B. Fang et al. (2014) "Coherent–Constructible Correspondence for Toric DM Stacks,"
International Mathematics Research Notices, Vol. 2014, No. 4, pp. 914–954
Advance Access Publication November 2, 2012
doi:10.1093/imrn/rns235

The Coherent–Constructible Correspondence for Toric Deligne–Mumford Stacks

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We extend our previous work [8] on coherent–constructible correspondence for toric varieties to toric Deligne–Mumford (DM) stacks. Following Borisov et al. [3], a toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}}$ is described by a "stacky fan" $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$, where N is a finitely generated abelian group and Σ is a simplicial fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. From $\boldsymbol{\Sigma}$, we define a conical Lagrangian $\Lambda_{\boldsymbol{\Sigma}}$ inside the cotangent $T^*M_{\mathbb{R}}$ of the dual vector space $M_{\mathbb{R}}$ of $N_{\mathbb{R}}$, such that torus-equivariant, coherent sheaves on $\mathcal{X}_{\boldsymbol{\Sigma}}$ are equivalent to constructible sheaves on $M_{\mathbb{R}}$ with singular support in $\Lambda_{\boldsymbol{\Sigma}}$. The microlocalization theorem of Nadler and the last author [18, 19] provides an algebro-geometrical description of the Fukaya category of a cotangent bundle $T^*M_{\mathbb{R}}$ in terms of constructible sheaves on the base $M_{\mathbb{R}}$. This allows us to interpret the main theorem stated earlier as an equivariant version of homological mirror symmetry for toric DM stacks.

1 Introduction

In [8], the familiar assignment in toric geometry that associates polytopes with line bundles on toric varieties was extended to an equivalence of categories between equivariant coherent sheaves on a toric n-fold and constructible sheaves on $\mathbb{R}^n$. The equivalence,

Received April 22, 2012; Revised September 21, 2012; Accepted September 26, 2012

the coherent–constructible correspondence (CCC), is therefore a "categorification" of Morelli's description of the K-theory of toric varieties in terms of a polytope algebra [17]. In this paper, we extend CCC to toric Deligne–Mumford (DM) stacks.

The aforementioned main result has an application to homological mirror symmetry (HMS). The microlocalization theorem of Nadler and the last author [18, 19] provides an algebro-geometric description of the Fukaya category of a cotangent bundle $T^*\mathbb{R}^n$ in terms of constructible sheaves on the base $\mathbb{R}^n$. This allows us to interpret the main theorem as an equivariant version of HMS for toric DM stacks.

In Section 1.1, we recall for the reader the CCC for toric varieties. We state our main result (CCC for toric DM stacks) and its corollary (equivariant HMS for toric DM stacks) in Sections 1.2 and 1.3, respectively. We illustrate our main result in the simple example of a weighted projective plane in Section 1.4. An outline of the paper follows.

In this paper, Σ is always a *finite* fan, that is, Σ consists of finitely many cones. Therefore all the toric varieties and toric DM stacks in this paper are *of finite type*.

1.1 CCC for toric varieties

Let $N \cong \mathbb{Z}^n$ be a rank n lattice, and let Σ be a complete fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let X_Σ be the toric variety defined by Σ, and let $T \cong (\mathbb{C}^*)^n$ be the torus acting on X_Σ. In [8], we defined a conical Lagrangian Λ_Σ in the cotangent bundle $T^*M_{\mathbb{R}}$ of $M_{\mathbb{R}}$, where $M_{\mathbb{R}}$ is the dual real vector space of $N_{\mathbb{R}}$, and established a quasi-equivalence of triangulated dg-categories (The results in [8] are valid over any commutative Noetherian base ring. The nonequivariant CCC has been studied by Bondal [2], and more recently by the third author [22].):

$$\kappa : \mathcal{P}\mathrm{erf}_T(X_\Sigma) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_\Sigma) \tag{1}$$

where $\mathcal{P}\mathrm{erf}_T(X_\Sigma)$ is the category of T-equivariant perfect complexes on X_Σ, and $\mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_\Sigma)$ is the category of complexes of sheaves on $M_{\mathbb{R}}$ with bounded, constructible, compactly supported cohomology, with singular support in Λ_Σ. Moreover, Equation (1) is a quasi-equivalence of *monoidal* dg categories, with respect to the tensor product on $\mathcal{P}\mathrm{erf}_T(X_\Sigma)$ and the convolution product on $\mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_\Sigma)$.

When X_Σ is a smooth complete toric variety (so that X_Σ can be viewed as a compact complex manifold), taking the cohomology of (1) yields an equivalences of tensor triangulated categories:

$$H(\kappa) : D_T(X_\Sigma) \xrightarrow{\sim} D_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_\Sigma), \tag{2}$$

where $D_T(X_\Sigma) = DCoh_T(X_\Sigma)$ is the bounded derived category of T-equivariant coherent sheaves on X_Σ, and $D_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_\Sigma) = DSh_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_\Sigma)$ is the derived category of $\mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_\Sigma)$.

1.2 CCC for toric DM stacks

Now let N be a finitely generated abelian group, and let Σ be a complete *simplicial* fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan in the sense of Borisov et al. [3]. Then $\boldsymbol{\Sigma}$ defines a complete toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}}$, and the complete toric variety X_Σ is the coarse moduli space of $\mathcal{X}_{\boldsymbol{\Sigma}}$. A toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}}$ is a toric orbifold (i.e., a toric DM stack with trivial generic stabilizer) if N is torsion-free. Given any stacky fan $\boldsymbol{\Sigma}$, there is a rigidification $\boldsymbol{\Sigma}^{\mathrm{rig}}$ which defines a toric orbifold $\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$, known as the rigidification of the toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}}$ (see Section 3.4).

A toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}}$ contains a DM torus $\mathcal{T} = T \times \mathcal{B}G$ as a dense open subset, where $T \cong (\mathbb{C}^*)^{\dim_{\mathbb{R}} N_{\mathbb{R}}}$ is a torus and G is the generic stabilizer. We define a conical Lagrangian $\Lambda_{\boldsymbol{\Sigma}}$ of $T^*M_{\mathbb{R}}$ (Λ_Σ is a subset of $\Lambda_{\boldsymbol{\Sigma}}$) and establish a quasi-equivalence of monoidal triangulated dg-categories

$$\kappa : \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}), \tag{3}$$

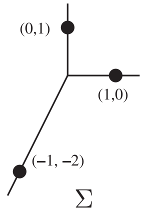
which is the stacky version of (1). We remark that (i) $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$ depends only on the rigidification $\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$ of $\mathcal{X}_{\boldsymbol{\Sigma}}$, which is not the case for the nonequivariant category $\mathcal{P}\mathrm{erf}(\mathcal{X}_{\boldsymbol{\Sigma}})$ (see Proposition 3.1 for precise statements), and (ii) $\Lambda_{\boldsymbol{\Sigma}} = \Lambda_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$. Taking cohomology of (3) yields an equivalence of tensor triangulated categories, exactly as in (2).

1.3 Application: equivariant HMS for toric DM stacks

By the microlocalization theorem of Nadler and the last author [18, 19], there is a quasi-equivalence of triangulated A_∞-categories:

$$\mu : \mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}) \xrightarrow{\sim} F(T^*M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}), \tag{4}$$

where $F = \mathrm{TrFuk}$ is the triangulated envelope of the Fukaya category. The objects in $\mathrm{Fuk}(T^*M_{\mathbb{R}}; \Lambda_\Sigma)$ are Lagrangian branes that are bounded in the $M_{\mathbb{R}}$ direction, with boundary at infinity contained in Λ_Σ. The quasi-equivalence (4) provides an algebro-geometrical description of the symplectic category $F(T^*M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$ in terms of



$$\beta\left(\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Fig. 1. Stacky fan for $\mathbb{P}(1, 1, 2)$.

constructible sheaves on the base $M_{\mathbb{R}}$. Combining our main result (3) with the microlocalization (4), we obtain a quasi-equivalence of triangulated A_∞-categories:

$$\tau := \mu \circ \tau : \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \xrightarrow{\sim} F(T^* M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}). \tag{5}$$

Taking cohomology of (5), we obtain an equivalence of triangulated categories

$$H(\tau) : D_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \xrightarrow{\sim} DF(T^* M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}), \tag{6}$$

which can be viewed as an equivariant version of HMS for toric DM stacks. Moreover, there is a product structure on $DF(T^* M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$ such that the equivalence (6) is tensorial (see Section 8 for details).

The traditional version for HMS of toric DM stacks relates the bounded derived category of coherent sheaves on a toric DM stack to the Fukaya–Seidel category of the Landau–Ginzburg mirror of the toric DM stack. This version has been proved for weighted projective planes by Auroux et al. [1], for toric orbifolds of toric del Pezzo surfaces by Ueda and Yamazaki [23], and more recently, for toric orbifolds of projective spaces (of any dimension) by Futaki and Ueda [12]. Of course, there are many other works on HMS for *nonstacky* toric varieties. See, for example, [9, Section 1.3] for a review of these results and further references.

1.4 Example: the weighted projective plane $\mathbb{P}(1, 1, 2)$

Let $N = \mathbb{Z}^2$. The weighted projective plane $\mathcal{X}_{\boldsymbol{\Sigma}} = \mathbb{P}(1, 1, 2)$ is defined from the stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ in Figure 1.

We first describe the derived category of $\mathbb{P}(1, 1, 2)$, following [1, Section 2]. Define a graded polynomial algebra $S = \mathbb{C}[x_0, x_1, x_2]$ graded by setting the degrees of x_0, x_1, x_2 to be $1, 1, 2$, respectively. The derived category of $\mathbb{P}(1, 1, 2)$ is the category $\mathrm{qgr}(S) = \mathrm{gr}(S)/\mathrm{tors}(S)$, where $\mathrm{gr}(S)$ is the category of finitely generated graded right

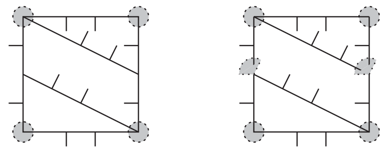


Fig. 2. Microlocal supports for constructible sheaves: $\mathbf{P}(1,1,2)$ (left) and $\mathbb{P}(1,1,2)$ (right).

S-modules and tors(S) is the full subcategory of gr(S) consisting of S-modules with finite dimensions over $\mathbb{C}$. Under this identification, the line bundle $\mathcal{O}(l)$ over $\mathbb{P}(1,1,2)$ is $\widetilde{S(l)}$. Here $S(l)$ is the module S in gr(S) shifted in l degrees, and passing to qgr(S), we denote it by $\widetilde{S(l)}$ (Figure 2).

Decompose $S=\oplus_{i=0}^{\infty}S_i$ via degrees. The weighted projective plane $\mathbb{P}(1,1,2)$, as a stack, has a full strong exceptional collection $\{\mathcal{O},\mathcal{O}(1),\mathcal{O}(2),\mathcal{O}(3)\}$, with the morphism

$$\mathrm{Ext}^i(\mathcal{O}(k),\mathcal{O}(l)) = \begin{cases} S_{l-k} & l\geq k, i=0,\\ 0 & \text{otherwise,}\end{cases}$$

and the obvious composition in the polynomial algebra S.

If we consider the singular toric variety (defined as GIT quotient) given by the fan Σ in Figure 1, we are dealing with the singular variety $X_\Sigma=\mathbf{P}(1,1,2)$, not the stack. The result of [8] says that the category of equivariant coherent sheaves on $\mathbf{P}(1,1,2)$ is quasi-equivalent to a subcategory of constructible sheaves on the plane $\mathbb{R}^2$. More precisely, the subcategory of constructible sheaves, denoted by $Sh_{cc}(\mathbb{R}^2,\Lambda_\Sigma)$, is generated by the pushing-forward with compact support of constant sheaves on the triangles with vertices (a,b), $(a,b+h)$, $(a+2h,b)$ with $a,b,h\in\mathbb{Z}$, $h\geq 0$. Comparing with the line bundles on the stack $\mathbb{P}(1,1,2)$, we are only recovering the (all possible equivariant versions of) sheaves $\mathcal{O},\mathcal{O}(2),\mathcal{O}(4),\ldots$. However, if we refine the lattice points, and allow h to take values in half integers, we have the whole derived category on $\mathbb{P}(1,1,2)$. The conical Lagrangians Λ_Σ and $\Lambda_{\boldsymbol{\Sigma}}$ are sketched in Figure 2.

1.5 Outline

In Section 2, we set the notation and describe the categories of sheaves we will employ. In Section 3, we review the definition of stacky fans and the construction of toric DM stacks, following [3, 10]. In Section 4, we describe the relation between equivariant line bundles and (twisted) polytopes. In Section 5, we construct the basic equivalence between equivariant $\mathcal{O}$-modules on toric charts and $\mathbb{C}$-modules on shifted dual cones. In Section 6, we give intrinsic characterizations of the categories introduced in Section 5. In Section 7, we derive our main result, the CCC, which is an equivalence between equivariant perfect complexes on the toric DM stack and compactly supported constructible sheaves on $M_{\mathbb{R}}$ with singular support in a conical Lagrangian determined by the stacky fan. In Section 8, we combine CCC and microlocalization to obtain equivariant HMS for toric DM stacks.

2 Notation and Conventions

2.1 Categories

We will use the language of dg and A_∞ categories throughout. All categories that appear will be $\mathbb{C}$-linear. If C is a dg or A_∞ category, then the graded $\mathbb{C}$-vector space $\hom(x, y)$ denotes the chain complex of homomorphisms between objects x and y of C. We will use $\mathrm{Hom}(x, y)$ to denote hom sets as $\mathbb{C}$-vector spaces in non-dg or non-A_∞ settings. We will regard the differentials in all chain complexes as having degree $+1$, that is, $d: K^i \to K^{i+1}$. If K is a chain complex (of vector spaces or sheaves, usually), then $h^i(K)$ will denote its ith cohomology object.

If C is a dg (or A_∞) category, then $\mathrm{Tr}(C)$ denotes the triangulated dg (or A_∞) category generated by C, and $D(C)$ denotes the cohomology category $H^0(\mathrm{Tr}(C))$.(Here is one construction of the triangulated envelope. The Yoneda embedding $\mathcal{Y}: C \to \mathrm{mod}(C)$ maps an object $L \in C$ to the A_∞ right C-module $\hom_C(-, L)$. The functor $\mathcal{Y}$ is a quasi-embedding of C into the triangulated category $\mathrm{mod}(C)$. Let $\mathrm{Tr}(C)$ denote the category of twisted complexes of representable modules in $\mathrm{mod}(C)$. Then $\mathrm{Tr}(C)$ is a triangulated envelope of C.) The triangulated category $H^0(\mathrm{Tr}(C))$ is sometimes called the *derived category* of C.

2.2 Constructible and microlocal geometry

We refer to [16] for the microlocal theory of sheaves. If X is a topological space, $\mathrm{Sh}(X)$ denotes the dg category of bounded chain complexes of sheaves of $\mathbb{C}$-vector spaces on X, localized with respect to acyclic complexes (see **[?]** for localizations of dg categories). If X is a real-analytic manifold, $\mathrm{Sh}_c(X)$ denotes the full subcategory of $\mathrm{Sh}(X)$ of objects whose cohomology sheaves are constructible with respect to a real-analytic stratification of X. Denote by $\mathrm{Sh}_{cc}(X) \subset \mathrm{Sh}_c(X)$ the full subcategory of objects that have compact support. We denote by $D_c(X)$ and $D_{cc}(X)$ the derived categories $D(\mathrm{Sh}_c(X))$ and $D(\mathrm{Sh}_{cc}(X))$, respectively.

