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The Coherent-Constructible Correspondence and Fourier–Mukai Transforms

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Dedicated to Professor Hua Loo-keng on his 100th birth anniversary

Abstract As evidence for his conjecture in birational log geometry, Kawamata constructed a family of derived equivalences between toric orbifolds. In a previous paper, the authors showed that the derived category of a toric orbifold is naturally identified with a category of polyhedrally-constructible sheaves on $\mathbb{R}^n$. In this paper we investigate and reprove some of Kawamata's results from this perspective.

Keywords Toric orbifolds, coherent sheaves, constructible sheaves, Fourier–Mukai transforms

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Contents

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1 Introduction

1.1 Coherent-Constructible Correspondence

The coherent-constructible correspondence (CCC), defined in [1, 2] and first described by Bondal [3], is an equivalence between a category of coherent sheaves on a toric n-fold and a category of constructible sheaves on a compact n-torus $(S^1)^n$. An equivariant version of this correspondence can be viewed as a "categorification" of Morelli's description of the equivariant K-theory of a toric variety in terms of a polytope algebra [4]. Generalizing the familiar correspondence in toric geometry between ample equivariant line bundles and moment polytopes, the equivariant CCC provides an equivalence between torus-equivariant coherent sheaves on a toric n-fold and constructible sheaves on $\mathbb{R}^n$, the universal cover of $(S^1)^n$.

The CCC was extended to toric Deligne–Mumford (DM) stacks in [5]. Toric DM stacks were defined by Borisov–Chen–Smith in terms of stacky fans [6]. In this paper we consider toric orbifolds, which are toric DM stacks with generically trivial stabilizers. Let $\mathcal{X}_{\mathbf{\Sigma}}$ be a complete toric orbifold defined by a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, where $N \cong \mathbb{Z}^n$, Σ is a simplicial fan in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. It contains the torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset. The CCC for the toric orbifold $\mathcal{X}_{\mathbf{\Sigma}}$ is the following quasi-equivalence of triangulated dg categories:

$$\kappa_{\mathbf{\Sigma}} : \mathcal{P}\mathrm{erf}_T(\mathcal{X}_{\mathbf{\Sigma}}) \xrightarrow{\sim} Sh_{cc}(M_{\mathbb{R}}; \Lambda_{\mathbf{\Sigma}}) \tag{1.1}$$

where $\mathcal{P}\mathrm{erf}_T(\mathcal{X}_{\mathbf{\Sigma}})$ is the category of T-equivariant perfect complexes on $\mathcal{X}_{\mathbf{\Sigma}}$, and $Sh_{cc}(M_{\mathbb{R}}; \Lambda_{\mathbf{\Sigma}})$ is a category of constructible sheaves on $M_{\mathbb{R}}$ (the dual space of $N_{\mathbb{R}}$) characterized by a conical Lagrangian $\Lambda_{\mathbf{\Sigma}} \in T^* M_{\mathbb{R}}$ determined by the stacky fan $\mathbf{\Sigma}$. (The precise definitions of the categories in (1.1) will be given in Section 3.) Moreover, the functor $\kappa_{\mathbf{\Sigma}}$ is monoidal with respect to the tensor product of coherent sheaves on $\mathcal{X}_{\mathbf{\Sigma}}$ and the convolution product of constructible sheaves on $M_{\mathbb{R}}$. Please note that since toric orbifolds are *smooth* DM stacks, the category $\mathcal{P}\mathrm{erf}_T(\mathcal{X})$ is the same as the category $\mathrm{Coh}_T(\mathcal{X})$, and we will use both notations interchangeably throughout the paper.

1.2 Fourier–Mukai Transforms

The coarse moduli space of the toric orbifold $\mathcal{X}_{\mathbf{\Sigma}}$ is the toric variety X_Σ defined by the simplicial fan Σ. The toric orbifold $\mathcal{X}_{\mathbf{\Sigma}}$ is the DM stack associated with a log pair (X_Σ, B) in the sense of Kawamata [7, Definition 2.1]. Kawamata considered pairs (X, B) of varieties and $\mathbb{Q}$-divisors which have smooth local coverings. For such a pair he associated a DM stack $\mathcal{X}$, such that $p^*(K_X + B) = K_{\mathcal{X}}$, where $p : \mathcal{X} \to X$ is the canonical map to the coarse moduli space. He conjectured that if there is an equivalence of log canonical divisors between birationally equivalent pairs, then there is an equivalence of derived categories of the associated DM stacks [7, Conjecture 2.2]. He proved his conjecture for quasi-smooth toroidal pairs. Here we briefly describe his results in the toric case; please see [7, Theorem 4.2] for the precise statements, in the toroidal case (which includes the toric case as a special case). Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be toric orbifolds associated with projective toric log pairs (X_1, B) and (X_2, C), respectively. Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are K-equivalent [8] in the sense that there exists an orbifold $\mathcal{W}$ and proper birational morphisms $\mu_i : \mathcal{W} \to \mathcal{X}_i$ of orbifolds such that $\mu_1^* K_{\mathcal{X}_1} = \mu_2^* K_{\mathcal{X}_2}$. Then the Fourier–Mukai functor

$$F'_{12} = \mu_{1*} \circ \mu_2^* : D^b\mathrm{Coh}(\mathcal{X}_2) \to D^b\mathrm{Coh}(\mathcal{X}_1)$$

is an equivalence of triangulated categories.[1] Indeed, Kawamata proved that if $\mu_1^* K_{\mathcal{X}_1} \geq \mu_2^* K_{\mathcal{X}_2}$ and the birational map $f : X_1 \dashrightarrow X_2$ is (1) the identity, (2) a divisorial contraction, (3) the inverse of a divisorial contraction, or (4) a flip, then the Fourier–Mukai functor F'_{12} is fully faithful; it is an equivalence when $\mu_1^* K_{\mathcal{X}_1} = \mu_2^* K_{\mathcal{X}_2}$. In all of these four cases, $\mathcal{W}$ in $\mathcal{X}_1 \xleftarrow{\mu_1} \mathcal{W} \xrightarrow{\mu_2} \mathcal{W}_2$ is also a toric orbifold. The open tori in $\mathcal{X}_1$ and $\mathcal{X}_2$ are identified by the birational map, and we denote them by the same T. The McKay correspondence for abelian quotient singularities is a special case of (2) or (3).

1.3 CCC and Fourier–Mukai Transforms

It is natural to expect that Kawamata's theorem [7, Theorem 4.2] holds in the equivariant setting, and that one can use CCC to give an elementary proof. The following square of functors commutes up to natural isomorphisms

$$\begin{tikzcd}
\mathcal{P}\mathrm{erf}_T(\mathcal{X}_2) \arrow[r, "\kappa_2"] \arrow[d, "F'_{12}"'] & Sh_{cc}(M_{\mathbb{R}}; \Lambda_{\mathbf{\Sigma}_2}) \arrow[d, "F_{12}"'] \\
\mathcal{P}\mathrm{erf}_T(\mathcal{X}_1) \arrow[r, "\kappa_1"] & Sh_{cc}(M_{\mathbb{R}}; \Lambda_{\mathbf{\Sigma}_1}),
\end{tikzcd}$$

where κ_i are quasi-equivalences. So it suffices to prove that F_{12} is cohomologically full and faithful in various cases (1)–(4). In this paper, we provide such elementary proofs in cases (1), (2) and (3). (One may also obtain an elementary proof of (4) in a similar way. Details will

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[1] We use F'_{12} instead of F_{12} since the notation without prime is reserved for the induced functor on constructible sheaves.

appear elsewhere.) We describe F_{12} and F'_{12} explicitly in terms of theta sheaves (see Section 3.3) which are building blocks of the CCC. Our proofs do not rely on [9, Theorem 1-2-5].

Note that F_{12} and F'_{12} do not preserve the monoidal structures.

Remark 1.1 Although we are restricting to *toric orbifolds*, i.e. toric DM stacks with a generically trivial stabilizer, Proposition 3.1 of [5] shows that for any toric DM stack $\mathcal{X}$, the dg category $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}) \cong \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}^{\mathrm{rig}})$ where $\mathcal{T}$ is the DM torus acting on $\mathcal{X}$, and the orbifold $\mathcal{X}^{\mathrm{rig}}$ is the rigidification of $\mathcal{X}$. (See [10] for definitions.) Thus the result of this paper implies that the functor $\mathcal{P}\mathrm{erf}_{\mathcal{T}_2}(\mathcal{X}_2) \to \mathcal{P}\mathrm{erf}_{\mathcal{T}_1}(\mathcal{X}_1)$ is a quasi-embedding (cohomologically full and faithful) in cases (1), (2) and (3).

Remark 1.2 We show that the Fourier–Mukai functor F'_{12} is a quasi-embedding of the dg categories of T-equivariant perfect sheaves in Section 4. Every line bundle on $\mathcal{X}_1$ or $\mathcal{X}_2$ admits an equivariant structure, and the set of such equivariant structures is (non-canonically) parameterized by $M = \mathrm{Hom}(N, \mathbb{Z})$, the weight lattice of the torus. The nature of our construction of F'_{12} respects twisting by a lattice point: $F'_{12}(\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}_2,\chi}) = F'_{12}(\mathcal{F}) \otimes \mathcal{O}_{\mathcal{X}_1,\chi}$, where $\mathcal{O}_{\mathcal{X}_1,\chi}$ and $\mathcal{O}_{\mathcal{X}_2,\chi}$ are structure sheaves twisted by a character $\chi \in M$. Since every coherent sheaf on $\mathcal{X}_1$ admits a bounded resolution by line bundles (see e.g. Section 4 of [11]), the equivariant quasi-embedding functor F'_{12} passes to a quasi-embedding functor $\mathcal{P}\mathrm{erf}(\mathcal{X}_2) \to \mathcal{P}\mathrm{erf}(\mathcal{X}_1)$.

1.4 A Simple Example: McKay Correspondence for the A_1-singularity

$X_2 = \mathbb{C}^2/\mathbb{Z}_2$ is the A_1-singularity, $X_1 = \mathcal{O}_{\mathbb{P}^1}(-2)$ is its crepant resolution, and $\mathcal{X}_1 = X_1$ and $\mathcal{X}_2 = [\mathbb{C}^2/\mathbb{Z}_2]$ are the canonical toric orbifolds associated with X_1 and X_2 respectively. In this case both F'_{12} and F'_{21} are equivalences, as shown in Figure 1.

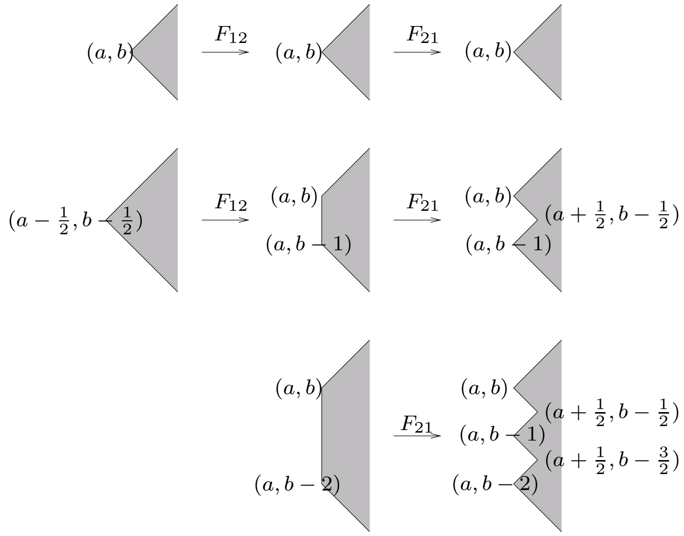


Figure 1 a, b are integers. The sheaves are costandard sheaves supported on the shaded area (see Subsection 3.2 for the definition)

1.5 Outline

In Section 2, we give a brief introduction to toric orbifolds. In Section 3, we give a leisure exposition of the CCC of toric orbifolds; we also state the CCC for toric varieties. In Section 4, we elaborate Kawamata's theorem in the equivariant setting from the perspective of constructible sheaves.

2 Toric Orbifolds

In [6], Borisov–Chen–Smith introduced toric DM stacks. In this paper we will consider the case of *toric orbifolds*. A toric orbifold is a toric DM stack with trivial generic stabilizer.

2.1 The Stacky Fan

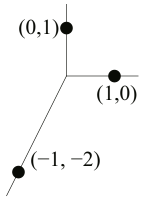
Let $N \cong \mathbb{Z}^n$ be a free abelian group, and let Σ be a simplicial fan in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. The pair (N, Σ) defines a simplicial toric variety X_Σ of dimension n (see [12]). Let $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ be the set of 1-dimensional cones in the fan Σ, and let $v_i \in N$ be the unique generator of the semigroup $\rho_i \cap N$, so that $\rho_i \cap N = \mathbb{Z}_{\geq 0} v_i$.

A *stacky fan* $\mathbf{\Sigma}$ is defined as the data (N, Σ, β), where

$$\beta : \widetilde{N} := \bigoplus_{i=1}^{r} \mathbb{Z}\widetilde{b}_i \cong \mathbb{Z}^r \to N \cong \mathbb{Z}^n$$

is a group homomorphism sending $\widetilde{b}_i$ to $b_i = n_i v_i \in \mathbb{Z}_{>0} v_i$. If $n_i = 1$ for $i = 1, \ldots, r$ then the corresponding map is denoted by β_{can}. We assume that $\{v_1, \ldots, v_r\}$ span $N_{\mathbb{R}}$, which implies that the cokernel of $\beta : \widetilde{N} \to N$ is finite.

Example 2.1 $N = \mathbb{Z}^2$, $\widetilde{N} = \mathbb{Z}^3$. The fan Σ and the map β are shown in Figure 2.



$$\widetilde{N} = \mathbb{Z}^3 \xrightarrow{\beta=\beta_{\mathrm{can}}=\begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} N = \mathbb{Z}^2.$$

Figure 2 The stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, with Σ and β shown above. The dots are $b_1 = (1, 0)$, $b_2 = (-1, -2)$, and $b_3 = (0, 1)$

We next consider a 1-dimensional example in which $\beta \neq \beta_{\mathrm{can}}$.

Example 2.2 $N = \mathbb{Z}$, $\widetilde{N} = \mathbb{Z}$. The fan Σ and the map β are shown in Figure 3.

$$\widetilde{N} = \mathbb{Z} \xrightarrow{\beta=[\,3\ -1\,]} N = \mathbb{Z}.$$

Figure 3 The stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, with Σ and β shown above. The larger dots are $b_1 = 3$ and $b_2 = -1$

2.2 Construction of the Toric Orbifold

Let $M = \mathrm{Hom}(N; \mathbb{Z})$ be the dual lattice of N, and let $\widetilde{M} = \mathrm{Hom}(M; \mathbb{Z})$ be the dual lattice of $\widetilde{N}$. Since $\beta : \widetilde{N} \to N$ has a finite cokernel, the dual map $\beta^* : M \to \widetilde{M}$ is injective. Applying $\mathrm{Hom}(-, \mathbb{C}^*)$ to the following short exact sequence

$$0 \to M \xrightarrow{\beta^*} \widetilde{M} \xrightarrow{\beta^\vee} \mathrm{coker}(\beta^*) \to 0,$$

we obtain another short exact sequence

$$1 \to G_{\boldsymbol{\Sigma}} \to \widetilde{T} \to T \to 1,$$

where $G_{\boldsymbol{\Sigma}} := \mathrm{Hom}(\mathrm{coker}(\beta^*), \mathbb{C}^*)$ is isomorphic to the direct product of $(\mathbb{C}^*)^{r-n}$ and a finite abelian group, and

$$\widetilde{T} = \mathrm{Hom}(\widetilde{M}, \mathbb{C}^*) \cong (\mathbb{C}^*)^r, \quad T = \mathrm{Hom}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^n.$$

The torus $\widetilde{T} \cong (\mathbb{C}^*)^r$ acts on $\mathbb{C}^r = \mathrm{Spec}\mathbb{C}[z_1, \dots, z_r]$. Let $I_\Sigma \subset \mathbb{C}[z_1, \dots, z_r]$ be the ideal generated by $\{\prod_{\rho_i \not\subset \sigma} z_i \mid \sigma \in \Sigma\}$. (Note that the ideal I_Σ depends on N and Σ but not on β.) Let $Z(I_\Sigma) \subset \mathbb{C}^r$ be the closed subscheme defined by I_Σ, and let $U_\Sigma = \mathbb{C}^r - Z(I_\Sigma)$, which is a Zariski open set of $\mathbb{C}^r$. We define $\mathcal{X}_{\boldsymbol{\Sigma}}$ to be the quotient stack:

$$\mathcal{X}_{\boldsymbol{\Sigma}} = [U_\Sigma / G_{\boldsymbol{\Sigma}}].$$

The simplicial toric variety X_Σ defined by Σ can be identified with the geometric quotient

$$X_\Sigma = U_\Sigma / G_{\boldsymbol{\Sigma}^{\mathrm{can}}},$$

where $\boldsymbol{\Sigma}^{\mathrm{can}} = (N, \Sigma, \beta_{\mathrm{can}})$. We have a 2-cartesian diagram

$$\begin{tikzcd} U_\Sigma \arrow[r, "\widetilde{p}"] \arrow[d] & U_\Sigma \arrow[d] \\ \mathcal{X}_{\boldsymbol{\Sigma}} \arrow[r, "p"] & X_\Sigma, \end{tikzcd}$$

where

$$\widetilde{p}(z_1, \dots, z_r) = (z_1^{n_1}, \dots, z_r^{n_r}). \tag{2.1}$$

We have the following properties regarding $\mathcal{X}_{\boldsymbol{\Sigma}}$:

- $\mathcal{X}_{\boldsymbol{\Sigma}}$ is an orbifold, i.e. it is a smooth DM stack with generically trivial stabilizers.
- There is an open dense embedding $T = \widetilde{T}/G_{\boldsymbol{\Sigma}} \hookrightarrow \mathcal{X}_{\boldsymbol{\Sigma}} = [U_\Sigma / G_{\boldsymbol{\Sigma}}]$, and the action on T on itself extends to a T-action on $\mathcal{X}_{\boldsymbol{\Sigma}}$.
- The coarse moduli space of the orbifold $\mathcal{X}_{\boldsymbol{\Sigma}}$ is the simplicial toric variety X_Σ. The projection $p : \mathcal{X}_{\boldsymbol{\Sigma}} \to X_\Sigma$ restricts to the identity map from T to itself.

