CLUSTER VARIETIES FROM LEGENDRIAN KNOTS

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ABSTRACT. We show that many interesting spaces — including all positroid strata and wild character varieties — are moduli of constructible sheaves on a surface with microsupport in a Legendrian link. We show that the existence of cluster structures on these spaces may be deduced in a uniform, systematic fashion by constructing a set of exact Lagrangian fillings in correspondence with alternating representatives of the link, then applying sheaf quantization. It follows that results in cluster algebra may be used to construct and distinguish exact Lagrangian fillings of Legendrian links in the standard contact three space.

CONTENTS

1. Introduction

The moduli space of decorated local systems on a punctured surface admits a cluster structure — which is to say, a space of *nonabelian* representations of the fundamental group can be built out of algebraic tori, which are objects of an *abelian* nature. ①This remarkable fact was first demonstrated by gluing together cluster structures constructed by combinatorial and representation theoretic arguments in a manner prescribed by a triangulation of a surface. ②③In this paper we show that these varieties can be constructed globally as moduli of objects in a category. Conceptually, this category is the Fukaya category of Lagrangian branes in the cotangent bundle of a surface, with boundary conditions defined by an appropriate Legendrian knot (or link) — but employing the microlocal philosophy, we work entirely with the category constructible sheaves in this paper. Our global formulation means, for example, that the form of cluster transformations arise from computations, not axioms. This new linkage also allows us to exploit cluster theory to prove symplectic results about Lagrangian fillings of Legendrians. We explain these results below.

Cluster algebra has the following previously known and rather suggestive connections to symplectic geometry. The building block associated to a triangle can be understood in terms of Postnikov's theory of bicolored graphs. In this theory, certain graph moves correspond to cluster transformations. D. Thurston showed that under such transformations, one encoding of the fundamental combinatorial data — the alternating strand diagrams — transformed as if front projections of Legendrian isotopies. Goncharov and Kenyon showed that the 'boundary measurement map' could be used to identify the cluster charts with spaces of rank one local systems on 'conjugate surfaces' whose boundaries could be identified with the alternating strand diagrams. Finally, it is known in symplectic geometry that cluster coordinate transformations appear in 'wall crossing'. More precisely, given a family of exact Lagrangians, smooth away from the appearance in one instant of a single double point, the moduli spaces of objects defined by rank one local systems before and after the critical moment are each complex tori, and they are related by a cluster transformation.④⑤⑥

Here we will give an account of the existence of cluster structures on moduli spaces of local systems which begins with the above geometric structures — legendrian knots and surfaces which fill them — and arrives at combinatorial formulas only as the result of calculations, never as prescriptions. The knots and fillings will live in the cotangent bundle of the base surface. To go from Lagrangians to sheaves we use the functor of "sheaf quantization", which takes local systems on Lagrangians in a cotangent to constructible sheaves on the base. We will identify our construction with the existing theory by showing that sheaf quantization is, a posteriori in these examples, computed by Postnikov's boundary measurement map.

To explain both the etymology of the 'sheaf quantization' and the significance of this functor, recall that quantization in the usual sense is (one wishes) a functor from symplectic geometry to analysis, which takes L^2 functions on lagrangians in the cotangent bundle to distributions on the base, and symplectomorphisms to unitary operators given by integral kernels.⑦⑧

Sheaf quantization (which can be viewed as arising from the topologically A-twisted superstring, aka Floer-Fukaya theory [NZ, Nad], but which can be constructed internally in sheaf theory [GKS, Gui]) is a functor taking locally constant sheaves on lagrangians in the cotangent bundle to constructible sheaves on the base, and conic symplectomorphisms to autoequivalences of the sheaf category given by integral kernels. Because constructible sheaves are ultimately combinatorial in nature, sheaf quantization is a functor from symplectic geometry to combinatorics.

The foundational microlocal study of sheaves was begun in [KS]; the recent works [Tam, GKS, Gui] have provided the essential theorems on sheaf quantization. The key results we need are the following. In [KS], the microsupport of a sheaf on a manifold is defined; it is a conical co-isotropic locus in the cotangent bundle measuring the failure of local propagation of sections. For the sheaves of interest here, this locus is a stratifiable (generally singular) conical Lagrangian. One can study the subcategory of sheaves with fixed microsupport. In [GKS], it is shown that 'contact isotopies quantize,' i.e. that given a contact isotopy there's a unique family of sheaf integral kernels, such that the corresponding autoequivalences of the sheaf category act on microsupports by the specified isotopy. In [Gui] it is shown that 'Lagrangians quantize', i.e. that given an eventually conical exact Lagrangian $L \subset T^*M$, there's a fully faithful functor from locally constant (Maslov-twisted) sheaves on L to sheaves on M whose microsupport at infinity is ∂L.

We can now describe our construction. Moduli of local systems with invariant flags at punctures can be understood as moduli spaces of sheaves with fixed microsupport in Legendrian circles near the boundary punctures. This is almost tautological; what is less so is that a similar description applies to moduli of 'wild' local systems (Section 3.3), and also to positroid varieties (Theorem 3.9). The combinatorial description of the sheaves microsupported in such circles give the natural definitions of these objects; e.g. for the positroids, the description exhibits their embedding in a Grassmannian.

In fact we introduce a larger class of moduli spaces, unifying and generalizing all these examples, to which our methods will apply. These are described by the data of a surface, together with a positive braid at each puncture. The braids which occur for local systems and positroid varieties are a tiny subset of all possible braids. ⑨⑩

Rank one local systems on exact Lagrangian filling L of this Legendrian boundary are parameterized by the complex torus $\hom(\pi_1(L), \mathbb{C}^*) = (\mathbb{C}^*)^{b_1(L)}$. By sheaf quantization, this determines a chart on moduli. Quantization of contact isotopies means that we may move the boundary around until we are best able to see such a filling.

We show that when the Legendrian has been isotoped to be an alternating link — equivalently, to look like the alternating strand diagram of a bicolored graph — it is possible to construct a filling Lagrangian (Theorem 4.9). This filling retracts onto a bicolored graph embedded in the base surface, and is an exact Lagrangian embedding of the conjugate surface introduced by Goncharov and Kenyon. In the case of the positroid varieties, we show that the isotopy from the defining braid of the positroid to a given alternating representative is unique up to homotopy (Proposition 5.3). We
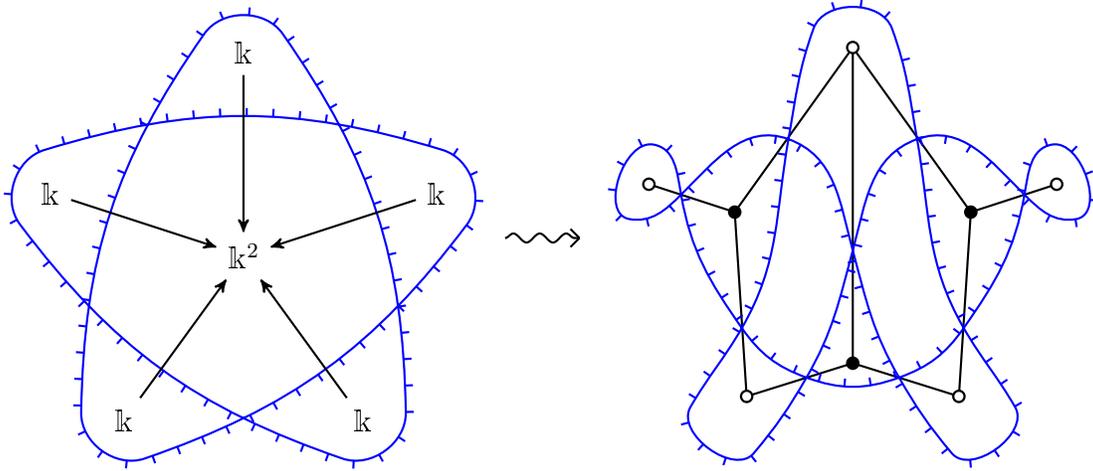
FIGURE 1. On the left, a Legendrian braid satellite whose rank-one moduli space recovers the open positroid stratum in $\mathrm{Gr}(2,5)$. The Legendrian lives in the cocircle bundle of the page; the hairs drawn along the immersed curve indicate the conormal directions of the lift of the projection. On the right, a bipartite graph whose associated alternating Legendrian is related to the knot on the left by Legendrian isotopy. If we restrict our attention to a disk whose boundary passes through the five white vertices we obtain a reduced plabic graph in the terminology of [Pos].

calculate that the sheaf quantization of this homotopy, when written in coordinates, is the boundary measurement map of Postnikov (Theorem 5.17).

As observed by D. Thurston, the so-called *square move* on graphs corresponds to a Legendrian isotopy from one alternating Legendrian to another. The corresponding Lagrangian fillings differ by a Lagrangian surgery. We calculate that the resulting charts differ by a cluster transformation (Theorem 5.8). (This could also be deduced from the fact that such surgery induce cluster mutations [Sei2] if one is willing to accept [NZ, Nad].) ⑪⑫

We also show that, more generally, braids around punctures can be isotoped to alternating Legendrians.

We emphasize the following novel aspect of this approach. Each bicolored graph (equivalently, each alternating Legendrian) determines **at once** both the algebraic torus of local systems on the filling, and the larger moduli space of sheaves with microsupport in the Legendrian. This latter moduli space has a direct global definition which is a priori invariant under Legendrian isotopy — in particular, under the square move — rather than being defined via some gluing procedure of the algebraic tori. Instead, the fact that it possesses an atlas of toric charts related by cluster transformations is deduced from the presence of Legendrian fillings.

The fact that the fillings of a knot give rise to a cluster structure can be used to deduce geometric consequences. We conclude the paper with one example. Note that sheaf quantization (or Floer

11. **EZ:** Remov sentence. The word "cluster" does not appear in Seidel's paper, and anyway I couldn't countenance the latter half of this sentence. (I claim this A_n issue is nonexistent and equivalent to the transverse perturbation problem in Fukaya theory.)

12. **VS:** We can remove this particular sentence, but somewhere we
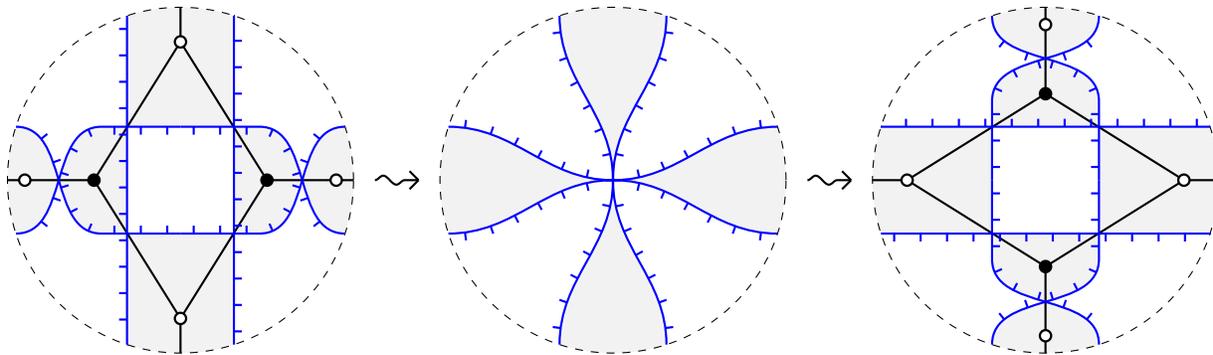
FIGURE 2. The left and right frames show the front projections of alternating Legendrians related by a square move of their associated bipartite graphs. There is a canonical Legendrian isotopy between them, which passes through alternating Legendrians except at one moment pictured in the middle. This Legendrian has a singular exact filling which meets the cotangent space of the origin in the union of the conormal lines to the x- and y-axes. Thus the isotopy of Legendrians is associated to a family of exact fillings that undergoes a Lagrangian surgery.

theory) implies that Lagrangian fillings of Legendrians determine objects in a certain category; one can in principle distinguish them by computing Homs. In practice this can be somewhat unwieldy.

Here we observe that the cluster structures we have constructed allow a different argument. Each Lagrangian filling determines a chart on a moduli space, and we know that the relation of charts is governed by cluster algebra. That is, we can use cluster algebra to show that certain Lagrangians give different charts. But, we know that Hamiltonian isotopic fillings must give the same chart.

We apply this to the $(2, n)$ torus link. In [EHK], a Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ of fillings was constructed; however, it was only possible to show these were non-pairwise-equivalent for $n = 3$. The above construction provides the same number of fillings (presumably the same fillings), but moreover distinguishes them one from another — their distinctness corresponds to the combinatorial fact that the A_{n-1} cluster structure has the Catalan number C_n of clusters [FZ2]. In particular, the answer to question 8.7 of [EHK] is "no". We also construct and distinguish fillings for torus knots more generally.

2. CONSTRUCTIBLE SHEAVES, MICROLOCALIZATION, AND MODULI

In this section we review the relevant background on constructible sheaves, including their singular support, microlocalization, moduli spaces, and invariance under contact transformations. We refer the reader to [KS] for detailed foundations. We note for the expert that here we adopt certain conventions adapted to the fact that we work throughout with Legendrians whose front projection is an immersion and we can therefore canonically trivialize all Maslov obstructions.

Throughout we fix a commutative ring $\Bbbk$. For a manifold X write $Sh(X) := Sh(X; \Bbbk)$ for the differential graded derived category of constructible sheaves on X – the dg category of constructible sheaves of perfect $\Bbbk$-modules on X, localized at the acyclic complexes. We refer to [Kel, Toë3] for background on dg categories. We write e.g. isomorphism instead of quasi-isomorphism when no confusion should arise.

2.1. **Singular support.** Given a sheaf $\mathcal{F} \in Sh(X)$, the **singular support** (or **microsupport**) $SS(\mathcal{F})$ is a closed, conic, Lagrangian subset of T^*X. The singular support at infinity of $\mathcal{F}$ is the Legendrian image of $SS(\mathcal{F})$ in the cocircle bundle $T^\infty X := (T^*X \setminus 0_X)/\mathbf{R}_+$, where $0_X \subset T^*X$ denotes the zero section. These notions are meant to capture the locus in T^*M of obstructions to the propagation of sections of $\mathcal{F}$ (see Examples 2.4 and 2.5). For instance, if $f : M \to \mathbb{R}$ is a function such that the graph of df avoids $SS(\mathcal{F})$ over the locus $f^{-1}((a, b])$, then the restriction of sections is an isomorphism [KS, Prop. 5.2.1]:

$$H^*(f^{-1}(-\infty, b], \mathcal{F}) \xrightarrow{\sim} H^*(f^{-1}(-\infty, a], \mathcal{F})$$

The formal definition is a local version of the above criterion:

Definition 2.1. [KS, Chap. 5] A point $p = (x, \xi) \in T^*X$ is in the microsupport of a sheaf $\mathcal{F}$ if there are points (x', ξ') arbitrarily close to (x, ξ) and functions $f : M \to \mathbb{R}$ with $f(x') = 0, df(x') = \xi'$, such that the following property holds: if $c_f : \{x \,|\, f(x) \geq 0\} \to M$ is the inclusion, then $(c_f^! \mathcal{F})_{x'} \neq 0$.

Shriek pullback to a closed subset gives the local sections supported on that subset. Thus the statement $(c_f^! \mathcal{F})_{x'} \neq 0$ is informally read as: "there is a section of $\mathcal{F}$ beginning at x' and propagating in the direction along which f increases." Note that, taking the zero function, the support of $\mathcal{F}$ is contained in its microsupport.

Definition 2.2. Given a Legendrian $\Lambda \subset T^\infty X$, we write $Sh_\Lambda(X) := Sh_\Lambda(X; \Bbbk)$ for the full subcategory of $Sh_\Lambda(X)$ consisting of sheaves whose singular support at infinity is contained in Λ. Note that every locally constant sheaf belongs to $Sh_\Lambda(X)$. If σ is a set of points in X not meeting the front projection of Λ, we write $Sh_\Lambda(X, \sigma)$ for the full subcategory of $Sh_\Lambda(X)$ consisting of sheaves that vanish on σ. (13)

13. **VS:** I am considering the alternate notation $Sh_{\Lambda - \sigma}(X)$

The subcategories $Sh_\Lambda(X), Sh_\Lambda(X, \sigma)$ are triangulated, since given a triangle $A \to B \to C \xrightarrow{[1]}$ we have $SS(C) \subset SS(A) \cup SS(B)$. Any sheaf in $Sh_\Lambda(X, \sigma)$ vanishes not only on σ, but on each component of $\Sigma \setminus \pi(\Lambda)$ containing a point in σ.