The *standard constructible sheaf* on the submanifold $i_Y: Y \hookrightarrow X$ is defined as the push-forward of the constant sheaf on Y, that is, $i_{Y*}\mathbb{C}_Y$, as an object in $Sh_c(X)$. The Verdier duality functor $\mathcal{D}: \mathrm{Sh}_c^{\circ}(X) \to \mathrm{Sh}_c(X)$ takes $i_{Y*}\mathbb{C}_Y$ to the *costandard constructible sheaf* on X. We know $\mathcal{D}(i_{Y*}\mathbb{C}_Y) = i_{Y!}\mathcal{D}(\mathbb{C}_Y) = i_{Y!}\omega_Y$. Here $\omega_Y = \mathcal{D}(\mathbb{C}_Y) = \mathfrak{or}_Y[\dim Y]$, where $\mathfrak{or}_Y$ is the orientation sheaf of Y (with respect to the base ring $\mathbb{C}$).

We denote the singular support of a complex of sheaves F by $SS(F) \subset T^*X$. If X is a real-analytic manifold and $\Lambda \subset T^*X$ is an $\mathbb{R}_{\geq 0}$-invariant Lagrangian subvariety, then $\mathrm{Sh}_c(X;\Lambda)$ (resp. $\mathrm{Sh}_{cc}(X;\Lambda)$) denotes the full subcategory of $\mathrm{Sh}_c(X)$ (resp. $\mathrm{Sh}_{cc}(X)$) whose objects have singular support in Λ.

2.3 Coherent and quasicoherent sheaves

All schemes and stacks that appear will be over $\mathbb{C}$.

2.3.1 *Sheaves on a scheme*

If X is a scheme, then $\mathcal{Q}(X)^{\text{naive}}$ denotes the dg category of bounded complexes of quasicoherent sheaves on X and $\mathcal{Q}(X)$ denotes the localization of this category with respect to acyclic complexes. If G is an algebraic group acting on X, $\mathcal{Q}_G(X)^{\text{naive}}$ denotes the dg category of complexes of G-equivariant quasicoherent sheaves. We denote by $\mathcal{Q}_G(X)$ the localization of this category with respect to acyclic complexes. We use $\mathcal{P}\mathrm{erf}(X) \subset \mathcal{Q}(X)$ and $\mathcal{P}\mathrm{erf}_G(X) \subset \mathcal{Q}_G(X)$ to denote the full dg subcategories consisting of *perfect* objects—that is, objects that are quasi-isomorphic to bounded complexes of vector bundles. If $u: X \to Y$ is a morphism of schemes, we have natural dg functors $u_*: \mathcal{Q}(X) \to \mathcal{Q}(Y)$ and $u^*: \mathcal{Q}(Y) \to \mathcal{Q}(X)$. Note that the functor u^* carries $\mathcal{P}\mathrm{erf}(Y)$ to $\mathcal{P}\mathrm{erf}(X)$. Suppose G and H are algebraic groups, X is a scheme with a G-action, and Y is a scheme with an H-action. If a morphism $u: X \to Y$ is equivariant with respect to a homomorphism of groups $\phi: G \to H$, then we will often abuse notation and write u_* and u^* for the equivariant pushforward and pullback functors $u_*: \mathcal{Q}_G(X) \to \mathcal{Q}_H(Y)$ and $u^*: \mathcal{Q}_H(Y) \to \mathcal{Q}_G(X)$.

2.3.2 *Sheaves on a DM stack*

We refer to [25, Definition 7.18] for the definitions of quasicoherent sheaves, coherent sheaves, and vector bundles on a DM stack. If $\mathcal{X}$ is a DM stack, then we denote by $\mathcal{Q}(\mathcal{X})^{\text{naive}}$ the dg category of bounded complexes of quasicoherent sheaves on $\mathcal{X}$ and by $\mathcal{Q}(\mathcal{X})$ the localization of this category with respect to acyclic complexes. By

$\mathcal{P}\mathrm{erf}(\mathcal{X}) \subset \mathcal{Q}(\mathcal{X})$, we denote the full dg subcategories consisting of *perfect* objects—that is, objects that are quasi-isomorphic to bounded complexes of vector bundles.

2.3.3 *Sheaves on a global quotient*

We now spell out the aforementioned definitions when $\mathcal{X}$ is a global quotient. Let G be an algebraic group acting on a scheme U such that the stabilizers of the geometric points of U are finite and reduced. By Vistoli [25, Example 7.17], the quotient stack $\mathcal{X} = [U/G]$ is a DM stack. By Vistoli [25, Example 7.21], the category of coherent sheaves on $\mathcal{X}$ is equivalent to the category of G-equivariant coherent sheaves on U. Similarly, the category of quasicoherent sheaves on $\mathcal{X}$ is equivalent to the category of G-equivariant quasicoherent sheaves on U. Therefore,

$$\mathcal{Q}(\mathcal{X})^{\mathrm{naive}} = \mathcal{Q}_G(U)^{\mathrm{naive}}, \quad \mathcal{Q}(\mathcal{X}) = \mathcal{Q}_G(U), \quad \mathcal{P}\mathrm{erf}(\mathcal{X}) = \mathcal{P}\mathrm{erf}_G(U).$$

Now suppose that, in addition, G is an *abelian* group, and there is an abelian group $\tilde{H}$ acting on U such that the G-action on U factors through a group homomorphism $\phi : G \to \tilde{H}$. Then $\mathcal{H} = [\tilde{H}/G]$ is a Picard stack acting on $\mathcal{X} = [U/G]$ in the sense of [10]. We define the category of $\mathcal{H}$-equivariant coherent (resp. quasicoherent) sheaves on $\mathcal{X}$ to be the category of $\tilde{H}$-equivariant coherent (resp. quasicoherent) sheaves on U:

$$\mathcal{Q}_{\mathcal{H}}(\mathcal{X}) = \mathcal{Q}_{\tilde{H}}(U), \quad \mathcal{P}\mathrm{erf}_{\mathcal{H}}(\mathcal{X}) = \mathcal{P}\mathrm{erf}_{\tilde{H}}(U).$$

3 Preliminaries on Toric DM Stacks

In [3], Borisov, Chen, and Smith defined toric DM stacks in terms of stacky fans. Toric DM stacks are smooth DM stacks and their coarse moduli spaces are simplicial toric varieties. A toric DM stack is called a *reduced toric DM stack* (in [3]) or a *toric orbifold* (in [10]) if its generic stabilizer is trivial. Later, more geometric definitions of toric orbifolds and toric DM stacks are given by Iwanari [13, 14] and by Fantechi et al. [10], respectively.

3.1 Stacky fans

In this subsection, we recall the definition of stacky fans. Let N be a finitely generated abelian group, and let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We have a short exact sequence of abelian groups:

$$1 \to N_{\mathrm{tor}} \to N \to \bar{N} = N/N_{\mathrm{tor}} \to 1,$$

where N_{tor} is the subgroup of torsion elements in N. Then N_{tor} is a finite abelian group, and $\bar{N} \cong \mathbb{Z}^n$, where $n = \dim_{\mathbb{R}} N_{\mathbb{R}}$. The natural projection $N \to \bar{N}$ is denoted by $b \mapsto \bar{b}$.

Let Σ be a simplicial fan in $N_{\mathbb{R}}$ (see [11]) and let $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ be the set of one-dimensional cones in the fan Σ. We assume that $\rho_1, \ldots, \rho_r$ span $N_{\mathbb{R}}$, and fix $b_i \in N$ such that $\rho_i = \mathbb{R}_{\geq 0}\bar{b}_i$. Define a rank r lattice $\tilde{N} := \bigoplus_{i=1}^{r} \mathbb{Z}\tilde{b}_i \cong \mathbb{Z}^r$, with $\{\tilde{b}_i\}$ as a standard integral basis in $\tilde{N}$. A *stacky fan* $\boldsymbol{\Sigma}$ is defined as the data (N, Σ, β), where $\beta : \tilde{N} \to N$ is a group homomorphism defined by $\tilde{b}_i \mapsto b_i$. By assumption, the cokernel of β is finite.

We introduce some notation.

(1) $M = \operatorname{Hom}(N, \mathbb{Z}) = \operatorname{Hom}(\bar{N}, \mathbb{Z}) \cong (\mathbb{Z}^n)^*$.

(2) $\tilde{M} = \operatorname{Hom}(\tilde{N}, \mathbb{Z}) \cong (\mathbb{Z}^r)^*$.

(3) Let $\Sigma(d)$ be the set of d-dimensional cones in Σ. Given $\sigma \in \Sigma(d)$, let $N_\sigma \subset N$ be the subgroup generated by $\{b_i \mid \rho_i \subset \sigma\}$, and let $\bar{N}_\sigma$ be the rank d sublattice of $\bar{N}$ generated by $\{\bar{b}_i \mid \rho_i \subset \sigma\}$. Let $M_\sigma = \operatorname{Hom}(\bar{N}_\sigma, \mathbb{Z})$ be the dual lattice of $\bar{N}_\sigma$.

Given $\sigma \in \Sigma(d)$, the surjective group homomorphism $N_\sigma \to \bar{N}_\sigma$ induces an injective group homomorphism $\operatorname{Hom}(\bar{N}_\sigma, \mathbb{Z}) \to \operatorname{Hom}(N_\sigma, \mathbb{Z})$, which is indeed an isomorphism. So $\operatorname{Hom}(N_\sigma, \mathbb{Z}) \cong M_\sigma \cong \mathbb{Z}^d$.

3.2 The Gale dual

The finite abelian group N_{tor} is of the form $\bigoplus_{j=1}^{l} \mathbb{Z}_{a_j}$. We choose a projective resolution of N:

$$0 \to \mathbb{Z}^l \xrightarrow{Q} \mathbb{Z}^{n+l} \to N \to 0.$$

Choose a map $B : \tilde{N} \to \mathbb{Z}^{n+l}$ lifting $\beta : \tilde{N} \to N$. Let $i_1 : \tilde{N} \to \tilde{N} \oplus \mathbb{Z}^l$ and $i_2 : \mathbb{Z}^l \to \tilde{N} \oplus \mathbb{Z}^l$ be injections of the first and second factors, respectively. We have the following commutative diagram:

$$\begin{tikzcd}
& & \tilde{N} \oplus \mathbb{Z}^l \arrow[d, "B \oplus Q"] & \tilde{N} \arrow[l, "i_1"'] \arrow[dl, "B"] \arrow[d, "\beta"] & \\
0 \arrow[r] & \mathbb{Z}^l \arrow[ur, "i_2"] \arrow[r, "Q"] & \mathbb{Z}^{n+l} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Define the dual group $\mathrm{DG}(\beta)$ to be the cokernel of $B^* \oplus Q^* : (\mathbb{Z}^{n+l})^* \longrightarrow \tilde{M} \oplus (\mathbb{Z}^l)^*$. The Gale dual of the map $\beta : \tilde{N} \to N$ is $\beta^\vee : \tilde{M} \to DG(\beta)$.

$$\begin{tikzcd}
 & 0 & \\
 & DG(\beta) \arrow[u] & \\
 & \tilde{M} \oplus (\mathbb{Z}^l)^* \arrow[u] \arrow[r, "i_1^*"] \arrow[dl, "i_2^*"'] & \tilde{M} \arrow[ul, "\beta^\vee"'] \\
\mathbb{Z}^l & (\mathbb{Z}^{n+l})^* \arrow[l, "Q^*"'] \arrow[u, "B^* \oplus Q^*"'] \arrow[ur, "B^*"'] &
\end{tikzcd}$$

3.3 Construction of the toric DM stack

We follow [3, Section 3]. Applying $\mathrm{Hom}(-, \mathbb{C}^*)$ to $\beta^\vee : \tilde{M} \to DG(\beta)$, one obtains

$$\phi : G_{\boldsymbol{\Sigma}} := \mathrm{Hom}(\mathrm{DG}(\beta), \mathbb{C}^*) \to \tilde{T} := \mathrm{Hom}(\tilde{M}, \mathbb{C}^*).$$

Let $G = \mathrm{Ker}\phi$. Then $G \cong \prod_{j=1}^{l} \mu_{a_j}$, where $\mu_{a_j} \subset \mathbb{C}^*$ is the group of a_jth roots of unity, which is isomorphic to $\mathbb{Z}_{a_j}$. Let $\mathcal{B}G$ denote the quotient stack $[\{0\}/G]$. The algebraic torus $\tilde{T}$ acts on $\mathbb{C}^r$ by

$$(\tilde{t}_1, \ldots, \tilde{t}_r) \cdot (z_1, \ldots, z_r) = (\tilde{t}_1 z_1, \ldots, \tilde{t}_r z_r), \quad (\tilde{t}_1, \ldots, \tilde{t}_r) \in \tilde{T}, \quad (z_1, \ldots, z_r) \in \mathbb{C}^r.$$

Let $G_{\boldsymbol{\Sigma}}$ act on $\mathbb{C}^r$ by $g \cdot z := \phi(g) \cdot z$, where $g \in G_{\boldsymbol{\Sigma}}$, $z \in \mathbb{C}^r$. Let $\mathcal{O}(\mathbb{C}^r) = \mathbb{C}[z_1, \ldots, z_r]$ be the coordinate ring of $\mathbb{C}^r$. Let I_Σ be the ideal of $\mathcal{O}(\mathbb{C}^r)$ generated by

$$\left\{ \prod_{\rho_i \not\subset \sigma} z_i : \sigma \in \Sigma \right\}$$

and let $Z(I_\Sigma)$ be the closed subscheme of $\mathbb{C}^r$ defined by I_Σ. Then $U := \mathbb{C}^r - Z(I_\Sigma)$ is a quasi-affine variety over $\mathbb{C}$. The toric DM stack associated with the stacky fan $\boldsymbol{\Sigma}$ is defined to be the quotient stack

$$\mathcal{X}_{\boldsymbol{\Sigma}} := [U/G_{\boldsymbol{\Sigma}}].$$

It is a smooth DM stack whose generic stabilizer is G, and its coarse moduli space is the toric variety X_Σ defined by the simplicial fan Σ. There is an open dense immersion

$$\iota : \mathcal{T} = [\tilde{T}/G_{\boldsymbol{\Sigma}}] \hookrightarrow \mathcal{X}_{\boldsymbol{\Sigma}} = [U/G_{\boldsymbol{\Sigma}}],$$

where $\mathcal{T} \cong (\mathbb{C}^*)^n \times \mathcal{B}G$ is a DM torus. The action of $\mathcal{T}$ on itself extends to an action a: $\mathcal{T} \times \mathcal{X}_{\boldsymbol{\Sigma}} \to \mathcal{X}_{\boldsymbol{\Sigma}}$.

3.4 Rigidification

We define the *rigidification* of $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ to be the stacky fan $\boldsymbol{\Sigma}^{\mathrm{rig}} := (\bar{N}, \Sigma, \bar{\beta})$, where $\bar{\beta}$ is the composition of $\beta : \tilde{N} \to N$ with the projection $N \to \bar{N}$. Note that M, $\bar{N}_\sigma$, and M_σ defined in Section 3.1 depend only on $\boldsymbol{\Sigma}^{\mathrm{rig}}$. The generic stabilizer of the toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$ is trivial because $\bar{N} \cong \mathbb{Z}^n$ is torsion-free. So $\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$ is a toric orbifold. There is a morphism of stacky fans $\boldsymbol{\Sigma} \to \boldsymbol{\Sigma}^{\mathrm{rig}}$ which induces a morphism of toric DM stacks $r : \mathcal{X}_{\boldsymbol{\Sigma}} \to \mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$. The toric orbifold $\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$ is called the *rigidification* of the toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}}$. The morphism $r : \mathcal{X}_{\boldsymbol{\Sigma}} \to \mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$ makes $\mathcal{X}_{\boldsymbol{\Sigma}}$ an abelian gerbe over $\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$.

$G_{\boldsymbol{\Sigma}^{\mathrm{rig}}} = G_{\boldsymbol{\Sigma}}/G$ is a subgroup of $\tilde{T}$. Let $T := \tilde{T}/G_{\boldsymbol{\Sigma}^{\mathrm{rig}}} \cong (\mathbb{C}^*)^n$. There is an open dense immersion

$$\iota^{\mathrm{rig}} : T = [\tilde{T}/G_{\boldsymbol{\Sigma}^{\mathrm{rig}}}] \hookrightarrow \mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}} = [U/G_{\boldsymbol{\Sigma}^{\mathrm{rig}}}].$$

We have the following statements (cf. Section 2.3.3).

