note that $\tau^{\geq \text{cod } \bar{C} - k + 1}\mathcal{G} \in \hat{q}\mathcal{D}_G^b(C)_{\leq k - \text{cod } \bar{C} - 1}$. Over the single orbit C , the functions \bar{q} and \hat{q} differ simply by the constant $\text{cod } \bar{C}$, and thus $\hat{q}\mathcal{D}_G^b(C)_{\leq k - \text{cod } \bar{C} - 1} = \hat{q}\mathcal{D}_G^b(C)_{\leq k - 1}$. Since $\mathbb{D}\mathcal{F}|_C \in \hat{q}\mathcal{D}_G^b(C)_{\geq k}$, the first term above vanishes, and hence so does the middle term.

We have shown that $\mathbb{D}\mathcal{F}|_C \in {}^q\mathcal{D}_G^+(C)^{\geq 0}$. Next, let $\mathcal{G} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$, and consider the exact sequence

$$\lim_{\substack{\longrightarrow \\ Z'}} \text{Hom}(Li_{Z'}^* \mathcal{G}[1], Ri_{Z'}^! \mathbb{D}\mathcal{F}) \longrightarrow \text{Hom}(\mathcal{G}[1], \mathbb{D}\mathcal{F}) \longrightarrow \text{Hom}(\mathcal{G}[1]|_C, \mathbb{D}\mathcal{F}|_C),$$

where $i_{Z'} : Z' \hookrightarrow X$ ranges over all closed subscheme structures on the complement of C . We have just seen that the last term vanishes. Also, $Li_{Z'}^* \mathcal{G} \in {}^q\mathcal{D}_G^-(Z')^{\leq 0}$ and $Ri_{Z'}^! \mathbb{D}\mathcal{F} \cong \mathbb{D}(Li_{Z'}^* \mathcal{F}) \in {}^q\mathcal{D}_G^+(Z')^{\geq 0}$ by Lemma 8.4 and the inductive assumption, so the first term vanishes as well. Therefore, the middle term vanishes, and we conclude that for $\mathcal{F} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}$ concentrated in a single degree, we have $\mathbb{D}\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\geq 0}$. It follows by induction on the number of nonzero cohomology sheaves that \mathbb{D} also takes all objects of the bounded category ${}^q\mathcal{D}_G^b(X)^{\leq 0}$ to ${}^q\mathcal{D}_G^b(X)^{\geq 0}$.

Finally, let us consider a general object $\mathcal{F} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}$. We wish to show that $\text{Hom}(\mathcal{G}[1], \mathbb{D}\mathcal{F}) = 0$ for all $\mathcal{G} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$. By the previous paragraph, $\mathbb{D}\mathcal{G} \in {}^q\mathcal{D}_G^b(X)^{\geq 0} \subset {}^q\mathcal{D}_G^+(X)^{\geq 0}$, so

$$\text{Hom}(\mathcal{G}[1], \mathbb{D}\mathcal{F}) \cong \text{Hom}(\mathcal{F}, \mathbb{D}(\mathcal{G}[1])) \cong \text{Hom}(\mathcal{F}[1], \mathbb{D}\mathcal{G}) = 0.$$

Thus, $\mathbb{D}({}^q\mathcal{D}_G^-(X)^{\leq 0}) \subset {}^q\mathcal{D}_G^+(X)^{\geq 0}$.

The argument for the opposite inclusion is similar, and we again use the Noetherian induction, but we cannot begin with the case of an object concentrated in one degree, since ${}^q\mathcal{D}_G^+(X)^{\geq 0}$ is not stable under the standard truncation functors. The bounded category ${}^q\mathcal{D}_G^b(X)^{\geq 0}$ is, however, stable under the baric truncation functors ${}_{\hat{q}}\beta_{\leq k}$ and ${}_{\hat{q}}\beta_{\geq k}$. Suppose, then, that $\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\geq 0}$ is “baric-pure”: that is, $\mathcal{F} \in {}_{\hat{q}}\mathcal{D}_G^b(X)_{\leq k} \cap {}_{\hat{q}}\mathcal{D}_G^b(X)_{\geq k}$ for some k . If we prove that $\mathbb{D}\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$, then it will follow by induction on “baric length” that \mathbb{D} takes all objects of ${}^q\mathcal{D}_G^b(X)^{\geq 0}$ to ${}^q\mathcal{D}_G^b(X)^{\leq 0}$.