Example 2.3 ($\mathbb{P}(1,1,2)$: Example 2.1 continued) The stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ is defined as in Example 2.1. Taking $\mathrm{Hom}(-, \mathbb{C}^*)$ of the following exact sequence

$$0 \to M = (\mathbb{Z}^2)^* \xrightarrow{\beta^* = \left[\begin{smallmatrix} 1 & 0 \\ -1 & -2 \\ 0 & 1 \end{smallmatrix}\right]} \widetilde{M} = (\mathbb{Z}^3)^* \xrightarrow{\beta^\vee = [1\ 1\ 2]} \mathbb{Z} \to 0.$$

produces

$$1 \to G_{\boldsymbol{\Sigma}} = \mathbb{C}^* \longrightarrow \widetilde{T} = (\mathbb{C}^*)^3 \longrightarrow T = (\mathbb{C}^*)^2 \to 1.$$

The map $G_{\boldsymbol{\Sigma}} \to \widetilde{T}$ is given by $\lambda \mapsto (\lambda, \lambda, \lambda^2)$. Therefore, the toric orbifold $\mathcal{X}_{\boldsymbol{\Sigma}}$ is defined as

$$\mathcal{X}_{\boldsymbol{\Sigma}} = \left[(\mathbb{C}^3 - \{(0,0,0)\})/\mathbb{C}^*\right] = \mathbb{P}(1,1,2),$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^3$ by $\lambda \cdot (z_1, z_2, z_3) = (\lambda z_1, \lambda z_2, \lambda^2 z_3)$. The corresponding coarse moduli space is the following simplicial toric variety

$$X_\Sigma = (\mathbb{C}^3 - \{(0,0,0)\})/\mathbb{C}^* = \mathbf{P}(1,1,2).$$

It has a unique singularity at $[0,0,1]$.

Example 2.4 ($\mathbb{P}(1,3)$: Example 2.2 continued) The stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ is defined as in Example 2.2. Then $\beta_{\text{can}} = [1 \ \ -1] : \mathbb{Z}^2 \to \mathbb{Z}$. There is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & G_{\mathbf{\Sigma}} = \mathbb{C}^* \arrow[r, "\phi"] \arrow[d, "\hat{p}"] & \widetilde{T} = (\mathbb{C}^*)^2 \arrow[r, "\pi"] \arrow[d, "\widetilde{p}"] & T = \mathbb{C}^* \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & G_{\mathbf{\Sigma}_{\text{can}}} = \mathbb{C}^* \arrow[r, "\phi_{\text{can}}"] & \widetilde{T} = (\mathbb{C}^*)^2 \arrow[r, "\pi_{\text{can}}"] & T = \mathbb{C}^* \arrow[r] & 1,
\end{tikzcd} \tag{2.2}$$

where the rows are short exact sequences of abelian groups. The arrows are group homomorphisms given explicitly as follows:

$$\phi(\lambda) = (\lambda, \lambda^3), \quad \pi(\widetilde{t}_1, \widetilde{t}_2) = \widetilde{t}_1^3 t_2^{-1}, \quad \phi_{\text{can}}(\lambda) = (\lambda, \lambda), \quad \pi_{\text{can}}(\widetilde{t}_1, \widetilde{t}_2) = \widetilde{t}_1 \widetilde{t}_2^{-1},$$
$$\hat{p}(\lambda) = \lambda^3, \quad \widetilde{p}(\widetilde{t}_1, \widetilde{t}_2) = (\widetilde{t}_1^3, \widetilde{t}_2), \quad p(t) = t.$$

The toric orbifold defined by $\mathbf{\Sigma}$ is a weighted projective line:

$$\mathcal{X}_{\mathbf{\Sigma}} = \left[(\mathbb{C}^2 - \{(0,0)\})/\mathbb{C}^*\right] = \mathbb{P}(1,3),$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^3 z_2)$. The coarse moduli space is the projective line:

$$X_{\Sigma} = (\mathbb{C}^2 - \{(0,0)\})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)$.

We say $X_{\mathbf{\Sigma}}$ is a *complete* toric orbifold if Σ is a complete fan in $N_{\mathbb{R}}$, or equivalently, the coarse moduli space X_{Σ} is a complete toric variety. For example, $\mathbb{P}(1,1,2)$ and $\mathbb{P}(1,3)$ are complete toric orbifolds.

2.3 Divisors and Line Bundles

Let $\widetilde{D}_i \subset U_{\Sigma}$ be the divisor defined by $z_i = 0$, and let D_i and $\mathcal{D}_i$ denote the corresponding T divisors in X_{Σ} and $\mathcal{X}_{\mathbf{\Sigma}}$, respectively. Let $p : \mathcal{X}_{\mathbf{\Sigma}} \to X_{\Sigma}$ be the canonical map to the coarse moduli space. By (2.1), $p^* D_i = n_i \mathcal{D}_i$.

In general, D_i is a $\mathbb{Q}$-Cartier divisor: there exists some positive integer k such that $\mathcal{O}_{X_{\Sigma}}(kD_i)$ is a line bundle on X_{Σ}. On the other hand, $\mathcal{O}_{\mathcal{X}_{\mathbf{\Sigma}}}(\mathcal{D}_i)$ is always a line bundle on the toric orbifold $\mathcal{X}_{\Sigma}$. Indeed, any T-equivariant line bundle on $\mathcal{X}_{\mathbf{\Sigma}}$ is of the form

$$\mathcal{L}_{\vec{c}} = \mathcal{O}_{\mathcal{X}_{\mathbf{\Sigma}}}\left(\sum_{i=1}^{r} c_i \mathcal{D}_i\right), \quad \vec{c} = (c_1, \ldots, c_r) \in \mathbb{Z}^r.$$

2.4 Relation to Log Pairs

In [7], Kawamata considers pairs of varieties with $\mathbb{Q}$-divisors which have local covering by smooth varieties, and associates a Deligne–Mumford stack with such a pair [7, Definition 2.1]. We now relate toric orbifolds to the Deligne–Mumford stacks in [7, Definition 2.1].

Let $\mathcal{X}_{\mathbf{\Sigma}}$ be the toric orbifold defined by a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, and let X_{Σ} be the simplicial toric variety defined by the simplicial fan $\Sigma \subset N_{\mathbb{R}}$. Let $p : \mathcal{X}_{\mathbf{\Sigma}} \to X_{\Sigma}$ be the canonical map to the coarse moduli space. Let $n_i \in \mathbb{Z}_{>0}$ be defined as before. Define a $\mathbb{Q}$-divisor

$$B = \sum_{i=1}^{r} \left(1 - \frac{1}{n_i}\right) D_i$$

on X. Then the pair (X_Σ, B) satisfies the condition in [7, Definition 2.1], and the associated Deligne–Mumford stack with this pair is exactly $\mathcal{X}_{\mathbf{\Sigma}}$. Let $K_{\mathcal{X}_{\mathbf{\Sigma}}}$ and K_{X_Σ} denote the canonical divisors on $\mathcal{X}_{\mathbf{\Sigma}}$ and X_Σ, respectively. We have the following identities:

$$K_{X_\Sigma} = -\sum_{i=1}^{r} D_i, \quad K_{X_\Sigma} + B = -\sum_{i=1}^{r} \frac{1}{n_i} D_i, \quad p^*(K_{X_\Sigma} + B) = -\sum_{i=1}^{r} \mathcal{D}_i = K_{\mathcal{X}_{\mathbf{\Sigma}}}. \tag{2.3}$$

In particular, when $n_1 = \cdots = n_r = 1$, $B = 0$, and the Deligne–Mumford associated with the pair $(X_\Sigma, 0)$ is $\mathcal{X}_{\mathbf{\Sigma}_{\mathrm{can}}}$.

3 Coherent-Constructible Correspondence

The coherent-constructible correspondence relates equivariant coherent sheaves on a toric orbifold of dimension n to certain constructible sheaves on a real vector space of dimension n. Before we give the precise statement of the coherent-constructible correspondence, we need to review some definitions.

We will use the language of dg categories throughout. If C is a dg category, then $\hom(x, y)$ denotes the chain complex of homomorphisms between objects x and y of C. We will continue to use $\mathrm{Hom}(x, y)$ to denote hom sets in non-dg settings. We will regard the differentials in all chain complexes as having degree $+1$, i.e. $d : K^i \to K^{i+1}$. If K is a chain complex (of vector spaces or sheaves, usually) then $h^i(K)$ will denote its ith cohomology object. If C is a dg category, then $\mathrm{Tr}(C)$ denotes the triangulated dg category generated by C, and $D(C)$ denotes the cohomology category $H(\mathrm{Tr}(C))$. The triangulated category $H(\mathrm{Tr}(C))$ is sometimes called the *derived category* of C.

3.1 Coherent and Quasicoherent Sheaves on Toric Orbifolds

We refer to [13, Definition 7.18] for the definitions of quasicoherent sheaves, coherent sheaves, and vector bundles on a general Deligne-Mumford stack. If $\mathcal{X}$ is a Deligne–Mumford stack, let $\mathcal{Q}(\mathcal{X})^{\mathrm{naive}}$ denote the dg category of bounded complexes of quasicoherent sheaves on $\mathcal{X}$, and let $\mathcal{Q}(\mathcal{X})$ denote the localization of this category with respect to acyclic complexes. We use $\mathcal{P}\mathrm{erf}(\mathcal{X}) \subset \mathcal{Q}(\mathcal{X})$ to denote the full dg subcategories consisting of *perfect* objects — that is, objects which are quasi-isomorphic to bounded complexes of vector bundles.

We now spell out the above definitions for a toric orbifold $\mathcal{X}_{\mathbf{\Sigma}} = [U_\Sigma / G_{\mathbf{\Sigma}}]$. By [13, Example 7.21], the category of coherent sheaves on $\mathcal{X}_{\mathbf{\Sigma}}$ is equivalent to the category of $G_{\mathbf{\Sigma}}$-equivariant coherent sheaves on U_Σ. Similarly, the category of quasicoherent sheaves on $\mathcal{X}_{\mathbf{\Sigma}}$ is equivalent to the category of $G_{\mathbf{\Sigma}}$-equivariant quasicoherent sheaves on U_Σ. Therefore,

$$\mathcal{Q}(\mathcal{X}_{\mathbf{\Sigma}}) = \mathcal{Q}_{G_{\mathbf{\Sigma}}}(U_\Sigma), \quad \mathcal{P}\mathrm{erf}(\mathcal{X}_{\mathbf{\Sigma}}) = \mathcal{P}\mathrm{erf}_{G_{\mathbf{\Sigma}}}(U_\Sigma). \tag{3.1}$$

We define the category of T-equivariant coherent (resp. quasicoherent) sheaves on $\mathcal{X}$ to be equivalent to the category of $\widetilde{T}$-equivariant coherent (resp. quasicoherent) sheaves on U:

$$\mathcal{Q}_T(\mathcal{X}_{\mathbf{\Sigma}}) = \mathcal{Q}_{\widetilde{T}}(U_\Sigma), \quad \mathcal{P}\mathrm{erf}_T(\mathcal{X}_{\mathbf{\Sigma}}) = \mathcal{P}\mathrm{erf}_{\widetilde{T}}(U_\Sigma). \tag{3.2}$$

There is a monoidal product structure $\otimes$ on these various dg categories of sheaves on $\mathcal{X}_{\mathbf{\Sigma}}$, simply given by the tensor product of quasi-coherent sheaves on U_Σ.

3.2 Constructible Sheaves

We refer to [14] for the microlocal theory of sheaves. If X is a topological space we let $Sh(X)$ denote the dg category of chain complexes of sheaves of $\mathbb{C}$-vector spaces on X, localized with respect to acyclic complexes (see [15] for localizations of dg categories). If X is a real-analytic manifold, $Sh_c(X)$ denotes the full subcategory of $Sh(X)$ of objects whose cohomology sheaves are bounded and constructible with respect to a real-analytic stratification of X. Denote by $Sh_{cc}(X) \subset Sh_c(X)$ the full subcategory of objects which have compact support. We use $D_c(X)$ and $D_{cc}(X)$ to denote the derived categories $D(Sh_c(X))$ and $D(Sh_{cc}(X))$ respectively.

The *standard constructible sheaf* on the submanifold $i_Y : Y \hookrightarrow X$ is defined as the push-forward of the constant sheaf on Y, i.e. $i_{Y*}\mathbb{C}_Y$, as an object in $Sh_c(X)$. The Verdier duality functor $\mathcal{D} : Sh_c^{\circ}(X) \to Sh_c(X)$ takes $i_{Y*}\mathbb{C}_Y$ to the *costandard constructible sheaf* on X. We know $\mathcal{D}(i_{Y*}\mathbb{C}_Y) = i_{Y!}\mathcal{D}(\mathbb{C}_Y) = i_{Y!}\omega_Y$. Here $\omega_Y = \mathcal{D}(\mathbb{C}_Y) = \mathfrak{or}_Y[\dim Y]$, where $\mathfrak{or}_Y$ is the orientation sheaf of Y (with respect to the base ring $\mathbb{C}$).

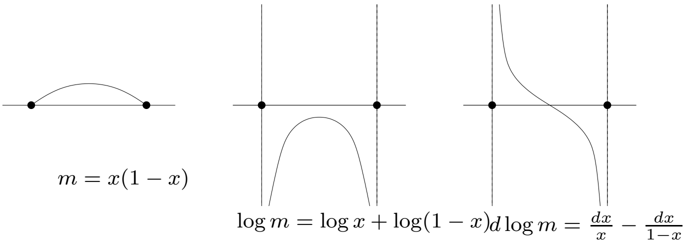
We denote the singular support of a complex of sheaves F by $SS(F) \subset T^*X$. If X is a real-analytic manifold and $\Lambda \subset T^*X$ is an $\mathbb{R}_{\geq 0}$-invariant Lagrangian subvariety, then $Sh_c(X;\Lambda)$ (resp. $Sh_{cc}(X;\Lambda)$) denotes the full subcategory of $Sh_c(X)$ (resp. $Sh_{cc}(X)$) whose objects have singular support in Λ. For any open subset $U \subset X$, the singular support of the associated standard and costandard sheaves are given by the following theorem.

Theorem 3.1 (Schmid–Vilonen)

$$SS(i_!\omega_U) = \lim_{\epsilon\to 0^+} \Gamma_{-\epsilon d\log m}, \qquad SS(i_*\mathbb{C}_U) = \lim_{\epsilon\to 0^+} \Gamma_{\epsilon d\log m},$$

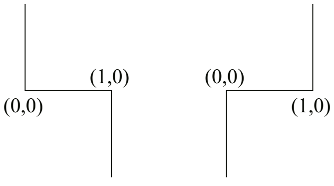
where $m : M_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$, $m|_U > 0$, $m|_{\partial U} = 0$.

Example 3.2 Let $U = (0,1) \subset \mathbb{R}$. Figure 4 depicts the standard Lagrangian on U in $T^*\mathbb{R} \cong \mathbb{R}^2$, while Figure 5 depicts the singular supports of standard and costandard constructible sheaves supported on this interval.



$m = x(1-x)$ $\quad$ $\log m = \log x + \log(1-x)$ $\quad$ $d\log m = \frac{dx}{x} - \frac{dx}{1-x}$

Figure 4 The graphs of m, $\log m$ and $d\log m$.
The graph of $d\log m$ is the standard Lagrangian over U



$SS(i_*\mathbb{C}_U)$ $\quad$ $SS(i_!\omega_U)$

Figure 5 Singular supports of standard and costandard sheaves associated with $U = (0,1)$

Given a submanifold $Y \subset X$, let T_Y^*X denote the conormal bundle of Y in X. T_Y^*X is a Lagrangian submanifold of T^*X.

Example 3.3 (Open sets with smooth boundaries) Let U be an open subset of $\mathbb{R}^n$, and suppose that the boundary ∂U is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. (This includes Example 3.3 as a special case.)

Let $\nu : \partial U \to T_{\partial U}^*\mathbb{R}^n$ be a nowhere zero section such that $\nu_x(v_x) > 0$ if v_x is an outward normal at $x \in \partial U$. Then

$$\begin{aligned}
&T_U^*\mathbb{R}^n = U \times \{0\} \subset T^*\mathbb{R}^n, \quad T_{\partial U}^*\mathbb{R}^n = \{(x, t\nu_x) \mid x \in \partial U, t \in \mathbb{R}\},\\
&SS(i_*\mathbb{C}_U) = T_U^*\mathbb{R}^n \cup \{(x, t\nu_x) \mid x \in \partial U, t \leq 0\},\\
&SS(i_!\omega_U) = T_U^*\mathbb{R}^n \cup \{(x, t\nu_x) \mid x \in \partial U, t \geq 0\}.
\end{aligned}$$

For example, let D be an open disk in $\mathbb{R}^2$, and identify conormal vectors with normal vectors. The singular supports of $i_*\mathbb{C}_D$ and $i_!\omega_D$ are depicted in Figure 6 below.

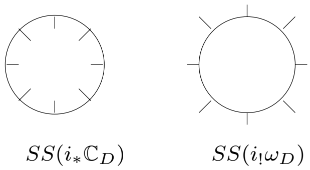


Figure 6 Singular supports of standard and costandard sheaves associated with an open disk $D \subset \mathbb{R}^2$

Example 3.4 (Manifold with corners) We can also consider an open set U in $\mathbb{R}^n$ such that the closure $\overline{U}$ of U is a manifold with corners. For example, an open square R in $\mathbb{R}^2$.

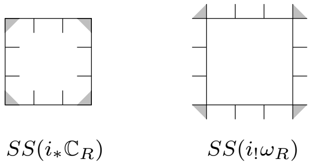


Figure 7 Singular supports of standard and costandard sheaves associated with an open square $R \subset \mathbb{R}^2$

There is a monoidal structure $\star$ on the dg category $Sh_c(X)$ when X is an abelian group, given by the convolution product of constructible sheaves:

$$\mathcal{F} \star \mathcal{G} = a_!\Big(\underbrace{\mathcal{F} \boxtimes \mathcal{G}}_{\in Sh(X\times X)}\Big),$$

where $a : X \times X \to X$ is the addition map of the abelian group X. This product turns out to be a tensor product (commutative monoidal).

3.3 Theta Sheaves

In [5] we have introduced the concept of *theta sheaves*, as building blocks of coherent-constructible correspondence. Let $\mathcal{X}_{\boldsymbol{\Sigma}}$ be the toric orbifold defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$.

3.3.1 Quasicoherent Theta Sheaves

The quasicoherent theta sheaves are certain T-equivariant quasicoherent sheaves on $\mathcal{X}_{\mathbf{\Sigma}}$ that arise in the Čech resolution with respect to an equivariant open cover of $\mathcal{X}_{\mathbf{\Sigma}}$. We first describe this open cover. Given a d-dimensional cone $\sigma \in \Sigma$, let $z_\sigma = \prod_{\rho_i \not\subset \sigma} z_i$. Then

$$U_\sigma = \{(z_1, \ldots, z_r) \in \mathbb{C}^r \mid z_\sigma \neq 0\} \cong \mathbb{C}^d \times (\mathbb{C}^*)^{r-d}$$

is a Zariski open subset of

$$U_{\mathbf{\Sigma}} = \bigcup_{\sigma \in \Sigma} U_\sigma.$$

The open embedding $U_\sigma \hookrightarrow U_{\mathbf{\Sigma}}$ descends to an open embedding of stacks:

$$j_\sigma : \mathcal{X}_\sigma := [U_\sigma / G_{\mathbf{\Sigma}}] \hookrightarrow \mathcal{X}_{\mathbf{\Sigma}} = [U_{\mathbf{\Sigma}} / G_{\mathbf{\Sigma}}].$$

Then $\{\mathcal{X}_\sigma \mid \sigma \in \Sigma\}$ is an open cover of $\mathcal{X}_{\mathbf{\Sigma}}$.