A key principle of [KS] is that a sheaf $\mathcal{F}$ localizes not just over X, but "microlocalizes" over the cotangent bundle T^*X and its own singular support $SS(\mathcal{F})$ in particular. There is a dg category $\mu loc(\Lambda)$, the category of microlocal sheaves on Λ, and a functor

$$\mu : Sh_\Lambda(X) \to \mu loc(\Lambda).$$

The category $\mu loc(\Lambda)$ is defined as follows. Following [KS], for $\Omega \subset T^*X$ one takes the category $\mu sh^{pre}(\Omega) := Sh(X)/Sh_{T^*X\setminus\Omega}(X)$. This is a presheaf of dg categories on T^*X, we write μsh for its sheafification. One shows that the (micro) support of an object in $\mu sh(\Omega)$ is a well defined conical Lagrangian in Ω, and that this construction is respected by restriction. It follows that there's a subsheaf μsh_Λ on full subcategories on objects supported in Λ; we write $\mu loc(\Lambda) := \mu sh_\Lambda(\Lambda)$.

The local sections μsh_Λ can be understood explicitly in certain cases. When $\Lambda \to X$ is finite, and $U \subset X$ is sufficiently small, then $\mu loc_\Lambda(U)$ is the quotient of $Sh_\Lambda(U)$ by local systems on U. This quotient is in turn equivalent to the subcategory of $Sh_\Lambda(U)$ of objects with stalk zero at any specified point (though the equivalence depends on the point). I.e., in this case the presheaf-of-categories description remains true after sheafifying. A sufficient condition that a chart is small enough is that the stratification is C^1-conical. One generally computes $\mu loc(\Lambda)$ by finding enough such charts and gluing. (In case $\Lambda \to X$ is not finite, one can use the local invariance of sheaves under contact transformations to make it so.)

In particular, when Λ is smooth, $\mu loc(\Lambda)$ is itself locally equivalent to $Loc(\Lambda)$. This is seen at a point by applying a contact transformation to make Λ locally the conormal to a smooth hypersurface. In general, there is only a 'Maslov' obstruction to globalizing this equivalence. A trivialization $\mu loc(\Lambda) \cong Loc(\Lambda)$ is determined by a Maslov potential on Λ; see [STZ] for a detailed discussion in the case where Λ is one dimensional, or [Gui] for a general account.

In this paper we generally do not consider front projections with cusps. As such we can always take the zero Maslov potential, and identify $\mu loc(\Lambda) \cong Loc(\Lambda)$. Given a sheaf $\mathcal{F} \in Sh_\Lambda(X)$, we write $\mathcal{F}|_\Lambda$ for its image in $Loc(\Lambda)$.

Definition 2.3. A sheaf $\mathcal{F} \in Sh_\Lambda(X)$ has **microlocal rank** n if the microlocalization of $\mathcal{F}$ along Λ is a local system of locally free $\Bbbk$-modules of rank n supported in degree zero. We denote by $\mathcal{C}_n(X, \Lambda)$ the full subcategory of $Sh_\Lambda(X)$ consisting of microlocal rank-n sheaves, similarly for $\mathcal{C}_n(\Lambda, \sigma)$. When X is fixed or clear from context, we omit it from the notation.

If $(x, \xi) \in \Lambda$ is a point, the **microlocal stalk** $\mathcal{F}|_{(x,\xi)}$ of $\mathcal{F} \in Sh_\Lambda(X)$ is by definition the stalk of $\mathcal{F}|_\Lambda$ at (x, ξ). For a point *at which $\Lambda \to X$ is an immersion* it can be computed directly as follows. Pick a function f defined in a neighborhood of x so that $\xi = df(x)$, as well as a small ball U around x and $\epsilon > 0$. Then $\mathcal{F}|_{(x,\xi)}$ is the cone over the restriction map from $\Gamma(U \cap \{f < \epsilon\}; \mathcal{F})$ to $\Gamma(U \cap \{f < -\epsilon\}; \mathcal{F})$. This does not depend on the precise choice of a sufficiently small U and ϵ.

For the most part our Legendrians will be smooth 1-dimensional submanifolds of the cosphere bundle of a surface, which we denote Σ rather than X. Except in Section 6, it will also be true that
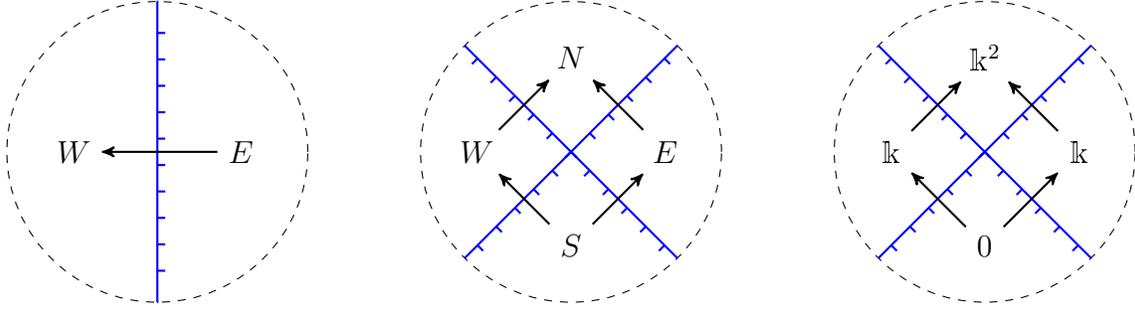
FIGURE 3. The local models of the sheaf categories we consider. On the left is an open disk where $\Lambda \to D^2$ is a single embedded strand, and $Sh_\Lambda(D^2) \cong \Bbbk A_2$-mod as described in Example 2.4. In the middle $\Lambda \to D^2$ is two embedded strands crossing, and $Sh_\Lambda(D^2) \cong \Bbbk A_3$-mod as described in Example 2.5. The rightmost picture illustrates a microlocal rank-one sheaf in this case.

the projection $\Lambda \to \Sigma$ is an immersion. Thus the sheaves we need to work with are locally of one of the following forms:

Example 2.4. Let D^2 be the open unit disk in $\mathbb{R}^2$ and $\Lambda = dx|_{\{x=0\}}$ the Legendrian whose front projection is the y-axis, cooriented to the right. Then $Sh_\Lambda(D^2)$ is equivalent to $\Bbbk A_2$-mod, the (dg derived) category of (perfect) representations of the A_2 quiver, as follows. We write W and E for any stalks in the open left half-disk $\{x < 0\}$ and closed right half-disk $\{x \geq 0\}$, respectively (all stalks in either region are canonically isomorphic up to homotopy). There is a generization map $E \to W$ given by restricting from a neighborhood of a point on the y-axis to a smaller open set lying entirely to the left of the y-axis (note the non-isomorphic restriction maps go "against the grain" of the covector in general). The microlocal stalk at a point of Λ is the cone over this map. An example of a sheaf of microlocal rank one is $i_!\Bbbk_{\{x<0\}}$, the extension by zero of the constant sheaf on the open left half-disk, which corresponds to $W = \Bbbk$, $E = 0$.

Example 2.5. Let D^2 be the open unit disk in $\mathbb{R}^2$ and $\Lambda = (dx - dy)|_{\{x=y\}} \cup (-dx - dy)|_{\{x=-y\}}$ the Legendrian whose front projection is the union of the lines $x = y$ and $x = -y$, co-oriented downwards (see Figure 3). Then $Sh_\Lambda(D^2)$ can be described in terms of the dg category of quadruples N, W, E, S of perfect complexes of $\Bbbk$-modules, with a commuting square of maps as pictured. Such data gives rise to an object of $Sh_\Lambda(D^2)$ under the following **crossing condition**: the total complex $S \to W \oplus E \to N$ must be acyclic [STZ, Theorem 3.12].

The restrictions $Sh_\Lambda(D^2) \to Sh_\Lambda(D^2 \cap \{y > \epsilon\})$, $Sh_\Lambda(D^2) \to Sh_\Lambda(D^2 \cap \{y < -\epsilon\})$ to the regions above and below the x-axis are equivalences. These categories are equivalent to the representation categories of two different orientations of the A_3 quiver, which we obtain by forgetting S, N, respectively. The induced equivalence $Sh_\Lambda(D^2 \cap \{y > \epsilon\}) \cong Sh_\Lambda(D^2 \cap \{y < -\epsilon\})$ is a reflection functor.

An example of a sheaf of microlocal rank one is the direct sum $i_!\mathbb{k}_{\{x+y>0\}} \oplus i_!\mathbb{k}_{\{y-x>0\}}$, which has $S = 0$, $W = E = \mathbb{k}$, and $N = \mathbb{k}^2$. The crossing condition here says that N is the direct sum of the images of W and E. If σ is any point in the bottom quadrant, this sheaf is in $Sh_\Lambda(D^2, \sigma)$. This is the only object of $\mathcal{C}_1(\Lambda, \sigma; \mathbb{k})$ if $\mathbb{k}$ has no nontrivial invertible modules.

2.2. **Invariance under contact transformations.** Invariance of the category $\mathcal{C}_1(\Lambda, \sigma; \mathbb{k})$ under Legendrian isotopy follows from the main theorem of [GKS]:

Theorem 2.6. [GKS] *Let M be a manifold and $\phi_t : T^\infty M \to T^\infty M$ be a contact isotopy. Then there's a unique sheaf Φ on $M \times M \times [0,1]$ which restricts to the constant sheaf on the diagonal on $M \times M \times 0$ and whose microsupport at infinity is the graph of the contact isotopy.*

We will mostly use the following consequence:

Proposition 2.7. *A Legendrian isotopy $\Lambda \to \Lambda'$ supported in the complement of the conormal to σ induces an equivalence of categories $Sh_\Lambda(M, \sigma) \cong Sh_{\Lambda'}(M, \sigma)$. The equivalence induces by a composition of isotopies is the composition of the equivalences induced by the isotopies.*

Proof. We recall also from contact geometry that given an isotopy of smooth Legendrians, there's an ambient contact isotopy which induces it. One can chose such contact isotopy supported in neighborhood of the Legendrian isotopy, and with this requirement said contact isotopy is unique up to homotopy. Let Φ_t be its sheaf quantization of one such isotopy.

The theory of sheaf integral transforms developed in [KS] implies that using Φ_1 as an integral kernel gives an equivalence of categories $Sh_\Lambda(M) \cong Sh_{\phi_1(\Lambda)}(M)$, and that (due to the support assumption) this transform preserves the stalk at σ.

To see that this functor does not depend on the choice of contact isotopy inducing the Legendrian isotopy, consider two and connect them by an homotopy of contact isotopies. The sheaf quantization provides a kernel $\Phi_{t,s}$ on $M \times M \times I \times I$. This is *not* constant in the final I direction; however when applied to elements in $Sh_\Lambda(M)$ it produces sheaves on $M \times I \times I$ whose microsupport is in the movie of $\phi_t(\Lambda)$ times a trivial factor in the final direction. It follows that these sheaves are constant in the final direction, and hence that the functors induced by $\Phi_{t,0}$ and $\Phi_{t,1}$ are the same.

Having shown this, the uniqueness of Φ_t for a given contact isotopy implies the functoriality of this correspondence asserted above. $\square$

Reidemeister Moves. Since the GKS equivalence is compatible with composition of isotopies, in order to compute the equivalence $\mathcal{C}_1(\Lambda, \sigma; \mathbb{k}) \cong \mathcal{C}_1(\Lambda', \sigma; \mathbb{k})$ associated to general Legendrian isotopy, it is enough to determine the equivalences associated to Legendrian Reidemeister moves. The ones relevant to our immediate purposes are pictured in Figures 4 and 5. Like all isotopy equivalences, these are determined by the kernels constructed in [GKS]. However, in these simple cases the equivalences are determined by the property that they restrict to the identity on the boundary of the picture, and can be described explicitly in terms of quiver representations.
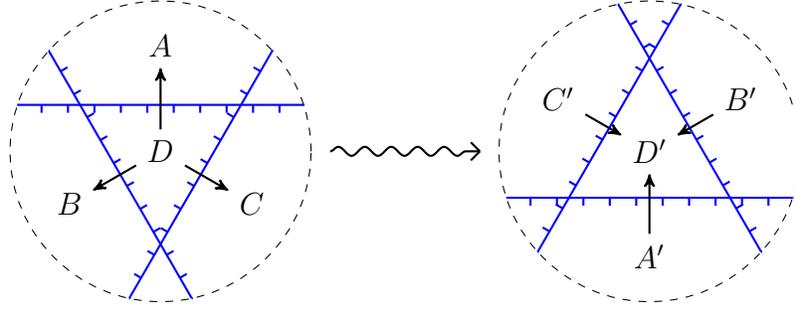
FIGURE 4. The Legendrian Reidemeister-III considered in Lemma 2.9. Sheaves microsupported on the respective Legendrians are equivalent to representations of oppositely-oriented D_4-quivers. Writing s_A, etc. for reflection functors, the equivalence between the two sides is given by the composition $s_D s_A s_B s_C s_D$.

Proposition 2.8. [STZ] *Let Λ, Λ' be a pair of Legendrians in $T^\infty D^2$ differing by a Legendrian Reidemeister move, as in Figures 4 and 5 or [STZ, Sec. 4.4]. There is a unique equivalence $Sh_\Lambda(D^2) \cong Sh_{\Lambda'}(D^2)$ that restricts to the identity of the boundary of the disk.*

Proof. In all cases, the restrictions to the boundary of $Sh_\Lambda(D^2)$ and $Sh_{\Lambda'}(D^2)$ are fully faithful with the same essential image. This follows from the fact that restriction from sheaves on a neighborhood of the crossing pictured in Figure 3 to the top and bottom regions is an equivalence; see Example 2.5. $\square$

We will be interested in having explicit descriptions of the monodromy of the microlocal stalks along our Legendrians, in order to show that certain Legendrian isotopies give rise to cluster transformations in Theorem 5.13. To follow these through a Reidemeister-III move, we note the following:

Lemma 2.9. *In the Legendrian Reidemeister-III, on a given component of the knot, all microstalks on the component before the move are canonically identified; all microstalks after the move are canonically identified, and there is a canonical identification of these canonical identifications.*

Proof. Before and after, each component of the microsupport is contractible, hence carries only trivial local systems, hence all microstalks are identified. To identify the before and after stalks, consider the quantization of the isotopy. It produces a sheaf on a cylinder $D^2 \times I$. Each component of the microsupport of this sheaf is contractible. $\square$

2.3. **Quantization of Lagrangians.** Let $L \subset T^*M$ be an eventually conical exact Lagrangian with vanishing Maslov obstruction. Lift to a Legendrian $\widetilde{L} \subset J^1 M$ and then embedding $J^1 M \hookrightarrow T^\infty(M \times R)$. Microlocalization gives a map $Sh_{\widetilde{L}}(M \times \mathbb{R}) \to \mu loc(\widetilde{L}) \cong Loc(L)$, and restriction to $M = M \times r$ for $r \gg 0$ gives a map $Sh_{\widetilde{L}}(M \times R) \to Sh_{\partial L}(M)$.
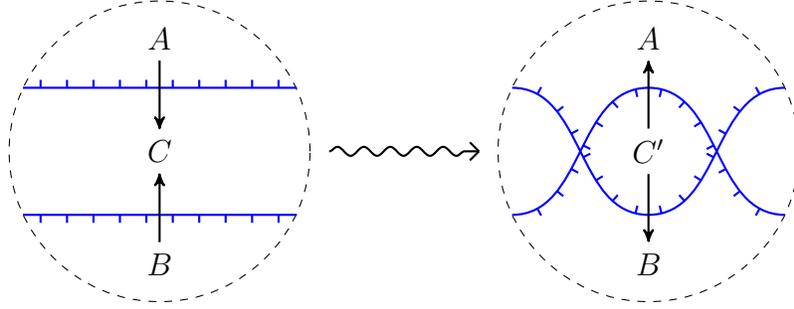
FIGURE 5. Sheaves microsupported on the Legendrians on either side of Reidemeister-II are equivalent to representations of oppositely-oriented A_3-quivers. The equivalence between the two sides is given by the reflection functor s_C; in other words, $C' = Cone(A \oplus B \to C)[-1]$.

(Unlike the Legendrians we usually work with, $\widetilde{L}$ will generally not have locally trivial Maslov obstruction. However except in this section we won't explicitly discuss this Legendrian lift, so no confusion should arise.)