The assumptions on \mathcal{F} imply that $\mathcal{F} \in \mathcal{D}_G^b(X)^{\geq -k}$. Once again, let $C \subset X$ be an open orbit. It follows from [1, Lemma 6.6] that $\mathbb{D}\mathcal{F}|_C \in \mathcal{D}_G^b(C)^{\leq \text{cod } \bar{C} + k}$. We also know that $\mathbb{D}\mathcal{F} \in {}_{\hat{q}}\mathcal{D}_G^b(X)_{\leq -k}$, where

$$\hat{q}(C) = \text{alt } \bar{C} - (\text{alt } \bar{C} + \text{cod } \bar{C} - q(C)) = q(C) - \text{cod } \bar{C}.$$

In particular, ${}_{\hat{q}}\mathcal{D}_G^b(C)_{\leq -k} = {}^q\mathcal{D}_G^b(C)_{\leq -\text{cod } \bar{C} - k}$. Since $\mathbb{D}\mathcal{F}|_C \in \mathcal{D}_G^b(C)^{\leq \text{cod } \bar{C} + k}$, we see that $\mathbb{D}\mathcal{F}|_C \in {}^q\mathcal{D}_G^b(C)^{\leq 0}$.

To show that $\mathbb{D}\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$, it suffices, by Proposition 2.10, to show that $\mathrm{Hom}(\mathbb{D}\mathcal{F}[1], \mathcal{G}) = 0$ for all $\mathcal{G} \in {}^q\mathcal{D}_G^b(X)^{\geq 0}$. Consider the exact sequence

$$\lim_{\substack{\longrightarrow \\ Z'}} \mathrm{Hom}(Li_{Z'}^* \mathbb{D}\mathcal{F}[1], Ri_{Z'}^! \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathbb{D}\mathcal{F}[1], \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathbb{D}\mathcal{F}[1]|_C, \mathcal{G}|_C),$$

where $i_{Z'} : Z' \hookrightarrow X$ ranges over all closed subscheme structures on the complement of C . The last term above vanishes because $\mathbb{D}\mathcal{F}|_C \in {}^q\mathcal{D}_G^b(C)^{\leq 0}$. We also have $Li_{Z'}^* \mathbb{D}\mathcal{F} \cong \mathbb{D}(Ri_{Z'}^! \mathcal{F}) \in {}^q\mathcal{D}_G^-(Z')^{\leq 0}$ and $Ri_{Z'}^! \mathcal{G} \in {}^q\mathcal{D}_G^+(Z')^{\geq 0}$ by Lemma 8.4 and the inductive assumption. Hence, the first term in the sequence above vanishes, so the middle term vanishes as well, and we conclude that $\mathbb{D}\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$. Thus, $\mathbb{D}({}^{\bar{q}}\mathcal{D}_G^b(X)^{\geq 0}) \subset {}^q\mathcal{D}_G^b(X)^{\leq 0}$.

Finally, we must consider a general object $\mathcal{F} \in {}^{\bar{q}}\mathcal{D}_G^+(X)^{\geq 0}$. Showing that $\mathbb{D}\mathcal{F} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}$ is equivalent to showing that $\tau^{\geq k} \mathbb{D}\mathcal{F} \in {}^q\mathcal{D}_G^b(X)^{\leq 0}$ for all k . If the latter condition fails for some k , then there exists an object $\mathcal{G} \in {}^q\mathcal{D}_G^b(X)^{\geq 1}$ such that $\mathrm{Hom}(\tau^{\geq k} \mathbb{D}\mathcal{F}, \mathcal{G}) \neq 0$. By replacing k by a smaller integer if necessary, we may assume that $\mathcal{G} \in \mathcal{D}_G^b(X)^{\geq k}$. We then have

$$\mathrm{Hom}(\tau^{\geq k} \mathbb{D}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathbb{D}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathbb{D}\mathcal{G}, \mathcal{F}) \neq 0.$$