Proposition 3.1.

(a) (Nonequivariant sheaves on toric DM stacks) The morphism $r : \mathcal{X}_{\boldsymbol{\Sigma}} \to \mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}$ induces the following quasi-embeddings of dg categories:

$$r^* : \mathcal{P}\mathrm{erf}(\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}) \cong \mathcal{P}\mathrm{erf}_{G_{\boldsymbol{\Sigma}^{\mathrm{rig}}}}(U) \hookrightarrow \mathcal{P}\mathrm{erf}(\mathcal{X}_{\boldsymbol{\Sigma}}) \cong \mathcal{P}\mathrm{erf}_{G_{\boldsymbol{\Sigma}}}(U),$$

$$r^* : \mathcal{Q}(\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}) \cong \mathcal{Q}_{G_{\boldsymbol{\Sigma}^{\mathrm{rig}}}}(U) \hookrightarrow \mathcal{Q}(\mathcal{X}_{\boldsymbol{\Sigma}}) \cong \mathcal{Q}_{G_{\boldsymbol{\Sigma}}}(U).$$

They are quasi-equivalences if and only if $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{\mathrm{rig}}$.

(b) (Equivariant sheaves on toric DM stacks) We have the following quasi-equivalences of dg categories:

$$\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}) \cong \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \cong \mathcal{P}\mathrm{erf}_{\tilde{T}}(U),$$

$$\mathcal{Q}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}^{\mathrm{rig}}}) \cong \mathcal{Q}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \cong \mathcal{Q}_{\tilde{T}}(U).$$

(c) (Forgetting the equivariant structure) The forgetful functors

$$\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \to \mathcal{P}\mathrm{erf}(\mathcal{X}_{\boldsymbol{\Sigma}}), \quad \mathcal{Q}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \to \mathcal{Q}(\mathcal{X}_{\boldsymbol{\Sigma}})$$

are essentially surjective if and only if $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{\mathrm{rig}}$. □

3.5 Lifting the fan

Let $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan, where $N \cong \mathbb{Z}^n$. Let U be defined as in Section 3.3. The open embedding $U \hookrightarrow \mathbb{C}^r$ is $\tilde{T}$-equivariant and can be viewed as a morphism between smooth toric varieties. More explicitly, consider the r-dimensional cone

$$\tilde{\sigma}_0 = \{y_1\tilde{b}_1 + \cdots + y_r\tilde{b}_r \mid y_1, \ldots, y_r \in \mathbb{R}_{\geq 0}\} \subset \tilde{N}_{\mathbb{R}} = \tilde{N} \otimes_{\mathbb{Z}} \mathbb{R},$$

and let $\tilde{\Sigma}_0 \subset \tilde{N}_{\mathbb{R}}$ be the fan that consists of all the faces of $\tilde{\sigma}_0$. Then $\mathbb{C}^r$ is the smooth toric variety defined by the fan $\tilde{\Sigma}_0$. We define a subfan $\tilde{\Sigma} \subset \tilde{\Sigma}_0$ as follows. Given $\sigma \in \Sigma(d)$ such that $\sigma \cap \{\bar{b}_1, \ldots, \bar{b}_r\} = \{\bar{b}_{i_1}, \ldots, \bar{b}_{i_d}\}$, let

$$\tilde{\sigma} = \{y_1\tilde{b}_{i_1} + \cdots + y_d\tilde{b}_{i_d} \mid y_1, \ldots, y_d \in \mathbb{R}_{\geq 0}\} \subset \tilde{N}_{\mathbb{R}}.$$

Then there is a bijection $\Sigma \to \tilde{\Sigma}$ given by $\sigma \mapsto \tilde{\sigma}$, and U is the smooth toric variety defined by $\tilde{\Sigma}$.

For any d-dimensional cone $\tilde{\sigma} \in \tilde{\Sigma}$, let $I = \{i \mid \rho_i \subset \sigma\}$, and define

$$\begin{aligned} U_{\tilde{\sigma}} &= \mathrm{Spec}\,\mathbb{C}[\tilde{\sigma}^{\vee} \cap \tilde{M}] = \mathbb{C}^r - \left\{\prod_{i \notin I} z_i = 0\right\} \\ &= \{(z_1, \ldots, z_r) \in \mathbb{C}^r \mid z_i \neq 0 \text{ for } i \notin I\} \cong \mathbb{C}^d \times (\mathbb{C}^*)^{r-d}, \\ \tilde{T}_{\tilde{\sigma}} &= \{(\tilde{t}_1, \ldots, \tilde{t}_r) \in \tilde{T} \mid \tilde{t}_i = 1 \text{ for } i \notin I\} \cong (\mathbb{C}^*)^d. \end{aligned}$$

Then $U_{\tilde{\sigma}}$ is a Zariski open subset of U and $\tilde{T}_{\tilde{\sigma}}$ is a subtorus of $\tilde{T}$. The action of $G_{\boldsymbol{\Sigma}}$ on $U_{\tilde{\sigma}}$ gives rise to a stack $[U_{\tilde{\sigma}}/G_{\boldsymbol{\Sigma}}]$ denoted by $\mathcal{X}_\sigma$, which is a substack of $\mathcal{X}_{\boldsymbol{\Sigma}}$. We have $\tilde{T}$-equivariant open embeddings

$$\tilde{T} \hookrightarrow X_{\tilde{\Sigma}} = U \hookrightarrow X_{\tilde{\Sigma}_0} = \mathbb{C}^r.$$

The $\tilde{T}$-equivariant line bundles on $U_{\tilde{\sigma}} = \mathrm{Spec}\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$ are in one-to-one correspondence with the characters in $\mathrm{Hom}(\tilde{T}_{\tilde{\sigma}}, \mathbb{C}^*)$. Moreover, we have canonical isomorphisms

$$\mathrm{Hom}(\tilde{T}_{\tilde{\sigma}}, \mathbb{C}^*) \cong \tilde{M}/(\tilde{\sigma}^\perp \cap \tilde{M}) \cong M_\sigma.$$

Given $\chi \in M_\sigma$, let $\mathcal{O}_{U_{\tilde{\sigma}}}(\chi)$ denote the $\tilde{T}$-equivariant line bundle on $U_{\tilde{\sigma}}$ associated with $\chi \in M_\sigma$, and let $\mathcal{O}_{\mathcal{X}_\sigma}(\chi)$ denote the corresponding $\mathcal{T}$-equivariant line bundle on $\mathcal{X}_\sigma = [U_{\tilde{\sigma}}/G_{\boldsymbol{\Sigma}}]$. Let $\tilde{\chi} \in \tilde{M}$ be any representative of the coset $\chi \in \tilde{M}/(\tilde{\sigma}^\perp \cap \tilde{M}) \cong M_\sigma$. Then the $\tilde{T}$-weights of $\Gamma(U_{\tilde{\sigma}}, \mathcal{O}_{U_{\tilde{\sigma}}}(\chi)) = \Gamma(\mathcal{X}_\sigma, \mathcal{O}_{\mathcal{X}_\sigma}(\chi))$ are in one-to-one correspondence with points in $\tilde{\chi} + (\tilde{\sigma}^\vee \cap \tilde{M})$.

4 Equivariant Line Bundles and Twisted Polytopes

In this section, we describe equivariant line bundles on a toric DM stack $\mathcal{X}$ defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$.

4.1 Equivariant line bundles

Let U, $G_{\boldsymbol{\Sigma}}$, $\tilde{\Sigma}$ be defined as in Section 3 so that $U = X_{\tilde{\Sigma}}$ and $\mathcal{X} = [U/G_{\boldsymbol{\Sigma}}]$. For $i = 1, \ldots, r$, let $\tilde{D}_i \subset U$ (resp. $\tilde{D}'_i \subset \mathbb{C}^r$) be the $\tilde{T}$-divisor defined by $z_i = 0$. For any $\vec{c} = (c_1, \ldots, c_r) \in \mathbb{Z}^r$, let

$$\tilde{L}_{\vec{c}} = \mathcal{O}_U\left(\sum_{i=1}^r c_i \tilde{D}_i\right), \quad \tilde{L}'_{\vec{c}} = \mathcal{O}_{\mathbb{C}^r}\left(\sum_{i=1}^r c_i \tilde{D}'_i\right).$$

Then $\tilde{L}_{\vec{c}}$ and $\tilde{L}'_{\vec{c}}$ are $\tilde{T}$-equivariant line bundles on U and on $\mathbb{C}^r$, respectively, and $\tilde{L}_{\vec{c}} = \tilde{L}'_{\vec{c}}|_U$. The $\tilde{T}$-equivariant line bundle $\tilde{L}_{\vec{c}}$ descends to a $\mathcal{T}$-equivariant line bundle $\mathcal{L}_{\vec{c}}$ on $\mathcal{X}$. Any $\mathcal{T}$-equivariant line bundle on $\mathcal{X}$ is of the form $\mathcal{L}_{\vec{c}}$ for some $\vec{c} = (c_1, \ldots, c_r) \in \mathbb{Z}^r$.

4.2 Twisted polytopes

In this subsection, we make the following assumptions on the fan Σ.

$$
\begin{aligned}
&\bullet \text{ All the maximal cones in } \Sigma \text{ are } n\text{-dimensional, where } n = \dim_{\mathbb{R}} N_{\mathbb{R}}.\\
&\bullet \text{ We fix a total ordering } C_1, \ldots, C_v \text{ of the maximal cones in } \Sigma.
\end{aligned}
\tag{7}
$$

Under the aforementioned assumptions, we will give an alternative description of equivariant line bundles on $\mathcal{X}$.

For each C_i we have

$$\bar{N}_{C_i} \subset \bar{N} \subset N_{\mathbb{R}},$$

where $\bar{N}_{C_i} \subset \bar{N}$ is a sublattice of finite index. Then

$$M := \mathrm{Hom}(\bar{N}, \mathbb{Z}), \quad M_{C_i} = \mathrm{Hom}(\bar{N}_{C_i}, \mathbb{Z})$$

can be identified with subgroups of $M_{\mathbb{R}}$:

$$
\begin{aligned}
M &= \{x \in M_{\mathbb{R}} \mid \langle x, v\rangle \in \mathbb{Z} \text{ for all } v \in \bar{N}\},\\
M_{C_i} &= \{x \in M_{\mathbb{R}} \mid \langle x, v\rangle \in \mathbb{Z} \text{ for all } v \in \bar{N}_{C_i}\},
\end{aligned}
$$

where $\langle , \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ is the natural pairing. For each C_i we have

$$M \subset M_{C_i} \subset M_{\mathbb{R}}.$$

Definition 4.1 (Twisted polytope)**.** Let $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan satisfying (7). A *twisted polytope* for $\boldsymbol{\Sigma}$ is an ordered v-tuple $\underline{\chi} = (\chi_1, \ldots, \chi_v)$, where $\chi_i \in M_{C_i}$, with the property that for any $1 \le i \le v$ and $1 \le j \le v$, the linear forms $\langle \chi_i, -\rangle$ and $\langle \chi_j, -\rangle$ agree when restricted to $C_i \cap C_j$. □

The terminology is motivated by [15].

Lemma 4.2. Let $\mathcal{X}$ be the toric DM stack defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ satisfying (7), and let $p: \mathcal{X} \to X$ be the map to the coarse moduli space $X = X_{\Sigma}$. Let $T = p(\mathcal{T}) \cong (\mathbb{C}^*)^n$.

(a) For each twisted polytope $\underline{\chi} = (\chi_1, \ldots, \chi_v)$ for $\boldsymbol{\Sigma}$, there is up to isomorphism a unique $\mathcal{T}$-equivariant line bundle $\mathcal{O}_{\mathcal{X}}(\underline{\chi})$ with the property that $\mathcal{O}_{\mathcal{X}}(\underline{\chi})|_{\mathcal{X}_{C_i}} \cong \mathcal{O}_{\mathcal{X}_{C_i}}(\chi_i)$, where $\mathcal{O}_{\mathcal{X}_{C_i}}(\chi_i)$ is defined as in Section 3.5.

(b) If $\chi_i \in M \subset M_{C_i}$ for $i = 1, \ldots, v$, then there is a T-equivariant line bundle $\mathcal{O}_X(\underline{\chi})$ on the coarse moduli space X such that $\mathcal{O}_{\mathcal{X}}(\underline{\chi}) = p^*\mathcal{O}_X(\underline{\chi})$. □

Proof. (a) For each $i, j \in \{1, \ldots, r\}$, we define

$$C_{ij} = C_i \cap C_j, \quad U_i = U_{\tilde{C}_i}, \quad U_{ij} = U_{\tilde{C}_{ij}} = U_i \cap U_j.$$

Then $U_{ij} = U_{ji}$, and $U_{ii} = U_i$. It suffices to show that there is a unique $\tilde{T}$-equivariant line bundle $\mathcal{O}_U(\underline{\chi})$ on U with the property that $\mathcal{O}_U(\underline{\chi})|_{U_i} \cong \mathcal{O}_{U_i}(\chi_i)$, where $\mathcal{O}_{U_i}(\chi_i)$ is defined as in Section 3.5.

There is an inclusion $N_{C_{ij}} \to N_{C_i}$ which induces a surjective map $f_{ij} : M_{C_i} \to M_{C_{ij}}$. Note that $\langle \chi_i, - \rangle$ and $\langle \chi_j, - \rangle$ agree when restricted to C_{ij} if and only if $f_{ij}(\chi_i) = f_{ji}(\chi_j) \in M_{C_{ij}}$. We define $\chi_{ij} = f_{ij}(\chi_i) = f_{ji}(\chi_j) \in M_{C_{ij}}$. Note that f_{ii} is the identity map and $\chi_{ii} = \chi_i$. Then $\mathcal{O}_{U_i}(\chi_i)\big|_{U_{ij}}$ is isomorphic to $\mathcal{O}_{U_{ij}}(\chi_{ij})$ as $\tilde{T}$-equivariant line bundles on U_{ij}. For each i, j, We fix an isomorphism of $\tilde{T}$-equivariant line bundles:

$$\lambda_{ij} : \mathcal{O}_{U_i}(\chi_i)|_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{U_{ij}}(\chi_{ij}) = \mathcal{O}_{U_{ji}}(\chi_{ji}).$$

In particular, we take λ_{ii} to be the identity map. For each i, j, we define an isomorphism of $\tilde{T}$-equivariant line bundles:

$$\phi_{ij} = \lambda_{ji}^{-1} \circ \lambda_{ij} : \mathcal{O}_{U_i}(\chi_i)|_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{U_j}(\chi_j)|_{U_{ji} = U_{ij}}.$$

Then

(i) for each i, ϕ_{ii} is the identity map and

(ii) for each i, j, k, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$.

Therefore, there exists up to isomorphism a unique $\tilde{T}$-equivariant line bundle $\mathcal{O}_U(\underline{\chi})$ on U with isomorphisms $\psi_i : \mathcal{O}_U(\underline{\chi})|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}(\chi_i)$ of $\tilde{T}$-equivariant line bundles such that $\psi_j = \phi_{ij} \circ \psi_i$ on U_{ij}.

(b) The construction of the T-equivariant line bundle $\mathcal{O}_X(\underline{\chi})$ on the simplicial toric variety $X = X_\Sigma$ from such $\chi_1, \ldots, \chi_r \in M$ is well known, see, for example, [11, Section 3.4]. It is clear from construction that $p^*\mathcal{O}_X(\underline{\chi}) = \mathcal{O}_{\mathcal{X}}(\underline{\chi})$. ■

Let $C_1, \ldots, C_v$ be defined as just seen. Given $1 \le i_0 < \cdots < i_k \le v$, define $C_{i_0 \cdots i_k} = C_{i_0} \cap \cdots \cap C_{i_k}$. Given a twisted polytope $\underline{\chi} = (\chi_1, \ldots, \chi_v)$ of $\boldsymbol{\Sigma}$, for each $1 \le i_0 < \cdots < i_k \le v$ define $\chi_{i_0 \cdots i_k} \in M_{C_{i_0 \cdots i_k}}$ to be the image of $\chi_{i_0} \in M_{C_{i_0}}$ under the group homomorphism $M_{C_{i_0}} \to M_{C_{i_0 \cdots i_k}}$. Then

$$\{\chi_{i_0 \cdots i_k} \in M_{C_{i_0 \cdots i_k}} \mid 1 \le i_0 < \cdots < i_k \le v\}$$

satisfies the following properties:

(1) If $1 \le j_0 < \cdots < j_l \le v$ refines $1 \le i_0 < \cdots < i_k \le v$, then $\chi_{i_0 \cdots i_k} \mapsto \chi_{j_0 \cdots j_l}$ under the group homomorphism $M_{C_{i_0 \cdots i_k}} \to M_{C_{j_0 \cdots j_l}}$.