Theorem 2.10. [Gui] *The morphisms* $Loc(L) \leftarrow Sh_{\widetilde{L}}(M \times \mathbb{R}, M \times -\infty) \to Sh_{\partial L}(M)$ *are isomorphisms.*

The composite gives a fully faithful embedding of the category of local systems on L into the category $Sh_{\partial L}(M)$. Note that the functor transforms naturally under compactly supported Hamiltonian isotopy of L, since such becomes a Legendrian isotopy of $\widetilde{L}$. In particular:

Lemma 2.11. *The image of* $Loc(L)$ *in* $Sh_{\partial L}(M)$ *is invariant under compactly supported Hamiltonian isotopy.*

It is useful to observe the following additional property

Lemma 2.12. *Let* $\mathcal{F}$ *be the sheaf on* M *corresponding to a rank one local system on* L. *If* $m \in M$ *is a point over which the projection* $\pi : L \to M$ *is locally a cover, then* $\mathcal{F}_m$ *can be computed by a complex* $\#\pi^{-1}(m)$ *generators.*

Proof. Let $\widetilde{\mathcal{F}}$ be the corresponding sheaf on $M \times \mathbb{R}$. Consider the line $m \times \mathbb{R}$. This line meets the front projection of $\widetilde{L}$ transversely. These intersections are in bijection with the intersection of the cotangent fiber of M. At an intersection point $m \times l$, one has a triangle

$$\widetilde{F}_{m \times (l \pm \epsilon)} \to \widetilde{F}_{m \times (l \mp \epsilon)} \to k[?] \xrightarrow{[1]}$$

We have $\widetilde{F}_{m \times -\infty} = 0$, so the result follows by induction. □

In fact we will only use the following consequence, which doesn't require the induction step:

Corollary 2.13. *Let* $\mathcal{F}$ *be the sheaf on* M *corresponding to a rank one local system on* L. *Let* $\pi : L \to M$ *be the projection. If* $\pi^{-1}(m) = \emptyset$, *then* $\mathcal{F}_m = 0$, *and if* $\pi^{-1}(m) = 1$ *then* $\mathcal{F}_m \cong k[?]$.

2.4. **Moduli spaces.** In studying microlocal rank-one objects of $Sh_\Lambda(\Sigma)$, we necessarily consider objects which are honestly complexes of sheaves, rather than simply sheaves. The resulting subcategory is not abelian in general — in particular, objects may have negative self-extensions, even for Legendrian knots in the standard contact $\mathbb{R}^3$; examples can be found in [STZ]. The correct setting for studying moduli spaces of objects in such dg categories is derived algebraic geometry [TVa]. For background we refer to the excellent survey [Toë2] and the foundational works [Lur, Toë1, TVe1, TVe2, TVe3]. However, in a sense the purpose of this section is to explain why these issues can be ignored for present purposes. The reader can skip all discussion of derived geometry by going directly to Remark 2.18.

Definition/Proposition 2.14. *Let X be the interior of a compact manifold with boundary, $\Lambda \subset T^\infty X$ a Legendrian contained in the spherically projectivized conormals of a Whitney stratification that extends to the boundary, and σ a collection of points in X. We write $\mathbb{R}\mathcal{M}(X, \Lambda, \sigma)$ for the moduli of objects in $Sh_\Lambda(X, \sigma)$. It is a locally geometric derived stack. We write $\mathbb{R}\mathcal{M}_n(\Sigma, \Lambda, \sigma)$ for the substack parameterizing sheaves of microlocal rank n.*

Proof. The existence of these spaces is guaranteed by [TVa], which constructs derived moduli stacks of pseudoperfect modules of finite-type dg categories (that is, of functors from finite-type dg categories to categories of perfect modules). The finite-type category in question is that of wrapped constructible sheaves on X microsupported on Λ; this is the full subcategory of compact objects in the cocomplete dg category of all sheaves microsupported on Λ (i.e. with not-necessarily perfect stalks) [Nad6]. Taking Hom spaces identifies the category $Sh_\Lambda(X, \sigma)$ of sheaves with perfect stalks as pseudoperfect modules of the wrapped category [Nad6]. The assumptions on X and Λ, together with the results of [Nad5] adapted to the setting of exact Lagrangians, imply that the wrapped sheaf category in question is a finite colimit of finite-type dg categories (specifically, of categories of perfect representations of acyclic quivers). The claim now follows since a finite colimit of finite-type dg categories is again finite type. $\square$

The higher and derived structures on the spaces $\mathbb{R}\mathcal{M}_n(X, \Lambda, \sigma)$ are essential from various points of view: for instance, to get meaningful point counts [NRSS] and to construct symplectic structures on these spaces [ShT] as was done for moduli of local systems in [PTVV]. An important point is that the infinitesimal study of derived moduli spaces is generally more accessible than that of ordinary moduli spaces. For example, we have following consequence of Theorem **??**:

Proposition 2.15. *Let $L \subset T^*X$ be an exact Lagrangian with $\partial L = \Lambda$ and whose projection is disjoint from $\sigma \subset X$. Let $\mathbb{R}Loc(L)$ denote the derived moduli space of local systems on L. Given a choice of brane structure on L, the functor $\mathcal{N}$ induces an open inclusion $\mathbb{R}Loc_n(L) \to \mathbb{R}\mathcal{M}_n(X, \Lambda, \sigma)$.*

Proof. This follows formally from the fact that the map on moduli spaces is induced by a fully faithful inclusion of dg categories. Indeed, it follows from this that the morphism is injective on

points, and since the tangent complexes to the moduli spaces are given by self-ext algebras [TVa, Thm 0.2], it follows that the map is étale. □

More classically, the fact that the self-ext algebra is carried to itself means that we identify obstruction theories and hence the solutions to Maurer-Cartan equations defining the classical points of the stacks.

When L is a Lagrangian *surface* and Λ nonempty, L is homotopy equivalent to a wedge of circles. In this case the derived stack $\mathbb{R}Loc_n(L)$ is truncated: it is isomorphic to its truncation $t_0\mathbb{R}Loc_n(L)$ – which is simply the classical Artin stack $Loc_n(L)$ of local systems – regarded as a derived stack. Thus if we are only interested in branes supported on Lagrangians of this kind, and the relations among them, we lose no information by working at the level of Artin stacks in the classical sense. Following [TVa, Sec. 3.4] we have the following classical moduli spaces:

Definition/Proposition 2.16. *Let Σ be a surface, $\Lambda \subset T^\infty\Sigma$ a nonempty Legendrian, σ a collection of points in Σ. We write $\mathcal{M}(\Sigma, \Lambda, \sigma)$ for the 1-rigid locus of $t_0\mathbb{R}Loc(\Sigma, \Lambda, \sigma)$; that is, the locus parametrizing objects without negative self-extensions. It is an Artin stack in the classical sense. We write $\mathcal{M}_n(\Sigma, \Lambda, \sigma)$ for the substack parametrizing sheaves of microlocal rank n.*

In many cases of interest the objects parameterized by $\mathbb{R}\mathcal{M}_n(\Sigma, \Lambda, \sigma)$ are sheaves, as opposed merely to complexes of sheaves. For example, this holds for the β-filtered local systems studied in Section 3, see Proposition 3.4. Hence these objects live in an abelian category, and in particular have no negative self-extensions to begin with — in which case $\mathcal{M}_n(\Sigma, \Lambda, \sigma)$ is equal to $t_0\mathbb{R}\mathcal{M}_n(\Sigma, \Lambda, \sigma)$. Since the truncation of an étale map is étale [TVe3, Sec. 2.2.4], we also have the underived analogue of Proposition 2.15:

Proposition 2.17. *Let $L \subset T^*\Sigma$ be an exact Lagrangian surface with $\partial L = \Lambda$ and whose projection is disjoint from $\sigma \subset \Sigma$. Given a choice of brane structure on L, the functor $\mathcal{N}$ induces an open inclusion $Loc_n(L) \to \mathcal{M}_n(\Sigma, \Lambda, \sigma)$.*

Remark 2.18. Explicitly, $\mathcal{M}_n(\Sigma, \Lambda, x)$ represents the functor from commutative rings to groupoids taking $\Bbbk$ to $\mathcal{C}_n(\Lambda, \sigma; \Bbbk)^{gpd}$, the groupoid whose objects are sheaves in $\mathcal{C}_n(\Lambda, \sigma; \Bbbk)$ without negative self-extensions and whose morphisms are quasi-isomorphisms up to homotopy, with pullback defined by base change.

We also need to consider moduli spaces of framed sheaves, constructed as follows. Let U be an open subset of S and $\mathcal{F}_\mathbb{Z}$ an object of $Sh_{\Lambda|_U}(U, \sigma \cap U; \mathbb{Z})$. The sheaf $\mathcal{F}_\mathbb{Z}$ defines a map $\operatorname{Spec}\mathbb{Z} \to \mathcal{M}_1(U, \Lambda|_U)$.

Definition 2.19. The moduli space $\mathcal{M}_1^{fr}(\Sigma, \Lambda, \sigma)$ of framed sheaves of microlocal rank one is the fiber product

$$
\begin{array}{ccc}
\mathcal{M}_1^{fr}(\Sigma, \Lambda, \sigma) & \longrightarrow & \mathcal{M}_1(\Sigma, \Lambda, \sigma) \\
\downarrow & & \downarrow \\
\operatorname{Spec}\mathbb{Z} & \longrightarrow & \mathcal{M}_1(U, \Lambda|_U),
\end{array}
$$

where the right-hand map is restriction to U and the bottom is the inclusion of $\mathcal{F}$.

The $\Bbbk$-points of $\mathcal{M}_1^{fr}(\Sigma, \Lambda, \sigma)$ are thus objects of $\mathcal{C}_1(\Lambda, \sigma; \Bbbk)$ together with an isomorphism of their restriction to U with $\mathcal{F}_{\Bbbk}$, the object of $Sh_{\Lambda|_U}(U, \sigma \cap U; \Bbbk)$ obtained from $\mathcal{F}_{\mathbb{Z}}$ by base change. In practice $\mathcal{F}$ will always be chosen in some trivial way, hence we omit it from the notation for $\mathcal{M}_1^{fr}(\Sigma, \Lambda, \sigma)$. Proposition 2.7 extends in the obvious way to the framed moduli spaces.

3. MICROLOCALLY ABELIAN MODULI PROBLEMS

Here we focus our attention on a class of moduli problems $\mathcal{M}_1(\Sigma, \Lambda, \{\sigma_i\})$ in which the link Λ is a disjoint union of positive n-strand braids, one placed in a neighborhood of a co-circle over each σ_i. As we detail in this section, these spaces include various ones of current interest, in particular:

- Positroid strata in the Grassmannian [Pos] occur when a single braid is placed on a sphere, for particular choices of braid. See Sec. 3.2.
- More generally, placing a single braid on a sphere gives rise to a certain moduli of flag configurations, whose point-count gives a term of the HOMFLY polynomial of the braid [STZ, Sec. 6].
- Moduli spaces of local systems with monodromy-invariant filtrations in the case when all the braids are trivial (see Ex. 3.3), and more generally any moduli space of monodromy and Stokes data, i.e., any wild character variety (see Sec. 3.3).

Moreover, if we are ultimately interested in cluster structures related to the moduli space of rank n local systems on a punctured surface, we are forced to consider spaces of the above kind. Most naturally, rank-n local systems correspond to sheaves with microlocal rank n along a collection of circles around the punctures, and vanishing at the punctures. However, our expected sources of cluster charts are *abelian* Lagrangian branes, which determine sheaves of microlocal rank one along their Legendrian boundary. Thus as a preliminary to abelianization of the rank n local systems, we must perform a microlocal abelianization of the boundary condition — i.e., replacing the circle labelled by n with an n-strand braid.

3.1. **Microlocal abelianization.** Consider a surface Σ with a set $\sigma = \{\sigma_i\}$ of marked points. Let Λ_i be a small circle around σ_i, co-oriented inward, and D_i the disk around σ_i whose boundary is the front projection of Λ_i. Consider the inclusions

$$\Sigma \smallsetminus \sigma \overset{r}{\hookleftarrow} \Sigma \smallsetminus \bigcup \overline{D}_i \overset{j}{\hookrightarrow} \Sigma$$

These induce an equivalence

$$j_! r^* : Loc(\Sigma \smallsetminus \sigma; \Bbbk) \to Sh_{\bigcup \Lambda_i}(\Sigma, \sigma; \Bbbk)$$

between the categories of local systems on the punctured curve and of sheaves on the complete curve with microsupport in the circles and vanishing stalks at the points. The equivalence carries the rank of the local system to the rank of the microstalk on any Λ_i.

It is the condition that the sheaves should have *rank-one microstalks* that gives rise to cluster structures. The moduli space corresponding to local systems of rank n does not have this property. To get ones which do by replacing the circle labelled by n with a suitable n-strand satellite.

By definition, the satellite construction takes as input data a triple (V, Λ, β) where V is a contact manifold, Λ is a Legendrian and β is a Legendrian in the 1-jet bundle $J^1(\Lambda)$. The output is a new Legendrian $\beta \hookrightarrow \Lambda$ in the same contact manifold V, formed by replacing a standard neighborhood of Λ by the $J^1(\Lambda)$ containing β. The Legendrian β is the *pattern* of the satellite construction. For some discussion and examples, see [Ng].

Lemma 3.1. *Let M be a manifold, $\Lambda \subset T^\infty M$ a Legendrian, and $\beta \subset J^1(\Lambda)$ a Legendrian. Assume that $\beta \to \Lambda$ is a covering map. Then there is a natural morphism*

$$\pi : Sh_{\beta \hookrightarrow \Lambda}(M) \to Sh_\Lambda(M)$$

such that

$$rank_\Lambda(\pi(\mathcal{F})) = \deg(\beta \to \Lambda) \cdot rank_{\beta \hookrightarrow \Lambda}(\mathcal{F})$$

We omit the (easy) proof, as we only use this proposition in the case when M is a surface and the Legendrian Λ is a union of circles around the punctures. In this case the result is evident; we just include the above formulation for the sake of clarity.

We can associate a Legendrian in $J^1(\Lambda)$ to any positive (annular) braid. Thus a choice of a positive n-strand braid β_i at the i'th puncture determines a morphism

(3.1.1) $$\mathcal{M}_1(\cup_i (\beta_i \hookrightarrow \Lambda_i), \sigma) \to \mathcal{M}_n(\cup_i \Lambda_i, \sigma) \cong Loc_n(\Sigma \smallsetminus \sigma).$$

That is, we draw n-stranded braids around the points and consider sheaves microsupported along these braids.

Definition 3.2. Let Σ be a surface and $\sigma = \{\sigma_i\}$ a collection of points. Let $\sigma_i \mapsto \beta_i \in Br_n^+$ be an assignment of a positive braid to each point of σ; by abuse of notation we also write β_i for $\beta_i \hookrightarrow \Lambda_i$ where Λ_i is an inward-co-oriented circle around σ_i. Writing $\beta = \coprod \beta_i$, we refer to the points of $\mathcal{M}_1(\Sigma, \beta, \sigma)$ as β-**filtered local systems**.

For the trivial braid, this recovers exactly the notion of filtered local system:

Example 3.3. Let D^2 be a disk, and $\bigcirc^n \subset T^\infty(D^2 \smallsetminus 0)$ be the link whose front projection is n concentric circles. Then $Sh_{\bigcirc^n}(D^2, 0)$ is the category of pairs $(0 = K_0 \to K_1 \to \cdots \to K_n = K; m : K \to K)$ where K is a filtered complex and ϕ is an endomorphism preserving the filtration. The correspondence is that K_i is the stalk between the ith and $i + 1$st strands away from 0, and m is the monodromy.

Fixing the microlocal rank to equal one forces K to be (quasi-isomorphic to) a locally free $\Bbbk$-module of rank n, and the filtration to be the same as a full flag. Thus, $\mathcal{M}_1(\bigcirc^n, 0)$ is the total space of the Grothendieck-Springer resolution: $\mathcal{M}_1(\bigcirc^n, 0) \cong \widetilde{GL}_n/GL_n$. The resolution morphism $\widetilde{GL}_n/GL_n \to GL_n/GL_n$ itself is the map $\mathcal{M}_1(\bigcirc^n, 0) \to Loc_n(D^2 \smallsetminus 0)$ of Equation (3.1.1).

Example 3.3 illustrates a general feature of β-filtered local systems: up to quasi-isomorphism they are sheaves in the usual sense rather than merely complexes of sheaves.