By exchanging the roles of q and \bar{q} in the previous paragraph, we see that $\mathbb{D}\mathcal{G} \in {}^{\bar{q}}\mathcal{D}_G^b(X)^{\leq -1}$, but this contradicts the fact that $\mathcal{F} \in {}^{\bar{q}}\mathcal{D}_G^+(X)^{\geq 0}$. Therefore, $\mathbb{D}\mathcal{F} \in {}^q\mathcal{D}_G^-(X)^{\leq 0}$, and $\mathbb{D}({}^{\bar{q}}\mathcal{D}_G^+(X)^{\geq 0}) = {}^q\mathcal{D}_G^-(X)^{\leq 0}$, as desired. ■

The next theorem follows immediately from the last proposition.

Theorem 8.6. The dual of the staggered t -structure $({}^q\mathcal{D}_G^b(X)^{\leq 0}, {}^q\mathcal{D}_G^b(X)^{\geq 0})$ is the staggered t -structure $({}^{\bar{q}}\mathcal{D}_G^b(X)^{\leq 0}, {}^{\bar{q}}\mathcal{D}_G^b(X)^{\geq 0})$. In particular, in the case where every orbit $C \subset X$ has even staggered codimension, and q is the function $q(C) = \frac{1}{2} \mathrm{scod} C$, the t -structure $({}^q\mathcal{D}_G^b(X)^{\leq 0}, {}^q\mathcal{D}_G^b(X)^{\geq 0})$ is self-dual. □

We conclude with a study of simple objects in ${}^q\mathcal{M}(X)$. The statements below and their proofs are very similar to those in [7, Section 3.2] or [1, Section 9], and most details of the proofs will be omitted. Instead, each statement is followed by brief remarks clarifying the relationship to statements in [7] or [1].

Proposition 8.7. Let $j : U \hookrightarrow X$ be a dense open subscheme. Given a function $q : X^{\mathrm{gen}} \rightarrow \mathbb{Z}$, define ${}^bq, {}^{\sharp}q : X^{\mathrm{gen}} \rightarrow \mathbb{Z}$ as in (8.1), and define a full subcategory ${}^q\mathcal{M}^{!*}(X) \subset {}^q\mathcal{M}(X)$

by ${}^q\mathcal{M}^{!*}(X) = {}^bq\mathcal{D}_G^b(X)^{\leq 0} \cap {}^{\#q}\mathcal{D}_G^b(X)^{\geq 0}$. The functor j^* induces an equivalence of categories ${}^q\mathcal{M}^{!*}(X) \rightarrow {}^q\mathcal{M}(U)$. Moreover, objects of ${}^q\mathcal{M}^{!*}(X)$ have no subobjects or quotients in ${}^q\mathcal{M}(X)$ that are supported on $X \setminus U$. □

Remarks on proof. This statement corresponds to [7, Theorem 2] or [1, Proposition 9.2], but both those statements impose a condition on the function q (denoted p in [1, 7]) that is not imposed here. The reason is that the proof requires that the categories $({}^bq\mathcal{D}_G^b(X)^{\leq 0}, {}^bq\mathcal{D}_G^b(X)^{\geq 0})$ and $({}^{\#q}\mathcal{D}_G^b(X)^{\leq 0}, {}^{\#q}\mathcal{D}_G^b(X)^{\geq 0})$ associated to bq and ${}^{\#q}$ (denoted p^+ and p^- in [1, 7]) actually constitute t -structures. In the present paper, Theorem 8.1 tells us that this is the case with no assumptions, whereas in both [1, 7], the t -structure is constructed only for p obeying certain inequalities. ■