(2) $\mathcal{O}_{\mathcal{X}}(\underline{\chi})|_{\mathcal{X}_{C_{i_0 \cdots i_k}}} \cong \mathcal{O}_{\mathcal{X}_{C_{i_0 \cdots i_k}}}(\chi_{i_0 \cdots i_k})$.

4.3 Equivariant $\mathbb{Q}$-ample line bundles

In this subsection, we assume that the toric DM stack $\mathcal{X}$ is *complete*, that is, it is defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$, where Σ is a complete fan in $N_{\mathbb{R}}$.

Given a twisted polytope $\underline{\chi} = (\chi_1, \ldots, \chi_r)$ of $\boldsymbol{\Sigma}$ and $n \in \mathbb{Z}$, let $n\underline{\chi} = (n\chi_1, \ldots, n\chi_r)$. Then $n\underline{\chi}$ is a twisted polytope of $\boldsymbol{\Sigma}$, and $\mathcal{O}_{\mathcal{X}}(n\underline{\chi}) = \mathcal{O}_{\mathcal{X}}(\underline{\chi})^{\otimes n}$. Given any twisted polytope $\underline{\chi} = (\chi_1, \ldots, \chi_r)$ of $\boldsymbol{\Sigma}$, there exists a positive integer n such that $n\chi_i \in M$ for $i = 1, \ldots, v$. Then $n\underline{\chi}$ defines an equivariant line bundle $\mathcal{O}_X(n\underline{\chi})$ on the coarse moduli space $X = X_\Sigma$, and $\mathcal{O}_{\mathcal{X}}(\underline{\chi})^{\otimes n} = p^* \mathcal{O}_X(n\underline{\chi})$.

Definition 4.3 ($\mathbb{Q}$-ample)**.** Let $\mathcal{X}$ be a complete toric DM stack and let $p : \mathcal{X} \to X$ be the morphism to the coarse moduli space. We say that a line bundle $\mathcal{L}$ on $\mathcal{X}$ is $\mathbb{Q}$-*ample* if there exists a positive integer $n > 0$ and an ample line bundle L on X such that $\mathcal{L}^{\otimes n} = p^* L$. □

Theorem 4.4. Let $\mathcal{X} = \mathcal{X}_{\boldsymbol{\Sigma}}$ be a complete toric DM stack defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ and let $\underline{\chi}$ be a twisted polytope of $\boldsymbol{\Sigma}$. The line bundle $\mathcal{O}_{\mathcal{X}}(\underline{\chi})$ is $\mathbb{Q}$-ample precisely when $\underline{\chi}$ satisfies the following two conditions:

(i) The set $\{\chi_1, \ldots, \chi_v\}$ is strictly convex, in the sense that its convex hull is strictly larger than the convex hull of any subset $\{\chi_{i_1}, \ldots, \chi_{i_w}\}$.

(ii) The convex hull of $\{\chi_1, \ldots, \chi_v\}$ coincides with the set of all $\xi \in M_{\mathbb{R}}$ satisfying

$$\langle \xi, \gamma \rangle \ge \langle \chi_i, \gamma \rangle \quad \text{for all } i \text{ and all } \gamma \in C_i. \qquad \square$$

Proof. Let $p: \mathcal{X} \to X$ be the morphism to the coarse moduli space $X = X_\Sigma$. We have seen that there exists $n \in \mathbb{Z}_{>0}$ such that $n\underline{\chi}$ defines an equivariant line bundle $\mathcal{O}_X(n\underline{\chi})$ on the coarse moduli space X and that $\mathcal{O}_{\mathcal{X}}(\underline{\chi})^{\otimes n} = p^*\mathcal{O}_X(n\underline{\chi})$. By Fulton [11, Section 3.4], $\mathcal{O}_X(n\underline{\chi})$ is ample if and only if $n\underline{\chi}$ satisfies the above conditions (i) and (ii). The proof is completed by the following observation: $\underline{\chi}$ satisfies conditions (i) and (ii) $\Leftrightarrow$ $n\underline{\chi}$ satisfies conditions (i) and (ii) for all $n \in \mathbb{Z}_{>0}$ $\Leftrightarrow$ $n\underline{\chi}$ satisfies conditions (i) and (ii) for some $n \in \mathbb{Z}_{>0}$. ■

5 Standard Quasicoherent Sheaves and Costandard Constructible Sheaves

Let $\mathcal{X}_\Sigma$ be a toric DM stack defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$. In this section, we introduce a useful class $\{\Theta(\sigma, \chi)\}$ of constructible sheaves on $M_{\mathbb{R}}$, and a corresponding class $\{\Theta'(\sigma, \chi)\}$ of $\mathcal{T}$-equivariant quasicoherent sheaves on $\mathcal{X}_{\boldsymbol{\Sigma}} = [U/G_{\boldsymbol{\Sigma}}]$, and show that the dg categories they generate are quasi-equivalent to each other. These two classes of sheaves are closely related to the class of constructible sheaves $\{\Theta(\tilde{\sigma}, \chi)\}$ on $\tilde{M}_{\mathbb{R}}$ and the class of $\tilde{T}$-equivariant quasicoherent sheaves $\{\Theta'(\tilde{\sigma}, \chi)\}$ on the smooth toric variety U introduced in [8, Section 3].

5.1 The poset $\boldsymbol{\Gamma}(\boldsymbol{\Sigma})$

Definition 5.1. Let $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan. Define

$$\Gamma(\boldsymbol{\Sigma}) = \{(\sigma, \chi) \mid \sigma \in \Sigma, \chi \in M_\sigma\}.$$

(1) Given a pair $(\sigma, \chi) \in \Gamma(\boldsymbol{\Sigma})$, where $\sigma \in \Sigma(d)$, we have

$$\sigma \cap \{\bar{b}_1, \ldots, \bar{b}_r\} = \{w_1, \ldots, w_d\},$$

where $\bar{b}_1, \ldots, \bar{b}_r \in \bar{N} \subset N_{\mathbb{R}}$ are define as in Section 3.1.

$$\sigma_\chi^\vee = \{x \in M_{\mathbb{R}} \mid \langle x, w_i \rangle \geq \langle \chi, w_i \rangle_\sigma\},$$

$$(\sigma_\chi^\vee)^\circ = \{x \in M_{\mathbb{R}} \mid \langle x, w_i \rangle > \langle \chi, w_i \rangle_\sigma\},$$

$$\sigma_\chi^\perp = \{x \in M_{\mathbb{R}} \mid \langle x, w_i \rangle = \langle \chi, w_i \rangle_\sigma\},$$

where

$$\langle\,,\,\rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}, \quad \langle\,,\,\rangle_\sigma : M_\sigma \times \bar{N}_\sigma \to \mathbb{Z}$$

are the natural pairings.

(2) Give the set $\Gamma(\boldsymbol{\Sigma})$ of ordered pairs (σ, χ) a partial order by setting

$$(\sigma_1, \chi_1) \leq (\sigma_2, \chi_2),$$

whenever $(\sigma_1)^\vee_{\chi_1} \subset (\sigma_2)^\vee_{\chi_2}$. □

On the basis of the partially ordered set $\Gamma(\boldsymbol{\Sigma})$ we introduce the following dg category.

Definition 5.2. Let $\Gamma(\boldsymbol{\Sigma})_{\mathbb{C}}$ be the $\mathbb{C}$-linear dg category whose objects are the elements of $\Gamma(\boldsymbol{\Sigma})$, with

$$\hom((\sigma_1, \chi_1), (\sigma_2, \chi_2)) = \begin{cases} \mathbb{C}[0] & (\sigma_1, \chi_1) \leq (\sigma_2, \chi_2), \\ 0 & \text{otherwise.} \end{cases}$$

The composition rule is evident. It is a dg category with the Hom complexes concentrated in degree zero. □

It is clear from the definitions that

$$\Gamma(\boldsymbol{\Sigma}) = \Gamma(\boldsymbol{\Sigma}^{\mathrm{rig}}), \quad \Gamma(\boldsymbol{\Sigma})_{\mathbb{C}} = \Gamma(\boldsymbol{\Sigma}^{\mathrm{rig}})_{\mathbb{C}}.$$

The smooth toric variety U is defined by the fan $\tilde{\Sigma} \subset \tilde{N}_{\mathbb{R}}$. In [8, Section 3.1], we define a poset

$$\Gamma(\tilde{\Sigma}, \tilde{M}) = \{(\tilde{\sigma}, \chi) \mid \sigma \in \tilde{\Sigma}, \chi \in \tilde{M}/(\sigma^\perp \cap \tilde{M})\}.$$

Recall that $\tilde{M}/(\sigma^\perp \cap \tilde{M}) \cong M_\sigma$. For any $(\tilde{\sigma}_1, \chi_1), (\tilde{\sigma}_2, \chi_2) \in \Gamma(\tilde{\Sigma}, \tilde{M})$, $(\tilde{\sigma}_1, \chi_1) \leq (\tilde{\sigma}_2, \chi_2)$ in $\Gamma(\tilde{\Sigma}, \tilde{M})$ if and only if $(\sigma_1, \chi_1) \leq (\sigma_2, \chi_2)$ in $\Gamma(\boldsymbol{\Sigma})$. We conclude the following lemma.

Lemma 5.3. The bijective map $\Gamma(\tilde{\Sigma}, \tilde{M}) \to \Gamma(\boldsymbol{\Sigma})$, $(\tilde{\sigma}, \chi) \mapsto (\sigma, \chi)$ is an isomorphism of posets and induces a quasi-equivalence of dg categories

$$\Gamma(\tilde{M}, \tilde{\Sigma})_{\mathbb{C}} \cong \Gamma(\boldsymbol{\Sigma})_{\mathbb{C}}.$$ □

5.2 Costandard sheaves on cones

In this section, we introduce constructible sheaves $\Theta(\sigma, \chi)$ on $M_{\mathbb{R}}$, indexed by elements of $\Gamma(\boldsymbol{\Sigma})$. They are *costandard* in the sense of [19].

Definition 5.4. Given $\sigma \in \Sigma$ and $\chi \in M_\sigma$, let $j = j_{(\sigma_\chi^\vee)^\circ} : (\sigma_\chi^\vee)^\circ \hookrightarrow M_\mathbb{R}$ be the inclusion map, and define

$$\Theta(\sigma, \chi) := j_! \omega_{(\sigma_\chi^\vee)^\circ} \in \mathrm{Ob}(\mathrm{Sh}_c(M_\mathbb{R})).$$ □

We recall the definition of similar constructible sheaves $\Theta(\tilde{\sigma}, \chi)$ on $\tilde{M}_\mathbb{R}$ for the fan $\tilde{\Sigma}$ [8, Definition 3.1]. Given $\tilde{\sigma} \in \tilde{\Sigma}$ and $\chi \in \tilde{M}/(\tilde{\sigma}^\perp \cap \tilde{M}) \cong M_\sigma$, define

$$\Theta(\tilde{\sigma}, \chi) = \tilde{j}_! \omega_{(\chi + \tilde{\sigma}^\vee)^\circ} \in Ob(Sh_c(\tilde{M}_\mathbb{R})),$$

where $(\chi + \tilde{\sigma}^\vee)^\circ$ is the interior of the shifted dual cone $\chi + \tilde{\sigma}^\vee \subset \tilde{M}_\mathbb{R}$, and $\tilde{j} : (\chi + \tilde{\sigma}^\vee)^\circ \hookrightarrow M_\mathbb{R}$ is the inclusion.

The group homomorphism $\beta : \tilde{N} \to N$ induces

$$\beta^* : M = \mathrm{Hom}(N, \mathbb{Z}) = \mathrm{Hom}(\bar{N}, \mathbb{Z}) \longrightarrow \tilde{M} = \mathrm{Hom}(\tilde{N}, \mathbb{Z}).$$

Let $\hat{\beta} = \beta^* \otimes \mathbb{R} : M_\mathbb{R} \to \tilde{M}_\mathbb{R}$. Then $\hat{\beta}$ is an injective $\mathbb{R}$-linear map.

Lemma 5.5.

$$\hat{\beta}^! \Theta(\tilde{\sigma}, \chi) = \Theta(\sigma, \chi). \tag{8}$$ □

Proof. Let $j : (\sigma_\chi^\vee)^\circ \to M_\mathbb{R}$ and $\tilde{j} : (\chi + \tilde{\sigma}^\vee)^\circ \to \tilde{M}_\mathbb{R}$ be defined as just seen, and let $\hat{\beta}' : (\sigma_\chi^\vee)^\circ \to (\chi + \tilde{\sigma}^\vee)^\circ$ be the restriction of $\hat{\beta}$. Then j and $\tilde{j}$ are open embeddings, $\hat{\beta}$ and $\hat{\beta}'$ are closed embeddings, and $\tilde{j} \circ \hat{\beta}' = \hat{\beta} \circ j$.

$$\hat{\beta}^! \tilde{j}_! \omega_{(\chi + \tilde{\sigma}^\vee)^\circ} = j_! (\hat{\beta}')^! \omega_{(\chi + \tilde{\sigma}^\vee)^\circ} = j_! \omega_{(\sigma_\chi^\vee)^\circ}.$$ ■

Proposition 5.6. For any $(\sigma, \phi), (\tau, \psi) \in \Gamma(\mathbf{\Sigma})$

$$\mathrm{Ext}^i(\Theta(\sigma, \phi), \Theta(\tau, \psi)) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } \sigma_\phi^\vee \subset \tau_\psi^\vee, \\ 0 & \text{otherwise.} \end{cases}$$ □

Proof. The proof is the same as the proof of [8, Proposition 3.3 (1)]. ■

5.3 Standard equivariant quasicoherent sheaves on $\mathcal{X}_{\boldsymbol{\Sigma}}$

In this section, we introduce $\mathcal{T}$-equivariant quasicoherent sheaves $\Theta'(\sigma, \chi)$ on the toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}} = [U/G_{\boldsymbol{\Sigma}}]$, indexed by $\Gamma(\boldsymbol{\Sigma})$. Under the quasi-equivalence $\mathcal{Q}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \cong \mathcal{Q}_{\tilde{T}}(U)$, they correspond to $\tilde{T}$-equivariant quasicoherent sheaves $\Theta'(\tilde{\sigma}, \chi)$ on U, indexed by $\Gamma(\tilde{\Sigma}, \tilde{M})$.

We first recall the definition of $\Theta'(\tilde{\sigma}, \chi)$ in [8, Section 3.2], using the notation in this paper. Given $\tilde{\sigma} \in \tilde{\Sigma}$ and $\chi \in \tilde{M}/(\sigma^{\perp} \cap \tilde{M}) \cong M_{\sigma}$, define $\mathcal{O}_{U_{\tilde{\sigma}}}(\chi)$ as in Section 3.5. Let $\iota_{\tilde{\sigma}} : U_{\tilde{\sigma}} \to U$ be the open embedding. We define

$$\Theta'(\tilde{\sigma}, \chi) = \iota_{\tilde{\sigma}*}\mathcal{O}_{U_{\sigma}}(\chi) \in Ob(\mathcal{Q}_{\tilde{T}}(U)).$$

We now define the sheaves $\Theta'(\sigma, \chi)$ on $\mathcal{X}_{\boldsymbol{\Sigma}}$.