Proposition 3.4. *Let Σ, β, and σ be as in Definition 3.2. Every microlocal rank-one object of $Sh_\beta(\Sigma, \sigma; \Bbbk)$ is isomorphic to an object supported in cohomological degree zero. In particular, no such objects have negative self-extensions, so $\mathcal{M}_n(\Sigma, \beta, \sigma)$ is exactly the truncation $t_0 \mathbb{R} \mathcal{M}_n(\Sigma, \beta, \sigma)$.*

We omit the proof, which is a straightforward generalization of [STZ, Prop. 5.19].

3.2. **Positroid strata and the Grassmannian.** The positroid stratification of the Grassmannian is the common refinement of the Schubert stratification and its cyclic shifts, and arises naturally from the perspective of total positivity [Pos]. The positroid strata of $\mathrm{Gr}(k, n)$ are indexed by a number of equivalent combinatorial objects, the most relevant of which for us will be cyclic rank matrices [KLS, Cor. 3.12]: in this section we use these to give microlocal descriptions of positroid strata.

Definition 3.5. A **cyclic rank matrix** of type (k, n) is a $\mathbb{Z} \times \mathbb{Z}$ integer matrix r such that

    C1: $r_{ij} = 0$ for $j < i$
    C2: $r_{ij} = k$ for $j \geq i + n - 1$
    C3: $r_{ij} - r_{(i+1)j} \in \{0, 1\}$ and $r_{ij} - r_{i(j-1)} \in \{0, 1\}$ for all i, j
    C4: if $r_{(i+1)(j-1)} = r_{(i+1)j} = r_{i(j-1)}$ then $r_{ij} = r_{(i+1)(j-1)}$
    C5: $r_{(i+n)(j+n)} = r_{ij}$ '

For each $V \in \mathrm{Gr}(k, n)$ there is an associated cyclic rank matrix $r(V)$ of type (k, n), and the positroid strata will be the level sets of this assignment. Let $c_1, \ldots, c_n \in \mathbb{C}^k$ be the columns of any matrix representative of V, and for arbitrary $i \in \mathbb{Z}$ define c_i so that $c_i = c_{i+n}$ for all i. Then we set $r(V)_{ij}$ to be the dimension of the span of the columns $c_i \cdots c_j$. Note that for $j < i$, we have the empty collection of columns, hence $r(V)_{ij} = 0$, and that for $j > i + n - 1$ we have all the columns, hence $r(V)_{ij} = k$. The conditions **C1**-**C5** exactly characterize the matrices that arise from $\mathrm{Gr}(k, n)$ in this fashion [KLS].

Definition 3.6. Given a cyclic rank matrix r of type (k, n), the associated **positroid stratum** is

$$\Pi_r = \{V \in \mathrm{Gr}(k, n) | r(V) = r\}.$$

In our context the most natural cyclic rank matrices are those such that $r_{ii} \neq 0$ for all i, and we assume this from now on. That is, we assume the columns of any matrix representative of $V \in \Pi_r$ are all nonzero. No generality is lost in the sense that any positroid stratum Π_r not satisfying this condition be embedded into a smaller Grassmannian as a positroid stratum that does.

We record loci where the entries of r jump as a Legendrian Λ_r in $T^\infty D^2$ as follows. The basic idea is to regard r as an actual geometric object in $\mathbb{R}^2$, and build Λ_r in such a way that the faces of its front projection correspond to patches of r where its entries are constant.

- We first define a Legendrian Λ'_r in $T^\infty \mathbb{R}^2$ lying over a neighborhood of the square grid $\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$. Consider the union of the segments $\{i\} \times (j, j+1)$ with $r_{i(j-1)} < r_{ij}$ and the segments $(i, i+1) \times \{j\}$ with $r_{ij} > r_{(i+1)j}$. We co-orient the former leftward and the latter downward. Its closure is a collection of pairwise-transverse immersed co-oriented curves with corners at the points $(i, j) \in \mathbb{Z}^2$ such that

$$r_{ij} = r_{(i+1)j} = r_{i(j-1)} = r_{(i+1)(j-1)} + 1.$$

We smooth all such corners and let Λ'_r be the Legendrian lift of the resulting collection of smooth immersed curves.

- Consider the restriction of Λ'_r to the infinite strip

$$\mathcal{S} = \{(x, y) | \frac{1}{2} < y + x < k + \frac{1}{2}\}.$$

By **C5**, the restriction is invariant under the translation $T_n : (x, y) \mapsto (x + n, y - n)$ of $\mathcal{S}$, hence gives rise to a Legendrian Λ_r in $T^\infty \mathcal{A}$, where $\mathcal{A}$ is the annulus $\mathcal{S}/\langle T_n \rangle$. Since Λ_r does not meet the boundary component whose preimage is the line $y + x = k + \frac{1}{2}$, we can embed $\mathcal{A}$ in a disk to regard Λ_r as a Legendrian in $T^\infty D^2$.

Example 3.7. Let us compute the moduli space of microlocal rank-one sheaves associated to the Legendrian whose front projection is pictured on the left in Figure 1. This Legendrian is the satellite formed by taking the braid on two strands, twisted five times, and inserting it in a neighborhood of the cocircle over the North pole of S^2; the front projection of this lands in the complement of the North pole, which we identify with the page.

A sheaf with microsupport in this Legendrian restricts to a local system on each component of the complement of the front projection. We are considering sheaves vanishing at the north pole, so these local systems are only nonzero on the six components which are bounded in the picture. Since these are contractible, the local systems are just the data of six vector spaces (for now we work over a ground field $\Bbbk$). The sheaf is then determined by the data of these vector spaces together with linear maps associated to paths going against the hairs. If the sheaf has microlocal rank-one, the dimensions increase by one as we move against the direction of the hairs, hence are as indicated in the picture. Choosing bases for the vector spaces involved, we can can encode the linear maps as the columns of a matrix:

$$(3.2.1) \qquad \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \end{bmatrix}$$

The condition that the pictured sheaf has singular support on Λ — as opposed to the union of Λ with the cocircle fibers over the crossings in the front projection — translates to the condition that any two cyclically adjacent columns of (3.2.1) are linearly independent (see [STZ], Sec. 5).

Two such matrices correspond to isomorphic sheaves if they are related by a combination of left multiplication by GL_2 and right multiplication by the diagonal subgroup of GL_5. Thus the moduli

space $\mathcal{M}_1(\Lambda)$ is the quotient of the space of 2×5 matrices satisfying the crossing conditions by these symmetries. In other words, $\mathcal{M}_1(\Lambda)$ is the configuration space of 5 cyclically ordered points in $\mathbb{P}^1$, with the condition that cyclic neighbors are distinct.

We can also frame this story by considering sheaves with fixed trivializations of their one-dimensional stalks. These are still represented by matrices as in (3.2.1), but now two matrices are equivalent if and only if they are related by the left GL_2 action. The moduli space of so-framed microlocal rank-one sheaves is an open subset of the Grassmannian of two-planes in five-space. The crossing conditions above are exactly the conditions that define the *big positroid stratum* of $\mathrm{Gr}(2,5)$ [Pos].

Example 3.8. Let V be the point of $\mathrm{Gr}(3,5)$ represented by the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The cyclic rank matrix $r(V)$ is shown below left, with the front projection of $\Lambda'_{r(V)}$ shown in red (without smoothed corners). The identification of top and bottom sides by the action of T_5 is indicated by the dashed green line. At right is the front of the associated (smooth) Legendrian knot on the cylinder, the horizontal dashed line indicating where ∂D^2 cuts $\Lambda'_{r(V)}$.
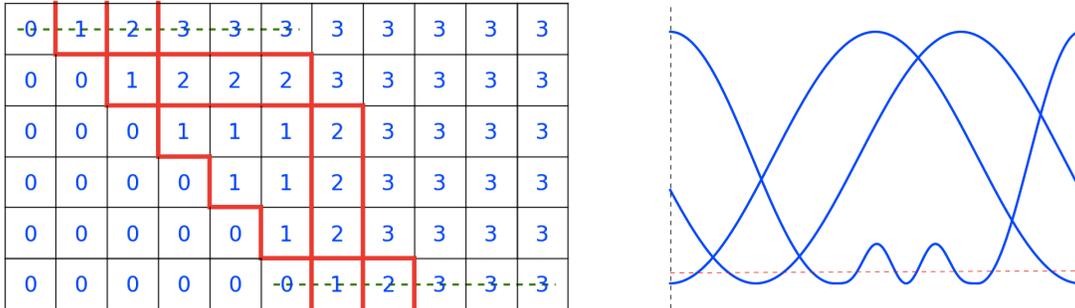
Returning to the general case, we fix a set of points $\sigma \subset D^2$, one in each of the n boundary regions of $D^2 \smallsetminus \pi(\Lambda_r)$ for which Λ_r is co-oriented into the given region. In the construction of Λ_r, these regions come from the "corners" of the subdiagonal entries of r, which are equal to zero. The pair Λ_r, σ effectively satisfy the hypotheses of Proposition 3.4, so the objects of $\mathcal{C}_1(\Lambda, \sigma)$ are ordinary sheaves rather than complexes.

Let U be an open collar of ∂D^2 containing no crossings of Λ_r, and V the union of the components of $D^2 \smallsetminus \pi(\Lambda_r)$ that do not contain points in σ. We let $\mathcal{F}_{\Bbbk} = i_* \Bbbk_{U \cap V} \in Sh_{\Lambda_r|_U}(U)$; this is a sheaf of microlocal rank one whose stalks are alternately 0 and $\Bbbk$ around the boundary of D^2. We let $\mathcal{M}_1^{fr}(\Lambda_r)$ denote the space of sheaves framed by $\mathcal{F}_{\mathbb{Z}}$, as in Definition 2.19 (since σ is fixed throughout the section, we omit it from the notation). The $\Bbbk$-points of $\mathcal{M}_1^{fr}(\Lambda_r)$ are objects in $\mathcal{C}_1(\Lambda, \sigma; \Bbbk)$ equipped with an isomorphism between their restriction to U and $\mathcal{F}_{\Bbbk}$.

Theorem 3.9. *For any cyclic rank matrix r of type (k, n), there is a canonical isomorphism between the positroid stratum Π_r and the framed moduli space $\mathcal{M}_1^{fr}(\Lambda_r)$.*

Proof. We first describe a canonical embedding of $\mathcal{M}_1^{fr}(\Lambda_r)$ into $\mathrm{Gr}(k, n)$, then show its image is exactly Π_r. By construction the connected components of $D^2 \smallsetminus \pi(\Lambda_r)$ are labeled by entries of r. Given the definition of σ and the co-orientation of Λ_r, these labels are also the ranks of the stalks of any object of $\mathcal{C}_1(\Lambda_r, \sigma; \Bbbk)$ in these components. Fix points $p_1, \ldots, p_n$, one in each boundary component of $D^2 \smallsetminus \pi(\Lambda_r)$ which does not meet σ (the indexing by $1, \ldots, n$ comes from the indexing of the corresponding rows of r), and a point x in the middle region where stalks have rank k.

A $\Bbbk$-point of $\mathcal{M}_1^{fr}(\Lambda_r)$ is a framed sheaf F, which has stalks $F_1, \ldots, F_n$ at $p_1, \ldots, p_n$, each equipped with a trivialization $F_i \cong \Bbbk$, and a stalk F_x at x. Choose characteristic paths from each p_i to x (that is, paths that only cross strands of $\pi(\Lambda_r)$ going against their co-orientations; the construction will be independent of the choice). Each defines an inclusion $F_i \hookrightarrow F_x$, the composition of the generization maps along the path (see Figure 3.2). The crossing conditions (see Example 2.5) guarantee that the images of the F_i together generate F_x. Thus from F we obtain a locally free $\Bbbk$-module F_x of rank k with a quotient map from $\oplus F_i \cong \Bbbk^n$, which is the data of a $\Bbbk$-point of $\mathrm{Gr}(k, n)$. This is clearly compatible with base change, so we obtain a map $\mathcal{M}_1^{fr}(\Lambda_r) \to \mathrm{Gr}(k, n)$. The claim that this is faithful is equivalent to the claim that restriction from $\mathcal{M}_1^{fr}(\Lambda_r)$ to the framed moduli space of a neighborhood of the union of the paths from the p_i to x is faithful. This follows by inductively applying the fact that in a neighborhood of a crossing, restriction to the upper or lower regions as pictured in Figure 3 is an equivalence. That is, we can expand the neighborhood of the paths to include a new crossing and meet a new region of $D^2 \smallsetminus \pi(\Lambda_r)$ one at a time, each time yielding an equivalence of sheaf categories, until the whole disk is covered.

On the other hand, we have defined Λ_r exactly so that its crossing conditions imply that the span in F_x of any cyclically adjacent subset of $F_1, \ldots, F_n$ has the rank specified by the corresponding entry of r. Thus the image of $\mathcal{M}_1^{fr}(\Lambda_r)$ is contained in Π_r. Conversely, given a $\Bbbk$-point of Π_r, we can directly define a sheaf whose sections over a given region are just the column span associated to the relevant entry or entries of r (note that while a $\Bbbk$-point of $\mathrm{Gr}(k, n)$ is a quotient map $\Bbbk^n \twoheadrightarrow E$ onto a locally free $\Bbbk$-module, $\Bbbk$-points of Π_r are described by quotients where E is in fact free). Finally, it follows from the fact that the images of the F_i generate F_x that all points of $\mathcal{M}_1^{fr}(\Lambda_r)$ have trivial stabilizers, hence $\mathcal{M}_1^{fr}(\Lambda_r) \cong \Pi_r$. $\square$

Remark 3.10. Many other objects in algebraic geometry and representation theory can be identified with positroid strata or their closures, see for example the discussion in [KLS, Section 6]. One class of examples are the double Bruhat cells $GL_n^{u,v}$ of GL_n. Here u and v are elements of the Weyl group of GL_n and $GL_n^{u,v}$ is the intersection of the double cosets $B_- u B_-$, $B_+ v B_+$ of u, v with respect to opposite Borel subgroups. These can be embedded as positroid strata in $\mathrm{Gr}(n, 2n)$; on the level of matrix representatives this is just concatenation with an identity matrix. Another class of examples
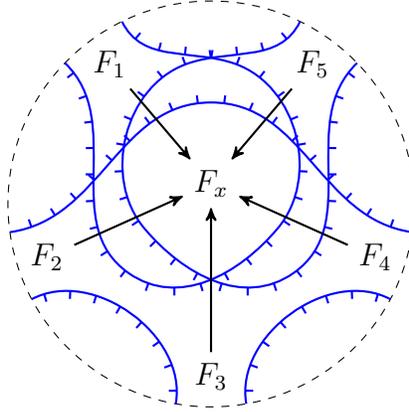
FIGURE 6. A depiction of a microlocal rank-one sheaf $F \in \mathcal{C}(\Lambda_r, \sigma; \Bbbk)$, where Λ_r is as in Example 3.8. As in the proof of Theorem 3.9, F has five rank one stalks at the boundary of the picture, each of which includes into the rank 3 stalk F_x in the middle. If F is framed, so we have isomorphisms $F_i \cong \Bbbk$, then the quotient map $\Bbbk^5 \twoheadrightarrow F_x$ defines a $\Bbbk$-point of the positroid stratum Π_r.

come from subvarieties of the full or partial flag varieties of GL_n that map isomorphically onto their images in $\mathrm{Gr}(k, n)$ under the natural projection. These include the moduli spaces of triples of flags in generic position considered in [FG1], which form building blocks in associating cluster charts on moduli spaces of local systems to triangulations of surfaces (see Construction 5.7).

Remark 3.11. The construction of $\Lambda_r \subset T^\infty D^2$ naturally produces a Legendrian $\beta_r \subset T^\infty \mathbb{R}^2$ whose intersection with $T^\infty D^2$ is Λ_r. The front projection of β_r is obtained from that of Λ_r by adding caps around the outside of D^2 as pictured in Figure 1 or Example 3.8. Moreover, β_r is a Legendrian braid satellite of a circle around ∞, as considered in Section 3.1. The unframed moduli spaces $\mathcal{M}_1(\Lambda_r)$ and $\mathcal{M}_1(\beta_r)$ are isomorphic, and this space of β_r-filtered local systems is a configuration space of points in $\mathbb{P}^k$ satisfying open conditions. The projection $\mathcal{M}_1^{fr}(\Lambda_r) \to \mathcal{M}_1(\Lambda_r)$ is a torus quotient, and this relationship is a version of Gelfand-Macpherson duality subject to the conditions imposed by r [GM].

3.3. **Wild character varieties.** In its simplest form, the Riemann-Hilbert correspondence asserts an equivalence of categories between integrable meromorphic connections on a complex analytic space, with regular singularities along a normal crossings boundary divisor, and locally constant sheaves in the complement of the divisor [Del]. In particular, the parameter space of such regular connections, considered up to gauge equivalence, can be identified with the space parameterizing representations of the fundamental group of the complement, up to isomorphism.