Definition 8.8. The inverse equivalence to that of the preceding proposition, denoted $j_{!*} : {}^q\mathcal{M}(U) \rightarrow {}^q\mathcal{M}^{!*}(X)$, is known as the *intermediate-extension functor*. □

Definition 8.9. Let Y be a locally closed subscheme of X . Let $h : Y \hookrightarrow \bar{Y}$ and $\kappa : \bar{Y} \hookrightarrow X$ denote the inclusion maps. For any $\mathcal{F} \in {}^q\mathcal{M}(Y)$, we define an object of ${}^q\mathcal{M}(X)$ by

$$\mathcal{IC}(\bar{Y}, \mathcal{F}) = \kappa_*(h_{!*}\mathcal{F}).$$

This is called the *(staggered) intersection cohomology complex* associated to \mathcal{F} . □

Recall that the *step* of a coherent sheaf is defined to be the unique integer w (if such an integer exists) such that the sheaf belongs $\mathcal{C}_G(X)_{\leq w} \cap \mathcal{C}_G(X)_{\geq w}$. An irreducible vector bundle on an orbit always has a well-defined step.

Proposition 8.10. Let $\mathcal{F} \in {}^q\mathcal{M}(X)$. \mathcal{F} is a simple object if and only if $\mathcal{F} \cong \mathcal{IC}(\bar{C}, \mathcal{L}[-q(C) + \text{step } \mathcal{L}])$ for some orbit $C \subset X$ and some irreducible vector bundle $\mathcal{L} \in \mathcal{C}_G(C)$. □

Remarks on proof. This statement is analogous to [7, Corollary 4] and to [1, Theorem 9.7]. The main difference is that in [1], \mathcal{F} is assumed at the outset to be supported on (a possibly nonreduced subscheme structure on) the closure of one orbit. (The statement of [1, Theorem 9.7] also imposes conditions on q , but those are unnecessary here for reasons explained in the remarks following Proposition 8.7.) In [7], it is shown that a simple object must be supported on an orbit closure using Rosenlicht’s Theorem, but that argument cannot be used here for the reasons given in [1, Remark 9.8].

To reduce this statement to one where the proof of [1, Theorem 9.7] can be repeated verbatim, we must show by other means that the support of a simple object is an (a priori possibly nonreduced) orbit closure. Since X is assumed to consist of finitely many G -orbits, it suffices to show that the support of a simple object is irreducible. Let $\kappa : X' \hookrightarrow X$ be the scheme-theoretic support of \mathcal{F} ; that is, $\mathcal{F} \cong \kappa_* \mathcal{F}'$, and the restriction of \mathcal{F}' to any open subscheme of X' is nonzero. Assume X' is reducible; let $i : Z \hookrightarrow X'$ and $i' : Z' \rightarrow X'$ be proper closed subschemes such that $Z \cup Z' = X'$. Let $U = Z \setminus (Z \cap Z')$ and $U' = Z' \setminus (Z \cap Z')$. Clearly, U and U' are disjoint open subschemes of X' . Let $V = U \cup U'$. The natural morphism

$$i_* Ri^! \mathcal{F}'|_V \longrightarrow \mathcal{F}'|_V$$

is the inclusion of the direct summand of $\mathcal{F}|_V$ supported on U . In particular, the above morphism is neither 0 nor an isomorphism. But it is also the restriction to V of the natural morphism

$${}^q h^0(i_* Ri^! \mathcal{F}') \longrightarrow \mathcal{F}',$$

so this latter is also neither 0 nor an isomorphism. Therefore, \mathcal{F}' is not simple, and hence neither is \mathcal{F} . ■

Theorem 8.11. ${}^q \mathcal{M}(X)$ is a finite-length category. □

Remarks on proof. This statement and its proof are identical to those of [7, Corollary 5] or of [1, Theorem 9.9], except that here, as in Propositions 8.7 and 8.10, we impose no restrictions on q . ■

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