Definition 5.7. Given $(\sigma, \chi) \in \boldsymbol{\Sigma}$, let $j_{\sigma} : \mathcal{X}_{\sigma} \to \mathcal{X}_{\boldsymbol{\Sigma}}$ be the open embedding, and define

$$\Theta'(\sigma, \chi) = j_{\sigma*}\mathcal{O}_{\mathcal{X}_{\sigma}}(\chi) \in Ob(\mathcal{Q}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})).$$

□

Lemma 5.8. Let q denote the dg functor $\mathcal{Q}_{\tilde{T}}(U) \to \mathcal{Q}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$ (which is a quasi-equivalence of dg categories). Then

$$q(\Theta'(\tilde{\sigma}, \chi)) = \Theta'(\sigma, \chi). \tag{9}$$

□

Proof. We have a 2-cartesian diagram

$$\begin{tikzcd} U_{\tilde{\sigma}} \arrow[r, "\iota_{\tilde{\sigma}}"] \arrow[d, "\pi_{\sigma}"'] & U \arrow[d, "\pi"] \\ \mathcal{X}_{\sigma} \arrow[r, "j_{\sigma}"] & \mathcal{X}_{\boldsymbol{\Sigma}} \end{tikzcd}$$

where $\iota_{\tilde{\sigma}}$ and j_{σ} are open embeddings. We need to show $\pi^*\Theta'(\sigma, \chi) = \Theta'(\tilde{\sigma}, \chi)$.

$$\pi^*\Theta'(\sigma, \chi) = \pi^* j_{\sigma*}\mathcal{O}_{\mathcal{X}_{\sigma}}(\chi) = \iota_{\tilde{\sigma}*}\pi_{\sigma}^*\mathcal{O}_{\mathcal{X}_{\sigma}}(\chi) = \iota_{\tilde{\sigma}*}\mathcal{O}_{U_{\tilde{\sigma}}}(\chi) = \Theta'(\tilde{\sigma}, \chi).$$

■

Proposition 5.9. For any $(\sigma, \phi), (\tau, \psi) \in \Gamma(\boldsymbol{\Sigma})$, we have

$$\mathrm{Ext}^i(\Theta'(\sigma, \phi), \Theta'(\tau, \psi)) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } \sigma_\phi^\vee \subset \tau_\psi^\vee, \\ 0 & \text{otherwise,} \end{cases}$$

where the Ext group is taken in the category $\mathcal{Q}_T(\mathcal{X}_{\boldsymbol{\Sigma}})$. □

Proof. Applying [8, Proposition 3.3(2)] to the smooth toric variety U, we obtain the following statement:

- For any $(\tilde{\sigma}, \phi), (\tilde{\tau}, \psi) \in \Gamma(\tilde{\Sigma}, \tilde{M})$,

$$\mathrm{Ext}^i(\Theta'(\tilde{\sigma}, \phi), \Theta'(\tilde{\tau}, \psi)) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } \phi + \tilde{\sigma}^\vee \subset \psi + \tilde{\tau}^\vee, \\ 0 & \text{otherwise.} \end{cases}$$

The proposition follows from Lemma 5.8 and the aforementioned statement. ■

5.4 Equivalence between categories of Θ-sheaves

Definition 5.10. Let $\langle\Theta\rangle_{\boldsymbol{\Sigma}}$, $\langle\Theta'\rangle_{\boldsymbol{\Sigma}}$, $\langle\Theta\rangle_{\tilde{\Sigma}}$, $\langle\Theta'\rangle_{\tilde{\Sigma}}$ be the full triangulated dg subcategories of $\mathrm{Sh}_c(M_{\mathbb{R}})$, $\mathcal{Q}_T(\mathcal{X}_{\boldsymbol{\Sigma}})$, $\mathrm{Sh}_c(\tilde{M}_{\mathbb{R}})$, $\mathcal{Q}_{\tilde{T}}(U)$ generated by

$$\{\Theta(\sigma, \chi)\}_{(\sigma,\chi)\in\Gamma(\boldsymbol{\Sigma})}, \quad \{\Theta'(\sigma, \chi)\}_{(\sigma,\chi)\in\Gamma(\boldsymbol{\Sigma})},$$
$$\{\Theta(\tilde{\sigma}, \chi)\}_{(\sigma,\chi)\in\Gamma(\tilde{\Sigma},\tilde{M})}, \quad \{\Theta'(\tilde{\sigma}, \chi)\}_{(\sigma,\chi)\in\Gamma(\tilde{\Sigma},\tilde{M})},$$

respectively. □

Theorem 5.11. The following square of functors commutes up to natural isomorphism:

$$\begin{tikzcd} \langle\Theta'\rangle_{\tilde{\Sigma}} \arrow[r, "\kappa_{\tilde{\Sigma}}"] \arrow[d, "q"'] & \langle\Theta\rangle_{\tilde{\Sigma}} \arrow[d, "\hat{\beta}^!"'] \\ \langle\Theta'\rangle_{\boldsymbol{\Sigma}} \arrow[r, "\kappa_{\boldsymbol{\Sigma}}"] & \langle\Theta\rangle_{\boldsymbol{\Sigma}} \end{tikzcd} \tag{10}$$

where $\hat{\beta}^!$ is given by (8), q is given by (9), and $\kappa_{\tilde{\Sigma}}$ and $\kappa_{\boldsymbol{\Sigma}}$ are given by

$$\kappa_{\tilde{\Sigma}}(\Theta'(\tilde{\sigma}, \chi)) = \Theta(\sigma, \chi), \quad \kappa_{\boldsymbol{\Sigma}}(\Theta'(\sigma, \chi)) = \Theta(\sigma, \chi).$$

Moreover, the dg functors $\hat{\beta}^!$, q, $\kappa_{\tilde{\Sigma}}$, $\kappa_{\boldsymbol{\Sigma}}$ in diagram (10) are quasi-equivalences of triangulated dg categories. □

Proof. This follows from Proposition 5.6, Proposition 5.9, and [8, Proposition 3.3]. The four categories in diagram (10) are quasi-equivalent to

$$\mathrm{Tr}(\Gamma(\boldsymbol{\Sigma})_{\mathbb{C}}) \cong \mathrm{Tr}(\Gamma(\tilde{\Sigma}, \tilde{M})_{\mathbb{C}}). \qquad \blacksquare$$

5.5 Coherent–constructible dictionary: line bundles

Let $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$ denote the dg category of perfect complexes of $\mathcal{T}$-equivariant coherent sheaves on $X_{\boldsymbol{\Sigma}}$. The dg functor $\mathcal{Q}_{\tilde{T}}(U) \to \mathcal{Q}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$ restricts to the dg functors

$$\mathcal{P}\mathrm{erf}_{\tilde{T}}(U) \to \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}), \quad \langle \Theta' \rangle_{\tilde{\Sigma}} \to \langle \Theta' \rangle_{\boldsymbol{\Sigma}},$$

which are quasi-equivalences of triangulated dg categories. The proof of [8, Corollary 3.5] shows that $\mathcal{P}\mathrm{erf}_{\tilde{T}}(U) \subset \langle \Theta' \rangle_{\tilde{\Sigma}}$. Therefore, $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \subset \langle \Theta' \rangle_{\boldsymbol{\Sigma}}$, and we have the following corollary.

Corollary 5.12. The functor $\kappa_{\boldsymbol{\Sigma}}$ defines a full embedding of $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$ into $\mathrm{Sh}_c(M_{\mathbb{R}})$. □

Thus, with each vector bundle we can associate a complex of sheaves on $M_{\mathbb{R}}$. For the rest of this section, we assume that Σ is *complete*, and we investigate this association in more detail for line bundles. Given a twisted polytope $\underline{\chi} = (\chi_1, \ldots, \chi_v)$ of $\boldsymbol{\Sigma}$, defined as in Section 4.2, define

$$\{\chi_{i_0 \cdots i_k} \in M_{C_{i_0 \cdots i_k}} \mid 1 \le i_0 < \cdots < i_k \le v\}$$

as in the last paragraph of Section 4.2. Then whenever $1 \le j_0 < j_1 < \cdots < j_\ell \le v$ refines $1 \le i_0 < \cdots < i_k \le v$, we have a well-defined inclusion map $\Theta(C_{i_0 \ldots i_k}, \chi_{i_0 \ldots i_k}) \hookrightarrow \Theta(C_{j_0 \cdots j_\ell}, \chi_{j_0 \ldots j_\ell})$.

Definition 5.13. For each twisted polytope $\underline{\chi}$, let $P(\underline{\chi}) \in Sh_c(M_{\mathbb{R}})$ be the following cochain complex

$$\bigoplus_{i_0} \Theta(C_{i_0}, \chi_{i_0}) \to \bigoplus_{i_0 < i_1} \Theta(C_{i_0 i_1}, \chi_{i_0 i_1}) \to \cdots, \tag{11}$$

where the differential is the alternating sum of inclusion maps. □

Naively, the first term $\bigoplus_{i_0} \Theta(C_{i_0}, \chi_{i_0})$ of (11) would be in degree zero, but because $\Theta(\sigma, \chi) = j_! \omega_{(\sigma_\chi^\vee)^\circ} = j_! \mathfrak{or}_{(\sigma_\chi^\vee)^\circ}[\dim M_\mathbb{R}]$, $P(\underline{\chi})$ is isomorphic to a complex of sheaves whose first term is in degree $-\dim M_\mathbb{R}$.

Theorem 5.14. Let $\mathcal{X} = \mathcal{X}_{\boldsymbol{\Sigma}}$ be a complete toric DM stack defined by a stacky fan $\boldsymbol{\Sigma}$. Let $\mathcal{O}_\mathcal{X}(\underline{\chi})$ denote the $\mathcal{T}$-equivariant line bundle on $\mathcal{X}$ associated with a twisted polytope $\underline{\chi}$ of $\boldsymbol{\Sigma}$, and let $P(\underline{\chi}) \in \mathrm{Sh}_c(M_\mathbb{R})$ be as in Definition 5.13. Then:

(1) $\kappa_{\boldsymbol{\Sigma}}(\mathcal{O}_\mathcal{X}(\underline{\chi})) \cong P(\underline{\chi})$.

(2) Denote the convex hull of $\{\chi_1, \ldots, \chi_v\}$ by $\mathbf{P}$ and its interior by $\mathbf{P}^\circ$. If $\mathcal{O}_\mathcal{X}(\underline{\chi})$ is ample, then $P(\underline{\chi}) \cong j_! \omega_{\mathbf{P}^\circ}$, where $j : \mathbf{P}^\circ \hookrightarrow M_\mathbb{R}$ is the inclusion map. Therefore, the embedding functor $\kappa_{\boldsymbol{\Sigma}} : \mathcal{P}\mathrm{erf}_\mathcal{T}(\mathcal{X}_{\boldsymbol{\Sigma}}) \hookrightarrow \mathrm{Sh}_c(M_\mathbb{R})$ maps the sheaf $\mathcal{O}_{\mathcal{X}_{\boldsymbol{\Sigma}}}(\underline{\chi})$ to $j_! \omega_{\mathbf{P}^\circ}$, the costandard constructible sheaf on $\mathbf{P}^\circ$. □

Proof. To see (1), apply $\kappa_{\boldsymbol{\Sigma}}$ to the Čech resolution of $\mathcal{O}_\mathcal{X}(\underline{\chi})$ (cf. the proof of [8, Corollary 3.5]). The proof of (2) is a minor modification of the proof of [8, Theorem 3.7]. ■

5.6 Morphisms between toric DM stacks

Following [3, Remark 4.5], we introduce the following definition.

Definition 5.15. Let $\boldsymbol{\Sigma}_1 = (N_1, \Sigma_1, \beta_1)$ and $\boldsymbol{\Sigma}_2 = (N_2, \Sigma_2, \beta_2)$ be stacky fans. A *morphism* $f : \boldsymbol{\Sigma}_1 \to \boldsymbol{\Sigma}_2$ is a group homomorphism $f : N_1 \to N_2$ such that

(1) For any cone $\sigma_1 \in \Sigma_1$, there exists a cone $\sigma_2 \in \Sigma_2$ such that $f_\mathbb{R}(\sigma_1) \subset \sigma_2$, where $f_\mathbb{R} = f \otimes_\mathbb{Z} \mathbb{R} : N_{1,\mathbb{R}} \to N_{2,\mathbb{R}}$.

(2) If $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$, and $f_\mathbb{R}(\sigma_1) \subset \sigma_2$, then $f(N_{\sigma_1}) \subset N_{\sigma_2}$, where N_{σ_i} is the subgroup of N_i defined as in Section 3.1. □

A morphism $f : \boldsymbol{\Sigma}_1 \to \boldsymbol{\Sigma}_2$ induces (see [3, Remark 4.5])

(1) A map $\mathcal{T}_1 \to \mathcal{T}_2$.

(2) A map $u_{f,\sigma_1,\sigma_2} : \mathcal{X}_{\sigma_1} \to \mathcal{X}_{\sigma_2}$ for a pair of cones σ_1, σ_2 such that $f_\mathbb{R}(\sigma_1) \subset \sigma_2$.

(3) A map $u = u_f : \mathcal{X}_1 = \mathcal{X}_{\mathbf{\Sigma}_1} \to \mathcal{X}_2 = \mathcal{X}_{\mathbf{\Sigma}_2}$ assembled from u_{f,σ_1,σ_2}, which extends the map $\mathcal{T}_1 \to \mathcal{T}_2$, and is equivariant; we have the following 2-cartesian diagram:

$$\begin{tikzcd} \mathcal{X}_1 \times \mathcal{T}_1 \arrow[r, "(u, u|_{\mathcal{T}_1})"] \arrow[d, "a_1"'] & \mathcal{X}_2 \times \mathcal{T}_2 \arrow[d, "a_2"'] \\ \mathcal{X}_1 \arrow[r, "u"] & \mathcal{X}_2 \end{tikzcd}$$

where a_i is the $\mathcal{T}_i$-action on $\mathcal{X}_i$.

(4) A linear map $v = v_f : M_{2,\mathbb{R}} \to M_{1,\mathbb{R}}$ of real vector spaces.

Remark 5.16. In [13], Iwanari established an equivalence between the 2-category of toric stacks and the 1-category of stacky fans. In [13, Definition 2.1], N is a free abelian group, so toric stacks in [13] are toric orbifolds. □

When the source $\mathcal{X}_1$ is a *complete* toric DM stack, that is, the coarse moduli space X_1 of $\mathcal{X}_1$ is a complete simplicial toric variety, Perroni [21, Theorem 5.1] gave a description of morphisms $\mathcal{X}_1 \to \mathcal{X}_2$ in terms of homogeneous polynomials. This description is similar to Cox's description of morphisms from a complete toric variety to a smooth toric variety [5, Theorem 3.2].

Suppose that $\mathcal{X}_i = [U_i / G_{\mathbf{\Sigma}_i}]$, where $U_i = \mathbb{C}^{r_i} - Z(I_{\Sigma_i})$. The map $u: \mathcal{X}_1 \to \mathcal{X}_2$ is the toric morphism induced from f as already discussed. By Perroni [21, Theorem 5.1], there exists a map

$$F : \mathbb{C}^{r_1} \to \mathbb{C}^{r_2}, \quad z = (z_1, \ldots, z_{r_1}) \mapsto (P_1(z), \ldots, P_{r_2}(z)),$$

where $P_1, \ldots, P_{r_2} \in \mathcal{O}(\mathbb{C}^{r_1}) = \mathbb{C}[z_1, \ldots, z_{r_1}]$ are homogeneous polynomials such that

- $F(U_1) \subset U_2$;
- the following diagram is 2-commutative

$$\begin{tikzcd} U_1 \arrow[r, "\tilde{u}"] \arrow[d] & U_2 \arrow[d] \\ \mathcal{X}_1 \arrow[r, "u"] & \mathcal{X}_2 \end{tikzcd}$$

where $\tilde{u}$ is the restriction of F and the vertical arrows are the quotient maps.