The moduli space of connections is called the *de Rham* moduli space, and the moduli space of locally constant sheaves is called the *Betti* moduli space. The Riemann-Hilbert correspondence asserts that these have the same points; in fact they are complex-analytically isomorphic, but have naturally different algebraic structures – passing from connections to their monodromy involves an

exponential. We restrict attention to the case where the space on which we study connections is a smooth Riemann surface.

The notion of regular singularities is essential in the above equivalence. One formulation is that the connection matrix can, analytically locally, be expressed with poles of order at most one. Equivalently, the local solutions exhibit polynomial growth as one approaches the singular point. A consequence of this is that the classification of such connections up to analytic local gauge equivalence is the same as the classification up to formal local gauge equivalence; the local form of the Riemann-Hilbert correspondence is then just the statement that both of these are characterized by the conjugacy class of the exponential of the singular term of the connection.

To classify connections with possibly irregular singularities, one records *Stokes data*, i.e., information about the growth rates of solutions [Mal, DMR, BV1, BV2, BV3]. This is often formulated in the following way: given a meromorphic connection on a disk D^2, analytic away from zero, the space of solutions forms a locally constant sheaf Sol on $D^2 \smallsetminus 0$, hence equivalently on the real oriented blowup $\pi : \widetilde{D^2} \to D^2$ at 0. Let $\mathcal{I}$ be the totally ordered set of all possible growth rates of the absolute value of the solution to a linear meromorphic ODE, modulo polynomial growth rates – we discuss what $\mathcal{I}$ is more explicitly later. Then the sheaf $Sol|_{\pi^{-1}(0)}$ carries a stalkwise filtration by $\mathcal{I}$, varying continuously in an appropriate sense. This filtration is termed the Stokes filtration.

The Riemann-Hilbert theorem in this possibly irregular case implies in particular the following three assertions: first, that connections up to analytic local gauge equivalence are classified by their solution sheaves equipped with Stokes filtrations; second, that connections up to formal local gauge equivalence are classified by their solution sheaves plus the associated graded of the Stokes filtration; and finally, that if a given associated graded arises from some connection, then there exist connections giving rise to any filtration with this associated graded [Mal].

The relation to our setting is obtained by projecting the $\mathcal{I}$ filtration to an $\mathbb{R}$ filtration, and then "turning it sideways" via the observation than a sheaf of $\mathbb{R}$-filtered objects on X is the same as a sheaf on $X \times \mathbb{R}$ with microsupport confined to covectors negative in the $\mathbb{R}$ direction. Recording the jumping locus of the filtration by passing to the associated graded is just the same as recording the microsupport of the sheaf.

Let us be more precise about how to produce the sheaf on $S^1 \times \mathbb{R}$. To this end we recall the formal classification of singularities of meromorphic ODE: any vector bundle on a disk equipped with a meromorphic connection ∇, analytic away from zero, is, over the universal cover of the disk, *formally* gauge equivalent to some $\bigoplus(\alpha \otimes \nabla_\alpha)$ where each α is an irregular connection *of rank one* and ∇_α is a regular connection. Note that the asymptotics of the holomorphic local solutions are controlled by the asymptotics of the formal local solutions. (The "main asymptotic existence theorem" asserts a converse; that one can lift formal solutions to holomorphic solutions with similar asymptotics. It is a key step in the proof of the Riemann-Hilbert correspondence, but in the black-box presentation we are giving it can be viewed as a consequence.)

Thus to understand the possible asymptotics of solutions, it suffices to consider the rank one equation

$$\frac{df}{dz} = \alpha \cdot f(z) \qquad\qquad \alpha \in \mathbb{C}((z^{1/\infty}))$$

Evidently the solution is $f = e^{\int \alpha dz}$. The regular part of the connection does not affect the growth rate of solutions *modulo polynomial growth rates*, i.e., the growth rate is determined by the class of α in $\mathbb{C}((z^{1/\infty}))/z^{-1}\mathbb{C}[[z^{1/\infty}]]$. We will thus take α to have no terms of degree greater than -1.

We return to our description of the sideways Stokes sheaf. Fix again some connection ∇, and after some gauge transformation defined over $\mathbb{C}((z^{1/\infty}))$, expand it as $\nabla = \bigoplus(\alpha \otimes \nabla_\alpha)$. Fix some $\epsilon \ll 1$, and plot, as a function of θ, the (multivalued-)functions $n_{\alpha,r}(\theta) := \log|f(\epsilon e^{i\theta})| = Re((\int \alpha dz)_{z=re^{i\theta}})$, for every α which appears in the above decomposition.

Consider the sheaf $Sol|_{\pi^{-1}(0)}$, and pull it back to $\mathbb{R} \times S^1$ under the projection of the $\mathbb{R}$ factor. Fix now some $\epsilon \ll 1$. Note that a stalk of this sheaf is in fact a function on the circle; form the subsheaf $\mathbb{S}^\epsilon$ whose stalk at (N, θ) consists of solutions which grow at most polynomially faster than any formal solution, the logarithm of whose evaluation at $\epsilon e^{i\theta}$ is at most N. That is:

$$\mathbb{S}_{N,\theta}^\epsilon := \{f \in Sol_{N,\theta} \,|\, N \leq n_{\alpha,\epsilon}(\theta) \implies \log|f(re^{i\theta})| \leq n_{\alpha,r}(\theta) + O_{r \to 0}(1)\}$$

By construction, the sheaf $\mathbb{S}^\epsilon$ has microsupport at infinity equal to the Legendrian link whose front projection is the union of the graphs of the $n_{\alpha,\epsilon}$, co-oriented towards $-\infty$ in $\mathbb{R} \times S^1$. We call this link the *Stokes Legendrian* of the connection, and term its front projection the *Stokes diagram*. Note that the Stokes diagram and Stokes Legendrian depend only on the formal type.

Remark 3.12. The fact that this filtration should be viewed as describing a Legendrian is mentioned in [KKP], its front projection having been drawn by Stokes himself [Sto] (we thank Philip Boalch for bringing this last reference to our attention).

We can now state more precisely the irregular Riemann-Hilbert correspondence. Let Σ be a surface, and $p_1, \ldots, p_k$ be points on Σ. Fix a formal type of irregular singularity τ_i at each p_i, i.e., choose some connection on a disk near each τ_i, meromorphic on the disk and holomorphic away from p_i, defined up to formal gauge equivalence, and up to changing the regular part of the connection. That is, for the moment we take our notion of formal type to mean that only the α are specified, and the ∇_α are left to vary. Let $C_{dR}(\Sigma, \{p_i\}, \{\tau_i\})$ be the category of connections with the prescribed formal types.

Let Λ_i be the Stokes Legendrian of the singularity τ_i. Draw the knot Λ_i on Σ by first passing to the real blowup $Bl_{p_i}\Sigma$, and then gluing the $\mathbb{R} \times S_1$ above to the inside of the boundary circle, with ∞ in the $\mathbb{R}$ factor facing 'into' the surface. One now has a punctured surface; the puncture can be filled in and re-labelled p_i.

The procedure we described locally before can now be performed globally over the surface. That is, if we write $C_B(\Sigma, \bigcup \Lambda_i, \bigcup p_i)$ for the subcategory of $Sh_{\bigcup \Lambda_i}(\Sigma)$ in which the stalk of the sheaf

vanishes at all p_i, then forming the global sideways Stokes sheaf of solutions defines a map

$$C_{dR}(\Sigma, \{p_i\}, \{\tau_i\}) \to C_B(\Sigma, \bigcup \Lambda_i, \bigcup p_i)$$

The irregular Riemann-Hilbert correspondence implies this map is an equivalence.

The ∇_α on the de Rham side integrate to the microlocal monodromies on the Betti side. In particular, when all the ∇_α have dimension one, the moduli space of objects in the above category is exactly of the sort we have been considering in this section.

Example 3.13. The ODEs $f'' = z^n f$ generalize the Airy equation ($n = 1$), and correspond to the $\mathcal{D}$-module $d - A$, with $A = \begin{pmatrix} 0 & 1 \\ z^n & 0 \end{pmatrix}$. This gives an SL_2-flat connection, and we will determine the formal type of the singularity by investigating the solutions $\begin{pmatrix} f \\ f' \end{pmatrix}$ at $z = \infty$. First put $x = z^{-1}$ to move the irregular singularity to the origin, then define the differential operator $\Theta = z\frac{d}{dz} = -x\frac{d}{dx}$, after which the equation becomes $Lf = 0$, with $L = \Theta^2 + \Theta - x^{-(n+2)}$. Newton's method instructs us how to look at the most singular terms and use gauge transformations to reduce the order of the singularity so as to develop the power-series parts of asymptotic solutions. In this case, we find $f_\pm = \exp(\mp\frac{2}{n+2}x^{-\frac{n+2}{2}})x^{\frac{n}{4}} \sum_m a_m x^{\frac{m}{2}}$. Wasow's "Main Asymptotic Existence Theorem" [Was, Section 14] states that these represent the singularity types of actual solutions. Which of the two $f_\pm(re^{i\theta})$ is most singular as $r \to 0$ changes at $n + 2$ values of θ, so we can read off the Stokes data as the $(2, n + 2)$ braid. Compare with Figure 1.

4. ALTERNATING LEGENDRIANS

In this section we construct exact Lagrangian fillings of alternating Legendrians. The data of such a Legendrian can be encoded by a bicolored graph on the surface; in the terminology of [Pos], the front projection is an *alternating strand diagram*. The surface we construct has the same homotopy type as the graph, and can be topologically identified with what has elsewhere been called the *conjugate surface* [GK]. After equipping the conjugate Lagrangian with the requisite brane data so as to give rise to objects in the infinitessimally wrapped Fukaya category of [NZ], we calculate the sheaves on the base surface which result from taking Floer homology with fibers.

4.1. **Alternating colorings.**

Definition 4.1. Let Σ be a surface, and let $\Lambda \subset T^\infty \Sigma$ be a Legendrian in its cocircle bundle whose front projection $\pi(\Lambda)$ has only transverse intersections as singularities. An **alternating coloring** for Λ is the data of, for each region in the complement of the front projection, a label *black, white, or null*, subject to the conditions:

- The boundary of a black region is co-oriented inward.
- The boundary of a white region is co-oriented outward.
- The boundary of a null region has co-orientations that alternate between inward and outward at each crossing.

- No black region shares a 1-d border with a white region, and no null region shares a 1-d border with another null region.

An **alternating Legendrian** Λ is a Legendrian equipped with an alternating coloring.

We term such colorings alternating because:

Proposition 4.2. *A link with an alternating coloring has the property that, following along any strand, successive crossing strands in the front projection have alternating co-orientations. For a one-component link, this is sufficient to guarantee the existence of an alternating coloring.*

Some Legendrians can admit more than one alternating coloring:

Example 4.3. Consider the link composed of two concentric circles in the plane, with the inner one co-oriented outward and the outer one co-oriented inward. There are three connected components of the complement of the front projection. This admits two alternating colorings: proceeding from inside to outside, the three components can be labelled (white, null, white) or (null, black, null).

However, this can be excluded by requiring sufficient crossings:

Proposition 4.4. *If every region of the complement of the front projection abuts a crossing, there is at most one alternating coloring, and moreover the fourth condition above follows from the first three.*

Proof. In the neighborhood of any crossing, there is at most one alternating coloring, which moreover verifies the fourth condition. $\square$

Remark 4.5. Even if one is only ultimately interested in diagrams satisfying the condition of the proposition, the extra flexibility in our definition of alternating coloring is still needed for it be a local notion on Σ.

We assume from now on that Σ is orientable; as such we can identify co-orientations with orientations, and do so by orienting the Legendrian such that the "hairs" indicating its co-orientation always point **to the left** when traveling in the direction of the orientation. In terms of orientations rather than co-orientations, the Legendrian travels counterclockwise around a black region, and clockwise around a white one.

label	co-orientation	orientation	sheaf type
black	inward	counterclockwise	standard
white	outward	clockwise	co-standard

TABLE 1. Labelling conventions for alternating diagrams.

The front projections of alternating Legendrians have been considered elsewhere in the context of bipartite graphs, where they are referred to as *alternating strand diagrams* and their components
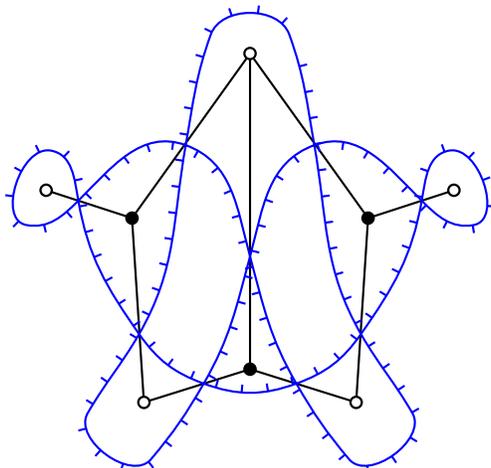
FIGURE 7. The front projection $\pi(\Lambda)$ of an alternating Legendrian and the associated bipartite graph Γ. Given $\pi(\Lambda)$, we recover Γ by placing a black/white vertex in each region whose boundary is co-oriented inward/outward, then connecting these by edges passing through crossings. Given Γ, we recover $\pi(\Lambda)$ by drawing paths going between midpoints of edges of Γ, co-orienting them away from white vertices and towards black vertices.

as *zig-zag paths* [Pos, GK]. A bicolored graph simply means one whose vertices are labeled white and black; if edges only connect vertices of distinct colors it is bipartite. The alternating strand diagram of an embedded bicolored graph $\Gamma \subset \Sigma$ is determined up to planar isotopy by the following conditions: it lies in an open set that retracts onto Γ, its crossings are in bijection with edges of the graph meeting a vertex of each color, with one crossing lying on each edge, and these crossings are the only points where it meets Γ. Thus bicolored graphs in Σ present alternating Legendrians in $T^\infty\Sigma$:

Proposition 4.6. *Let Γ be a bicolored graph. Then there is a unique Legendrian lift Λ of the alternating strand diagram of Γ such that the vertex coloring of Γ gives the labels for an alternating coloring of Λ.*

Conversely, from an alternating Legendrian we can produce a bicolored graph that gives rise to it in the above fashion. This is simplest when each standard and costandard region is contractible; in this case we simply place appropriately colored vertices in the white and black regions and connect them with edges across crossings. More generally, we further attach to each vertex a configuration of embedded self-loops onto which its black/white region retracts.

4.2. **The conjugate Lagrangian.** From an alternating coloring of $\Lambda \subset T^\infty\Sigma$, we now construct an exact Lagrangian filling and equip it with the necessary structures to provide an object of the infinitessimally-wrapped Fukaya category $Fuk(T^*\Sigma)$. We begin by describing the filling-to-be as an abstract topological surface, absent the embedding into the cotangent bundle. The desired surface
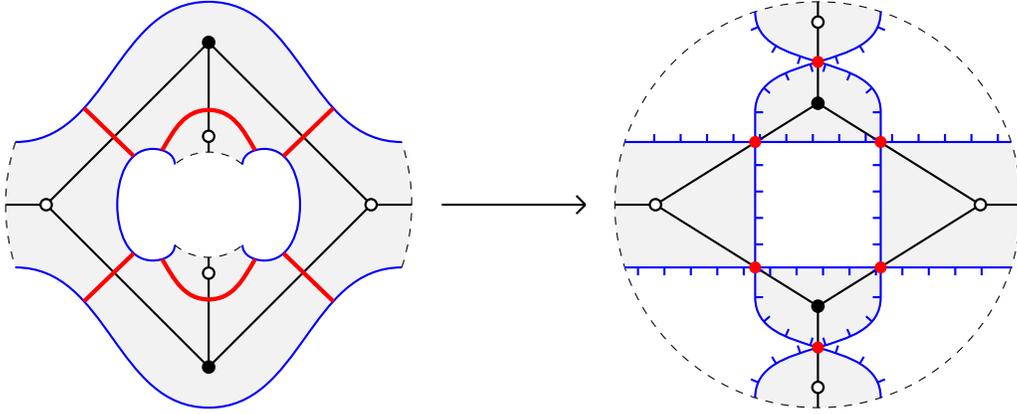
FIGURE 8.   The right picture shows an alternating Legendrian in $T^\infty D^2$ and the associated bicolored graph. The left shows its conjugate Lagrangian L together with the strict transform of the bicolored graph under $L \to D^2$. The shaded regions on the right indicate the image of $L \to D^2$. Note the embedding of L into the page, exhibiting it as a planar surface, is somewhat unnatural. The exceptional arcs and their images are indicated in red.

coincides with that associated to a bipartite graph in [FHKV, GK], where it is called the *conjugate surface*.