We now describe T-equivariant line bundles on $\mathcal{X}_\sigma$, or equivalently, the $\widetilde{T}$-equivariant line bundles on U_σ. We first introduce some notation.

- Let $M_{\mathbb{R}} := M \otimes \mathbb{R}$ be the dual vector space of $N_{\mathbb{R}}$, and let $\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ be the natural pairing.
- Given a d-dimensional cone $\sigma \in \Sigma$, let $N_\sigma \subset N$ be the subgroup generated by $\{b_i \mid \rho_i \subset \sigma\}$, and M_σ be the dual lattice $M_\sigma = \mathrm{Hom}(N_\sigma, \mathbb{Z})$. Then N_σ and M_σ are free abelian groups of rank d. Let $\langle -, - \rangle_\sigma : M_\sigma \times N_\sigma \to \mathbb{Z}$ be the natural pairing.

The T-equivariant line bundles on $\mathcal{X}_\sigma$ are in one-to-one correspondence with the elements in M_σ. Let $\mathcal{O}_{\mathcal{X}_\sigma}(\chi)$ denote the T-equivariant line bundle on $\mathcal{X}_\sigma$ associated with $\chi \in M_\sigma$.

We define the quasicoherent theta sheaf $\Theta'(\sigma, \chi)$ to be the pushforward of $\mathcal{O}_{\mathcal{X}_\sigma}(\chi)$ under the open embedding $j_\sigma : \mathcal{X}_\sigma \hookrightarrow \mathcal{X}_{\mathbf{\Sigma}}$,

$$\Theta'(\sigma, \chi) := j_{\sigma *} \mathcal{O}_{\mathcal{X}_\sigma}(\chi).$$

3.3.2 Constructible Theta Sheaves

For a cone $\sigma \in \Sigma$ and a character $\chi \in M_\sigma$, we fix the following notation:

$$\begin{aligned}
\sigma_\chi^\vee &= \{x \in M_{\mathbb{R}} \,|\, \langle x, v \rangle \geq \langle \chi, v \rangle_\sigma, v \in N_\sigma \cap \sigma\}, \\
(\sigma_\chi^\vee)^\circ &= \{x \in M_{\mathbb{R}} \,|\, \langle x, v \rangle > \langle \chi, v \rangle_\sigma, v \in N_\sigma \cap \sigma\}, \\
\sigma_\chi^\perp &= \{x \in M_{\mathbb{R}} \,|\, \langle x, v \rangle = \langle \chi, v \rangle_\sigma, v \in N_\sigma \cap \sigma\}.
\end{aligned}$$

We define the constructible theta sheaf $\Theta(\sigma, \chi)$ to be the costandard constructible sheaf associated with the open set $(\sigma_\chi^\vee)^\circ$ in $M_{\mathbb{R}}$.

$$\Theta(\sigma, \chi) := i_{(\sigma_\chi^\vee)^\circ !} \omega_{(\sigma_\chi^\vee)^\circ} \in Ob(Sh_c(M_{\mathbb{R}})),$$

where $i_{(\sigma_\chi^\vee)^\circ} : (\sigma_\chi^\vee)^\circ \hookrightarrow M_{\mathbb{R}}$ is the inclusion.

3.4 The Coherent-Constructible Correspondence

The theta sheaves are indexed by the set

$$\Gamma(\mathbf{\Sigma}) = \{(\sigma, \chi) | \sigma \in \Sigma, \chi \in M_\sigma\}.$$

We define a partial order on $\Gamma(\mathbf{\Sigma})$:

$$(\sigma_1,\chi_1)\le(\sigma_2,\chi_2) \text{ if and only if } (\sigma_1)^\vee_{\chi_1}\subset(\sigma_2)^\vee_{\chi_2}.$$

The "linearized" dg category $\Gamma(\mathbf{\Sigma})_{\mathbb{C}}$ consists of objects $(\sigma,\chi)\in\Gamma(\mathbf{\Sigma})$ and the following morphisms with obvious composition rules

$$\hom((\sigma_1,\chi_1),(\sigma_2,\chi_2))=\begin{cases}\mathbb{C}[0], & \text{if } (\sigma_1,\chi_1)\le(\sigma_2,\chi_2);\\ 0, & \text{otherwise.}\end{cases}$$

It is proved in [5] that

$$\begin{aligned}\hom_{\mathcal{Q}_T(\mathcal{X}_{\mathbf{\Sigma}})}(\Theta'(\sigma_1,\chi_1),\Theta'(\sigma_2,\chi_2))&=\hom_{Sh_c(M_{\mathbb{R}})}(\Theta(\sigma_1,\chi_1),\Theta(\sigma_2,\chi_2))\\ &=\hom((\sigma_1,\chi_1),(\sigma_2,\chi_2))\end{aligned}\tag{3.3}$$

for any $(\sigma_1,\chi_1),(\sigma_2,\chi_2)\in\Gamma(\mathbf{\Sigma})$.

Let $\langle\Theta\rangle_{\mathbf{\Sigma}}$ (resp. $\langle\Theta'\rangle_{\mathbf{\Sigma}}$) be the full triangulated subcategory of $Sh_c(M_{\mathbb{R}})$ (resp. $\mathcal{Q}_T(\mathcal{X}_{\mathbf{\Sigma}})$) generated by all $\Theta(\sigma,\chi)$ (resp. $\Theta'(\sigma,\chi)$). Then (3.3) implies that $\langle\Theta\rangle_{\mathbf{\Sigma}}$ and $\langle\Theta'\rangle_{\mathbf{\Sigma}}$ are quasi-equivalent as triangulated dg categories. We proved that this quasi-equivalence is monoidal:

Theorem 3.5 *There is a quasi-equivalence monoidal functor $\kappa_{\mathbf{\Sigma}}:\langle\Theta'\rangle_{\mathbf{\Sigma}}\to\langle\Theta\rangle_{\mathbf{\Sigma}}$, which sends $\Theta'(\sigma,\chi)$ to $\Theta(\sigma,\chi)$ for $(\sigma,\chi)\in\Gamma(\mathbf{\Sigma})$.*

By Čech resolution, the dg category of coherent sheaves $\mathcal{P}\mathrm{erf}_T(\mathcal{X}_{\mathbf{\Sigma}})$ is a full subcategory of $\langle\Theta'\rangle_{\mathbf{\Sigma}}$, as shown in [5]. Restricted to $\mathcal{P}\mathrm{erf}_T(\mathcal{X}_{\mathbf{\Sigma}})$, the functor $\kappa_{\mathbf{\Sigma}}$ is a quasi-embedding. We have characterized the image $\kappa_{\mathbf{\Sigma}}(\mathcal{P}\mathrm{erf}_T(\mathcal{X}_{\mathbf{\Sigma}}))$ as a full sub-category of $\langle\Theta\rangle_{\mathbf{\Sigma}}$, as in the following theorem.

Theorem 3.6 (Coherent-constructible correspondence for toric orbifolds) *Let $\mathcal{X}_{\mathbf{\Sigma}}$ be a complete toric orbifold defined by a stacky fan $\mathbf{\Sigma}=(N,\Sigma,\beta)$. Then there is a quasi-equivalence of monoidal triangulated dg categories*:

$$\kappa_{\mathbf{\Sigma}}:\mathcal{P}\mathrm{erf}_T(\mathcal{X}_{\mathbf{\Sigma}})\xrightarrow{\sim}Sh_{cc}(M_{\mathbb{R}},\Lambda_{\mathbf{\Sigma}}).$$

In the above theorem, the dg category $Sh_{cc}(M_{\mathbb{R}},\Lambda_{\mathbf{\Sigma}})$ is the full dg subcategory of $Sh_{cc}(M_{\mathbb{R}})$ on $\mathcal{X}_{\mathbf{\Sigma}}$ whose objects have singular support inside $\Lambda_{\mathbf{\Sigma}}$. It is closed under the monoidal product $\star$. The conical Lagrangian ($\mathbb{R}_{>0}$-invariant Lagrangian in $T^*M_{\mathbb{R}}$) is defined directly from the stacky fan $\mathbf{\Sigma}=(N,\Sigma,\beta)$,

$$\Lambda_{\mathbf{\Sigma}}=\bigcup_{\sigma\in\Sigma,\chi\in M_\sigma}\sigma_\chi^\perp\times(-\sigma)\subset M_{\mathbb{R}}\times N_{\mathbb{R}}=T^*M_{\mathbb{R}}.$$

By definition $\Lambda_{\mathbf{\Sigma}}$ is a conical Lagrangian in $T^*M_{\mathbb{R}}$, and is invariant under $(x,y)\mapsto(x+m,y)$, $m\in M$.

Remark 3.7 It is particularly easy to describe what the functor $\kappa_{\mathbf{\Sigma}}$ does to $\mathbb{Q}$-ample equivariant line bundles. Recall that any T-equivariant line bundle on $\mathcal{X}=\mathcal{X}_{\mathbf{\Sigma}}$ is of the form $\mathcal{L}_{\vec{c}}=\mathcal{O}_{\mathcal{X}}(c_1\mathcal{D}_1+\cdots+c_r\mathcal{D}_r)$, where $\mathcal{D}_i$ denotes the T-divisor associated with the ray $\rho_i\in\Sigma(1)$, and $c_1,\dots,c_r\in\mathbb{Z}$ are integers. We say $\mathcal{L}_{\vec{c}}$ is $\mathbb{Q}$-ample if there is some positive integer n such that $\mathcal{L}_{\vec{c}}^{\otimes n}$ is the pull back of an ample line bundle on the coarse moduli space. If $\mathcal{L}_{\vec{c}}$ is $\mathbb{Q}$-ample then

$$\triangle_{\vec{c}}=\{x\in M_{\mathbb{R}}\mid\langle x,b_i\rangle\ge-c_i\}$$

is a convex polytope in $M_{\mathbb{R}}$. The interior $\triangle_{\vec{c}}^{\circ}$ of $\triangle_{\vec{c}}$ is a bounded open set in $M_{\mathbb{R}}$. Let $i : \triangle_{\vec{c}}^{\circ} \hookrightarrow M_{\mathbb{R}}$ be the inclusion. Then

$$\kappa_{\boldsymbol{\Sigma}}(\mathcal{L}_{\vec{c}}) = i_! \omega_{\triangle_{\vec{c}}^{\circ}}.$$

We have also proved a coherent-constructible correspondence for the coarse moduli space X_Σ [1]. Indeed, we prove the following for any complete (not necessarily simplicial) toric varieties:

Theorem 3.8 (Coherent-constructible-correspondence for toric varieties) *Let X_Σ be a complete toric variety defined by a fan $\Sigma \subset N_{\mathbb{R}}$. Then there is a quasi-equivalence of monoidal triangulated dg categories*:

$$\kappa_{\boldsymbol{\Sigma}} : \mathcal{P}\mathrm{erf}_T(X_\Sigma) \xrightarrow{\sim} Sh_{cc}(M_{\mathbb{R}}, \Lambda_\Sigma).$$

The category $Sh_{cc}(M_{\mathbb{R}}; \Lambda_\Sigma)$ is similarly defined as the subcategory of $Sh_{cc}(M_{\mathbb{R}})$ whose objects have singular support in a conical Lagrangian

$$\Lambda_\Sigma := \bigcup_{\sigma \in \Sigma, \chi \in M} (\sigma^\perp + \chi) \times (-\sigma) \subset M_{\mathbb{R}} \times N_{\mathbb{R}} = T^* M_{\mathbb{R}},$$

where $\sigma^\perp = \{x \in M_{\mathbb{R}} \mid \langle x, v \rangle = 0 \text{ for all } v \in \sigma\}$.

The theorem above can be considered as a “categorification” of Morelli’s theorem [4]. Let $L_M(M_{\mathbb{R}})$ be the group of functions generated over $\mathbb{Z}$ by the indicator functions 1_P of convex lattice polyhedra P, and let $S_M(M_{\mathbb{R}})$ be the abelian group generated by rational convex cones in $M_{\mathbb{R}}$. Then $S_M(M_{\mathbb{R}})$ is the group of germs of functions in $L_M(M_{\mathbb{R}})$ at the origin (or any point in the lattice M). Let $S_\Sigma(M_{\mathbb{R}})$ be the subgroup of $S_M(M_{\mathbb{R}})$ generated by $\{\sigma^\vee \mid \sigma \in \Sigma\}$.

Theorem 3.9 (Morelli) *Let X_Σ be a smooth projective toric variety. Then there is a group isomorphism $K_T(X_\Sigma) \xrightarrow{\sim} S_\Sigma(M_{\mathbb{R}})$, $\mathcal{L}_{\vec{c}} \mapsto 1_{\triangle_{\vec{c}}}$, $\mathcal{L}_{\vec{c}}$ ample.*

Bondal has also proved a similar relation between (non-equivariant) coherent sheaves and constructible sheaves [3], characterized by a stratification.

Theorem 3.10 (Bondal) *Let X_Σ be a smooth projective variety defined by a fan Σ, with some additional assumption on Σ.*

$$D^b\mathrm{Coh}(X_\Sigma) \cong DSh_c(M_{\mathbb{R}}/M, \mathcal{S}),$$

where $\mathcal{S}$ is a stratification on the compact torus $M_{\mathbb{R}}/M \cong (S^1)^n$ determined by Σ.

Example 3.11 (Example 2.4 continued) Let the stacky fan $\boldsymbol{\Sigma}$ be as in Example 2.4, which defines the toric orbifold $\mathbb{P}(1, 3)$. The conical Lagrangians Λ_Σ and $\Lambda_{\boldsymbol{\Sigma}}$ are shown in Figure 8.

Λ_Σ $\Lambda_{\boldsymbol{\Sigma}}$

Figure 8 The conical Lagrangians Λ_Σ and $\Lambda_{\boldsymbol{\Sigma}}$ for $\boldsymbol{\Sigma}$ defined in Example 2.2. The horizontal direction is $M_{\mathbb{R}}$ and the vertical direction is $N_{\mathbb{R}}$

Example 3.12 (Example 2.3 continued) (1) T-equivariant ample line bundle on $\mathbb{P}^1$: $\mathcal{O}(c_1D_1 + c_2D_2)$, $c_1, c_2 \in \mathbb{Z}$, $c_1 + c_2 > 0$.

$$\triangle^{\circ}_{c_1,c_2} = \{x \in \mathbb{R} \mid x > -c_1, -x > -c_2\} = (-c_1, c_2).$$

(2) T-equivariant $\mathbb{Q}$-ample line bundles on $\mathbb{P}(1,3)$: $\mathcal{O}(c_1\mathcal{D}_1 + c_2\mathcal{D}_2)$, $c_1, c_2 \in \mathbb{Z}$, $\frac{c_1}{3} + c_2 > 0$. Let $p : \mathbb{P}(1,3) \to \mathbb{P}^1$ be the projection to the coarse moduli space. Then $p^*D_1 = 3\mathcal{D}_1$, $p^*D_2 = \mathcal{D}_2$.

$$\triangle^{\circ}_{c_1,c_2} = \{x \in \mathbb{R} \mid 3x > -c_1, -x > c_2\} = \Big(-\frac{c_1}{3}, c_2\Big).$$

4 Fourier–Mukai Transformation: A Constructible Perspective

In this section, X_1 and X_2 are always simplicial toric varieties defined by simplicial fans Σ_1 and Σ_2 in $N_{\mathbb{R}}$, respectively; B and C are effective toric $\mathbb{Q}$-divisors on X_1 and X_2, respectively, such that (X_1, B) and (X_2, C) are toric log pairs as in Section 2.4. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be the toric orbifolds associated with the pairs (X_1, B) and (X_2, C), respectively. Assume there are proper birational morphisms $\mu_1 : W \to X_1$ and $\mu_2 : W \to X_2$ for some variety W such that $\mu_1^*(K_{X_1} + B) \geq \mu_2 * (K_{X_2} + C)$. Kawamata conjectures that there exists a full and faithful functor of triangulated categories

$$F'_{12} = \mu_{1*} \circ \mu_2^* : D^b\mathrm{Coh}(\mathcal{X}_2) \to D^b\mathrm{Coh}(\mathcal{X}_1)$$

in [7], where $\mu_i : \mathcal{W} \to \mathcal{X}_i$ are the morphisms for the corresponding stacks, by an abuse of notation. If the inequality above becomes an equality, by invoking this conjecture in both directions, F'_{12} is then an equivalence of triangulated categories. Passing Kawamata's argument to the language of constructible sheaves via Theorem 3.6, this Fourier–Mukai fully faithful functor arises from intuitive combinatorial argument. This section elaborates Kawamata's theorem from the perspective of constructible sheaves, proving some cases discussed in [7], in the equivariant and dg setting.[2)]

We introduce some notation:

(1) The Fourier Mukai functors are $F'_{12} = \mu_{1*} \circ \mu_2^*$ and $F'_{21} = \mu_{2*} \circ \mu_1^*$.

(2) Let $D_{1,i}$ (resp. $\mathcal{D}_{1,i}$) denote the T-divisor on X_1 (resp. $\mathcal{X}_1$) associated with the 1-dimensional cone $\rho_i \in \Sigma_1(1)$.

(3) Let $D_{2,i}$ (resp. $\mathcal{D}_{2,i}$) denote the T-divisor on X_2 (resp. $\mathcal{X}_2$) associated with the 1-dimensional cone $\rho_i \in \Sigma_2(1)$.

(4) Let D'_i (resp. $\mathcal{D}'_i$) denote the T-divisor on W (resp. $\mathcal{W}$) associated with the 1-dimensional cone $\rho_i \in \Sigma'(1)$.

(5) Let $p_i : \mathcal{X}_i \to X_i$ be canonical map to the coarse moduli.

We have

$$B = \sum_{i=1}^{l_1} \Big(1 - \frac{1}{r_i}\Big) D_{1,i}, \quad C = \sum_{i=1}^{l_2} \Big(1 - \frac{1}{s_i}\Big) D_{2,i},$$

where r_i, s_i are positive integers. Then

$$p_1^* D_{1,i} = r_i \mathcal{D}_{1,i}, \quad p_2^* D_{2,i} = s_i \mathcal{D}_{2,i}.$$

[2)] Although not explicitly stated, Kawamata's proof is essentially equivariant in [7].

Note that from the construction of $\mathcal{X}_i$,

$$p_1^*(K_{X_1}+B)=K_{\mathcal{X}_1},\quad p_2^*(K_{X_2}+B)=K_{\mathcal{X}_2}.$$

4.1 Toric Orbifolds with the Same Coarse Moduli Space

In the first case of [7, Theorem 4.2], Kawamata shows that if $X_1=X_2=X$ and $K_{X_1}+B\geq K_{X_2}+C$, then the Fourier–Mukai functor

$$F'_{12}=\mu_{1*}\circ\mu_2^*:\mathrm{Coh}(\mathcal{X}_2)\to\mathrm{Coh}(\mathcal{X}_1)$$

is fully faithful.