Moreover, $\{P'_i\}$ and $\{P_i\}$ determine 2-isomorphic morphisms if and only if there exists $g \in G_{\boldsymbol{\Sigma}_2}$ such that

$$(P'_1, \dots, P'_{r_2}) = g \cdot (P_1, \dots, P_{r_2}).$$

Note that for a given choice of $\{P_i\}$, $\tilde{u}: U_1 \to U_2$ can be viewed as a morphism between smooth toric varieties. We have a cartesian diagram:

$$\begin{tikzcd} U_1 \times \tilde{T}_1 \arrow[r, "(\tilde{u}, \tilde{u}|_{\tilde{T}_1})"] \arrow[d, "\tilde{a}_1"'] & U_2 \times \tilde{T}_2 \arrow[d, "\tilde{a}_2"'] \\ U_1 \arrow[r, "\tilde{u}"] & U_2 \end{tikzcd}$$

where $\tilde{T}_i \cong (\mathbb{C}^*)^{r_i}$. This gives a group homomorphism $\tilde{f}: \tilde{N}_1 \to \tilde{N}_2$ which fits in the following commutative diagram.

$$\begin{tikzcd} \tilde{N}_1 \arrow[r, "\tilde{f}"] \arrow[d, "\beta_1"'] & \tilde{N}_2 \arrow[d, "\beta_2"'] \\ N_1 \arrow[r, "f"] & N_2 \end{tikzcd}$$

5.7 Coherent–constructible dictionary: functoriality and tensoriality

In this section, we show that the equivalence $\kappa_{\boldsymbol{\Sigma}}$ between coherent and constructible sheaves intertwines with appropriate pull-back and push-forward functors. We use the notation in Section 5.6.

Theorem 5.17 (functoriality). Let $f: \boldsymbol{\Sigma}_1 = (N_1, \Sigma_1, \beta_1) \to \boldsymbol{\Sigma}_2 = (N_1, \Sigma_2, \beta_2)$ be a morphism of stacky fans, where Σ_1 is a complete fan. Suppose that f furthermore satisfies the following conditions:

(i) The inverse image of any cone $\sigma_2 \subset \Sigma_2$ is a union of cones in Σ_1. (For instance, if both fans are complete, then f automatically satisfies this condition.)

(ii) f is injective.

Let u, v, $\mathcal{X}_i$, $\mathcal{T}_i$ be as in Section 5.6. Then

(1) The pullback $u^*: \mathcal{Q}_{\mathcal{T}_2}(\mathcal{X}_2) \to \mathcal{Q}_{\mathcal{T}_1}(\mathcal{X}_1)$ takes $\langle \Theta' \rangle_{\boldsymbol{\Sigma}_2}$ to $\langle \Theta' \rangle_{\boldsymbol{\Sigma}_1}$.

(2) The proper push-forward $v_!: Sh_c(M_{2,\mathbb{R}}) \to Sh_c(M_{1,\mathbb{R}})$ takes $\langle \Theta \rangle_{\boldsymbol{\Sigma}_2}$ to $\langle \Theta \rangle_{\boldsymbol{\Sigma}_1}$.

(3) The following square of functors commutes up to natural isomorphism:

$$\begin{tikzcd} \langle \Theta' \rangle_{\boldsymbol{\Sigma}_2} \arrow[r, "\kappa_2"] \arrow[d, "u^*"'] & \langle \Theta \rangle_{\boldsymbol{\Sigma}_2} \arrow[d, "v_!"] \\ \langle \Theta' \rangle_{\boldsymbol{\Sigma}_1} \arrow[r, "\kappa_1"] & \langle \Theta \rangle_{\boldsymbol{\Sigma}_1} \end{tikzcd}$$

where $\kappa_i = \kappa_{\boldsymbol{\Sigma}_i}$, $i = 1, 2$. □

Proof. As in Section 5.6, we choose liftings $\tilde{\Sigma}_i \subset \tilde{N}_{i,\mathbb{R}}$ and $\tilde{f}: \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$, which induces a morphism $\tilde{u}: U_1 \to U_2$ of smooth toric varieties such that

$$\begin{tikzcd} U_1 \arrow[r, "\tilde{u}"] \arrow[d] & U_2 \arrow[d] \\ \mathcal{X}_1 \arrow[r, "u"] & \mathcal{X}_2 \end{tikzcd}$$

where $U_i = X_{\tilde{\Sigma}_i}$ and $\mathcal{X}_i = [U_i / G_{\boldsymbol{\Sigma}_i}]$.

For each cone $\sigma_2 \in \Sigma_2$, we have a 2-cartesian square

$$\begin{tikzcd} u^{-1}(\mathcal{X}_{\sigma_2}) \arrow[r, "u'"] \arrow[d, "j"'] & \mathcal{X}_{\sigma_2} \arrow[d, "j_{\sigma_2}"] \\ \mathcal{X}_1 \arrow[r, "u"] & \mathcal{X}_2 \end{tikzcd} \tag{12}$$

where $u' = u|_{u^{-1}(\mathcal{X}_{\sigma_2})}$.

We have the following cartesian square, which corresponds to (12):

$$\begin{tikzcd} \tilde{u}^{-1}(U_{\sigma_2}) \arrow[r, "\tilde{u}'"] \arrow[d, "\tilde{j}"'] & U_{\sigma_2} \arrow[d, "\tilde{j}_{\sigma_2}"] \\ U_1 \arrow[r, "\tilde{u}"] & U_2 \end{tikzcd} \tag{13}$$

where $\tilde{u}' = \tilde{u}|_{\tilde{u}^{-1}(U_{\sigma_2})}$. Let $\mathcal{O}_{U_{\sigma_2}}(\chi_2)$ be defined as in Section 3.5. The vertical arrows in (13) are open inclusions, so by the flat base change formula we have

$$\tilde{u}^* \tilde{j}_{\sigma_2 *}\mathcal{O}_{U_{\sigma_2}}(\chi_2) \cong \tilde{j}_* \mathcal{L}$$

where $\mathcal{L} := (\tilde{u}')^* \mathcal{O}_{U_{\sigma_2}}(\chi_2)$ is a $\tilde{T}_1$-equivariant line bundle on $\tilde{u}^{-1}(U_{\sigma_2})$.

We fix a total order on the set of maximal cones $B_1, \ldots, B_w$ contained in $f^{-1}(\sigma_2)$. By assumption

$$u^{-1}(\mathcal{X}_{\sigma_2}) = \bigcup_{i=1}^{w} \mathcal{X}_{B_i}, \quad \tilde{u}^{-1}(U_{\sigma_2}) = \bigcup_{i=1}^{w} U_{B_i}.$$

For each B_i we have $f(N_{B_i}) \subset N_{\sigma_2}$. Let $f_{B_i,\sigma_2} : N_{B_i} \to N_{\sigma_2}$ be the restriction of f, and let $f^*_{B_i,\sigma_2} : M_{\sigma_2} \to M_{B_i}$ be the dual map of $\bar{f}_{B_i,\sigma_2} : \bar{N}_{B_i} \to \bar{N}_{\sigma_2}$. Let $\phi_i = f^*_{B_i,\sigma_2}(\chi_2) \in M_{B_i}$. Then

$$\mathcal{L}|_{U_{B_i}} = \mathcal{O}_{U_{B_i}}(\phi_i).$$

More generally, put

$$B_{i_0 \cdots i_k} = B_{i_0} \cap \cdots \cap B_{i_k}, \quad \phi_{i_0\cdots i_k} = f^*_{B_{i_0\cdots i_k},\sigma_2}(\chi_2) \in M_{B_{i_0\cdots i_k}}$$

then $\mathcal{L}|_{U_{B_{i_0\cdots i_k}}} = \mathcal{O}_{U_{B_i}}(\phi_{i_0\cdots i_k})$. Therefore $\tilde{j}_*\mathcal{L}$ is quasi-isomorphic to the Čech complex

$$\bigoplus_{i_0} \tilde{j}_{B_{i_0} *}\mathcal{O}_{U_{B_{i_0}}}(\phi_{i_0}) \to \bigoplus_{i_0<i_1} \tilde{j}_{B_{i_0 i_1} *}\mathcal{O}_{U_{B_{i_0 i_1}}}(\phi_{i_0 i_1}) \to \cdots .$$

Equivalently, $u^*\Theta'(\sigma_2, \chi_2)$ is quasi-isomorphic to the complex

$$\bigoplus_{i_0} \Theta'(B_{i_0}, \phi_{i_0}) \to \bigoplus_{i_0<i_1} \Theta'(B_{i_0 i_1}, \phi_{i_0 i_1}) \to \cdots .$$

This proves assertion (1).

After Theorem 5.11, assertions (2) and (3) follow from the commutativity of the following diagram:

$$\begin{tikzcd}
\langle \Theta' \rangle_{\boldsymbol{\Sigma}_2} \arrow[r, "\kappa_2"] \arrow[d, "u^*"'] & \mathrm{Sh}_c(M_{2,\mathbb{R}}) \arrow[d, "v_!"] \\
\langle \Theta' \rangle_{\boldsymbol{\Sigma}_1} \arrow[r, "\kappa_1"'] & \mathrm{Sh}_c(M_{1,\mathbb{R}})
\end{tikzcd}$$

We follow the strategy of the proof of [8, Theorem 3.8]. To construct a natural quasi-isomorphism $\iota : v_! \circ \kappa_2 \xrightarrow{\sim} \kappa_1 \circ u^*$, it suffices to give maps

$$\iota_{\sigma_2,\chi_2} : v_! \kappa_2(\Theta'(\sigma_2, \chi_2)) \to \kappa_1(u^* \Theta'(\sigma_2, \chi_2))$$

with the following properties.

(1) Each ι_{σ_2,χ_2} is a quasi-isomorphism.

(2) The following square commutes whenever $(\sigma_2, \chi_2) \le (\tau_2, \psi_2)$:

$$\begin{tikzcd}
v_! \kappa_2(\Theta'(\sigma_2, \chi_2)) \arrow[r] \arrow[d, "\iota_{\sigma_2,\chi_2}"'] & v_! \kappa_2(\Theta'(\tau_2, \psi_2)) \arrow[d, "\iota_{\tau_2,\psi_2}"'] \\
\kappa_1(u^* \Theta'(\sigma_2, \chi_2)) \arrow[r] & \kappa_1(u^* \Theta'(\tau_2, \psi_2))
\end{tikzcd}$$

As in the proof of [8, Theorem 3.8], we have a quasi-isomorphism

$$v_! \kappa_2(\Theta'(\sigma_2, \chi_2)) \xrightarrow{\sim} j_{v((\sigma_2)^\vee_{\chi_2})^\circ !}\omega, \quad \omega = \omega_{((\sigma_2^\vee)_{\chi_2})^\circ}.$$

Now let us compute $\kappa_1 u^* \Theta'(\sigma_2, \chi_2)$. We have already seen that $u^* \Theta'(\sigma_2, \chi_2)$ has the Čech resolution

$$\bigoplus_{i_0} \Theta'(B_{i_0}, \phi_{i_0}) \to \bigoplus_{i_0 < i_1} \Theta'(B_{i_0 i_1}, \phi_{i_0 i_1}) \to \cdots .$$

After applying κ_1, we have

$$\bigoplus_{i_0} \Theta(B_{i_0}, \phi_{i_0}) \to \bigoplus_{i_0 < i_1} \Theta(B_{i_0 i_1}, \phi_{i_0 i_1}) \to \cdots .$$

Now we define the map $\iota: v_!\kappa_2(\Theta'(\sigma_2,\chi_2)) \to \kappa_1 u^*\Theta'(\sigma_2,\chi_2)$ to be the morphism of complexes:

$$\begin{array}{ccccc}
j_{v((\sigma_2)^\vee_{\chi_2})^\circ !}\omega & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\bigoplus_{i_0} \Theta(B_{i_0}, \phi_{i_0}) & \longrightarrow & \bigoplus_{i_0<i_1} \Theta(B_{i_0 i_1}, \phi_{i_0 i_1}) & \longrightarrow & \cdots
\end{array}$$

where the nonzero vertical arrow is the direct sum of the maps induced by the inclusion of open sets

$$v((\sigma_2)^\vee_{\chi_2})^\circ \subset (B_{i_0})^\circ_{\phi_{i_0}}.$$

This map clearly has the desired naturality property. By the argument in the last part of the proof of [8, Theorem 3.8], it is a quasi-isomorphism. ∎

Example 5.18. The diagonal map $N \to N \oplus N$ satisfies the hypotheses of Theorem 5.17. The corresponding map $u: \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is also the diagonal map, and the corresponding map $v: M_{\mathbb{R}} \times M_{\mathbb{R}} \to M_{\mathbb{R}}$ is the addition map.

More generally, let $\boldsymbol{\Sigma} = (N, \boldsymbol{\Sigma}, \beta)$ be the stacky fan defining $\mathcal{X}$, where Σ is not necessarily complete. We have

$$\begin{array}{ccc}
U & \xrightarrow{\tilde{\Delta}} & U \times U \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}
\end{array}$$

where $\mathcal{X} = [U/G_{\boldsymbol{\Sigma}}]$ and $\mathcal{X} \times \mathcal{X} = [U/G_{\boldsymbol{\Sigma}}] \times [U/G_{\boldsymbol{\Sigma}}] \cong [U \times U/G_{\boldsymbol{\Sigma}} \times G_{\boldsymbol{\Sigma}}]$. Since we can lift the diagonal map $\Delta: \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ to the diagonal map $\tilde{\Delta}: U \to U \times U$ for any toric DM stack $\mathcal{X}$, the proof of Theorem 5.17 is valid, and the conclusions of Theorem 5.17 hold, when f is the diagonal map for any toric DM stack. □

Recall that the *convolution* of two sheaves F and G on a vector space $M_{\mathbb{R}}$ is given by the formula $F \star G = v_!(F \boxtimes G)$, where v denotes the addition map as in the example. Convolution defines a monoidal structure on $\mathrm{Sh}(M_{\mathbb{R}})$ and various subcategories, including $\langle \Theta \rangle_{\boldsymbol{\Sigma}}$ and $\mathrm{Sh}_{cc}(M_{\mathbb{R}})$. From Example 5.18, following the argument in the proof of [8, Corollary 3.13], we see the following corollary.

Corollary 5.19. For any stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$, the equivalence $\kappa_{\boldsymbol{\Sigma}} : \langle \Theta' \rangle_{\boldsymbol{\Sigma}} \xrightarrow{\sim} \langle \Theta \rangle_{\boldsymbol{\Sigma}}$ is an equivalence of monoidal dg categories, where the monoidal structure on $\langle \Theta' \rangle_{\boldsymbol{\Sigma}}$ is given by the tensor product of quasicoherent sheaves and the monoidal structure on $\langle \Theta \rangle_{\boldsymbol{\Sigma}}$ is given by convolution. □

6 Intrinsic Characterizations

In Section 5, we have given an equivalence $\kappa_{\boldsymbol{\Sigma}}$ between a certain dg category of quasi-coherent sheaves on the toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}}$—which we have called $\langle \Theta' \rangle_{\boldsymbol{\Sigma}}$—and a certain category of constructible sheaves on the real vector space $M_{\mathbb{R}}$—which we have called $\langle \Theta \rangle_{\boldsymbol{\Sigma}}$. The categories $\langle \Theta \rangle_{\boldsymbol{\Sigma}}$ and $\langle \Theta' \rangle_{\boldsymbol{\Sigma}}$ are defined by their set of generating objects $\{\Theta(\sigma, \chi)\}_{(\sigma,\chi)\in\Gamma(\boldsymbol{\Sigma})}$ and $\{\Theta'(\sigma, \chi)\}_{(\sigma,\chi)\in\Gamma(\boldsymbol{\Sigma})}$. In this section, we give intrinsic characterizations of these categories.