Let $\widehat{\Sigma}$ denote the real blow up of Σ at the finite set of crossings of the front projection of Λ. The blow-down map $\widehat{\Sigma} \to \Sigma$ is a diffeomorphism away from the crossings, and the fiber above a crossing is the $\mathbb{RP}^1$ of lines tangent to the the crossing. We denote by $W \subset \Sigma$ (resp. $B \subset \Sigma$) the union of the interiors of the white (resp. black) regions of the complement of the front projection.

Definition 4.7. The **conjugate surface** $\overline{L}$ is the closure of the preimage of $W \cup B$ in $\widehat{\Sigma}$. It is a smooth surface with boundary, and we write L for its interior.

The boundary of $\overline{L}$ is canonically homeomorphic to Λ. The blowdown map identifies the white and black regions of Σ with open subset of L, which we also refer to as white and black regions. Each exceptional $\mathbb{RP}^1$-curve on $\widehat{\Sigma}$ meets $\overline{L}$ in an line segment that separates a white region from a black region. We term such line segments "exceptional arcs", and sometimes indicate them in red as in Figure 8.

Definition 4.8. Let $U \subset \Sigma$ be a neighborhood of a crossing of the front projection of an alternating Legendrian Λ. We say a coordinatization $x, y : U \hookrightarrow \mathbb{R}^2$ is **adapted** to the coloring of Λ if, in these coordinates, the front projection is the union of the x- and y-axes, the black region is the first quadrant $\{x, y > 0\}$, and the white region is the third quadrant $\{x, y < 0\}$. (Note that the orientation induced on the front from the co-orientations implicit in the above choices points towards positive x and towards *negative* y.)

Denote the preimage of U in $\widehat{\Sigma}$ by $\widehat{U}$. We endow $\widehat{U} \cap \overline{L}$ with the coordinates s, t, where s takes values in $\mathbb{R}$ and t takes values in $[0, 1]$, so that the blowdown map is

$$x(s,t) = s(1-t) \qquad y(s,t) = st$$

We likewise refer to these as the adapted coordinates on $\overline{L}$ near the exceptional arc $s = 0$.

The following proposition asserts that L can be embedded into $T^*\Sigma$ as an exact Lagrangian in an essentially unique way. Thus from now on we regard L as embedded in $T^*\Sigma$ without comment.

Definition/Proposition 4.9. *Let $\Lambda \subset T^\infty\Sigma$ be a Legendrian equipped with an alternating coloring, and let $\overline{L}$ be the corresponding conjugate surface. Let $\bigcup_a \partial a \subset \partial\overline{L}$ denote the union of the boundaries of all exceptional arcs. Let m be a smooth function on L that takes values in $(0, \infty)$, and that extends to a smooth function*

$$\overline{L} - \bigcup_a \partial a \to [0, \infty]$$

taking the value 0 on the boundary of black regions and ∞ on the boundary of white regions. Suppose furthermore that in adapted coordinates s, t near an exceptional arc, $\partial m/\partial t$ is divisible by s. Then there is a unique embedding $L \to T^*\Sigma$ with the following properties:

(1) *The composition of $L \to T^*\Sigma$ with the projection $T^*\Sigma \to \Sigma$ coincides with the blowdown map.*

(2) *The image of L is an exact Lagrangian, and the pullback of the tautological one-form to L coincides with $d\log(m)$.*

(3) *The image of L is asymptotic to Λ.*

Moreover, given two choices of m the corresponding images of L are Hamiltonian isotopic. We call the image of L under any such embedding the **conjugate Lagrangian**.

Proof. Fix a coordinate patch $U \subset \Sigma$ with coordinates $x, y : U \hookrightarrow \mathbb{R}^2$. The cotangent bundle over U is then endowed with coordinates $x, y, \xi, \eta : T^*U \hookrightarrow \mathbb{R}^4$ in which the tautological one-form is $\xi dx + \eta dy$. Let $V \subset L$ be a coordinate patch contained in the inverse image of U under the blowdown map $\widehat{\Sigma} \to \Sigma$, with coordinates $s, t : V \hookrightarrow \mathbb{R}^2$. We then write $x(s,t), y(s,t)$ for the components of the blowdown map in these coordinates. Any Lagrangian embedding $V \hookrightarrow T^*U$ obeying conditions (1) and (2) must be of the form $(x(s,t), y(s,t), \xi(s,t), \eta(s,t))$ where ξ and η obey the formula

(4.2.1)
$$\begin{aligned} \xi x_s + \eta y_s &= m_s/m \\ \xi x_t + \eta y_t &= m_t/m \end{aligned}$$

i.e. the product of the Jacobian of $x(s,t)$, $y(s,t)$ and $(\xi, \eta)^T$ is equal to the gradient of $\log(m)$.

Away from the exceptional arcs the Jacobian is invertible, hence there is a unique such ξ, η, namely

$$\xi = m_x/m, \quad \eta = m_y/m.$$

Since the complement of the exceptional arcs is dense, if a global solution to (4.2.1) exists it is unique. Near the exceptional arcs we may take our coordinates to be adapted and rewrite (4.2.1) as

$$
\begin{pmatrix} 1-t & t \\ -s & s \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} m_s/m \\ m_t/m \end{pmatrix}.
$$

One can then check that the unique local solution is given by

$$
\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 1 & 1-t \end{pmatrix} \begin{pmatrix} m_s/m \\ m_t/sm \end{pmatrix}.
$$

Note that here we have used the hypothesis that m_t/s exists.

   The fact that condition (3) is satisfied away from the boundaries of the exceptional arcs is the same as the usual calculation with standard and costandard sheaves [NZ]. As for the boundaries of the exceptional arcs, they can be studied in a local model: Example 4.10 below therefore proves the remainder of the theorem.   □

Example 4.10. In adapted coordinates, an example of a function satisfying the hypotheses of Definition/Proposition 4.9 is $m = \exp(-\frac{s}{t(1-t)})$. Solving for ξ and η, we find

$$
\xi = \frac{3t-2}{t(1-t)^2}, \quad \eta = \frac{1-3t}{t^2(1-t)}
$$

As $t \to 0^+$, the vector (ξ, η) approaches $\mathbb{R}_+(0,1) \in T^\infty_{(0,0)}\mathbb{R}^2$, and as $t \to 1^-$ it approaches $\mathbb{R}_+(1,0) \in T^\infty_{(0,0)}\mathbb{R}^2$. In particular, this is consistent with the conclusion that L is asymptotic to Λ. As t goes from 0 to 1 while $s = 0$, the argument of (ξ, η) moves counterclockwise through an angle of $3\pi/2$ from $\pi/2$ to 0. The level set $m = e^{-C}$ in L is the curve $s = Ct(1-t)$, a horizontal parabola passing through $(0,0)$ and $(0,1)$ with vertex $(C/4, 1/2)$ as shown below.
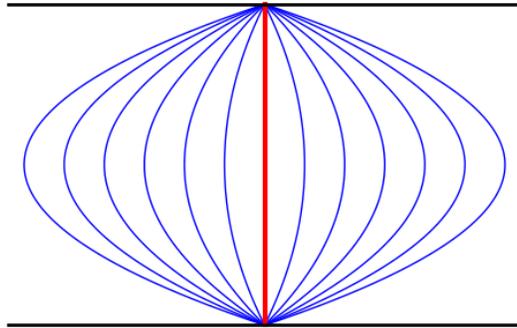
Remark 4.11. If we isotope Λ so that at each crossing both strands are tangent to the graph Γ to all orders, hence nontransverse to each other, we can arrange the conjugate Lagrangian L so that its intersection with the zero section is exactly Γ. To do this we choose m so that on the white regions $\log(m)$ is equal to zero along Γ and decreases monotonically to $-\infty$ towards the boundary of L (so the preimage of $(-R, 0]$) retracts onto Γ for all R). In particular the critical locus of $\log(m)$ on the white regions is exactly Γ. Likewise, we ask that on the black regions $\log(m)$ is zero along Γ

and increases monotonically to ∞ towards the boundary. We no longer insist $\log(m)$ be defined on the exceptional arcs, but still the complement of the exceptional arcs is embedded into $T^*\Sigma$ as the graph of $d\log(m)$. Then by the tangency conditions on the strands of Φ taking a closure embeds the exceptional arcs as the conormal lines to Γ at the crossings. It is clear that L is exact since it retracts onto its intersection with the zero section, where the Liouville form is zero.

4.3. **Alternating sheaves.** Given an alternating Legendrian $\Lambda \subset T^{\infty}\Sigma$ and its conjugate Lagrangian L, the structures constructed in Proposition **??** allow us to consider Lagrangian branes supported on L. Taking family Floer homology with fibers of the cotangent bundle as in [Nad, NZ] yields sheaves on Σ. We describe the resulting sheaves here.

Note first that since each fiber intersects L in a unique point above the white and black regions, and in no points above the null regions, the resulting sheaf will be supported on the closure of the union of the black and white regions. Moreover, the stalks of the sheaf on these regions will necessarily be locally constant and isomorphic to the stalk of the local system on L, with a degree shift between white and black regions corresponding to the shift in gradings described in the proof of **??**. Since $\partial L = \Lambda$, the resulting sheaf will have microsupport at infinity contained in Λ. That is, it is a sheaf of the following form:

Definition 4.12. Let $\Lambda \subset T^{\infty}\Sigma$ be equipped with an alternating coloring. An **alternating sheaf** is an object of $Sh_\Lambda(\Sigma; \mathbb{k})$ whose support is contained in the closure of the union of the white and black regions.

By a locally costandard sheaf on an open subset $j : U \hookrightarrow \Sigma$ we mean any sheaf of the form $j_!\mathcal{L}$ for a locally constant sheaf $\mathcal{L}$ of invertible $\mathbb{k}$-modules on U. Likewise, by a locally standard sheaf we mean any sheaf of the form $j_*\mathcal{L}$ for such an $\mathcal{L}$.

Proposition 4.13. *Let $\mathcal{F}$ be an alternating sheaf whose microstalks have cohomology vanishing outside of degree zero. Let $w : W \to \Sigma$ and $b : B \to \Sigma$ be the inclusion of the interior of the (open) white and black regions, respectively. Then $\mathcal{H}^0(\mathcal{F}) \cong w_!w^!\mathcal{F}$ is a locally costandard sheaf on the union of the white regions, $\mathcal{H}^1(\mathcal{F}) \cong b_*b^*\mathcal{F}[1]$ is a locally standard sheaf on the union of the black regions, and all other cohomology sheaves vanish.*

Proof. Note that the closure of a given white region is disjoint from all other white regions; similarly the closure of a given black region is disjoint from all other black regions.

At a neighborhood of a smooth point of the front projection, one has either a null region separated from a white region, or a black region separated from a null region. Denote the corresponding morphisms along the characteristic path by $\mathcal{F}_N \to \mathcal{F}_W$ or $\mathcal{F}_B \to \mathcal{F}_N$. In both cases, the cone has, by assumption, cohomology only in degree zero, and also by assumption, $\mathcal{F}_N = 0$. It follows that the stalk on the black region has cohomology only in degree 1, and the stalk in the white region has cohomology only in degree 0.

The microsupport prescribes that the generization maps from stalks at the boundary of the black regions into the black regions give isomorphisms, and that the generization maps from the stalks on

the smooth boundaries of the white regions to the nearby null regions are isomorphisms (to zero). In particular, the stalks of $\mathcal{H}^0(\mathcal{F})$ vanish outside the open white regions, and $\mathcal{H}^1(\mathcal{F})$ is a locally constant sheaf on the closure of the black regions. The result follows.   □

Otherwise stated, the alternating sheaves fit into exact triangles

$$\mathcal{F}_W[1] \to \mathcal{F} \to \mathcal{F}_B \xrightarrow{[1]}$$

where $\mathcal{F}_W$ is a locally costandard sheaf on W and $\mathcal{F}_B$ a locally standard sheaf on B. Such triangles are classified by $\mathscr{H}om(\mathcal{F}_B[-1], \mathcal{F}_W[1])$; we now recall from [STZ] which such extensions give objects in $Sh_\Lambda(\Sigma)$.

Proposition 4.14. *Let $\mathcal{L}_W$ and $\mathcal{L}_B$ be local systems on the interiors of the white and black regions of an alternating coloring, with generic stalks ℓ_W, ℓ_B. Then $\mathscr{H}om(b_*\mathcal{L}_B[-1], w_!\mathcal{L}_W[1])$ is a direct sum of skyscraper sheaves in degree zero at the crossings of $\pi(\Lambda)$, each isomorphic to $\mathrm{Hom}(\ell_W, \ell_B)$.*

The extension determined by a given class in $\mathrm{Ext}^1(b_*\mathcal{L}_B, w_!\mathcal{L}_W[1])$ has microsupport in Λ iff the corresponding local elements of $\mathrm{Hom}(\ell_W, \ell_B)$ are all isomorphisms.

Proof. Recall that $\mathscr{H}om(X, Y) = \mathbb{D}(\mathbb{D}Y \otimes X)$, where $\mathbb{D}$ denotes Verdier duality. So

$$\mathscr{H}om(\mathcal{L}_B[-1], \mathcal{L}_W[1]) = \mathbb{D}(\mathbb{D}(w_!\mathcal{L}_W[1])\otimes b_*\mathcal{L}_B[-1]) = \mathbb{D}(w_*\mathcal{L}_W^\vee[1]\otimes b_*\mathcal{L}_B[-1]) = \mathbb{D}(w_*\mathcal{L}_W^\vee\otimes b_*\mathcal{L}_B)$$

If p is any point, then $(w_*\mathcal{L}_W^\vee \otimes b_*\mathcal{L}_B)_p = (w_*\mathcal{L}_W^\vee)_p \otimes (b_*\mathcal{L}_B)_p$, which can only be nonzero for p in the intersection of the closures of the white and black regions, i.e., at a crossing. The above formula shows that the stalk here is evidently the hom space between nearby stalks of the local systems in the white and black regions. This proves the first statement.

The second statement follows from a direct computation as in [STZ, Theorem 3.12]. This calculation can be packaged as the statement that the above hom space is also the stalk of the Kashiwara-Schapira μhom sheaf (see [KS, Sec. 6]) along the interior of $ss(w_!\mathcal{L}_W[1]) \cap ss(b_*\mathcal{L}_B[-1])$ — the covectors pointing into the black region — and the cone over the stalk of the μhom becomes the microstalk of the cone.

Another way to see this is to perform a contact transformation, moving the Legendrian graph $ss(w_!\mathcal{L}_W[1])\cap ss(b_*\mathcal{L}_B[-1])$ to one whose front projection is locally an embedding near the desired microstalk, whereupon the desired μhom calculation reduces to the above $\mathscr{H}om$ calculation.   □

Theorem 4.15. *Fixing brane structures on L determines a morphism $Loc(L; \Bbbk) \hookrightarrow Fuk_\Lambda(T^*\Sigma; \Bbbk) \cong Sh_\Lambda(\Sigma; \Bbbk)$ whose essential image is the category of alternating sheaves.*

Proof. The fact that the image is an alternating sheaf follows from **??** together with the fact that L projects to the union of the white and black regions.

To show surjectivity, observe that according to Proposition 4.14, an alternating sheaf is equivalent data to the specification of a locally free module of some fixed rank at each black and white vertex, and an isomorphism along each edge — that is, the category of alternating sheaves is equivalent to

local systems on the bicolored graph, hence to the category of local systems on L, as the Lagrangian retracts to the graph.

It remains to show our Floer theoretically defined map realizes this isomorphism. Since we know full faithfulness, it suffices to show that the map is surjective on moduli spaces. Since the spaces are abstractly isomorphic, surjectivity follows from injectivity by the Ax-Grothendieck theorem [Gro, 10.4.11]. □

Recall that the space of rank-one Lagrangian branes supported on L is (up to stabilizers) an open algebraic torus in $\mathcal{M}_1(\Lambda)$. We can now describe it sheaf-theoretically:

Corollary 4.16. *In $\mathcal{M}_1(\Lambda)$, the subspace of rank-one Lagrangian branes supported on L coincides with the subspace of alternating sheaves.*

Since abstractly this subspace is also isomorphic to the space of local systems on Γ, it has natural coordinates described by holonomies around the faces of Γ (that is, around the contractible regions of $\Sigma \smallsetminus \Gamma$). However, there is one subtlety: the equivalence between local systems on L and alternating sheaves depends on choices of spin structure on L and Σ, and different choices lead to different signs.