Recall from Section 2 that $N=\mathbb{Z}^n$, and Σ is a simplicial fan in $N_{\mathbb{R}}$. The 1-cones $\Sigma(1)$ consists of rays $\rho_1,\dots,\rho_l$, and the generating set of $\Sigma(1)\cap N$ is $\{v_1,\dots,v_r\}$. Let β_1 and β_2 be maps (where v_i are regarded as column vectors below)

$$\beta_1=[r_1v_1\quad\cdots\quad r_lv_l]:\mathbb{Z}^l\longrightarrow N=\mathbb{Z}^n,$$

$$\beta_2=[s_1v_1\quad\cdots\quad s_lv_l]:\mathbb{Z}^l\longrightarrow N=\mathbb{Z}^n.$$

From the stacky fans $\mathbf{\Sigma}_i=(N,\Sigma,\beta_i)$ one defines two toric DM stacks $\mathcal{X}_1=\mathcal{X}_{\mathbf{\Sigma}_1}$ and $\mathcal{X}_2=\mathcal{X}_{\mathbf{\Sigma}_2}$. They have the same coarse moduli space $X=X_\Sigma$ given by the fan Σ as a toric variety.

Let $\mathcal{W}=\mathcal{X}_1\times_X\mathcal{X}_2$. It is the toric orbifold defined by the stacky fan $\mathbf{\Sigma}'=(N,\Sigma,\beta')$, where

$$\beta'=[t_1v_1\quad\cdots\quad t_lv_l]:\mathbb{Z}^l\longrightarrow N=\mathbb{Z}^n,\quad t_i=\mathrm{l.c.m.}(r_i,s_i).$$

We have the following diagram:

$$\begin{tikzcd}
 & \mathcal{W} \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_2"] & \\
\mathcal{X}_1 \arrow[dr, "p_1"] & & \mathcal{X}_2 \arrow[dl, "p_2"'] \\
 & X &
\end{tikzcd}$$

where p_i is the morphism from $\mathcal{X}_i$ to their common coarse moduli space X. Given a 1-dimensional cone $\rho_i\in\Sigma(1)$ let D_i, $\mathcal{D}_{1,i}$, $\mathcal{D}_{2,i}$, and $\mathcal{D}'_i$ denote the associated T-divisors on X, $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{W}$, respectively. Let $m_i=\frac{t_i}{r_i}\in\mathbb{Z}$ and $n_i=\frac{t_i}{s_i}\in\mathbb{Z}$.

$$p_1^*D_i=r_i\mathcal{D}_{1,i},\quad p_2^*D_i=s_i\mathcal{D}_{2,i},\quad \mu_1^*\mathcal{D}_{1,i}=m_i\mathcal{D}'_i,\quad \mu_2^*\mathcal{D}_{2,i}=n_i\mathcal{D}'_i,$$

$$K_X=-\sum_{i=1}^l D_i,\quad B=\sum_{i=1}^l\left(1-\frac{1}{r_i}\right)D_i,\quad C=\sum_{i=1}^l\left(1-\frac{1}{s_i}\right)D_i,$$

$$p_1^*(K_X+B)=-\sum_{i=1}^l\mathcal{D}_{1,i}=K_{\mathcal{X}_1},\quad p_2^*(K_X+C)=-\sum_{i=1}^l\mathcal{D}_{2,i}=K_{\mathcal{X}_2},$$

$$\mu_1^*K_{\mathcal{X}_1}=-\sum_{i=1}^l m_i\mathcal{D}'_i,\quad \mu_2^*K_{\mathcal{X}_2}=-\sum_{i=1}^l n_i\mathcal{D}'_i.$$

From the above calculations, we observe that

Lemma 4.1 $\mu_1^*K_{\mathcal{X}_1}\geq\mu_2^*K_{\mathcal{X}_2}\Leftrightarrow r_i\geq s_i,\ i=1,\dots,l\Leftrightarrow K_X+B\geq K_X+C.$

For any $\sigma \in \Sigma(d)$, let $\{v_{i_1}, \dots, v_{i_d}\} = \sigma \cap \{v_1, \dots, v_l\}$. There are injective group homomorphisms

$$\mu_{1,\sigma} : N'_\sigma = \bigoplus_{k=1}^{d} \mathbb{Z}(t_{i_k} v_{i_k}) \longrightarrow N_{1,\sigma} = \bigoplus_{k=1}^{d} \mathbb{Z}(r_{i_k} v_{i_k}),$$
$$\mu_{2,\sigma} : N'_\sigma = \bigoplus_{k=1}^{d} \mathbb{Z}(t_{i_k} v_{i_k}) \longrightarrow N_{2,\sigma} = \bigoplus_{k=1}^{d} \mathbb{Z}(s_{i_k} v_{i_k})$$

and surjective group homomorphisms

$$\mu^*_{i,\sigma} : M_{i,\sigma} := \mathrm{Hom}(N_{i,\sigma}, \mathbb{Z}) \to M'_\sigma := \mathrm{Hom}(N'_\sigma, \mathbb{Z}), \quad i = 1, 2.$$

We now introduce some notation. Given $\sigma \in \Sigma$, let $\langle \cdot, \cdot \rangle'_\sigma : M'_\sigma \times N'_\sigma \to \mathbb{Z}$ and $\langle \cdot, \cdot \rangle_{i,\sigma} : M_{i,\sigma} \times N_{i,\sigma} \to \mathbb{Z}$, $i = 1, 2$, be the natural pairing. Given $x \in \mathbb{R}$, define $\lceil x \rceil \in \mathbb{Z}$ by $\lceil x \rceil - 1 < x \leq \lceil x \rceil$. We define surjective maps (which is not a group homomorphism) $\mu_{i,\sigma *} : M'_\sigma \to M_{i,\sigma}$, $i = 1, 2$, by

$$\langle \mu_{1,\sigma *}(\chi), (r_{i_k} v_{i_k}) \rangle_{1,\sigma} = \left\lceil \frac{1}{m_{i_k}} \langle \chi, (t_{i_k} v_{i_k}) \rangle'_\sigma \right\rceil \in \mathbb{Z},$$
$$\langle \mu_{2,\sigma *}(\chi), (s_{i_k} v_{i_k}) \rangle_{2,\sigma} = \left\lceil \frac{1}{n_{i_k}} \langle \chi, (t_{i_k} v_{i_k}) \rangle'_\sigma \right\rceil \in \mathbb{Z},$$

where

$$\chi \in M'_\sigma, \quad \frac{1}{m_{i_k}} \langle \chi_i, (t_{i_k} v_{i_k}) \rangle_\sigma \in \frac{1}{m_{i_k}} \mathbb{Z}, \quad \frac{1}{n_{i_k}} \langle \chi_i, (t_{i_k} v_{i_k}) \rangle_\sigma \in \frac{1}{n_{i_k}} \mathbb{Z}.$$

For $i = 1, 2$, define (with an abuse of notation)

$$\mu^*_i : \Gamma(\mathbf{\Sigma}_i) \to \Gamma(\mathbf{\Sigma}'), \quad (\sigma, \chi) \mapsto (\sigma, \mu^*_{i,\sigma} \chi),$$
$$\mu_{i*} : \Gamma(\mathbf{\Sigma}') \to \Gamma(\mathbf{\Sigma}_i), \quad (\sigma, \chi) \mapsto (\sigma, \mu_{i,\sigma *} \chi).$$

Let $\Theta_1(\sigma, \chi)$ (resp. $\Theta_{\mathcal{W}}(\sigma, \chi)$, $\Theta_2(\sigma, \chi)$) be the constructible theta sheaves on $M_{1,\mathbb{R}}$ (resp. $M'_\mathbb{R}$, $M_{2,\mathbb{R}}$) for $\sigma \in \Sigma_1$ (resp. Σ', Σ_2) and $\chi \in M_{1,\sigma}$ (resp. M'_σ, $M_{2,\sigma}$). Similarly, let $\Theta'_1(\sigma, \chi)$ (resp. $\Theta'_{\mathcal{W}}(\sigma, \chi)$, $\Theta'_2(\sigma, \chi)$) be the quasi-coherent theta sheaves on $\mathcal{X}_1$ (resp. $\mathcal{X}_{\mathcal{W}}$, $\mathcal{X}_2$) for $\sigma \in \Sigma_1$ (resp. Σ', Σ_2) and $\chi \in M_{1,\sigma}$ (resp. M'_σ, $M_{2,\sigma}$).

Proposition 4.2 *For $i = 1, 2$, let $\mu^*_i : \mathcal{Q}_T(\mathcal{X}_i) \to \mathcal{Q}_T(\mathcal{W})$ and $\mu_{i*} : \mathcal{Q}_T(\mathcal{W}) \to \mathcal{Q}_T(\mathcal{X}_i)$ be the pullback and pushforward functors of equivariant quasicoherent sheaves. Then*

$$\mu^*_i \Theta'_i(\sigma, \chi) = \Theta'_{\mathcal{W}}(\mu^*_i(\sigma, \chi)), \quad (\sigma, \chi) \in \Gamma(\mathbf{\Sigma}_i),$$
$$\mu_{i*} \Theta'_{\mathcal{W}}(\sigma, \chi) = \Theta'_i(\mu_{i*}(\sigma, \chi)), \quad (\sigma, \chi) \in \Gamma(\mathbf{\Sigma}').$$

Proof It suffices to consider the case $i = 1$. The first statement follows directly from the functoriality property of CCC [5, Theorem 5.16]. For the second statement, the theta sheaf $\Theta'_{\mathcal{W}}(\sigma, \chi)$ is given by the module $\mathbb{C}[\sigma^\vee_\chi \cap M'_\sigma]$. The sections of the push-forward are the sections of $\Theta'_{\mathcal{W}}(\sigma, \chi)$ whose characters are in $M_{1,\sigma}$. Thus $\mu_{1*} \Theta'_{\mathcal{W}}(\sigma, \chi)$ is given by the module $\mathbb{C}[\sigma^\vee_\chi \cap M_{1,\sigma}]$. Notice that $\sigma^\vee_\chi \cap M_{1,\sigma} = \sigma^\vee_{\mu_{1*}(\chi)} \cap M_{1,\sigma}$, and the result follows. □

Proposition 4.3 *If $r_i \geq s_i$ for $i = 1, \dots, l$, then*

$$F := \mu_{1*} \circ \mu^*_2 : \Gamma(\mathbf{\Sigma}_2) \to \Gamma(\mathbf{\Sigma}_1)$$

is an injective map of posets:

$$(\sigma,\chi)\le(\sigma',\chi')\Leftrightarrow F(\sigma,\chi)\le F(\sigma',\chi').$$

Proof For any $\sigma\in\Sigma$, let $F_\sigma=\mu_{1,\sigma*}\mu_{2,\sigma}^*: M_{2,\sigma}\to M_{1,\sigma}$. By definition,

$$F(\sigma,\chi)=(\sigma,F_\sigma(\chi)),\quad F(\sigma',\chi')=(\sigma',F_{\sigma'}(\chi')).$$

The statements (i) and (ii) below also follow from the definitions.

(i) Suppose that $(\sigma,\chi),(\sigma,\chi')\in\Gamma(\mathbf{\Sigma}_2)$. Then

$$\begin{aligned}&(\sigma,\chi)\le(\sigma',\chi')\\ &\quad\Leftrightarrow \sigma\supset\sigma' \text{ and } \langle\chi,s_iv_i\rangle_{2,\sigma}\ge\langle\chi',s_iv_i\rangle_{2,\sigma} \text{ for all } v_i\in\sigma'\cap\{v_1,\dots,v_l\},\\ &(\sigma,F_\sigma(\chi))\le(\sigma',F_\sigma(\chi))\\ &\quad\Leftrightarrow \sigma\supset\sigma' \text{ and } \langle F_\sigma(\chi),r_iv_i\rangle_{1,\sigma}\ge\langle F_{\sigma'}(\chi_i'),r_iv_i\rangle_{1,\sigma} \text{ for all } v_i\in\sigma'\cap\{v_1,\dots,v_l\}.\end{aligned}$$

(ii) If $(\chi,\sigma)\in M_{2,\sigma}$ and $v_i\in\sigma\cap\{v_1,\dots,v_l\}$ then

$$\langle F_\sigma(\chi),r_iv_i\rangle_{1,\sigma}=\left\lceil\frac{r_i}{s_i}\langle\chi,s_iv_i\rangle_{2,\sigma}\right\rceil.$$

By (i) and (ii), it suffices to show that for any $k,k'\in\mathbb{Z}$,

$$k\ge k'\Longleftrightarrow\left\lceil\frac{r_i}{s_i}k\right\rceil\ge\left\lceil\frac{r_i}{s_i}k'\right\rceil.$$

$\Rightarrow$ is always true, and $\Leftarrow$ is true if $\frac{r_i}{s_i}\ge 1$. □

Theorem 4.4 *Suppose that* $\mu_1^*K_{\mathcal{X}_1}\ge\mu_2^*K_{\mathcal{X}_2}$ *(or equivalently,* $r_i\ge s_i$ *for* $i=1,\dots,l$*). Then the dg functor*

$$F_{12}':\mathrm{Coh}_T(\mathcal{X}_2)\to\mathrm{Coh}_T(\mathcal{X}_1)$$

is cohomologically full and faithful.

Proof The functor $F_{12}':\mathcal{Q}_T(\mathcal{X}_2)\to\mathcal{Q}_T(\mathcal{X}_1)$, restricted on $\langle\Theta_2'\rangle\subset\mathcal{Q}_T(\mathcal{X}_2)$, is a functor to $\langle\Theta_1'\rangle$, since it sends any theta sheaf to a theta sheaf. Therefore F_{12}' restricted on $\langle\Theta_2'\rangle\subset\mathcal{Q}_T(\mathcal{X}_2)$ is a full and faithful functor since $F_{12}:=\kappa_1\circ F_{12}'\circ\kappa_2^{-1}$ is full and faithful due to Lemma 4.1, Proposition 4.2, and Proposition 4.3. Further restricting this functor to coherent sheaves (i.e. perfect sheaves), we obtain a full and faithful functor $F_{12}':\mathrm{Coh}_T(\mathcal{X}_2)\to\mathrm{Coh}_T(\mathcal{X}_1)$. □

Example 4.5 Set $N=\mathbb{Z}$, $\Sigma=\{\mathbb{R}^+,\mathbb{R}^-\}$, $\beta_1=[3\ \ -1]$ and $\beta_2=[2\ \ -1]$. The stacks associated with $\mathbf{\Sigma}_1=(N,\Sigma,\beta_1)$ and $\mathbf{\Sigma}_2=(N,\Sigma,\beta_2)$ are

$$\mathcal{X}_1=\mathbb{P}(1,3),\quad\mathcal{X}_2=\mathbb{P}(1,2).$$

Both $\mathcal{X}_1$ and $\mathcal{X}_2$ have the same coarse moduli space $\mathbb{P}^1$. The fiber product $\mathcal{W}=\mathbb{P}(1,6)$ is constructed from the stacky fan $(N,\Sigma,[6\ \ -1])$. Let $\rho_1=\mathbb{R}^+$ and $\rho_2=\mathbb{R}^-$.

For $i=1,2$, let $D_i\subset X=\mathbb{P}^1$, $\mathcal{D}_{1,i}\subset\mathcal{X}_1$, $\mathcal{D}_{2,i}\subset\mathcal{X}_2$, and $D_i'\subset\mathcal{W}$ be defined as above. In particular, D_1 and D_2 are the two torus fixed points in $\mathbb{P}^1$. Any equivariant line bundle on $\mathcal{X}_1=\mathbb{P}(1,3)$ is of the form

$$\mathcal{L}_{1,(c_1,c_2)}:=\mathcal{O}_{\mathbb{P}(1,3)}(c_1\mathcal{D}_{1,1}+c_2\mathcal{D}_{1,2})=p_1^*\mathcal{O}_{\mathbb{P}^1}\left(\frac{c_1}{3}D_1+c_2D_2\right),\quad c_1,c_2\in\mathbb{Z},$$

whereas any equivariant line bundle on $\mathcal{X}_2 = \mathbb{P}(1,2)$ is of the form

$$\mathcal{L}_{2,(c_1,c_2)} := \mathcal{O}_{\mathbb{P}(1,2)}(c_1\mathcal{D}_{2,1} + c_2\mathcal{D}_{2,2}) = p_2^*\mathcal{O}_{\mathbb{P}^1}\left(\frac{c_1}{2}D_1 + c_2D_2\right), \quad c_1, c_2 \in \mathbb{Z}.$$

The Fourier–Mukai functor $F = \mu_{1*} \circ \mu_2^*$ is given by

$$F'_{12} : \mathcal{L}_{2,(c_1,c_2)} \mapsto \mathcal{L}_{1,(\lfloor\frac{3}{2}c_1\rfloor,c_2)}.$$

The line bundle $\mathcal{L}_{2,(c_1,c_2)}$ is $\mathbb{Q}$-ample iff $\frac{c_1}{2} + c_2 > 0$. In this case, it corresponds to the costandard sheaf supported on the open interval $(-\frac{c_1}{2}, c_2) \subset \mathbb{R}$. The constructible analogue of the Fourier–Mukai functor is

$$F_{12} : i_{(-\frac{c_1}{2},c_2)!}\mathbb{C}_{(-\frac{c_1}{2},c_2)}[1] \mapsto \begin{cases} i_{(-\frac{c_1}{2},c_2)!}\omega_{(-\frac{c_1}{2},c_2)} & \text{if } c_1 \text{ is even,} \\ i_{(-\frac{c_1}{2}+\frac{1}{6},c_2)!}\omega_{(-\frac{c_1}{2}+\frac{1}{6},c_2)} & \text{if } c_1 \text{ is odd,} \end{cases}$$

where $i_U : U \hookrightarrow \mathbb{R}$ is the embedding of the corresponding open subset. Note that $F'_{12}(\mathcal{L}_{2,(c_1,c_2)})$ is also $\mathbb{Q}$-ample.

4.2 Divisorial Contraction: Overview

Let $N = \mathbb{Z}^n$ and $\sigma_{X_2} \subset N_{\mathbb{R}}$ be a simplicial cone generated by rays $\rho_1, \ldots, \rho_n$. Let v_i be the primitive generator of $\rho_i \cap N$, and $v_{n+1} = a_1v_1 + \cdots + a_{n'}v_{n'}$ for some $n' \leq n$ with all $a_i \in \mathbb{Q}_{>0}$ such that $v_{n+1} \in N$ is primitive. Define σ_{X_1,i_0} to be the n-dimensional cone generated by v_i, $1 \leq i \leq n+1$ with $i \neq i_0$. Then

$$\sigma_{X_2} = \bigcup_{i_0=1}^{n+1} \sigma_{X_1,i_0}.$$

Let Σ_2 be the fan consisting of the top dimensional cone σ_{X_2} and its faces, and let Σ_1 be the fan consisting of top dimension cones σ_{X_1,i_0}, $1 \leq i_0 \leq n+1$, and their faces. Then there is a morphism of fans $\Sigma_1 \to \Sigma_2$, which induces a toric morphism $f : X_1 = X_{\Sigma_1} \to X_2 = X_{\Sigma_2}$. Note that X_2 is an affine simplicial toric variety, and f is a toric divisorial contraction.