6.1 Shard arrangements

Definition 6.1 (Shard arrangement)**.** Let $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan. A *shard arrangement* for $\boldsymbol{\Sigma}$ is a closed set $E \subset M_{\mathbb{R}} \times N_{\mathbb{R}}$ of the form

$$E = \bigcup_{i=1}^{k} (\sigma_i)^{\perp}_{\chi_i} \times -\sigma_i,$$

where for each $i = 1, \ldots, k$, $(\sigma_i, \chi_i) \in \Gamma(\boldsymbol{\Sigma})$. □

Definition 6.2. A sheaf F on $M_{\mathbb{R}}$ is called a *$\boldsymbol{\Sigma}$-shard sheaf* if it is cohomologically bounded and constructible, and has finite-dimensional fibers and its singular support $SS(F)$ is a subset of a shard arrangement. Let $\mathrm{Shard}(M_{\mathbb{R}}; \boldsymbol{\Sigma})$ denote the triangulated dg category of $\boldsymbol{\Sigma}$-shard sheaves on $M_{\mathbb{R}}$. □

The union of all possible shard arrangements is a conical Lagrangian $\Lambda_{\boldsymbol{\Sigma}}$ in $M_{\mathbb{R}} \times N_{\mathbb{R}} = T^* M_{\mathbb{R}}$:

Definition 6.3. Define the conical Lagrangian

$$\Lambda_{\boldsymbol{\Sigma}} = \bigcup_{\tau\in\Sigma} \bigcup_{\chi\in M_\tau} \tau^{\perp}_{\chi} \times -\tau \subset M_{\mathbb{R}} \times N_{\mathbb{R}} = T^* M_{\mathbb{R}}. \tag{14}$$

□

For example, we have $\mathrm{Sh}_{cc}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}) \subset \mathrm{Shard}(M_{\mathbb{R}}; \boldsymbol{\Sigma})$. More generally, the sheaves $\Theta(\sigma, \chi)$ belong to $\mathrm{Shard}(M_{\mathbb{R}}; \boldsymbol{\Sigma})$ but do not have compact support. Suppose that $\tau \subset \sigma \in \Sigma$, let $f_{\tau\sigma}$ denote the inclusion $N_\tau \to N_\sigma$ and let $f^*_{\tau\sigma} : M_\sigma \to M_\tau$ be the dual map of $\bar{f}_{\tau\sigma} : \bar{N}_\tau \to \bar{N}_\sigma$.

We refer the reader to [16] for the microlocal sheaf theory, including the specialization operation $\nu_Y : \mathrm{Sh}_c(M_{\mathbb{R}}) \to \mathrm{Sh}_{c,\mathbb{R}_{>0}}(T_Y M_{\mathbb{R}})$ and the microlocalization operation $\mu_Y : \mathrm{Sh}_c(M_{\mathbb{R}}) \to \mathrm{Sh}_{c,\mathbb{R}_{>0}}(T^*_Y M_{\mathbb{R}})$. The category $\mathrm{Sh}_{c,\mathbb{R}_{>0}}(V)$ is the dg category of constructible sheaves on the vector space V that are constants on the orbits of the scaling action by $\mathbb{R}_{>0}$. The proof of the following proposition is identical to the proof of [8, Proposition 5.1].

Proposition 6.4. After identifying $T^* M_{\mathbb{R}}$ with $M_{\mathbb{R}} \times N_{\mathbb{R}}$, the singular support of $\Theta(\sigma, \chi)$ is given by the following:

$$SS(\Theta(\sigma, \chi)) = \bigcup_{\tau \subset \sigma} (\tau^{\perp}_{\chi_\tau} \cap \sigma^{\vee}_{\chi}) \times -\tau,$$

where $\chi_\tau = f^*_{\tau\sigma}(\chi)$. □

Proof. If τ is a face of σ then for every $y \in \tau^\circ$, the specialization $\nu_y(\Theta(\sigma, \chi))$ coincides with the costandard sheaf on $\tau^\vee \subset M_{\mathbb{R}}$ under the identification of $M_{\mathbb{R}}$ with $T_y M_{\mathbb{R}}$. Thus $\mu_y(\Theta(\sigma, x))$ is the standard sheaf on $-\tau$ by Kashiwara and Schapira [16, Lemma 3.7.10]. In particular, $\mu_{y,\xi}$ vanishes if and only if (y, ξ) is not in $(\tau^{\perp}_{\chi_\tau} \cap \sigma^{\vee}_{\chi}) \times -\tau$ for some τ. It follows that

$$SS(\Theta(\sigma, \chi)) = \bigcup_{\tau \subset \sigma} (\tau^{\perp}_{\chi_\tau} \cap \sigma^{\vee}_{\chi}) \times -\tau.$$ ■

Theorem 6.5. The dg category $\mathrm{Shard}(M_{\mathbb{R}}; \boldsymbol{\Sigma})$ is quasi-equivalent to $\langle \Theta \rangle_{\boldsymbol{\Sigma}}$. In other words, every shard sheaf is quasi-isomorphic to a bounded complex of the form

$$\cdots \to \bigoplus_i \Theta(\sigma_i, \chi_i) \to \bigoplus_j \Theta(\sigma_j, \chi_j) \to \cdots,$$

where each sum is finite. □

Proof. This is an immediate consequence of [8, Theorem 5.2]. ■

Combining Theorems 5.5 and 6.5, and the results in [8, Section 5], we obtain the following proposition.

Proposition 6.6 (Intrinsic characterization of $\langle\Theta\rangle$). If F belongs to $\mathrm{Shard}(\tilde{M}_{\mathbb{R}}, \boldsymbol{\Sigma})$, $\hat{\beta}^! F$ belongs to $\mathrm{Shard}(M_{\mathbb{R}}, \boldsymbol{\Sigma})$. The following square of functors commutes up to natural isomorphism:

$$\begin{array}{ccc} \langle\Theta\rangle_{\tilde{\Sigma}} & \xrightarrow{\cong} & \mathrm{Shard}(\tilde{M}_{\mathbb{R}}, \tilde{\Sigma}) \\ {\scriptstyle \hat{\beta}^!}\downarrow & & {\scriptstyle \hat{\beta}^!}\downarrow \\ \langle\Theta\rangle_{\boldsymbol{\Sigma}} & \xrightarrow{\cong} & \mathrm{Shard}(M_{\mathbb{R}}, \boldsymbol{\Sigma}) \end{array}$$

where all the arrows are quasi-equivalences of dg categories. □

6.2 Quasicoherent sheaves with finite fibers

We first recall a definition from [8, Section 6].

Definition 6.7. A quasicoherent sheaf (or complex of sheaves) on a scheme X has *finite fibers* if for each closed point $x \in X$ we have

(1) $\mathrm{Tor}_i(\mathcal{O}_X/\mathfrak{m}_x, \mathcal{F}) := h^{-i}(\mathcal{O}_X/\mathfrak{m}_x \overset{\mathbf{L}}{\otimes} \mathcal{F})$ are finite-dimensional and
(2) $\mathrm{Tor}_i(\mathcal{O}_X/\mathfrak{m}_x, \mathcal{F}) = 0$ for all but finitely many $i \in \mathbb{Z}$. □

If $f : S_1 \to S_2$ is faithfully flat, then we may check whether a quasicoherent sheaf $\mathcal{F}$ on S_2 has finite fibers by showing that $f^*\mathcal{F}$ on S_1 does. Thus, we have a good notion of quasicoherent sheaves with finite fibers on quotient stacks:

Definition 6.8. Let U be a scheme on which a group scheme G acts and let $\mathcal{X} = [U/G]$ be the quotient stack. We say that a quasicoherent sheaf (or complex of sheaves) on $\mathcal{X}$ has *finite fibers* if the corresponding G-equivariant quasicoherent sheaf (or complex of quasicoherent sheaves) on U has finite fibers.

Suppose that the G-action comes from a group homomorphism $\phi : G \to \tilde{T}$ where $\tilde{T}$ acts on U, and both G and $\tilde{T}$ are *abelian*. Then the Picard stack $\mathcal{T} = [\tilde{T}/G]$ acts on $\mathcal{X}$. We say that a $\mathcal{T}$-equivariant quasicoherent sheaf on $\mathcal{X}$ has finite fibers if the corresponding $\tilde{T}$-equivariant quasicoherent sheaf on U has finite fibers. □

Remark 6.9.

(1) Let $\mathcal{X} = [U/G]$ be a toric DM stack. Then any coherent sheaf on $\mathcal{X}$ has finite fibers. In particular, all vector bundles and perfect complexes have finite fibers.

(2) It follows from the adjunction formula

$$\hom_{\mathcal{O}_x/\mathfrak{m}_x}(\mathcal{F} \otimes \mathcal{O}_x/\mathfrak{m}_x, \mathcal{O}_x/\mathfrak{m}_x) \cong \hom_X(\mathcal{F}, \mathcal{O}_x/\mathfrak{m}_x)$$

that $\mathcal{F}$ has finite fibers if and only if $\mathrm{Ext}^i(\mathcal{F}, \mathcal{O}_x/\mathfrak{m}_x)$ is finite-dimensional for all i and vanishes for almost all i. □

Recall that there is a quasi-equivalence of dg categories (see Theorem 5.11)

$$q : \langle \Theta' \rangle_{\tilde{\Sigma}} \xrightarrow{\cong} \langle \Theta' \rangle_{\Sigma},$$

where $\langle \Theta' \rangle_{\Sigma}$ (resp. $\langle \Theta' \rangle_{\tilde{\Sigma}}$) is a dg subcategory of $\mathcal{Q}^{\mathrm{fin}}_{\mathcal{T}}(\mathcal{X})$ (resp. $\mathcal{Q}^{\mathrm{fin}}_{\tilde{T}}(U)$). It follows from the definition that

$$q : \mathcal{Q}^{\mathrm{fin}}_{\tilde{T}}(U) \xrightarrow{\cong} \mathcal{Q}^{\mathrm{fin}}_{\mathcal{T}}(\mathcal{X}).$$

By Fang et al. [8, Theorem 6.3], $\langle \Theta' \rangle_{\tilde{\Sigma}}$ is quasi-equivalent to $\mathcal{Q}^{\mathrm{fin}}_{\tilde{T}}(U)$. We conclude that $\langle \Theta' \rangle_{\Sigma}$ is quasi-equivalent to $\mathcal{Q}^{\mathrm{fin}}_{\mathcal{T}}(\mathcal{X})$:

Proposition 6.10 (Intrinsic characterization of $\langle \Theta' \rangle$). The following square of functors commutes up to natural isomorphism:

$$\begin{tikzcd}
\langle \Theta' \rangle_{\tilde{\Sigma}} \arrow[r, "\cong"] \arrow[d, "q"'] & \mathcal{Q}^{\mathrm{fin}}_{\tilde{T}}(U) \arrow[d, "q"'] \\
\langle \Theta' \rangle_{\Sigma} \arrow[r, "\cong"] & \mathcal{Q}^{\mathrm{fin}}_{\mathcal{T}}(\mathcal{X})
\end{tikzcd}$$

where all the arrows are quasi-equivalence of dg categories. □

6.3 Finite fibers and shard arrangements

After the main results of Sections 6.1 and 6.2 and Theorem 5.11, we have the following theorem.

Theorem 6.11. Let $\mathcal{X}_{\boldsymbol{\Sigma}}$ be a toric DM stack defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ and let $\tilde{\Sigma}$ and U be defined as in Section 3. Then the following square of functors commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathcal{Q}^{\mathrm{fin}}_{\tilde{T}}(U) & \xrightarrow{\kappa_{\tilde{\Sigma}}} & \mathrm{Shard}(\tilde{M}_{\mathbb{R}}, \tilde{\Sigma}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \hat{\beta}^!} \\ \mathcal{Q}^{\mathrm{fin}}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) & \xrightarrow{\kappa_{\boldsymbol{\Sigma}}} & \mathrm{Shard}(M_{\mathbb{R}}, \boldsymbol{\Sigma}) \end{array}$$

where all the arrows are quasi-equivalence of monoidal dg categories. □

7 Perfect Complexes and Compactly Supported Constructible Sheaves

In this section, $\mathcal{X} = \mathcal{X}_{\boldsymbol{\Sigma}}$ is a *complete* DM stack, that is, it is defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$, where Σ is a complete fan in $N_{\mathbb{R}}$. The coarse moduli space $X = X_{\Sigma}$ of $\mathcal{X}$ is a complete simplicial toric variety.

7.1 Generating sets of line bundles

Let $U \subset \mathbb{C}^r$, $G_{\boldsymbol{\Sigma}}$ be defined as in Section 3, so that $\mathcal{X} = [U/G_{\boldsymbol{\Sigma}}]$. Let $L_{\vec{c}}$, $L'_{\vec{c}}$, and $\mathcal{L}_{\vec{c}}$ be defined as in Section 4.1.

Proposition 7.1. Let $\mathcal{F}$ be a $\mathcal{T}$-equivariant coherent sheaf on $\mathcal{X}$. Then there is a $\mathcal{T}$-equivariant free resolution:

$$0 \to \mathcal{V}_l \to \cdots \to \mathcal{V}_0 \to \mathcal{F} \to 0,$$

where each $\mathcal{V}_i$ is the direct sum of $\mathcal{T}$-equivariant line bundles $\{\mathcal{L}_{\vec{c}} \mid \vec{c} \in \mathbb{Z}^r\}$. □

Proof. This can be proved by a slight modification of the proof of [4, Theorem 4.6]. We outline the argument here and refer to [4, Section 4] for details.

Let $\tilde{\mathcal{F}}$ be the $\tilde{T}$-equivariant coherent sheaf on U which descends to the $\mathcal{T}$-equivariant coherent sheaf $\mathcal{F}$ on $\mathcal{X} = \mathcal{X}_{\boldsymbol{\Sigma}}$. It suffices to show that there exists a $\tilde{T}$-equivariant free resolution

$$0 \to \tilde{\mathcal{V}}_l \to \cdots \to \tilde{\mathcal{V}}_0 \to \tilde{\mathcal{F}} \to 0, \tag{15}$$

where each $\tilde{\mathcal{V}}_i$ is the direct sum of $\tilde{T}$-equivariant line bundles $\{\tilde{L}_{\vec{c}} \mid \vec{c} \in \mathbb{Z}^r\}$.

Let $S = H^0(U, \tilde{\mathcal{F}})$ and let $A = \mathbb{C}[z_1, \ldots, z_r]$. By Borisov and Horja [4, Lemma 4.7], S is a finitely generated A-module. Let $\tilde{\mathcal{F}}'$ be the coherent sheaf on $\mathbb{C}^r = \mathrm{Spec} A$ determined by the finitely generated A-module S. Then $\tilde{\mathcal{F}}'|_U = \tilde{\mathcal{F}}$.

The $\tilde{T}$-linearization on $\tilde{\mathcal{F}}$ gives rise to a $\tilde{T}$-action on S, which is compatible with the $\tilde{T}$-action on A. Therefore A and S are graded by $\tilde{M} = \mathrm{Hom}(\tilde{T}, \mathbb{C}^*)$. There is a surjection

$$F_0 \to S \to 0,$$

where F_0 is a direct sum of rank 1 A-modules generated by the eigenelements of $\tilde{T}$. Let $I \subset A$ be the maximal idea generated by $z_1, \ldots, z_r$. We may choose $F_0 \to S$ such that the surjective map $F_0/IF_0 \to S/IS$ of $\mathbb{C}$-vector spaces is an isomorphism. Then the kernel S_1 of $F_0 \to S$ is contained in IF_0. We replace S by S_1 and repeat the procedure, and we obtain an exact sequence of $\tilde{T}$-equivariant A-modules

$$F_1 \xrightarrow{\phi_1} F_0 \to S \to 0,$$

where F_1 is a direct sum of rank 1 A-modules generated by the eigenelements of $\tilde{T}$. We continue this procedure and obtain a $\tilde{T}$-equivariant, free resolution

$$\cdots \to F_i \xrightarrow{\phi_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to S \to 0,$$

where the image of each $\phi_i : F_i \to F_{i-1}$ is contained in IF_{i-1}. The aforementioned resolution is a minimal graded resolution of the finitely generated graded A-module S, so it has finite length $l \leq r$ [7, Chapter 19]. Therefore, we have a finite, $\tilde{T}$-equivariant, free resolution

$$0 \to F_l \xrightarrow{\phi_l} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to M \to 0, \tag{16}$$

where each F_i is a direct sum of rank 1 A-modules generated by the eigenelements of $\tilde{T}$. The eigenelements of $\tilde{T}$ are of the form $z_1^{-c_1} \cdots z_r^{-c_r}$, $c_1, \ldots, c_r \in \mathbb{Z}$. The resolution (16) defines a finite, $\tilde{T}$-equivariant, free resolution

$$0 \to \tilde{\mathcal{V}}_l' \xrightarrow{\phi_l} \cdots \xrightarrow{\phi_2} \tilde{\mathcal{V}}_1' \xrightarrow{\phi_1} \tilde{\mathcal{V}}_0' \to \tilde{\mathcal{F}}' \to 0, \tag{17}$$

where each $\tilde{\mathcal{V}}_i'$ is a direct sum of $\tilde{T}$-equivariant line bundles $\{\tilde{L}'_{\vec{c}} \mid \vec{c} \in \mathbb{Z}^r\}$. Restricting (17) to U, we obtain a finite, $\tilde{T}$-equivariant, free resolution of the desired form (15). ■

Corollary 7.2. Let $\mathcal{X}$ be a toric DM stack.