There is a convenient combinatorial way of fixing this ambiguity, following [STZ, Prop. 5.12]. Suppose $\mathcal{F}$ is an alternating sheaf and ℓ_W, ℓ_B stalks of $\mathcal{F}_W$, $\mathcal{F}_B$ in the neighborhood of a fixed crossing of $\pi(\Lambda)$. Picking one of the two components of Λ above the crossing picks out an isomorphism between ℓ_W and $\ell_B[1]$: each is isomorphic to the microstalk of $\mathcal{F}$ at a point of that component on either side of the crossing, and parallel transport in the microlocalization $\mathcal{F}_\Lambda$ defines an isomorphism between these microstalks. Choosing the other component changes the isomorphism $\ell_W \cong \ell_B[1]$ by a sign.

In particular, if we choose a component of Λ above every crossing these isomorphisms between stalks of $\mathcal{F}_W$ and $\mathcal{F}_B[1]$ assemble into a local system on L: the sheaves $\mathcal{F}_W$ and $\mathcal{F}_B[1]$ define a canonical local system on the complement of the exceptional arcs, and the construction above defines a parallel transport across the exceptional arcs. On the other hand, since Σ is oriented there is a consistent notion of which component of Λ is clockwise from the white/black regions and which is counterclockwise. The natural thing is then to choose components consistently:

Definition 4.17. Given an alternating Legendrian Λ, the **standard trivialization** of the space of alternating sheaves is the isomorphism with $Loc_1(L)$ induced by choosing the clockwise component of Λ at each crossing. The **standard face coordinates** on the space of alternating sheaves are the counterclockwise holonomies around the faces of Γ under the standard trivialization. The **positive face coordinates** are the negatives of the standard face coordinates.

Note that choosing counterclockwise components at crossings in the above definition results in the same coordinates. The positive coordinates are so-called because, as we will see in Section 5.2, their transformations are described by subtraction-free expressions. We use the term coordinate

somewhat loosely: depending on the number of contractible regions of $\Sigma \smallsetminus \Gamma$ their boundaries may not form a basis of $H_1(\Gamma; \mathbb{Z})$.

5. CLUSTER COMBINATORICS FROM LEGENDRIAN ISOTOPY

Thus far, we have considered structures which arise from the geometry of a Legendrian link in a fixed position. We turn now to comparisons between these structures arising from Legendrian isotopies.

At the level of categories or of moduli spaces, isotopies give rise to equivalences: given an isotopy $\Lambda \to \Lambda'$, one gets by [GKS] an equivalence $Sh_\Lambda(\Sigma) \to Sh_{\Lambda'}(\Sigma)$, and a corresponding isomorphism of the moduli spaces. However, different isotopy representatives of Λ present different structures on the moduli space. In particular, we saw in Section 3 that an isotopy representative in which Λ is presented as a union of positive braids has, in some cases, a canonical identification with a positroid stratum or a wild character variety. On the other hand, we saw in Section 4 that an isotopy representative which is alternating comes with a natural filling L, hence its moduli space has an abelian chart $Loc_1(L) \hookrightarrow \mathcal{M}_1(\Sigma, \Lambda)$.

This raises a series of questions:

(1) Which Legendrians have alternating representatives; how many are there and what are the isotopies between them?

(2) Given an isotopy between alternating Legendrians, what is the change of coordinates between the corresponding abelian charts?

(3) Given an isotopy from an alternating Legendrian to a localized positive braid, can the coordinates of the abelian chart from the filling be written in terms of some coordinates natural from the nonabelian point of view of the positive braid?

The first question is one of topological combinatorics, and the foundational results in this direction are due to D. Thurston [Thu]. We survey and extend his results in Section 5.1, showing in particular that the alternating Legendrian of a reduced plabic graph admits a homotopically unique isotopy to a Legendrian of positroid type. Following ideas of [GK] we show that this leads to alternating representatives of the Legendrian braid satellites of Section 3.

Though the second and third questions are implicitly Floer theoretic in nature, they can be reduced to combinatorics given the results collected so far. In the previous section, we established a sheaf-theoretic description of the abelian charts, and the constructible sheaves under consideration can be described locally in terms of quiver representation theory. Any isotopy can be factored into a sequence of Reidemeister moves, and the isomorphism induced by [GKS] factors accordingly; each term in this factorization can be described explicitly as reviewed in Section 2.2.

Towards the second question, we show that the *square move* on bicolored graphs is interpolated by a Legendrian isotopy of their corresponding Legendrians. This is move is fundamental, for example any isotopy between reduced alternating Legendrians in $T^\infty D^2$ can be factored into a sequence of square moves. We show that the abelian charts on either side of a square move are

related by the cluster $\mathcal{X}$-transformation classically associated to the square move. The conjugate Lagrangians themselves are related by Lagrangian surgery, a perspective which we develop more systematically in [STW].

In the direction of the third question, we consider the unique isotopy from the alternating Legendrian associated to a reduced plabic graph to the corresponding positroid braid. Since this isotopy gives abelian coordinates on the Grassmannian, the natural question is how to express these in terms of Plücker coordinates. We identify the resulting expression with the boundary measurement map of Postnikov [Pos], which describes the answer in terms of the combinatorics of flows or perfect matchings on the graph.

5.1. **Alternating Legendrians from braids.** We consider here the existence of alternating isotopy representatives of the Legendrians studied in Section 3. These were braid satellites of cocircle fibers of $T^{\infty}\Sigma$, and their rank-one moduli spaces were spaces of filtered local systems on Σ. The main result is that essentially all such Legendrians have alternating representatives. From our point of view this accounts for the appearance of bicolored graphs in the study of such spaces. After proving the general statement we explain how in various examples alternating representatives can be constructed explicitly, bringing us into contact with the combinatorics of triangulations and double wiring diagrams familiar in cluster theory.

We begin our discussion with a class of particularly simple Legendrians:

Definition 5.1. A Legendrian $\Lambda \subset T^{\infty}D^2$ is **reduced** if it satisfies the following conditions:

   (1) Along ∂D^2 the strands of $\pi(\Lambda)$ have alternating orientations.
   (2) The front projection $\Lambda \to D^2$ is an immersion (that is, $\pi(\Lambda)$ has no cusps).
   (3) There are no parallel crossings: if p_1, p_2 are intersection points of two strands, one is oriented from p_1 to p_2 and the other from p_2 to p_1.
   (4) No strands have self-intersections.
   (5) All strands meet the boundary of the disk.

Our terminology is modeled that of [Pos] for bicolored graphs: an embedded bicolored graph $\Gamma \subset D^2$ is a **reduced plabic graph** if its associated Legendrian is reduced. Note that our conventions implicitly allow us to assume that a reduced plabic graph has white vertices where it meets the boundary of D^2.

If $\Lambda \subset T^{\infty}D^2$ is reduced the set $\pi(\Lambda) \cap \partial D^2$ of intersections between its front projection and the boundary of the disk are divided into sets of incoming and outgoing points (we freely pass between orientations and co-orientations following Section 4.1). Each strand of $\pi(\Lambda)$ has one incoming endpoint and one outgoing endpoint, hence Λ defines a matching between these two sets. Conversely, we can fix a set of points on ∂D^2, label them alternatively incoming and outgoing, choose a matching between those of opposite labels, and ask for reduced Legendrians realizing this matching. In this direction we have the following reformulation of a fundamental result of D. Thurston:

Proposition 5.2. [Thu] *Fix a set of points on ∂D^2 alternatively labeled as incoming and outgoing. Every matching between incoming and outgoing points is realized by a reduced alternating Legendrian in $T^\infty D^2$. Moreover, any two reduced alternating Legendrians with the same matching are Legendrian isotopic through a series of square moves.*

We note in passing that while in applications this fact is often used as a purely combinatorial statement (e.g. [Pos]), its relevance to Legendrian knot theory was specifically anticipated in [Thu]. We can complement the part of Proposition 5.2 dealing with isotopies as follows:

Proposition 5.3. *Suppose $\Lambda, \Lambda' \subset T^\infty D^2$ are reduced Legendrians such that $\pi(\Lambda) \cap \partial D^2 = \pi(\Lambda') \cap \partial D^2$ compatibly with incoming/outgoing labels. If Λ and Λ' define the same matching of boundary points, they are Legendrian isotopic. This isotopy can be chosen so that it is stationary above ∂D^2, and only passes through Legendrians whose front projections are immersions. Moreover, the space of such isotopies is contractible.*

Proof. We construct an isotopy $\Lambda \to \Lambda'$ of the stated kind as follows. We notate it as a family of Legendrian embeddings $f_t : \Lambda \to T^\infty D^2$ depending smoothly on $t \in [0,1]$ such that f_0 is the identity map on Λ and f_1 is a diffeomorphism from Λ to Λ'. Number the components of Λ' (hence also Λ) 1 through m, denoting the kth components by Λ'_k, Λ_k. As t varies from $(\ell - 1)/m$ to ℓ/m, we take f_t to be independent of t except along Λ_ℓ.

For $(\ell - 1)/m \le t \le \epsilon + (\ell - 1)/m =: t_0$ we let f_t be a small perturbation such that the part of the front projection of Λ'_ℓ that does not meet the front projection of $f_{t_0}(\Lambda_\ell)$ (that is, $\pi(\Lambda'_\ell) \smallsetminus \pi(f_{t_0}(\Lambda_\ell)) \cap \pi(\Lambda'_\ell)$) has finitely many components. We define f_t for $t_0 \le t \le \ell/m$ inductively as follows. Suppose t_i is such that $\pi(\Lambda'_\ell) \smallsetminus \pi(f_{t_i}(\Lambda_\ell)) \cap \pi(\Lambda'_\ell)$ has finitely many components. If there is only one such component, let $t_{i+1} = \ell/m$, otherwise let t_{i+1} be between t_i and ℓ/m. Let C' be the component of $\pi(\Lambda'_\ell) \smallsetminus \pi(f_{t_i}(\Lambda_\ell)) \cap \pi(\Lambda'_\ell)$ closest to one end of $\pi(\Lambda'_\ell)$. Let C be the segment of $\pi(f_{t_i}(\Lambda_\ell))$ which has the same endpoints as the closure of C'. Together C' and C form the boundary of an embedded disk, since by assumption there are no self-loops in either. For $t_i \le t \le t_{i+1}$ we let f_t act on the front projections by retracting this disk onto the part of its boundary lying along C'. That this can be done so that it lifts to a Legendrian isotopy follows from the assumption that there are no parallel crossings or cusps (the part of $\pi(\Lambda_\ell)$ just past the end of C should also be perturbed in order to not create a corner in the front projection).

To show the space of such isotopies is contractible it suffices to show contractiblity of the group $\mathrm{Aut}(\Lambda)$ of Legendrian isotopies from Λ to itself that are stationary at the boundary and pass through Legendrians whose front projections are immersions. To do this it suffices to describe, for any element g_s of $\mathrm{Aut}(\Lambda)$ and any $s \in [0,1]$, an isotopy f_t from $g_s(\Lambda)$ to Λ which is smooth in s, is the stationary isotopy at $s = 0, 1$, and which itself only passes through Legendrians with immersed front projections. But this can be done using the same prescription we used to construct an isotopy from Λ to Λ'. It is clearly continuous in s, and f_t limits to the stationary isotopy at $s = 0, 1$ since the process of retracting the embedded disks does not increase their size. $\qquad\square$

This guarantees the existence of alternating isotopy representatives of reduced Legendrians in $T^\infty D^2$. We have the following more general existence theorem, which is proved by reduction to Proposition 5.2 following a strategy employed in [GK] for a different class of alternating strand diagrams on T^2. We follow the notation of Section 3.1, including the use of β_i for both a choice of abstract braid at σ_i and for the associated Legendrian satellite in $T^\infty(\Sigma \smallsetminus \sigma)$.

Theorem 5.4. *Let Σ be a closed surface, $\sigma = \{\sigma_1, \ldots, \sigma_k\}$ a nonempty collection of k marked points, and $\sigma_i \mapsto \beta_i \in Br_n^+$ a choice of positive braid at each marked point. If Σ has genus zero and $k = 1$, assume β_1 can be written as $\beta'\Delta^2$ where Δ denotes a half-twist. Then in $T^\infty(S \smallsetminus \sigma)$ the associated Legendrian $\beta = \coprod \beta_i$ is Legendrian isotopic to an alternating Legendrian.*

Proof. If Σ has genus zero and $k < 3$, this follows from Constructions 5.5 and 5.6 (described after the proof), so from now on we assume $k \geq 3$ in the genus zero case. The key point in general is to cut Σ apart into a polygon in such a way that Proposition 5.2 may be applied. While spelling this out in detail is regrettably tedious, it is ultimately an elementary construction. If g is the genus of Σ, fix a $(2g + 2)$-gon P with a gluing map $p : P \twoheadrightarrow S$; that is, Σ is obtained from P by gluing pairs of edges together. The image in Σ of the boundary ∂P is an embedded graph C, either an interval if $g = 0$ or a bouquet of circles. We choose C so that:

(1) The points of σ lie in order along on a single component of the smooth locus of C.
(2) The front projection of each β_i intersects C in $2n$ points, so that C separates it into a pair of n-strand braids.
(3) On one side of C all such braids are trivial.

The preimage $p^{-1}(\pi(\beta))$ of the front projection of β then consists of k disjoint braids attached to ∂P along each of two edges. We call these edges A and A', letting A' denote the edge where all the braids are trivial by item (3).

We now subdivide P further into a union $P = P' \cup B_i \cup \cdots \cup B_k$ of smaller polygons with pairwise disjoint interiors. The role of each B_i will be to isolate the nontrivial part of each braid β_i. That is, we choose them to satisfy the following:

(1) Each B_i is a quadrilateral such that B_i meets ∂P only along A, and $B_i \cap \partial P$ is an edge of B_i.
(2) The image in Σ of $B_i \cap \partial P$ does not meet $\pi(\beta)$, and contains σ_i but no other points of σ.
(3) The interior of B_i contains all crossings in $p^{-1}(\pi(\beta_i))$, and $B_i \cap p^{-1}(\pi(\beta))$ is an n-strand braid diagram with all strands going from one edge of B_i to its opposite (hence each of these edges shares an endpoint with $B_i \cap \partial P$).

We next isotope β so that neighboring front projections $\pi(\beta_i)$ and $\pi(\beta_{i+1})$ overlap in an alternating fashion, stretching each one out along C. That is, we require the following:

(1) The isotopy is only nontrivial on the complement of $p(\sqcup B_i)$.
(2) On the interval of C lying between $p(B_i \cap \partial P)$ and $p(B_{i+1} \cap \partial P)$, the intersections of $\pi(\beta_i)$ and $\pi(\beta_{i+1})$ with C alternate co-orientations ater the isotopy.
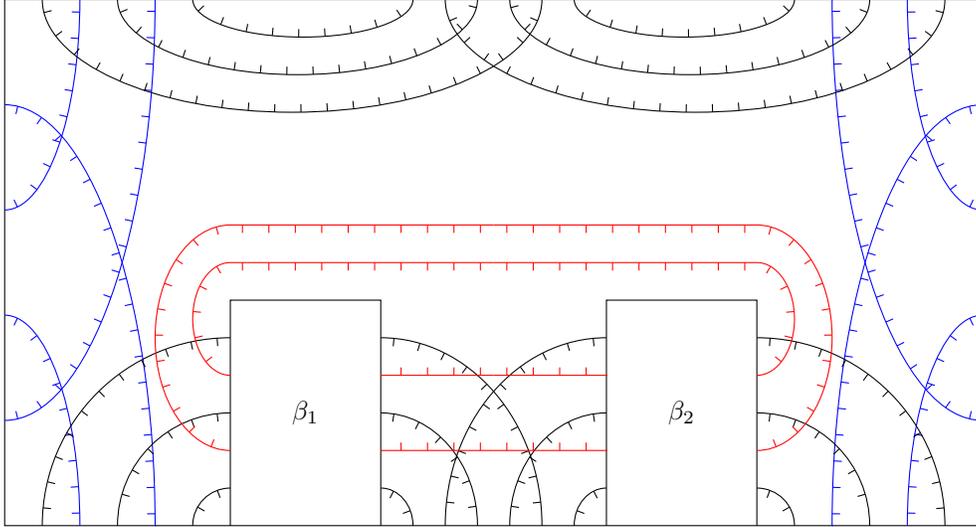
FIGURE 9.   The proof of Theorem 5.4 when $\Sigma = T^2$ and σ consists of two points. The polygon P is pictured with two squares B_1 and B_2 cut out to yield the crenellated polygon P'. The front projection of $\beta = \beta_1 \sqcup \beta_2$ is in black, the nontrivial part of each braid lying in B_1, B_2, respectively. The front projections of the auxiliary Legendrians Λ_B and Λ_C are in red and blue, respectively. Taking their connect sums with β, we obtain a new Legendrian which is isotopic to β and whose co-orientations alternate along the boundary of P'.