Define

$$\beta_1 = [r_1v_1 \quad \cdots \quad r_nv_n \quad r_{n+1}v_{n+1}] : \mathbb{Z}^{n+1} \longrightarrow N = \mathbb{Z}^n,$$

$$\beta_2 = [r_1v_1 \quad \cdots \quad r_nv_n] : \mathbb{Z}^n \longrightarrow N = \mathbb{Z}^n.$$

For $i = 1, 2$, one associates a toric orbifold $\mathcal{X}_i$ with the stacky fan $\boldsymbol{\Sigma}_i = (N, \Sigma_i, \beta_i)$. Let ρ_{n+1} be the ray $\mathbb{R}^+ \cdot v_{n+1}$, and denote $r'_{n+1}v_{n+1}$ to be generator of $\mathbb{Z} \cdot r_{n+1}v_{n+1} \cap N_{2,\sigma}$. Then

$$r'_{n+1}v_{n+1} = \sum_{i=1}^{n'} a'_i(r_iv_i), \quad a'_i := \frac{r'_{n+1}a_i}{r_i} \in \mathbb{Z}.$$

Setting $b_i = r_iv_i$ for $i = 1, \ldots, n+1$ as in Section 2, $\alpha_i = \frac{r_{n+1}}{r_i}a_i \in \mathbb{Q}_{>0}$ for $i = 1, \ldots, n'$ and $\alpha_i = 0$ for $i = n'+1, \ldots, n$, we have $b_{n+1} = \alpha_1b_1 + \cdots + \alpha_{n'}b_{n'}$. Let $b'_{n+1} = r'_{n+1}v_{n+1} = mb_{n+1}$, there is a similar relation $b'_{n+1} = \beta_1b_1 + \cdots + \beta'_nb'_n$, where $\beta'_i = \alpha'_i \in \mathbb{Z}_{>0}$.

Let $\mathcal{W}$ be the toric orbifold given by the stacky fan

$$\boldsymbol{\Sigma}' = (N, \Sigma_1, \beta' = [r_1v_1 \quad \cdots \quad r_nv_n \quad r'_{n+1}v_{n+1}]).$$

The identity map $N \to N$ defines morphisms of stacky fans $\boldsymbol{\Sigma}' \to \boldsymbol{\Sigma}_i$, $i = 1, 2$, which induce morphisms $\mu_i : \mathcal{W} \to \mathcal{X}_i$ of toric orbifolds. For $j = 1, \ldots, n$, let $\mathcal{D}_{1,j}$, $\mathcal{D}_{2,j}$, and $\mathcal{D}'_j$ be T-divisors associated with ρ_j in $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{W}$, respectively; let $\mathcal{D}_{1,n+1}$ and $\mathcal{D}'_{n+1}$ be the T-divisors associated with ρ_{n+1} in $\mathcal{X}_1$ and $\mathcal{W}$, respectively. Then for $i = 1, \dots, n$ we have

$$\mu_1^*\mathcal{D}_{1,i} = \mathcal{D}'_i, \quad \mu_2^*\mathcal{D}_{2,i} = \mathcal{D}'_i + a'_i\mathcal{D}'_{n+1},$$

where $a'_i = 0$ for $n' < i \le n$. We also have

$$\begin{aligned} &\mu_1^*\mathcal{D}_{1,n+1} = \frac{r'_{n+1}}{r_{n+1}}\mathcal{D}'_{n+1}, \\ &K_{\mathcal{X}_1} = -\sum_{i=1}^{n+1}\mathcal{D}_{1,i}, \quad \mu_1^*K_{\mathcal{X}_1} = -\sum_{i=1}^{n}\mathcal{D}'_i - \frac{r'_{n+1}}{r_{n+1}}\mathcal{D}'_{n+1}, \\ &K_{\mathcal{X}_2} = -\sum_{i=1}^{n}\mathcal{D}_{2,i}, \quad \mu_2^*K_{\mathcal{X}_2} = -\sum_{i=1}^{n}\mathcal{D}'_i - \Big(\sum_{i=1}^{n'} a'_i\Big)\mathcal{D}'_{n+1}. \end{aligned}$$

Lemma 4.6 (a) $\mu_1^*K_{\mathcal{X}_1} \ge \mu_2^*K_{\mathcal{X}_2} \Leftrightarrow \sum_{i=1}^{n'}\frac{a_i}{r_i} \ge \frac{1}{r_{n+1}}$.

(b) $\mu_1^*K_{\mathcal{X}_1} \le \mu_2^*K_{\mathcal{X}_2} \Leftrightarrow \sum_{i=1}^{n'}\frac{a_i}{r_i} \le \frac{1}{r_{n+1}}$.

Proof From the above computation,

$$\mu_1^*K_{\mathcal{X}_1} - \mu_2^*K_{\mathcal{X}_2} = \Big(\sum_{i=1}^{n} a'_i - \frac{r'_{n+1}}{r_n}\Big)\mathcal{D}'_{n+1} = \Big(\sum_{i=1}^{n'} \frac{a_i}{r_i} - \frac{1}{r_{n+1}}\Big)(r'_{n+1}\mathcal{D}'_{n+1}).$$

Theorem 4.7 *Let $\mu_i : \mathcal{W} \to \mathcal{X}_i$ be defined as above, and let*

$$\begin{aligned} F'_{12} &:= \mu_{1*} \circ \mu_2^* : \mathcal{P}\mathrm{erf}_T(\mathcal{X}_2) \to \mathcal{P}\mathrm{erf}_T(\mathcal{X}_1), \\ F'_{21} &:= \mu_{2*} \circ \mu_1^* : \mathcal{P}\mathrm{erf}_T(\mathcal{X}_1) \to \mathcal{P}\mathrm{erf}_T(\mathcal{X}_2) \end{aligned}$$

be Fourier–Mukai functors.

(a) *If $\mu_1^*K_{\mathcal{X}_1} \ge \mu_2^*K_{\mathcal{X}_2}$ then F'_{12} is cohomologically full and faithful.*

(b) *If $\mu_1^*K_{\mathcal{X}_1} \le \mu_2^*K_{\mathcal{X}_2}$ then F'_{21} is cohomologically full and faithful.*

(c) *If $\mu_1^*K_{\mathcal{X}_1} = \mu_2^*K_{\mathcal{X}_2}$ then F'_{12} and F'_{21} are quasi-equivalences.*

In (c), F_{12} and F_{21} are not inverses of each other in general, as we will see in the following example.

Example 4.8 $N = \mathbb{Z}^2$,

$$\begin{aligned} &\beta_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}, \quad \beta' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & -2 \end{bmatrix}, \\ &v_1 = (1,0), \quad v_2 = (-1,-2), \quad v_3 = (0,-1), \\ &r_1 = r_2 = r_3 = 1, \quad r'_3 = 2, \quad a_1 = a_2 = \frac{1}{2}, \quad a'_1 = a'_2 = 1. \end{aligned}$$

$\mathcal{X}_1$ is the total space of $O_{\mathbb{P}^1}(-2)$, and $\mathcal{X}_2 = [\mathbb{C}^2/\mathbb{Z}_2]$. Given $c_1, c_2, c_3 \in \mathbb{Z}$, we define

$$\begin{aligned} \mathcal{L}_{1,(c_1,c_2,c_3)} &= \mathcal{O}_{\mathcal{X}_1}(c_1\mathcal{D}_{1,1} + c_2\mathcal{D}_{1,2} + c_3\mathcal{D}_{1,3}), \\ \mathcal{L}_{2,(c_1,c_2)} &= \mathcal{O}_{\mathcal{X}_2}(c_1\mathcal{D}_{2,1} + c_2\mathcal{D}_{2,2}). \end{aligned}$$

Then

$$F'_{12}(\mathcal{L}_{2,(c_1,c_2)}) = \begin{cases} \mathcal{L}_{1,(c_1,c_2,\frac{c_1+c_2}{2})}, & c_1 + c_2 \text{ is even}, \\ \mathcal{L}_{1,(c_1,c_2,\frac{c_1+c_2-1}{2})}, & c_1 + c_2 \text{ is odd}. \end{cases}$$

$$\begin{cases} F'_{21}(\mathcal{L}_{1,(c_1,c_2,\frac{c_1+c_2}{2})}) = \mathcal{L}_{2,(c_1,c_2)}, & c_1+c_2 \text{ is even,} \\ F'_{21}(\mathcal{L}_{1,(c_1,c_2,\frac{c_1+c_2+1}{2})} = \mathcal{L}_{2,(c_1+c_2)} & c_1+c_2 \text{ is odd.} \end{cases}$$

$$\begin{cases} F'_{12}\circ F'_{21}(\mathcal{L}_{1,(c_1,c_2,\frac{c_1+c_2}{2})}) = \mathcal{L}_{1,(c_1,c_2,\frac{c_1+c_2}{2})}, & c_1+c_2 \text{ is even,} \\ F'_{12}\circ F'_{21}(\mathcal{L}_{1,(c_1,c_2,\frac{c_1+c_2+1}{2})} = \mathcal{L}_{1,(c_1,c_2,\frac{c_1+c_2-1}{2})} & c_1+c_2 \text{ is odd.} \end{cases}$$

The corresponding functors for constructible sheaves are shown in Figure 9.

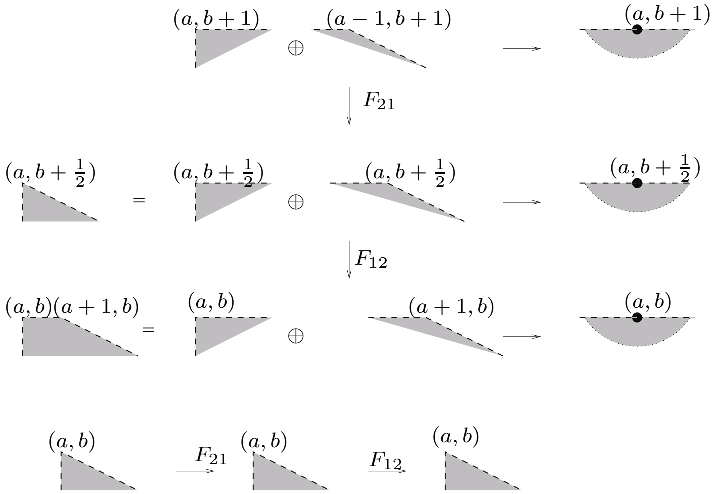


Figure 9 a,b are integers. Maps between constructible sheaves are induced by inclusion maps of open subsets of $\mathbb{R}^2$

4.3 Divisorial Contraction: $F'_{12}=\mu_{1*}\circ\mu_2^*$

Let $\mu_i:\mathcal{W}\to\mathcal{X}_i$ be defined as in Subsection 4.2. Recall that there is a toric morphism $f:X_1\to X_2$ which is a divisorial contraction. In this subsection, we study the Fourier–Mukai functor $F'_{12}=\mu_{1*}\circ\mu_2^*$ and the corresponding functor for constructible sheaves. We obtain an equivariant version of [7, Theorem 4.2 (2)].

The pullback map μ_2^* has been studied in [5]. The identity map $id:N\to N$ induces a morphism $\mathbf{\Sigma}'\to\mathbf{\Sigma}_2$ of stacky fans, which induces a morphism $\mu_2:\mathcal{W}\to\mathcal{X}_2$ of toric orbifolds. The following square of functors commutes up to natural isomorphism by [5, Theorem 5.16]:

$$\begin{tikzcd} \langle\Theta'_2\rangle \arrow[r, "\kappa_2"] \arrow[d, "\mu_2^*"'] & \langle\Theta_2\rangle \arrow[d, "id_!"'] \\ \langle\Theta'_{\mathcal{W}}\rangle \arrow[r, "\kappa'"] & \langle\Theta_{\mathcal{W}}\rangle, \end{tikzcd}$$

where $\kappa_2=\kappa_{\mathbf{\Sigma}_2}$ and $\kappa'=\kappa_{\mathbf{\Sigma}'}$. Since id is the identity map and $id_!$ is cohomologically full and faithful, μ_2^* is also a cohomologically full and faithful functor.

The toric orbifolds $\mathcal{W}$ and $\mathcal{X}_1$ have the same coarse moduli space $X_1=X_{\mathbf{\Sigma}_1}$. The pushforward functor μ_{1*} was described in terms of theta sheaves in Subsection 4.1. We now describe the composition $F'_{12}=\mu_{1*}\circ\mu_2^*$ and the corresponding functor F_{12} on constructible sheaves.

Let $\sigma\in\Sigma_2$ be a d-dimensional cone generated by the rays $\rho_{i_1},\dots,\rho_{i_k}$, $k=1,\dots,d$. Denote $t_k=\langle\chi,r_{i_k}v_{i_k}\rangle_{2,\sigma}\in\mathbb{Z}$ for a given $\chi\in N_{2,\sigma}$. The theta sheaf $\Theta_2(\sigma,\chi)\in\langle\Theta_2\rangle$ is the costandard constructible sheaf supported on the submanifold given by

$$(\sigma^\vee)^\circ_\chi = \left\{ x \in M_\mathbb{R} : \langle x, v_{i_k} \rangle > \frac{t_k}{r_{i_k}},\ k = 1, \ldots, d \right\}$$

where $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ is the natural pairing.

Proposition 4.9 *The Fourier–Mukai functor F'_{12} takes a theta sheaf in $\langle \Theta'_2 \rangle$ to $\langle \Theta'_1 \rangle$. Moreover, if $v_{n+1} \in \sigma$, then the analogous constructible functor $F_{12} = \kappa_1 \circ F'_{12} \circ \kappa_2^{-1} : \langle \Theta_2 \rangle \to \langle \Theta_1 \rangle$ takes $\Theta_2(\sigma, \chi)$ to the costandard constructible sheaf on*

$$F(\sigma^\vee_\chi)^\circ := \left\{ x \in M_\mathbb{R} : \langle x, v_{i_k} \rangle > \frac{t_k}{r_{i_k}},\ k = 1, \ldots, d; \langle x, v_{n+1} \rangle > \frac{t_{n+1}}{r_{n+1}} \right\},$$

where

$$t_{n+1} = \left\lceil r_{n+1} \sum_{i=1}^{n'} \frac{a_k t_k}{r_k} \right\rceil \in \mathbb{Z}.$$

Otherwise if $v_{n+1} \notin \sigma$ then $F_{12}(\Theta_2(\sigma, \chi)) = \Theta_1(\sigma, \chi)$.

Proof Suppose that $\sigma \in \Sigma_2(d)$ and $v_{n+1} \in \sigma$. Let $v_{i_1}, \ldots, v_{i_d}$ be defined as above. Then we may assume $i_k = k$ for $k = 1, \ldots, n'$, and

$$n' < i_{n'+1} < \cdots < i_d \leq n.$$

We have

$$\sigma = \bigcup_{k=1}^{n'} \sigma_k,$$

where $\sigma_k \in \Sigma_1(d)$ is the cone generated by

$$v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_{n'}, v_{i_{n'+1}}, \ldots, v_{i_d}, v_{n+1}.$$

For $1 \leq j_0 < \cdots < j_k \leq n'$, let $\sigma_{j_0 \cdots j_k} = \sigma_{j_0} \cap \cdots \cap \sigma_{j_k} \in \Sigma_1(d-k)$, and let $\chi_{j_0 \ldots j_k} \in M'_{\sigma_{j_0 \cdots j_k}}$ be the image of $\chi \in M_{2,\sigma}$ under the group homomorphism $M_{2,\sigma} \to M'_{\sigma_{j_0 \cdots j_k}}$. Let $P(\chi_1, \ldots, \chi_{n'}) \in Sh_c(M_\mathbb{R}; \Lambda_{\mathbf{\Sigma}'})$ be the following cochain complex:

$$\bigoplus_{1 \leq j_0 \leq n'} \Theta_\mathcal{W}(\sigma_{j_0}, \chi_{j_0}) \to \bigoplus_{1 \leq j_0 < i_1 \leq n'} \Theta_\mathcal{W}(\sigma_{j_0 j_1}, \chi_{j_0, j_1}) \to \cdots.$$

Then $P(\chi_1, \ldots, \chi_{n'})$ is quasi-isomorphic to $j_{(\sigma^\vee_\chi)^\circ !}\mathbb{C}_{(\sigma^\vee_\chi)^\circ}[n]$.

If $\tau, \tau' \in \Sigma_1$ and $\tau \subset \tau'$ then there are surjective group homomorphisms $f^*_{1,\tau\tau'} : M_{1,\tau'} \to M_{1,\tau}$ and $f'^*_{\tau\tau'} : M'_{\tau'} \to M'_\tau$. Recall that the pushforward map $\mu_{1*,\tau}$ is the pushforward map of the characters for a single cone defined in Section 4.1. These maps are compatible with the restriction map f^*:

$$\mu_{1,\tau *} \circ f'^*_{\tau\tau'} = f^*_{1,\tau\tau'} \circ \mu_{1,\tau' *}.$$

Let $\phi_{i_0 \cdots i_k} := \mu_{1, \sigma_{i_0 \ldots i_k} *}(\chi_{i_0 \cdots i_k}) \in M_{1, \sigma_{i_0 \cdots i_k}}$, and let $P(\phi_1, \ldots, \phi_{n'}) \in Sh_c(M_\mathbb{R}; \Lambda_{\mathbf{\Sigma}_1})$ be the following cochain complex:

$$\bigoplus_{1 \leq i_0 \leq n'} \Theta_1(\sigma_{i_0}, \phi_{i_0}) \to \bigoplus_{1 \leq i_0 < i_1 \leq n'} \Theta_1(\sigma_{i_0 i_1}, \phi_{i_0, i_1}) \to \cdots.$$

It remains to show that $P(\phi_1, \ldots, \phi_{n'})$ is quasi-isomorphic to $j_{F(\sigma^\vee_\chi)^\circ !}\mathbb{C}_{F(\sigma^\vee_\chi)^\circ}[n]$. It suffices to prove the following two statements:

(i) The piecewise linear function $\psi : \sigma \to \mathbb{R}$ defined by $\phi_i \in M_{1,\sigma_i}$ for $i = 1, \dots, n'$ is convex:

$$\psi(v_{n+1}) \geq \sum_{i=1}^{n'} a_i \psi(v_i).$$

(ii) $F(\sigma_\chi^\vee)^\circ = \{x \in M_\mathbb{R} \mid \langle x, v\rangle > \psi(v) \text{ for any } v \in \sigma \subset N_\mathbb{R}\}$.