(a) The cohomology category of $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X})$ is $D_{\mathcal{T}}(\mathcal{X})$, the bounded derived category of $\mathcal{T}$-equivariant coherent sheaves on $\mathcal{X}$.

(b) Any element in $K_{\mathcal{T}}(\mathcal{X}) = K(\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}))$ can be written as a finite sum $a_1\mathcal{L}_{\vec{c}_1} + \cdots + a_N\mathcal{L}_{\vec{c}_N}$, where $a_1, \ldots, a_N \in \mathbb{Z}$.

(c) The category $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X})$ is generated by $\mathcal{T}$-equivariant line bundles. □

7.2 Perfect complexes and compact support

In this section, we prove that for any complete toric DM stack $\mathcal{X}_{\boldsymbol{\Sigma}}$, there is a quasi-equivalence of monoidal dg categories

$$\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \cong \mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}).$$

As in [8, Section 7], the proof makes use of the monoidal structure, in particular, the fact that a complex of quasicoherent sheaves is perfect if and only if it is *dualizable*. For any sheaf $\mathcal{E} \in \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$, let $\mathcal{E}^\vee$ denote the hom sheaf $\underline{\mathrm{hom}}(\mathcal{E}, \mathcal{O})$. This assignment $\mathcal{E} \mapsto \mathcal{E}^\vee$ is a contravariant involution. The argument in the proof of [8, Theorem 7.3] shows the following.

Theorem 7.3. Suppose $\mathcal{X}$ is a complete toric DM stack and let $\kappa_{\boldsymbol{\Sigma}} : \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}) \hookrightarrow \mathrm{Sh}_{\mathrm{c}}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$ denote the functor defined in Section 5.5. There is a natural isomorphism (see Section 2.2 for the definition of the Verdier duality functor $\mathcal{D}$)

$$\kappa_{\boldsymbol{\Sigma}}(\mathcal{F}^\vee) \cong -\mathcal{D}(\kappa_{\boldsymbol{\Sigma}}(\mathcal{F})).$$ □

Lemma 7.4 (Toric Chow's lemma for toric DM stacks)**.** Let $\mathcal{X}_{\boldsymbol{\Sigma}}$ be a complete toric DM stack. Then there exists a stacky fan $\boldsymbol{\Sigma}' = (N, \Sigma', \beta')$ and a morphism of stacky fans $f : \boldsymbol{\Sigma}' \to \boldsymbol{\Sigma}$ (equivalently, a morphism of toric DM stacks $u : \mathcal{X}_{\boldsymbol{\Sigma}'} \to \mathcal{X}_{\boldsymbol{\Sigma}}$), where

(i) the toric variety $X_{\Sigma'}$ is projective,

(ii) the morphism $X_{\Sigma'} \to X_{\Sigma}$ is birational. □

Proof. By toric Chow's lemma (cf. [[6], [20, Proposition 2.17]]), there exists a refinement Σ' of Σ which induces a morphism $X_{\Sigma'} \to X_\Sigma$ of toric varieties such that (i) and (ii) hold. $\mathcal{X}_{\boldsymbol{\Sigma}'}$ is the toric DM stack obtained by taking the fiber product $X_{\Sigma'} \times_{X_\Sigma} \mathcal{X}_{\boldsymbol{\Sigma}}$. ■

Theorem 7.5 (CCC for toric DM stacks)**.** Suppose that $\mathcal{X} = \mathcal{X}_{\boldsymbol{\Sigma}}$ is a complete toric DM stack defined by a stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$. Then $\kappa_{\boldsymbol{\Sigma}}$ restricts to a quasi-equivalence of monoidal dg categories $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}) \cong \mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$. □

Proof. We first show that $\kappa_{\boldsymbol{\Sigma}}$ carries $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X})$ into $\mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$. By (c) of Corollary 7.2, it suffices to show that if $\mathcal{L}$ is a $\mathcal{T}$-equivariant line bundle on $\mathcal{X}$, then $\kappa(\mathcal{L})$ has compact support. By functoriality and by Lemma 7.4, we may assume that the coarse moduli space X_Σ is projective. Then there exist $\mathcal{T}$-equivariant, $\mathbb{Q}$-ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ on $\mathcal{X}$ such that $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$. Then

$$\kappa_{\boldsymbol{\Sigma}}(\mathcal{L}) = \kappa_{\boldsymbol{\Sigma}}(\mathcal{L}_1) \star \kappa_{\boldsymbol{\Sigma}}(\mathcal{L}_2^{-1}),$$

where $\kappa_{\boldsymbol{\Sigma}}(\mathcal{L}_2^{-1}) = -\mathcal{D}(\kappa_{\boldsymbol{\Sigma}}(\mathcal{L}_2))$ by Theorem 7.3. By Theorem 5.14, $\kappa_{\boldsymbol{\Sigma}}(\mathcal{L}_1)$ and $\kappa_{\boldsymbol{\Sigma}}(\mathcal{L}_2)$ have compact supports. Therefore, $\kappa_{\boldsymbol{\Sigma}}(\mathcal{L})$ has compact support.

It is clear from our earlier results that $\kappa_{\boldsymbol{\Sigma}} : \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \to \mathrm{Sh}_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$ is a fully faithful embedding of monoidal dg categories. To complete the proof of the theorem, it remains to be shown that $\kappa_{\boldsymbol{\Sigma}}$ is essentially surjective. Suppose that $F \in Sh_{\mathrm{cc}}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$. Then $F \in \mathrm{Shard}(M_{\mathbb{R}}; \boldsymbol{\Sigma})$, so there exists $\mathcal{G} \in \langle \Theta' \rangle_{\boldsymbol{\Sigma}}$ such that $\kappa_{\boldsymbol{\Sigma}}(\mathcal{G}) \cong F$. We also have $\mathcal{D}F \in \mathrm{Shard}(M_{\mathbb{R}}; \boldsymbol{\Sigma})$, so there exists $\mathcal{H} \in \langle \Theta' \rangle_{\boldsymbol{\Sigma}}$ such that $\kappa_{\boldsymbol{\Sigma}}(\mathcal{H}) \cong -\mathcal{D}F$. By Fang et al. [8, Lemma 7.4], $F \to F \star (-\mathcal{D}F) \star F \to F$ is the identity map. Therefore, $\mathcal{G} \to \mathcal{G} \otimes \mathcal{H} \otimes \mathcal{G} \to \mathcal{G}$ is the identity map. So $\mathcal{G}$ is strongly dualizable, and thus perfect (cf. [8, Proposition 7.2]). ■

8 Equivariant HMS for Toric DM Stacks

Recall that $\mathrm{Fuk}(T^*M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$ is a subcategory in the Fukaya category $\mathrm{Fuk}(T^*M_{\mathbb{R}})$, consisting of Lagrangian branes L whose boundary at infinity L^∞ is a subset of the infinity boundary of $\Lambda_{\boldsymbol{\Sigma}}$, as defined in [19]. We use $F(T^*M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$ to denote the A_∞ triangulated envelope of $\mathrm{Fuk}(T^*M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}})$. The following theorem is a direct consequence of Theorems 7.5 and 5.14, and the microlocalization functor in [18, 19].

Theorem 8.1. If $\mathcal{X}_{\boldsymbol{\Sigma}}$ is a complete DM stack, then there is a quasi-equivalence of triangulated A_∞-categories:

$$\tau : \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \xrightarrow{\sim} F(T^* M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}),$$

which is given by the composition

$$\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \xrightarrow{\kappa_{\boldsymbol{\Sigma}}} \mathrm{Sh}_{cc}(M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}) \xrightarrow{\mu_{M_{\mathbb{R}}}} F(T^* M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}).$$

If $\mathcal{O}_{\mathcal{X}_\Sigma}(\underline{\chi})$ is a $\mathcal{T}$-equivariant $\mathbb{Q}$-ample line bundle associated with a twisted polytope $\underline{\chi} = (\chi_1, \dots, \chi_r)$ of $\boldsymbol{\Sigma}$ then $\tau(\mathcal{O}_{\mathcal{X}_\Sigma}(\underline{\chi}))$ is a costandard brane over the interior of the convex hull $\mathbb{P}$ of $\chi_1, \dots, \chi_r \in M_{\mathbb{R}}$. □

We also have functoriality involving Fukaya categories [18]. Let Y_0 and Y_1 be real analytical manifolds. An object $\mathcal{K}$ in $\mathrm{Sh}_c(Y_0 \times Y_1)$ defines a functor

$$\Phi_{\mathcal{K}!} : \mathrm{Sh}_c(Y_0) \to \mathrm{Sh}_c(Y_1), \quad \mathcal{F} \mapsto p_{1!}(\mathcal{K} \otimes p_0^* \mathcal{F}), \tag{18}$$

where p_0 and p_1 are projections of $Y_0 \times Y_1$ to the corresponding components. For a Lagrangian brane L in $\mathrm{Fuk}(T^*(Y_0 \times Y_1))$, define a functor

$$\Psi_{L!} := \mu_{Y_1} \circ \Phi_{\mathcal{K}!} \circ \mu_{Y_0}^{-1} : F(T^* Y_0) \to F(T^* Y_1).$$

For two toric DM stacks $\mathcal{X}_1 = \mathcal{X}_{\boldsymbol{\Sigma}_1}$ and $\mathcal{X}_2 = \mathcal{X}_{\boldsymbol{\Sigma}_2}$ and a morphism of stacky fans $f : \boldsymbol{\Sigma}_1 = (N_1, \Sigma_1, \beta_1) \to \boldsymbol{\Sigma}_2 = (N_1, \Sigma_2, \beta_2)$, let $v : M_{2,\mathbb{R}} \to M_{1,\mathbb{R}}$ and $u : \mathcal{X}_1 \to \mathcal{X}_2$ be defined as in Section 5.7. Set the Lagrangian brane L_v to be the conormal bundle $T^*_{\Gamma_v}(M_{2,\mathbb{R}} \times M_{1,\mathbb{R}})$, where Γ_v is the graph of v in $M_{2,\mathbb{R}} \times M_{1,\mathbb{R}}$. The derivation of the following theorem is the same as that of [9, Theorem 3.7].

Theorem 8.2 (Functoriality). For two complete toric DM stacks $\mathcal{X}_1 = \mathcal{X}_{\boldsymbol{\Sigma}_1}$ and $\mathcal{X}_2 = \mathcal{X}_{\boldsymbol{\Sigma}_2}$ and a morphism of stacky fans

$$f : \boldsymbol{\Sigma}_1 = (N_1, \Sigma_1, \beta_1) \to \boldsymbol{\Sigma}_2 = (N_2, \Sigma_2, \beta_2)$$

and associated maps $u\colon \mathcal{X}_1 \to \mathcal{X}_2$, $u|_{\mathcal{T}_1}: \mathcal{T}_1 \to \mathcal{T}_2$, and $v: M_{2,\mathbb{R}} \to M_{1,\mathbb{R}}$, the following diagram commutes up to quasi-isomorphism:

$$
\begin{tikzcd}
\mathcal{P}\mathrm{erf}_{\mathcal{T}_2}(\mathcal{X}_2) \arrow[r, "\kappa_2"] \arrow[d, "u^*"'] & Sh_{cc}(M_{2,\mathbb{R}};\Lambda_{\boldsymbol{\Sigma}_2}) \arrow[r, "\mu_{M_{2,\mathbb{R}}}"] \arrow[d, "v_!"] & F(T^*M_{2,\mathbb{R}};\Lambda_{\boldsymbol{\Sigma}_2}) \arrow[d, "\Psi_{L_v!}"] \\
\mathcal{P}\mathrm{erf}_{\mathcal{T}_1}(\mathcal{X}_1) \arrow[r, "\kappa_1"] & Sh_{cc}(M_{1,\mathbb{R}};\Lambda_{\boldsymbol{\Sigma}_1}) \arrow[r, "\mu_{M_{1,\mathbb{R}}}"] & F(T^*M_{1,\mathbb{R}};\Lambda_{\boldsymbol{\Sigma}_1})
\end{tikzcd}
$$

where $\kappa_i = \kappa_{\boldsymbol{\Sigma}_i}$. □

In [9, Section 3.4], we define a product structure on the Fukaya category $F(T^*M_{\mathbb{R}})$ by

$$L_1 \diamond L_2 = \Psi_{L_v!}(L_1 \times L_2), \tag{19}$$

where $v: M_{\mathbb{R}} \times M_{\mathbb{R}}$ is the addition map. The following is a special case of [9, Proposition 3.9]:

Proposition 8.3 (The microlocalization intertwines the product structures)**.** The microlocalization functor $\mu_{M_{\mathbb{R}}}: \mathrm{Sh}_{cc}(M_{\mathbb{R}}) \xrightarrow{\sim} F(T^*M_{\mathbb{R}})$ intertwines the monoidal product on $\mathrm{Sh}_{cc}(M_{\mathbb{R}})$ given by the convolution and the product structure on $F(T^*M_{\mathbb{R}})$ given by the product $\diamond$ defined by (19), up to a quasi-isomorphism: the functors $\mu_{M_{\mathbb{R}}}(- \star -)$ and $\mu_{M_{\mathbb{R}}}(-) \diamond \mu_{M_{\mathbb{R}}}(-)$ are quasi-isomorphic in the category of A_∞-functors from $Sh_{cc}(M_{\mathbb{R}}) \times Sh_{cc}(M_{\mathbb{R}})$ to $F(T^*M_{\mathbb{R}})$. □

Corollary 8.4. The quasi-equivalence $\tau: \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \xrightarrow{\sim} F(T^*M_{\mathbb{R}};\Lambda_{\boldsymbol{\Sigma}})$ intertwines the monoidal product on $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$ given by the tensor product $\otimes$ of sheaves and the product structure on $\mathrm{Fuk}(T^*M_{\mathbb{R}};\Lambda_{\boldsymbol{\Sigma}})$ given by the product $\diamond$ defined by (19), up to a quasi-isomorphism: the functors $\tau(- \otimes -)$ and $\tau(-) \diamond \tau(-)$ are quasi-isomorphic in the category of A_∞-functors from $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \times \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$ to $\mathrm{Fuk}(T^*M_{\mathbb{R}};\Lambda_{\boldsymbol{\Sigma}})$—the category $\mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \times \mathcal{P}\mathrm{erf}_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}})$ carries a dg structure as the standard tensor product of dg categories. □

Theorem 8.5 (Equivariant HMS for toric DM stacks)**.** Let $\mathcal{X}_{\boldsymbol{\Sigma}}$ be a complete toric DM stack defined by a stack fan $\boldsymbol{\Sigma}$. Then there is an equivalence of tensor triangulated categories:

$$H(\tau): D_{\mathcal{T}}(\mathcal{X}_{\boldsymbol{\Sigma}}) \xrightarrow{\sim} DF(T^* M_{\mathbb{R}}; \Lambda_{\boldsymbol{\Sigma}}). \qquad \square$$

Acknowledgement

We thank Hsian-Hua Tseng for his helpful suggestions. We also thank Johan de Jong, Fabio Nironi, and Jason Starr for helpful conversations.

Funding

The work was supported in part by NSF/DMS-0707064 (to E.Z.).

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