(i) and (ii) follow from:

$$\psi(v_k) = \frac{t_k}{r_k}, \quad k = 1, \dots, n', i_1, \dots, i_{d-n'}, n+1,$$

$$\frac{t_{n+1}}{r_{n+1}} = \frac{1}{r_{n+1}} \left\lceil r_{n+1} \sum_{i=1}^{n'} \frac{a_i t_i}{r_i} \right\rceil \geq \sum_{i=1}^{n'} a_i \frac{t_i}{r_i}.$$

Proposition 4.10 *Suppose that* $\sum_{i=1}^n \frac{a_i}{r_i} \geq \frac{1}{r_{n+1}}$. *We have the following statements involving the map* F:

(1) $(\sigma_\chi^\vee)^\circ \subset (\sigma'^\vee_{\chi'})^\circ \Rightarrow F(\sigma_\chi^\vee)^\circ \subset F(\sigma'^\vee_{\chi'})^\circ$,

(2) $(\sigma_\chi^\vee)^\circ \not\subset (\sigma'^\vee_{\chi'})^\circ \Rightarrow F(\sigma_\chi^\vee)^\circ \not\subset F(\sigma'^\vee_{\chi'})^\circ$ *and* $F(\sigma_\chi^\vee)^\circ - F(\sigma'^\vee_{\chi'})^\circ$ *is contractible.*

Proof (1) We only need to show the case σ and σ' both contain $\rho_1, \dots, \rho_{n'}$. It is obvious that $\sigma \supset \sigma'$. Let $v_1, \dots, v_{i_d}$ be the generators of σ, and $v_1, \dots, v_{i_{d'}}$ generate σ' where $1 \leq d' \leq d$. Similarly to the definition of t_k, set $t'_k = \langle \chi', r_{i_k} v_{i_k}\rangle_{2,\sigma}$, and

$$t'_{n+1} = \left\lceil r_{n+1} \sum_{i=1}^{n'} \frac{a_k t'_k}{r_k} \right\rceil.$$

The inclusion $(\sigma_\chi^\vee)^\circ \subset (\sigma'^\vee_{\chi'})^\circ$ gives $t_k \geq t'_k$, and a straightforward calculation shows $t_{n+1} \geq t'_{n+1}$. It follows that $F(\sigma_\chi^\vee)^\circ \subset F(\sigma'^\vee_{\chi'})^\circ$ by definition.

(2) If $\sigma \not\supset \sigma'$ the statement is trivial. In case that $\sigma \supset \sigma'$, we must have some k_0 such that $t_{i_{k_0}} < t'_{i_{k_0}}$, which follows by $F(\sigma_\chi^\vee)^\circ \not\subset F(\sigma'^\vee_{\chi'})^\circ$. The only situation that $F(\sigma_\chi^\vee)^\circ - F(\sigma'^\vee_{\chi'})^\circ$ is not contractible is that $t'_k > t_k$ while $t'_{n+1} = t_{n+1}$; but this is impossible since

$$\begin{aligned} t'_{n+1} - t_{n+1} &= \left\lceil r_{n+1} \sum_{i=1}^{n'} \frac{a_k t'_k}{r_k} \right\rceil - \left\lceil r_{n+1} \sum_{i=1}^{n'} \frac{a_k t_k}{r_k} \right\rceil \\ &\geq \left\lceil r_{n+1} \sum_{i=1}^{n'} \frac{a_k (t'_k - t_k)}{r_k} \right\rceil \geq \left\lceil r_{n+1} \sum_{i=1}^{n'} \frac{a_k}{r_k} \right\rceil \geq \lceil 1 \rceil > 0. \end{aligned}$$

Proposition 4.9 and Proposition 4.10 give the following theorem:

Theorem 4.11 *If* $\mu_1^* K_{\mathcal{X}_1} \geq \mu_2^* K_{\mathcal{X}_2}$, *or equivalently,*

$$\sum_{i=1}^{n} \frac{a_i}{r_i} \geq \frac{1}{r_{n+1}},$$

then the Fourier–Mukai functor $F'_{12} = \mu_{1*} \circ \mu_2^* : \langle \Theta'_2 \rangle \to \langle \Theta'_1 \rangle$ *is a quasi-embedding. If restricted on the full dg subcategory* $\mathrm{Coh}_T(\mathcal{X}_2)$, F'_{12} *is a quasi-embedding of* $\mathrm{Coh}_T(\mathcal{X}_2)$ *into* $\mathrm{Coh}_T(\mathcal{X}_1)$.

Proof Passing to constructible sheaves via CCC, it suffices to work on the constructible theta sheaves since they are generators. The theorem follows from the simple facts

$$\mathrm{Ext}^*(\Theta_2(\sigma, \chi), \Theta_2(\sigma', \chi')) = \begin{cases} \mathbb{C}[0] & \text{if } (\sigma_\chi^\vee)^\circ \subset (\sigma'^\vee_{\chi'})^\circ, \\ 0 & \text{if } (\sigma_\chi^\vee)^\circ \not\subset (\sigma'^\vee_{\chi'})^\circ, \end{cases}$$

and

$$\begin{aligned}&\mathrm{Ext}^*(i_{F(\sigma_{\mathcal{X}}^\vee)^\circ !}(\sigma_{\mathcal{X}}^\vee)^\circ, i_{F(\sigma_{\mathcal{X}'}^{\prime\vee})^\circ !}(\sigma_{\mathcal{X}'}^{\prime\vee})^\circ)\\ &\quad=\begin{cases}\mathbb{C}[0] & \text{if } F(\sigma_{\mathcal{X}}^\vee)^\circ \subset F(\sigma_{\mathcal{X}'}^{\prime\vee})^\circ,\\ 0 & F(\sigma_{\mathcal{X}}^\vee)^\circ \not\subset F(\sigma_{\mathcal{X}'}^{\prime\vee})^\circ \text{ and } F(\sigma_{\mathcal{X}}^\vee)^\circ - F(\sigma_{\mathcal{X}'}^{\prime\vee})^\circ \text{ is contractible.}\end{cases}\end{aligned}$$

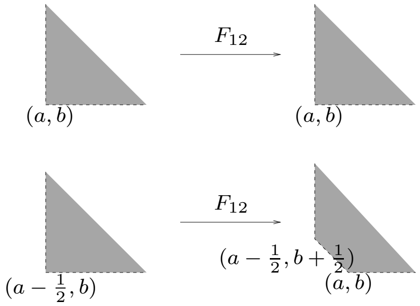


Figure 10 a, b are integers. Constructible sheaves are costandard constructible sheaves over shaded regions. Maps between constructible sheaves are induced by inclusion maps of open subsets of $\mathbb{R}^2$

Example 4.12 $N = \mathbb{Z}^2$,

$$\beta_1 = \begin{bmatrix} 2 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 2 & 0\\ 0 & 1\end{bmatrix}, \quad \beta' = \begin{bmatrix} 2 & 0 & 2\\ 0 & 1 & 2\end{bmatrix}.$$

$\mathcal{X}_2 = [\mathbb{C}/\mathbb{Z}_2] \times \mathbb{C}$, $X_2 = \mathbb{C}^2$.

$$\begin{aligned}&v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (1,1),\\ &r_1 = 2, \quad r_2 = r_3 = 1, \quad r_3' = 2, \quad a_1 = a_1' = 1, \quad a_2 = 1, \quad a_2' = 2.\end{aligned}$$

For $c_1, c_2, c_3 \in \mathbb{Z}$, define

$$\begin{aligned}&\mathcal{L}_{1,(c_1,c_2,c_3)} = \mathcal{O}_{\mathcal{X}_1}(c_1\mathcal{D}_{1,1} + c_2\mathcal{D}_{1,2} + c_3\mathcal{D}_{1,3}),\\ &\mathcal{L}_{2,(c_1,c_2)} = \mathcal{O}_{\mathcal{X}_2}(c_1\mathcal{D}_{2,1} + c_2\mathcal{D}_{2,2}) = p_1^*\mathcal{O}_{\mathbb{C}^2}\left(\frac{c_1}{2}D_{2,1} + D_{2,2}\right).\end{aligned}$$

Then

$$F_{12}'(\mathcal{L}_{2,(c_1,c_2)}) = \begin{cases}\mathcal{L}_{1,(c_1,c_2,\frac{c_1}{2}+c_2)}, & c_1 \text{ is even},\\ \mathcal{L}_{1,(c_1,c_2,\frac{c_1-1}{2}+c_2)}, & c_1 \text{ is odd}.\end{cases}$$

The corresponding functors for constructible sheaves are shown in Figure 10.

4.4 Divisorial Contraction: $F_{21}' = \mu_{2*} \circ \mu_1^*$

Let $\mu_i : \mathcal{W} \to \mathcal{X}_i$ be defined as in Section 4.2 and Section 4.3. In this section, we study the Fourier–Mukai functor $F_{21}' = \mu_{2*} \circ \mu_1^*$ and the corresponding functor for constructible sheaves. We obtain an equivariant version of [7, Theorem 4.2 (4)].

The pullback map μ_1^* has been studied in [5]. The identity map $id : N \to N$ induces a morphism $\mathbf{\Sigma}' = (N, \Sigma_1, \beta') \to \mathbf{\Sigma}_1 = (N, \Sigma_1, \beta_1)$ of stacky fans, which induces a morphism

$\mu_1 : \mathcal{W} \to \mathcal{X}_1$ of toric orbifolds. The following square of functors commutes up to natural isomorphism by [5, Theorem 5.16]:

$$\begin{tikzcd}
\langle \Theta_1' \rangle \arrow[r, "\kappa_1"] \arrow[d, "\mu_1^*"'] & \langle \Theta_1 \rangle \arrow[d, "id_!"'] \\
\langle \Theta_{\mathcal{W}}' \rangle \arrow[r, "\kappa'"] & \langle \Theta_{\mathcal{W}} \rangle,
\end{tikzcd}$$

where $\kappa_1 = \kappa_{\mathbf{\Sigma}_1}$ and $\kappa' = \kappa_{\mathbf{\Sigma}'}$. Since *id* is the identity map and $id_!$ is cohomologically full and faithful, μ_1^* is also a cohomologically full and faithful functor.

It remains to study the pushforward map μ_{2*}. Generally speaking, the image of $\mathcal{Q}_T^{\mathrm{fin}}(\mathcal{W}) = \langle \Theta_{\mathcal{W}}' \rangle$ under the pushforward map $\mu_{2*} : \mathcal{Q}_T(\mathcal{W}) \to \mathcal{Q}_T(\mathcal{X}_2)$ is *not* contained in $\mathcal{Q}_T^{\mathrm{fin}}(\mathcal{X}_2) = \langle \Theta_2' \rangle$.

4.4.1 Notation

By definition,

$$N_{2,\sigma_{X_2}} = \bigoplus_{i=1}^{n} \mathbb{Z} b_i \subset N_{\mathbb{R}}.$$

Define $b_1^*, \dots, b_n^* \in M_{\mathbb{R}}$ by $\langle b_i^*, b_j \rangle = \delta_{ij}$. Then the dual lattice of $N_{2,\sigma_{X_2}}$ is given by

$$M_{2,\sigma_{X_2}} = \bigoplus_{i=1}^{n} \mathbb{Z} b_i^* \subset M_{\mathbb{R}}.$$

Recall from Subsection 4.2 that

$$v_{n+1} = \sum_{i=1}^{n'} a_i v_i, \quad b_{n+1} = \sum_{i=1}^{n'} \alpha_i b_i, \quad b_{n+1}' = m b_{n+1} = \sum_{i=1}^{n'} \beta_i b_i,$$

where

$$a_i \in \mathbb{Q}_{>0}, \quad \alpha_i = \frac{r_{n+1}}{r_i} a_i \in \mathbb{Q}_{>0}, \quad m \in \mathbb{Z}_{>0}, \quad \beta_i = m\alpha_i \in \mathbb{Z}_{>0}$$

for $i = 1, \dots, n'$. We fix the following notation:

- Let $\bar{I} = \{1, \dots, n+1\}$, $I = \{1, \dots, n\}$, $I' = \{1, \dots, n'\}$.
- Given a proper subset J of $\bar{I}$, let σ_J denote the cone in $N_{\mathbb{R}}$ generated by

$$\{v_j \mid j \in J\}.$$

In particular, $\sigma_\emptyset = \{0\}$, where $\emptyset$ is the empty set.
With the above notation, we have

$$\begin{aligned}
&\sigma_{X_2} = \sigma_I, \quad \sigma_{i_0} = \sigma_{\bar{I} - \{i_0\}} \quad \text{if } i_0 \in I', \\
&\Sigma_1 = \{\sigma_J \mid I' \not\subset J \subset \bar{I}\}, \quad \Sigma_2 = \{\sigma_J \mid J \subset I\}.
\end{aligned}$$

We define a map $\Lambda := \{J \subset \bar{I} \mid I' \not\subset J\} \to 2^I = \{J' \subset I\}$, $J \mapsto J'$, such that $\sigma_{J'} \in \Sigma_2$ is the intersection of all cones in Σ_2 which contains $\sigma_J \in \Sigma_1$. More explicitly,

$$J' = \begin{cases} J & \text{if } n+1 \notin J, \\ (J - \{n+1\}) \cup I' & \text{if } n+1 \in J. \end{cases}$$

For $J \in \Lambda$, define $\mathcal{X}_{1,J} = \mathcal{X}_{1,\sigma_J}$ and $\mathcal{W}_J = \mathcal{W}_{\sigma_J}$; for $J \in 2^I$, define $\mathcal{X}_{2,J} = \mathcal{X}_{2,\sigma_J}$.

Lemma 4.13 *Suppose that $J \in \Lambda$, so that $\sigma_J \in \Sigma_1$. If $n+1 \notin J$ then*

$$F'_{21}\Theta'_1(\sigma_J, \phi) = \Theta'_2(\sigma_J, \phi)$$

for any $\phi \in M_{1,\sigma_J} = M_{2,\sigma_J}$.

Proof If $n+1 \notin J$ then $\mu_i : \mathcal{W} \to \mathcal{X}_i$ restricts to the identity map $\mathcal{W}_J \to \mathcal{X}_{i,J}$, $i = 1, 2$. □

We will consider the case $n+1 \in J$ later.

By definition, $N_{1,\sigma_{i_0}} = \bigoplus_{i \in \bar{I} - \{i_0\}} \mathbb{Z} b_i$. Since the cones σ_{X_2} and σ_{i_0} are n-dimensional, one may regard $M_{2,\sigma_{X_2}}$ and $M_{1,\sigma_{i_0}}$ as subsets in $M_{\mathbb{R}}$, embedded in a canonical way. Straightforward calculations show that

Lemma 4.14

$$M_{1,\sigma_{i_0}} = \bigoplus_{i \in I' - \{i_0\}} \mathbb{Z}\left(b_i^* - \frac{\alpha_i}{\alpha_{i_0}} b_{i_0}^*\right) \oplus \bigoplus \mathbb{Z} \frac{b_{i_0}^*}{\alpha_{i_0}} \oplus \bigoplus_{i \in I - I'} \mathbb{Z} b_i^*.$$

$$M'_{\sigma_{i_0}} = \bigoplus_{i \in I' - \{i_0\}} \mathbb{Z}\left(b_i^* - \frac{\alpha_i}{\alpha_{i_0}} b_{i_0}^*\right) \oplus \bigoplus \mathbb{Z} \frac{b_{i_0}^*}{m\alpha_{i_0}} \oplus \bigoplus_{i \in I - I'} \mathbb{Z} b_i^*.$$

4.4.2 Reduction

We fix $i_0 \in I'$. Define $\mu_{1,i_0} := \mu_1|_{\mathcal{W}_{\sigma_{i_0}}} : \mathcal{W}_{\sigma_{i_0}} \to \mathcal{X}_{1,\sigma_{i_0}}$ and $\mu_{2,i_0} := \mu_2|_{\mathcal{W}_{\sigma_{i_0}}} : \mathcal{W}_{\sigma_{i_0}} \to \mathcal{X}_{2,\sigma_{X_2}} = \mathcal{X}_2$. Define $F'_{21,i_0} = \mu_{2,i_0 *} \circ \mu^*_{1,i_0}$. Suppose that $J \in \Lambda$ and $\sigma_J \subset \sigma_{i_0}$. Let $j : \mathcal{X}_{1,J} \to \mathcal{X}_{1,\sigma_{i_0}}$ be the open embedding. For every $\phi \in M_{1,\sigma_J}$, define $\Theta_{1,i_0}(\sigma_J, \phi) := j_* \mathcal{O}_{\mathcal{X}_{1,J}}(\phi) \in \mathcal{Q}_T^{fin}(\mathcal{X}_{1,\sigma_{i_0}})$. Then

$$F_{21}\Theta'_1(\sigma_J, \phi) = j_{\sigma_{i_0} *} F_{21,i_0} \Theta'_{1,i_0}(\sigma_J, \phi),$$

where $j_{\sigma_{i_0}} : \mathcal{X}_{1,\sigma_{i_0}} \hookrightarrow \mathcal{X}_1$ is the embedding of $\mathcal{X}_{1,\sigma_{i_0}}$. Every $\sigma_J \in \Sigma_1$ is contained in σ_{i_0} for some $i_0 \in I'$, so it suffices to describe F_{12,i_0} for any $i_0 \in I'$.

4.4.3 Coordinate Rings

For some $i_0 \in I'$, we define the following notations.

- For $i \in I' - \{i_0\}$, define

$$x_i = \chi^{b_i^*} \in \mathbb{C}[M_{2,\sigma_{X_2}}], \quad y_i = \chi^{b_i^* - \frac{\alpha_i}{\alpha_{i_0}} b_{i_0}^*} \in \mathbb{C}[M_{1,\sigma_{i_0}}],$$

where $\chi^{b_i^*}$ is defined as in [12, Section 1.3].

- Define

$$y_{i_0} = \chi^{\frac{b_{i_0}^*}{\alpha_{i_0}}} \in \mathbb{C}[M_{1,\sigma_{i_0}}], \quad z = \chi^{\frac{b_{i_0}^*}{m\alpha_{i_0}}} \in \mathbb{C}[M'_{\sigma_{i_0}}].$$

In particular, $y_{i_0} = z^m$.

- For $i \in I - I'$, i.e. $n'+1 \leq i \leq n$, define $y_i = \chi^{b_i^*}$.
- Define rings

$$\begin{aligned} A_1 &:= \mathbb{C}[\sigma_{i_0}^\vee \cap M_{1,\sigma_{i_0}}] = \mathbb{C}[y_1, \ldots, y_n], \\ A' &:= \mathbb{C}[\sigma_{i_0}^\vee \cap M'_{\sigma_{i_0}}] = \mathbb{C}[y_1, \ldots, y_{i_0-1}, z, y_{i_0+1}, y_n], \\ A_2 &:= \mathbb{C}[\sigma_{X_2}^\vee \cap M_{2,\sigma_{X_2}}] = \mathbb{C}[x_1, \ldots, x_{n'}, y_{n'+1}, \ldots, y_n]. \end{aligned}$$

We define

$$U_1 = \mathrm{Spec} A_1, \quad U' = \mathrm{Spec} A', \quad U_2 = \mathrm{Spec} A_2.$$

Then U_1, U', and U_2 are isomorphic to $\mathbb{C}^n$. Define

$$\widetilde{T}_1 = \mathrm{Spec}\mathbb{C}[M_{1,\sigma_{i_0}}], \quad \widetilde{T}' = \mathrm{Spec}\mathbb{C}[M'_{\sigma_{i_0}}], \quad \widetilde{T}_2 = \mathrm{Spec}\mathbb{C}[M_{2,\sigma_{X_2}}], \quad T = \mathrm{Spec}\mathbb{C}[M].$$

Then $\widetilde{T}_1$, $\widetilde{T}'$, $\widetilde{T}_2$ and T are isomorphic to $(\mathbb{C}^*)^n$. $\widetilde{T}_1$, $\widetilde{T}'$ and $\widetilde{T}_2$ act on U_1, U' and U_2, respectively.

There are short exact sequence of abelian groups

$$1 \to G_1 \to \widetilde{T}_1 \to T \to 1, \quad 1 \to G' \to \widetilde{T}' \to T \to 1, \quad 1 \to G_2 \to \widetilde{T}_2 \to T \to 1,$$

where G_1, G', and G_2 are finite groups. We have

$$\mathcal{X}_{1,\sigma_{i_0}} = [U_1/G_2], \quad \mathcal{W}_{\sigma_0} = [U'/G'], \quad \mathcal{X}_2 = [U_2/G_2].$$

The morphism $\mu_{1,i_0} : \mathcal{W}_{\sigma_{i_0}} \to \mathcal{X}_{1,\sigma_{i_0}}$ lifts to $g_1 : U' \to U_1$, where

$$g_1(y_1, \ldots, y_{i_0-1}, z, y_{i_0+1}, \ldots, y_n) = (y_1, \ldots, y_{i_0-1}, z^m, y_{i_0+1}, \ldots, y_n).$$

The morphism $\mu_{2,i_0} : \mathcal{W}_{\sigma_{i_0}} \to \mathcal{X}_2$ lifts to $g_2 : U' \to U_2$, where

$$\begin{aligned} &g_2(y_1, \ldots, y_{i_0-1}, z, y_{i_0+1}, \ldots, y_n) \\ &\quad = (y_1 z^{\beta_1}, \ldots, y_{i_0-1} z^{\beta_{i_0-1}}, z^{\beta_{i_0}}, y_{i_0+1} z^{\beta_{i_0+1}}, \ldots, y_{n'} z^{\beta_{n'}}, y_{n'+1}, \ldots, y_n). \end{aligned}$$

Suppose that $J \in \Lambda$. Then $J \subset \sigma_{i_0}$ for some $i_0 \in I'$. We fix i_0 and J, and assume that $n+1 \in J$. Define

$$K_1 = \bar{I} - J - \{i_0\}, \quad K_2 = I - J'.$$

Define rings

$$B_1 = A_1[y_i^{-1}]_{i \in K_1}, \quad B' = A'[y_i^{-1}]_{i \in K_1}, \quad B_2 = A_2[y_i^{-1}]_{i \in K_2},$$

where $A_1[y_i^{-1}]_{i \in K_1}$ is the ring A_1 adjoint with y_i^{-1} for all $i \in K_1$, etc.

We define

$$V_1 = \mathrm{Spec}B_1, \quad V' = \mathrm{Spec}B', \quad V_2 = \mathrm{Spec}B_2.$$

The inclusion $A_1 \subset B_1$, $A' \subset B'$, and $A_2 \subset B_2$ induce open embeddings

$$V_1 \subset U_1, \quad V' \subset U', \quad V_2 \subset U_2.$$

We have

$$\mathcal{X}_{1,J} = [V_1/G_1], \quad \mathcal{W}_J = [V'/G'], \quad \mathcal{X}_{2,J'} = [V_2/G_2].$$

4.4.4 Sheaves and Modules

$\Theta'_{1,i_0}(\sigma_J, \phi)$ corresponds to a $\widetilde{T}_1$-equivariant quasicoherent sheaf $\widetilde{\Theta}'_{1,i_0}(\sigma_J, \phi)$ on U_1, and $F_{21,i_0}\Theta'_{1,i_0}(\sigma_J, \phi)$ corresponds to the $\widetilde{T}_2$-equivariant quasicoherent sheaf $g_{2*}g_1^*\widetilde{\Theta}'_{1,i_0}(\sigma_J, \phi)$ on U_2. Let H be the kernel of $\widetilde{T}' \to \widetilde{T}_2$. Define

$$\begin{aligned} &Q_1 = \Gamma(U_1, \widetilde{\Theta}'_{1,i_0}(\sigma_J, \phi)), \quad Q' = \Gamma(U', g_1^*\widetilde{\Theta}'_{1,i_0}(\sigma_J, \phi)), \\ &Q_2 = \Gamma(U_2, g_{2*}g_1^*\widetilde{\Theta}'_{1,i_0}(\sigma_J, \phi)) = \Gamma(U', g_1^*\widetilde{\Theta}'_{1,i_0}(\sigma_J, \phi))^H. \end{aligned}$$

Then

(1) $\widetilde{\Theta}'_{1,i_0}(\sigma_J, \phi)$ is the $\widetilde{T}_1$-equivariant quasicoherent sheaf on U_1 defined by the $\widetilde{T}_1$-equivariant A_1-module Q_1.

(2) $g_1^*\widetilde{\Theta}'_{1,i_0}(\sigma_J,\phi)$ is the $\widetilde{T}'$-equivariant quasicoherent sheaf on U' defined by the $\widetilde{T}'$-equivariant A'-module Q'.

(3) $g_{2*}g_1^*\widetilde{\Theta}'_{1,i_0}(\sigma_J,\phi)$ is the $\widetilde{T}_2$-equivariant quasicoherent sheaf on U_2 defined by the $\widetilde{T}_2$-equivariant A_2-module Q_2.

More explicitly, $\phi \in M_{1,\sigma_J}$ is determined by $c_i = \langle \phi, b_i\rangle \in \mathbb{Z}$, $i \in J$. We have

$$\begin{aligned}
Q_1 &= \mathbb{C}[(\sigma_J)_\phi \cap M_{1,\sigma_{i_0}}] = y_{i_0}^{c_{n+1}} \cdot \left(\prod_{j\in J-\{n+1\}} y_j^{c_j}\right)\cdot B_1\\
Q' &= \mathbb{C}[(\sigma_J)_\phi \cap M'_{\sigma_{i_0}}] = z^{mc_{n+1}} \cdot \left(\prod_{j\in J-\{n+1\}} y_j^{c_j}\right)\cdot B'\\
Q_2 &= \mathbb{C}[(\sigma_J)_\phi \cap M'_{\sigma_{i_0}}] \cap \mathbb{C}[M_{2,\sigma_{X_2}}]\\
&= x_{i_0}^{\frac{c_{n+1}}{\alpha_{i_0}}} \cdot \prod_{j\in J\cap I'} (x_j x_{i_0}^{-\alpha_j/\alpha_{i_0}})^{c_j} \cdot \prod_{j\in J'-I'} y_j^{c_j} \cdot g(B_1) \cap \mathbb{C}[M_{2,\sigma_{X_2}}],
\end{aligned}$$

where

$$\begin{aligned}
g(B_1) = {}& \mathbb{C}[x_j x_{i_0}^{-\alpha_j/\alpha_{i_0}}]_{j\in J\cap I'} \otimes_{\mathbb{C}} \mathbb{C}[x_{i_0}^{\frac{1}{m\alpha_{i_0}}}] \otimes_{\mathbb{C}} \mathbb{C}[x_j x_{i_0}^{-\alpha_j/\alpha_{i_0}}, x_j^{-1} x_{i_0}^{\alpha_j/\alpha_{i_0}}]_{j\in K_1\cap I'}\\
&\otimes_{\mathbb{C}} \mathbb{C}[y_j]_{j\in J'-I'} \otimes_{\mathbb{C}} \mathbb{C}[y_j, y_j^{-1}]_{j\in K_2}\\
\mathbb{C}[M_{2,\sigma_{X_2}}] = {}& \mathbb{C}[x_1, x_1^{-1}, \ldots, x_{n'}, x_{n'}^{-1}, y_{n'+1}, y_{n'+1}^{-1}, \ldots, y_n, y_n^{-1}].
\end{aligned}$$

Here we use $z = x_{i_0}^{\frac{1}{m\alpha_{i_0}}}$ and $y_i = x_i x_{i_0}^{-\frac{\alpha_i}{\alpha_{i_0}}}$ for $i \in I' - \{i_0\}$.

Finally, we remark that

(1) Q_1 is a free B_1-module of rank 1, and defines a line bundle $\mathcal{O}_{V_1}(\phi)$ on $V_1 = \mathrm{Spec}B_1$.

(2) Q' is a free B'-module of rank 1, and defines a line bundle $\mathcal{O}_{V'}(\phi)$ on $V' = \mathrm{Spec}B'$.

(3) Q_2 is a B_2-module, and defines a quasicoherent sheaf on $V_2 = \mathrm{Spec}B_2$.

4.4.5 Koszul Resolution

Q_2 is not finitely generated as a B_2-module. The goal of this section is to find a resolution of Q_2 by free B_2-modules. The following observations are useful:

(i) Let $B = \mathbb{C}[x_{i_0}, y_{n'+1}, \ldots, y_n] \otimes_{\mathbb{C}} \mathbb{C}[y_j^{-1}]_{j\in K_2}$. Then B is a subring of B_1, so Q_2 can be viewed as a B-module. We observe that Q_2 is a *free* B-module.

(ii) $B_2 = B[x_j]_{j\in I'-\{i_0\}}$ can be viewed as a B-module. We have the following exact sequence of B-modules (the Koszul complex):

$$\begin{aligned}
0 \to \left(\prod_{j\in I'-\{i_0\}} x_j\right) B_1 \longrightarrow \cdots \longrightarrow \bigoplus_{i,j\in I'-\{i_0\}, i<j} x_i x_j B_1\\
\longrightarrow \bigoplus_{j\in I'-\{i_0\}} x_j B_1 \longrightarrow B_1 \longrightarrow B \to 0.
\end{aligned}$$

Note that

- The set $I' - \{i_0\}$ is the disjoint union of $I' \cap J$ and $I' \cap K_1$.
- The set J' is the disjoint union of I' and $J' - I'$.

For any $\mathfrak{m} = (m_i)_{i\in I'-\{i_0\}}$, where $m_i \in \mathbb{Z}_{\geq 0}$ if $i \in I' \cap J$ and $m_i \in \mathbb{Z}$ if $I' \cap K_1$, we define

$\gamma(\mathfrak{m}) \in M_{2,\sigma_{J'}}$ as follows:

$$\gamma(\mathfrak{m}) := \left\lceil \frac{c_{n+1}}{\alpha_{i_0}} - \frac{1}{\alpha_{i_0}}\Bigg(\sum_{i\in I'\cap J} \alpha_i(c_i+m_i) + \sum_{i\in I'\cap K_1} \alpha_i m_i \Bigg) \right\rceil b^*_{i_0} + \sum_{i\in I'\cap J}(c_i+m_i)b^*_i + \sum_{i\in I'\cap K_1} m_i b^*_i + \sum_{i\in J'-I'} c_i b^*_i.$$

Define

$$\Gamma := \{\gamma(\mathfrak{m}) \mid m_i \in \mathbb{Z}_{\geq 0} \text{ if } i\in I'\cap J;\ m_i\in\mathbb{Z} \text{ if } i \in I'\cap K_1\} \subset M_{2,\sigma_{J'}}.$$

For any $\chi = \sum_{j\in J'} k_j b^*_j \in M_{2,\sigma_{J'}}$, denote the monomial

$$f_\chi = \prod_{j\in I'} x_j^{k_j} \cdot \prod_{j\in J'-I'} y_j^{k_j}.$$

Then Q_2 is a free B-module generated by $\{f_\chi \mid \chi\in\Gamma\}$:

$$Q_2 = \bigoplus_{\chi\in\Gamma} f_\chi B.$$

Multiplying the exact sequence in (ii) by f_χ, and taking the direct sum over all f_χ for $\chi\in\Gamma$, one arrives at the following resolution of Q_2 by free B_2-modules:

$$0 \longrightarrow \bigoplus_{\chi\in\Gamma}\Bigg(\prod_{i\in I'-\{i_0\}} x_i\Bigg) f_\chi B_2 \longrightarrow \cdots \longrightarrow \bigoplus_{\chi\in\Gamma}\bigoplus_{i,j\in I'-\{i_0\},\ i<j} x_i x_j f_\chi B_2 \longrightarrow \bigoplus_{\chi\in\Gamma}\bigoplus_{i\in I'-\{i_0\}} x_i f_\chi B_2 \longrightarrow \bigoplus_{\chi\in\Gamma} f_\chi B_2 \longrightarrow Q_2 \longrightarrow 0.$$

4.4.6 Resolution by Theta Sheaves

Lemma 4.15 *The Fourier–Mukai transformed sheaf $F_{21}\Theta'(\sigma_J,\phi)$ admits the following resolution*

$$0 \to \bigoplus_{\chi\in\Gamma}\Theta'_2\Bigg(\sigma_{J'},\chi+\sum_{i\in I'-\{i_0\}} b^*_i\Bigg) \longrightarrow \cdots \longrightarrow \bigoplus_{\chi\in\Gamma}\bigoplus_{i,j\in I'-\{i_0\},\ i<j}\Theta'_2(\sigma_{J'},\chi+b^*_i+b^*_j) \longrightarrow \bigoplus_{\chi\in\Gamma}\bigoplus_{i\in I'-\{i_0\}}\Theta'_2(\sigma_{J'},\chi+b^*_i) \longrightarrow \bigoplus_{\chi\in\Gamma}\Theta'_2(\sigma_{J'},\chi) \longrightarrow F'_{21}\Theta'_1(\sigma_J,\phi) \to 0.$$

Taking the coherent-constructible correspondence functor κ to the resolution, we obtain a chain complex of constructible theta sheaves on $M_\mathbb{R}$

$$0 \to \bigoplus_{\chi\in\Gamma}\Theta_2\Bigg(\sigma_{J'},\chi+\sum_{i\in I'-\{i_0\}} b^*_i\Bigg) \longrightarrow \cdots \longrightarrow \bigoplus_{\chi\in\Gamma}\bigoplus_{i,j\in I'-\{i_0\},\ i<j}\Theta_2(\sigma_{J'},\chi+b^*_i+b^*_j) \longrightarrow \bigoplus_{\chi\in\Gamma}\bigoplus_{i\in I'-\{i_0\}}\Theta_2(\sigma_{J'},\chi+b^*_i) \longrightarrow \bigoplus_{\chi\in\Gamma}\Theta_2(\sigma_{J'},\chi) \longrightarrow .$$

Although this complex is not finitely-generated by Θ_2-sheaves on $M_\mathbb{R}$ since it involves countably-many direct sums, it is a constructible sheaf on $M_\mathbb{R}$. Thus we have obtained a functor denoted by $F_{21} : \langle\Theta_1\rangle \to Sh_c(M_\mathbb{R};\Lambda_{\mathbf{\Sigma}_2})$.

For the given σ_J and $\phi\in M_{1,\sigma_J}$, define the "Fourier–Mukai transformed set"

$$F(\sigma_J)^\vee_\phi = \bigcup_{\chi\in\Gamma}(\sigma_{J'})^\vee_\chi,$$

while similarly we denote $F((\sigma_J)^\vee_\chi)^\circ$ to be the interior of the above set. (In case that $n+1\notin J$, we simply set $F(\sigma_J)^\vee_\phi=(\sigma_J)^\vee_\phi$.) We have the following proposition characterizing $F_{21}(\Theta_1(\sigma_J,\phi))$.

Proposition 4.16

$$F_{21}(\Theta_1(\sigma_J,\phi))\cong i_!\omega_{F((\sigma_J)^\vee_\chi)^\circ},$$

where $i:((\sigma_J)^\vee_\chi)^\circ\hookrightarrow M_\mathbb{R}$ *is the embedding of the open subset, and* $\omega_{F((\sigma_J)^\vee_\chi)^\circ}$ *is the costandard constructible sheaf on this set.*

Proof In order to prove the resolution of $F_{21}(\Theta_1(\sigma_J,\phi))$ is quasi-isomorphic to $\omega_{F((\sigma_J)^\vee_\chi)^\circ}$, we only need to show they are quasi-isomorphic at every stalk $p\in M_\mathbb{R}$. If $p\notin F((\sigma_J)^\vee_\chi)^\circ$ the stalk of the costandard sheaf $(i_!\omega_{F((\sigma_J)^\vee_\chi)^\circ})_p=0$, while the stalk $(F_{21}(\Theta_1(\sigma_J,\phi)))_p$ is also a zero complex. It remains to show that when $p\in F((\sigma_J)^\vee_\chi)^\circ$, the stalk $(F_{21}(\Theta_1(\sigma_J,\phi)))_p$ is quasi-isomorphic to $(i_!\omega_{F((\sigma_J)^\vee_\chi)^\circ})_p\cong\mathbb{C}[0]$.

Let $p=\sum_{i=1}^n p_ib_i^*\in M_\mathbb{R}$, and $\Gamma(p)=\{\chi\in\Gamma\mid p\in((\sigma_{J'})^\vee_\chi)^\circ\}$. Set $m_i^0=\lceil p_i\rceil-1-c_i$ for $i\in I'\cap J$, and $m_i^0=\lceil p_i\rceil-1$ if $i\in I'\cap K_1$. The character $\gamma^0:=\gamma(m_1^0,\dots,m_{i_0-1}^0,m_{i_0+1}^0,\dots,m_{n'}^0)$ is the unique element in $\Gamma(p)$ such that $\gamma^0-b_i^*\notin\Gamma(p)$, for any $i\in I'-\{i_0\}$. For any $\chi\in\Gamma(p)-\{\gamma^0\}$, denote

$$I'_\chi=\{i\in I'-\{i_0\}\mid\chi-b_i^*\in\Gamma(p)).$$

With these notations, the last two terms of the stalk $(F_{21}(\Theta_1(\sigma_J,\phi)))_p$ is

$$\longrightarrow\bigoplus_{\chi\in\Gamma(p)-\{\gamma^0\}}\bigoplus_{i\in I'_\chi}\mathbb{C}_\chi\longrightarrow\bigoplus_{\chi\in\Gamma(p)}\mathbb{C}_\chi\longrightarrow,$$

where $\mathbb{C}_\chi\cong\mathbb{C}$ is indexed by the character χ. The image of the middle arrow is $\bigoplus_{\chi\in\Gamma(p)-\{\gamma^0\}}\mathbb{C}_\chi$. Thus one defines a chain map $q:(F_{21}(\Theta_1(\sigma_J,\phi)))_p\to\mathbb{C}[0]$ where

$$q_0:\bigoplus_{\chi\in\Gamma(p)}\mathbb{C}_\chi\to\mathbb{C}$$
$$(k_\chi)_{\chi\in\Gamma(p)}\mapsto\sum_{\chi\in\Gamma(p)}k_\chi,$$

and $q_j=0$ for $j\neq0$. This map induces the cohomology map $H^*(q)=id:\mathbb{C}[0]\to\mathbb{C}[0]$, and it is a quasi-isomorphism. □

Remark 4.17 Although the definition of $F(\sigma_J)^\vee_\phi$ relies on the choice of some $i_0\in I'-J$, the above proposition shows that it is the support of the cohomology sheaf of $F_{21}(\Theta_1(\sigma_J,\phi))$, which is independent of the choice of i_0.

4.4.7 Full and Faithful Functor

Let $\sigma_{J_1},\sigma_{J_2}\in\Sigma_1$, $\phi_1\in M_{1,\sigma_{J_1}}$ and $\phi_2\in M_{1,\sigma_{J_2}}$, where $J_1,J_2\in\Lambda=\{J\subset\bar I\mid I'\not\subset J\}$. Define

$$c_{1,i}=\begin{cases}\langle\phi_1,b_i\rangle_{\sigma_{J_1}}, & i\subset J_1,\\ -\infty, & \text{otherwise};\end{cases}\qquad c_{2,i}=\begin{cases}\langle\phi_2,b_i\rangle_{\sigma_{J_2}}, & i\subset J_2,\\ -\infty, & \text{otherwise}.\end{cases}$$

Define the polyhedral set $C(t_1,\dots,t_{n+1})\subset M_\mathbb{R}$ to be

$$C(t_1,\dots,t_{n+1}):=\{x\in M_\mathbb{R}\mid\langle x,b_i\rangle>t_i\}.$$

In the remainder of this subsection, we let $\mathbf{c}_1$ and $\mathbf{c}_2$ denote $(c_{1,1},\dots,c_{1,n})$ and $(c_{2,1},\dots,c_{2,n})$, respectively, and write $\mathbf{t}$ for $(t_1,\dots,t_n)$.

It is obvious that $C(\mathbf{c}_1,c_{1,n+1})=((\sigma_{J_1})^\vee_{\phi_1})^\circ$, and $C(\mathbf{c}_2,c_{2,n+1})=((\sigma_{J_2})^\vee_{\phi_2})^\circ$. Furthermore, define $D(\mathbf{t},t_{n+1})$ to be

$$D(\mathbf{t},t_{n+1}):=\{x\in M_{\mathbb{R}}\mid \langle x,b_i\rangle>t_i,\ i=1,\dots,n;\langle x,b_{n+1}\rangle\geq t_{n+1}\}.$$

Lemma 4.18 *If $n+1\in J_1$, for any $\phi_1\in M_{1,\sigma_J}$ there is an $s_1\in[c_{1,n+1},c_{1,n+1}+1)$ such that $D(\mathbf{c}_1,s_1)\subset F((\sigma_{J_1})^\vee_{\phi_1})^\circ$. The same result holds for J_2 as well.*

Proof Let $x=x_1b_1^*+\cdots+x_{n+1}b_n^*\in D(\mathbf{c}_1,s_1)$, for some $s_1=c_{1,n+1}+\epsilon$ where $\epsilon>0$ will be determined below. Recall that in the definition of $F((\sigma_{J_1})^\vee_{\phi_1})^\circ$, we have chosen an $i_0\in I'-J_1$. Without loss of generality, in this proof we assume $i_0=1$. Set

$$m_i=\begin{cases}\lceil x_i\rceil-1-c_{1,i}, & i\in J_1\cap I',\\ \lceil x_i\rceil-1, & i\in I'-(J_1\cup\{1\}),\end{cases}$$

and $\mathfrak{m}=(m_2,\dots,m_{n'})\in\mathbb{Z}^{n'-1}$. It suffices to show that $x\in((\sigma_{J_1'})^\vee_{\gamma(\mathfrak{m})})^\circ$ after we specify a particular $\epsilon\in(0,1)$ (which depends on ϕ_1 but not on x). The coordinate x_1 satisfies

$$\begin{aligned}x_1&\geq\frac{1}{\alpha_1}\bigg(s_1-\sum_{i=2}^{n'}\alpha_ix_i\bigg)\\&\geq\frac{1}{\alpha_1}\bigg(s_1-\sum_{i\in J_1\cap I'}(m_i+c_{1,i}+1)\alpha_i-\sum_{i\in I'-(J_1\cup\{1\})}(m_i+1)\alpha_i\bigg)\\&\geq\frac{1}{\alpha_1}\bigg(c_{1,n+1}+(\epsilon-1)+1-\sum_{i\in J_1\cap I'}(m_i+c_{1,i}+1)\alpha_i-\sum_{i\in I'-(J_1\cup\{1\})}(m_i+1)\alpha_i\bigg)\\&\geq\frac{1}{\alpha_1}\bigg(c_{1,n+1}+(\epsilon-1)+\alpha_1-\sum_{i\in J_1\cap I'}(m_i+c_{1,i})\alpha_i-\sum_{i\in I'-(J_1\cup\{1\})}m_i\alpha_i\bigg).\end{aligned}$$

The last inequality depends on the fact $\alpha_1+\cdots+\alpha_{n'}\leq 1$. For any $\mathfrak{m}'=(m_2',\dots,m_{n'}')\in\mathbb{Z}^{n'-1}$, define

$$u(\mathfrak{m}')=\frac{1}{\alpha_1}\bigg(c_{1,n+1}-\sum_{i\in J_1\cap I'}(m_i'+c_{1,i})\alpha_i-\sum_{i\in I'-(J_1\cup\{1\})}m_i'\alpha_i\bigg).$$

Since $\alpha_1,\dots,\alpha_{n'}$ are rational numbers,

$$A(\phi_1):=\{u(\mathfrak{m}')+1-\lceil u(\mathfrak{m}')\rceil\mid\mathfrak{m}'\in\mathbb{Z}^{n'-1}\}$$

is a finite subset of $(0,1]$. Define

$$\epsilon:=1-\frac{\alpha_1}{2}\min A(\phi_1)\in(0,1).$$

Then

$$x_1\geq u(\mathfrak{m})+1+\frac{\epsilon-1}{\alpha_1}\geq\lceil u(\mathfrak{m})\rceil+\frac{1}{2}\min A(\phi_1)>\lceil u(\mathfrak{m})\rceil,$$

which implies that $x\in((\sigma_{J_1'})^\vee_{\gamma(\mathfrak{m})})^\circ$. □

The lemma above implies the relation $D(\mathbf{c}_1,s_1)\subset F((\sigma_{J_1})^\vee_{\phi_1})^\circ\subset C(\mathbf{c}_1,c_{1,n+1})$. Moreover, given

$$x=x_1b_1^*+\cdots+x_nb_n^*\in F((\sigma_{J_1})^\vee_{\phi_1})^\circ-D(\mathbf{c}_1,s_1)$$

and any $l \in I'$, there is a unique

$$r_{J_1,\phi_1,l}(x) = x_1 b_1^* + \cdots + x_{l-1} b_{l-1}^* + \hat{x}_l b_l^* + x_{l+1} b_{l+1}^* + \cdots + x_n b_n^*$$

such that $\langle r_{J_1,\phi_1,l}(x), b_{n+1} \rangle = s_1$, where $\hat{x}_l \geq x_l$. Meanwhile, given

$$x = x_1 b_1^* + \cdots + x_n b_n^* \in C(\mathbf{c}_1, c_{1,n+1}) - F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ}$$

and any $l \in I'$, there is also a unique

$$r'_{J_1,\phi_1,l} = x_1 b_1^* + \cdots + x_{l-1} b_{l-1}^* + \hat{x}'_l b_l^* + x_{l+1} b_{l+1}^* + \cdots + x_n b_n^*$$

such that $\langle r'_{J_1,\phi_1,l}(x), b_{n+1} \rangle = c_{1,n+1}$, where $\hat{x}'_l \leq x_l$.

Proposition 4.19 *Let* J_1 *and* J_2 *be two proper subsets of* $\bar{I}$ *such that* $n+1 \in J_1, J_2$. *If* $((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} \not\subset ((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}$, *then* $F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} \not\subset F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}$, *and* $F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}$ *is a contractible set.*

Proof Since $((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} \not\subset ((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}$, we have some $c_{1,l} < c_{2,l}$ for some l. Recall that in the definition of $F(\sigma_J)_{\chi}^{\vee}$ we have chosen an $i_0 \in I' - J$. Here we fix $i_{1,0} \in I' - J_1$ and $i_{2,0} \in I' - J_2$. We prove the statement in the following two cases.

Case $l = n+1$ Let $s_1 \in [c_{1,n+1}, c_{1,n+1}+1)$ be as given in Lemma 4.18, so that $D(\mathbf{c}_1, s_1) \subset F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ}$. By definition, $F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ} \subset C(\mathbf{c}_2, c_{2,n+1})$. Therefore, the non-empty set

$$D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1}) \subset F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}.$$

We will show that there is a deformation retract

$$h_t : (F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}) \times [0,1] \to F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ},$$

such that $h_0 = id$ and the image of h_1 is inside $D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1})$, while h_1 is the identity map on $D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1})$. Given $x = x_1 b_1^* + \cdots + x_n b_n^* \in M_{\mathbb{R}}$, the retract h_t is defined as

$$h_t(x) = \begin{cases} tx + (1-t) r_{J_1,\phi_1,i_{2,0}}(x), & \text{if } x \in F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - (D(\mathbf{c}_1, s_1) \cup F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}), \\ x, & \text{if } x \in D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1}), \\ tx + (1-t) r'_{J_2,\phi_2,i_{1,0}}(x), & \text{if } x \in (C(\mathbf{c}_2, c_{2,n+1}) - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}) \cap F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ}. \end{cases}$$

Since the closures of $D(\mathbf{c}_1, s_1)$ and $C(\mathbf{c}_2, c_{2,n+1})$ are dual cones of toric cones in a fan, $D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1})$ is contractible. Hence we conclude that $F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}$ is contractible.

Case $l \neq n+1$ In this case, one may assume that $c_{1,n+1} \geq c_{2,n+1}$ since otherwise we might let l to be $n+1$ and goes back the the previous case. Similarly, we define a deformation retract

$$h_t : (F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}) \times [0,1] \to F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ},$$

given as below:

$$h_t(x) = \begin{cases} tx + (1-t) r_{J_1,\phi_1,i_{2,0}}(x), & \text{if } x \in F((\sigma_{J_1})_{\phi_1}^{\vee})^{\circ} - (D(\mathbf{c}_1, s_1) \cup F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}); \\ x, & \text{if } x \in D(\mathbf{c}_1, s_1) - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ}. \end{cases}$$

Since $C(\mathbf{c}_2, s_2) \subset F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ} \subset C(\mathbf{c}_2, c_{2,n+1})$, we have

$$D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1}) \subset D(\mathbf{c}_1, s_1) - F((\sigma_{J_2})_{\phi_2}^{\vee})^{\circ} \subset D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, s_1).$$

The fact that $c_{1,n+1} \geq c_{2,n+1}$ implies $s_1 \geq s_2 \geq c_{2,n+1}$. Therefore

$$D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1}) = D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, s_2),$$

and then

$$D(\mathbf{c}_1, s_1) - F((\sigma_{J_2})^\vee_{\phi_2})^\circ = D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1})$$

is a non-empty contractible set. The above deformation retract shows that $F((\sigma_{J_1})^\vee_{\phi_1})^\circ - F((\sigma_{J_2})^\vee_{\phi_2})^\circ$ is also a contractible set. □

Proposition 4.20 *If* $n+1 \notin J_1$ *or* $n+1 \notin J_2$, *then*

$$((\sigma_{J_1})^\vee_{\phi_1})^\circ \not\subset ((\sigma_{J_2})^\vee_{\phi_2})^\circ \implies F((\sigma_{J_1})^\vee_{\phi_1})^\circ - F((\sigma_{J_2})^\vee_{\phi_2})^\circ \textit{ is a non-empty contractible set.}$$

Proof The proof is similar to Proposition 4.19. There are three cases.

Case $n+1 \notin J_1$ and $n+1 \in J_2$ Notice that $F((\sigma_{J_1})^\vee_{\phi_1})^\circ = C(\mathbf{c}_1, c_{1,n+1})$. Let $i_{1,0} \in I' - J_1$, since $I' \not\subset J_1$. Define the deformation retract between $F((\sigma_{J_1})^\vee_{\phi_1})^\circ - F((\sigma_{J_2})^\vee_{\phi_2})^\circ$ and $C(\mathbf{c}_1, c_{1,n+1}) - C(\mathbf{c}_2, c_{2,n+1})$ as

$$h_t(x) = \begin{cases} x, & \text{if } x \in C(\mathbf{c}_1, c_{1,n+1}) - C(\mathbf{c}_2, c_{2,n+1}), \\ tx + (1-t)r'_{J_2,\phi_2,i_{1,0}}(x), & \text{if } x \in (C(\mathbf{c}_2, c_{2,n+1}) - F((\sigma_{J_2})^\vee_{\phi_2})^\circ) \cap C(\mathbf{c}_1, c_{1,n+1}). \end{cases}$$

Case $n+1 \in J_1$ and $n+1 \notin J_2$ $F((\sigma_{J_2})^\vee_{\phi_2})^\circ = C(\mathbf{c}_2, c_{2,n+1})$. Let $i_{2,0} \in I' - J_2$. Define the deformation retract between $F((\sigma_{J_1})^\vee_{\phi_1})^\circ - F((\sigma_{J_2})^\vee_{\phi_2})^\circ$ and $D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1})$ as

$$h_t(x) = \begin{cases} tx + (1-t)r_{J_1,\phi_1,i_{2,0}}(x), & \text{if } x \in F((\sigma_{J_1})^\vee_{\phi_1})^\circ - (D(\mathbf{c}_1, s_1) \cup C(\mathbf{c}_2, c_{2,n+1})), \\ x, & \text{if } x \in D(\mathbf{c}_1, s_1) - C(\mathbf{c}_2, c_{2,n+1}). \end{cases}$$

Case $n+1 \notin J_1$ and $n+1 \notin J_2$ This is trivial since $F((\sigma_{J_2})^\vee_{\phi_2})^\circ = C(\mathbf{c}_2, c_{2,n+1})$ and $F((\sigma_{J_1})^\vee_{\phi_1})^\circ = C(\mathbf{c}_1, c_{1,n+1})$. □

Theorem 4.21 *The functors* F'_{21} *and* F_{21} *are quasi-embeddings. If restricted on* $\mathrm{Coh}_T(\mathcal{X}_1)$, F'_{21} *is a quasi-embedding of* $\mathrm{Coh}_T(\mathcal{X}_1)$ *into* $\mathrm{Coh}_T(\mathcal{X}_2)$.

Proof One only needs to show that F_{21} is a cohomologically full and faithful functor. Since we have

$$\mathrm{Ext}^*(\Theta_1(\sigma_{J_1}, \phi_1), \Theta_1(\sigma_{J_2}, \phi_2)) = \begin{cases} \mathbb{C}[0] & \text{if } ((\sigma_{J_1})^\vee_{\phi_1})^\circ \subset ((\sigma_{J_2})^\vee_{\phi_2})^\circ, \\ 0 & \text{if } ((\sigma_{J_1})^\vee_{\phi_1})^\circ \not\subset ((\sigma_{J_2})^\vee_{\phi_2})^\circ, \end{cases}$$

and

$$\begin{aligned} &\mathrm{Ext}^*(i_{F((\sigma_{J_1})^\vee_{\phi_1})^\circ !}F((\sigma_{J_1})^\vee_{\phi_1})^\circ, i_{F((\sigma_{J_2})^\vee_{\phi_2})^\circ !}F((\sigma_{J_2})^\vee_{\phi_2})^\circ \\ &\quad = \begin{cases} \mathbb{C}[0] & \text{if } F((\sigma_{J_1})^\vee_{\phi_1})^\circ \subset F((\sigma_{J_2})^\vee_{\phi_2})^\circ, \\ 0 & F((\sigma_{J_1})^\vee_{\phi_1})^\circ \not\subset F((\sigma_{J_2})^\vee_{\phi_2})^\circ \text{ and } F((\sigma_{J_1})^\vee_{\phi_1})^\circ - F((\sigma_{J_2})^\vee_{\phi_2})^\circ \text{ is contractible,} \end{cases} \end{aligned}$$

the desired result follows immediately from Proposition 4.16, Proposition 4.19 and Proposition 4.20, and the simple fact that

$$((\sigma_{J_1})^\vee_{\phi_1})^\circ \subset ((\sigma_{J_2})^\vee_{\phi_2})^\circ \implies F((\sigma_{J_1})^\vee_{\phi_1})^\circ \subset F((\sigma_{J_2})^\vee_{\phi_2})^\circ.$$

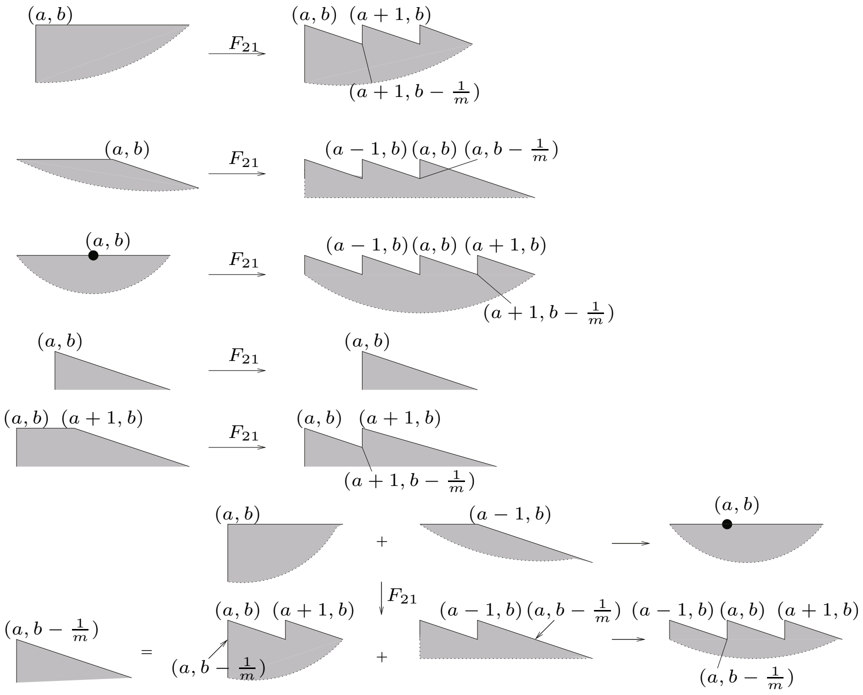


Figure 11 a and b are integers. The constructible sheaves are costandard sheaves over the shaded regions

Example 4.22 $N=\mathbb{Z}^2$,

$$\beta_1=\begin{bmatrix}1 & -1 & 0\\ 0 & -m & -1\end{bmatrix},\quad \beta_2=\begin{bmatrix}1 & -1\\ 0 & -m\end{bmatrix},\quad \beta'=\begin{bmatrix}1 & -1 & 0\\ 0 & -m & -m\end{bmatrix}.$$

$\mathcal{X}_1$ is the total space of $\mathcal{O}_{\mathbb{P}^1}(-m)$, and $\mathcal{X}_2=[\mathbb{C}^2/\mathbb{Z}_m]$.

$$v_1=b_1=(1,0),\quad v_2=b_2=(-1,-m),\quad v_3=b_3=(0,-1),$$

$$r_1=r_2=r_3=1,\quad r_3'=m,\quad a_1=a_2=\alpha_1=\alpha_2=\frac{1}{m},\quad a_1'=a_2'=\beta_1=\beta_2=1,$$

$$\frac{a_1}{r_1}+\frac{a_2}{r_2}\leq\frac{1}{r_3}\Leftrightarrow m\geq 2.$$

The Fourier–Mukai functors for constructible sheaves are shown in Figure 11.

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