(3) The number of intersections of $\pi(\beta)$ with C remains constant through the isotopy.
(4) After the isotopy there are $n(n-1)(k-1)$ new crossings of $\pi(\beta)$ (the minimal possible number created in order to satisfy (1)–(3)).

In particular, $\pi(\beta_i)$ is still disjoint from $\pi(\beta_j)$ unless $|i - j| \leq 1$. From now on β refers to the result of this isotopy.

We now define some auxiliary Legendrians $\Lambda_B, \Lambda_C \subset T^\infty \Sigma$. Their role will be to make sure the co-orientations of $p^{-1}(\pi(\beta \cup \Lambda_B \cup \Lambda_C))$ at its intersections with $\partial P'$ are alternating, setting up the application of Proposition 5.2. The front projection of each component of Λ_B, Λ_C will be an embedded loop, those of Λ_C encircling the vertex or vertices of C and those of Λ_B passing through each of the B_i but not meeting $p(\partial P)$.

We choose Λ_B so that the following properties hold:

(1) $\pi(\Lambda_B)$ consists of $n - 1$ pairwise disjoint embedded loops, and is disjoint from C.
(2) The intersection of each component of $p^{-1}(\pi(\Lambda_B))$ with $\partial P'$ consists of $2k$ points, one on each edge of the B_i that also intersects $p^{-1}(\pi(\beta))$.
(3) The co-orientations of $p^{-1}(\pi(\Lambda \cup \Lambda_B))$ at its intersections with ∂B_i alternate along ∂B_i.
(4) Inside a given B_i the number of intersections of $p^{-1}(\pi(\Lambda_B))$ with $p^{-1}(\pi(\beta))$ is twice the number of crossings of $p^{-1}(\pi(\beta))$ in B_i (the minimum possible number), and the intersection of $p^{-1}(\pi(\Lambda \cup \Lambda_B))$ with B_i is an alternating strand diagram.

We choose Λ_C so that the following properties hold:

(1) If $g > 0$, $\pi(\Lambda_C)$ consists of $n - 1$ pairwise disjoint embedded loops lying in a contractible set containing the vertex of C, each isotopic to a small loop around the vertex of C and intersecting each component of the smooth part of C exactly twice. If $g = 0$ and C is an embedded interval, $\pi(\Lambda_C)$ consists two sets of $n - 1$ pairwise disjoint embedded loops, each set surrounding either end of the interval, and each loop intersecting the interval once.

(2) $\pi(\Lambda_C)$ is disjoint from $p(\sqcup B_i)$

(3) The co-orientations of $p^{-1}(\pi(\Lambda \cup \Lambda_B \cup \Lambda_C))$ at its intersections with $\partial P'$ altnernate along $\partial P'$.

(4) $\pi(\Lambda_C)$ is disjoint from $\pi(\Lambda_B)$, and the strands of $\pi(\Lambda_C)$ intersect each other and the strands of $\pi(\beta)$ the minimal number of times such that the above properties hold.

In particular, though $\pi(\Lambda_B)$ and $\pi(\Lambda_C)$ intersect $\pi(\beta)$, Λ_B and Λ_C can be Legendrian isotoped through $T^\infty(\Sigma \smallsetminus \sigma) \smallsetminus \beta$ so that their front projections are disjoint from $\pi(\beta)$.

The setup so far is illustrated in Figure 9. We have arranged so that $p^{-1}(\pi(\beta \cup \Lambda_B \cup \Lambda_C)) \cap P'$ has alternating co-orientations along $\partial P'$ and no self-loops or parallel bigons, hence we may apply Proposition 5.2 to obtain an alternating Legendrian in $T^\infty P'$. What we really want, however, is an alternating representative of $p^{-1}(\pi(\beta))$. Thus the final step is to take connected sums between certain components of β and Λ_B, Λ_C. This will yield a new Legendrian $\beta' \subset T^\infty\Sigma$ which is Legendrian isotopic to β but retains the desirable combinatorial properties of $\beta \cup \Lambda_B \cup \Lambda_C$.

First we consider the case where Σ has positive genus. The preimage $p^{-1}(\pi(\Lambda_C)) \cap P'$ is a union of embedded intervals, $n - 1$ surrounding each corner of P. Each $p^{-1}(\pi(\beta_i)) \cap P'$ consists of three sets of n parallel embedded intervals, of which one set has both endpoints on A'. We form connected sums between the outer $n - 1$ strands of $p^{-1}(\pi(\beta_1)) \cap P'$ having both endpoints on A' and the strands of $p^{-1}(\pi(\Lambda_C)) \cap P'$ surrounding the corner of A neareast to σ_k. That is, we choose a path γ in P' between the outermost strands of each set, which only meets $p^{-1}(\pi(\beta \cup \Lambda_B \cup \Lambda_C))$ at the endpoints of γ. We now cut both outermost strands at the endpoints of γ, reattaching them to each other by following γ across P'. We repeat this for the remaining $n - 2$ strands, never increasing the total number of crossings. The result is to replace $p^{-1}(\pi(\beta \cup \Lambda_B \cup \Lambda_C)) \cap P'$ with a new collection of immersed co-oriented curves with the same set of crossings but $n - 1$ fewer smooth embedded components.

Next we perform a similar procedure with Λ_B. The components of $p^{-1}(\pi(\Lambda_B)) \cap P'$ we use in the connected sum are those closest to the middle of P; that is, those connecting the edge of B_1 closest to one end of A to the edge of B_k closest to the other end of A. We connect these to the strands of $p^{-1}(\pi(\Lambda_C)) \cap P'$ surrounding the corner of A nearest to σ_1.

Call $\beta' \subset T^\infty\Sigma$ the Legendrian lift of image of the resulting surgered front projection. It follows from the construction that β' is Legendrian isotopic to β: beforehand we could isotope each component of Λ_B, Λ_C so that its front projection is an embedded loop disjoint from the front projection of β. We could equivalently describe β' by doing this isotopy, then taking a connected
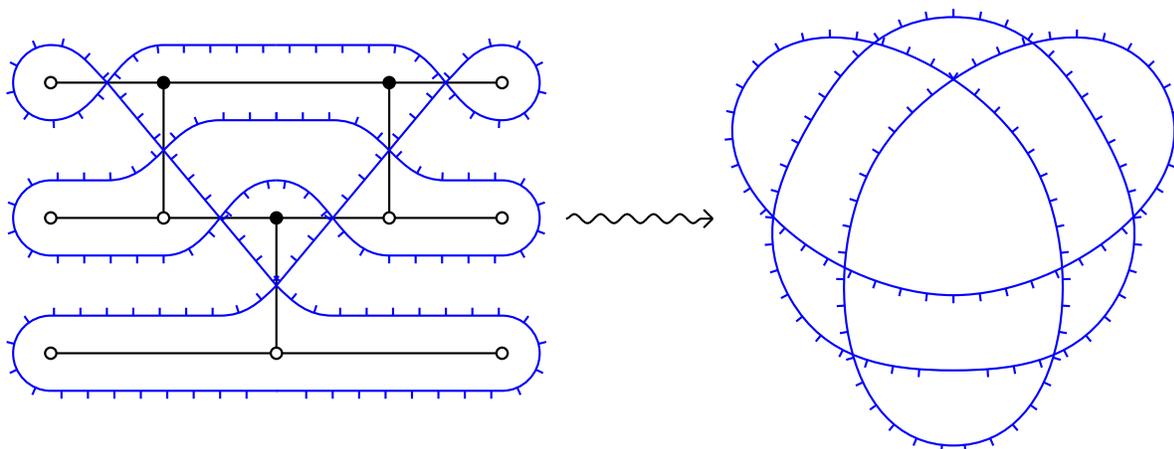
FIGURE 10. Construction 5.5 associates the bicolored graph on the left to the word $s_2 s_1 s_2$ for $\beta' = \Delta$. This produces an alternating representative of $\beta = \Delta^3$ on the right.

sum, then isotoping back, and a connected sum of a Legendrian with one whose front projection is a circle does not change its Legendrian isotopy class.

On the other hand, the connected sum we performed in P' did not create self-loops or parallel bigons (the step involving Λ_B harmlessly creates $n-1$ antiparallel bigons, as does the step involving Λ_C if $k = 1$). By construction the strands of $p^{-1}(\pi(\beta')) \cap P'$ have alternating co-orientations along the boundary of P', hence we can apply Proposition 5.2 to find an alternating Legendrian in $T^\infty P'$ which is isotopic to the lift of $p^{-1}(\pi(\beta')) \cap P'$. But β' was already alternating above the image of each B_i, so we obtain an alternating Legendrian isotopy representative of β', hence β, in $T^\infty \Sigma$. Note that since we apply Proposition 5.2 in P', and σ is disjoint from the image of the interior of P', the resulting isotopy from β to an alternating representative takes place in $T^\infty(\Sigma \smallsetminus \sigma)$.

In the genus zero case, the strategy is the same though we have to specify where to take connected sums differently. We have assumed $k > 2$, so for any choice of $1 < i < k$ the n components of $p^{-1}\pi(\beta_i) \cap P'$ whose endpoints lie on A' do not intersect $p^{-1}(\pi(\Lambda_C))$. As before, we take a connected sum with the $n-1$ outermost of these components with those of $p^{-1}(\pi(\Lambda_C))$ surrounding one corner of P'. We have to now separately take a connected sum of the same components of $p^{-1}(\pi(\beta_i)) \cap P'$ with those of $p^{-1}(\pi(\Lambda_C))$ surrounding the other corner of P' (since in the genus zero case Λ_C has $2(n-1)$ components). Finally, we take a connected sum of the same outermost $n-1$ components of $p^{-1}\pi(\beta_i) \cap P'$ with the components of $p^{-1}(\pi(\Lambda_B)) \cap P'$ closest to the middle of P'. Again the resulting collection of immersed co-oriented curves has no self-loops or parallel bigons, so we may apply Proposition 5.2 as above (if we took $i = 1$ or $i = k$ the above prescription would result in self-loops, so we have indeed used the assumption that $k > 2$).   □

We now describe several constructions of alternating representative for special cases of the Legendrian satellites appearing in Theorem 5.4. The discussion largely amounts to reinterpreting well-known constructions in combinatorics into the language of Legendrian knot theory. We write words for the annular braids β_i in letters $s_1, \ldots, s_{n-1}$, with the convention that s_1 is a crossing of the strands furthest from σ_i. We use Δ to denote the positive half-twist. The first two constructions are Legendrian reinterpretations of the notion of double wiring diagram introduced in [FZ1] and considered for general braids in [FG4]:

Construction 5.5. Let $\Sigma = S^2$, $\sigma = \{\infty\}$, and β a positive annular braid of the form $\beta'\Delta^2$. A word $\beta' = s_{i_1} \cdots s_{i_k}$ determines an alternating isotopy representative of β as follows. Begin with a bicolored graph in the plane consisting of n horizontal line segments running from $(0, i)$ to $(k+1, i)$ for $1 \le i \le n$, with white vertices at both ends. For $1 \le j \le k$ adjoin a vertical segment along the line $y = j$ connecting the line $x = i_j$ to the line $x = 1 + i_j$, with a black vertex at its top and a white vertex at its bottom. From the resulting alternating strand diagram, one obtains the front projection of β by sliding all upward co-oriented strands to the top of the picture and all downward co-oriented strands to the bottom; see Figure 10.

The above assumption that β be of the form $\beta'\Delta^2$ is present for good reason. For example, if β has no crossings at all then $\mathcal{M}_1(S^2, \beta, \infty)$ is a single point whose stabilizer is a Borel subgroup of GL_n. But if β had any smooth exact fillings at all, let alone one arising from an alternating representative, the moduli space would necessarily have a point with an abelian stabilizer. We do not know whether the requirement $\beta = \beta'\Delta^2$ is a necessary condition for β to have alternating representatives.

We note that when $s_{i_1} \cdots s_{i_k}$ is a reduced word for an element w of the symmetric group, a suitably-framed moduli space $\mathcal{M}_1^{fr}(S^2, \beta, \infty)$ recovers the double Bruhat cell $B_+ \cap B_- w B_-$ [FZ1] as well as a certain positroid stratum of $\mathrm{Gr}(n, 2n)$.

Construction 5.6. Let $\Sigma = S^2$, $\sigma = \{0, \infty\}$, and β_0, β_∞ any positive annular braids at 0 and ∞. A double word for (β_0, β_∞) is a shuffle of words for β_0 and β_∞ [FZ]. We encode a double word as a sequence $(s_{i_1}, \ldots, s_{i_k})$ and a function $\tau : \{1, \ldots, k\} \to \{0, \infty\}$ such that the ordered product $\prod_{\tau(j)=\ell} s_j$ is a word for β_ℓ. A double word determines an alternating isotopy representative as follows. Begin with a bicolored graph in the punctured plane consisting of n concentric circles centered at the origin, with white vertices where each intersects the positive x-axis. We adjoin a radial line segment of phase $j2\pi i/(k+1)$ for each $1 \le j \le k$. This segment connects the i_jth and $(i_j + 1)$th circles closest to $\tau(j)$, and has white/black vertices at its farthest/closest endpoint to $\tau(j)$, respectively. From the resulting alternating strand diagram, one obtains the front projection of β by sliding the strands co-oriented towards $0, \infty$ past each other towards $0, \infty$, respectively.

The combinatorics essential to the next construction is due to [FG1]. The cluster algebras associated to these examples when $n = 2$ are often just called cluster algebras from surfaces [FST]. In the literature one often starts with the surface Σ' with marked boundary appearing in the
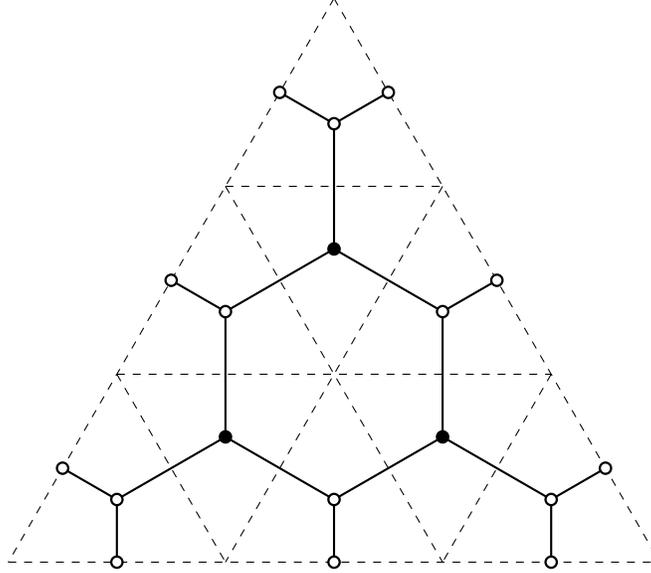
FIGURE 11. The graph Γ_3 of Construction 5.7. If we regard it as a freestanding graph in $\mathbb{R}^2$, rather than attaching it to other copies of itself following some ideal triangulation, it defines an alternating Legendrian isotopic to the Legendrian of Figure 10.

construction, but from our point of view this is simply a convenient way of encoding the number of half-twists.

Construction 5.7. Let Σ, σ be arbitrary and each β_i of the form Δ^{k_i} for some $k_i \in \mathbb{N}$. Alternating represenatatives of $\beta \subset T^\infty \Sigma$ can be constructed using triangulations. First cut out a disk D_i around each σ_i with $k_i > 0$ and let $\Sigma' \subset \Sigma$ denote the resulting surface with boundary. For each such σ_i we also mark k_i points points on the component of $\partial \Sigma'$ that surrounds it. We say an ideal triangulation of Σ' is a triangulation such that all triangles have vertices either on marked points of boundary components or on points σ_i for which $k_i = 0$.

There is a standard bicolored graph Γ_n we can embed in any triangle [FG1]. This is dual to a triangulation of the given triangle into n^2 smaller triangles as in Figure 11. If we are given an equilateral triangle in $\mathbb{R}^2$, we cut it by $n - 1$ equally-spaced lines parallel to each of its three sides. We label the triangles of the resulting triangulation as white or black so that every triangle on the boundary of the original one is white, and no triangles of the same color share an edge. The graph Γ_n has a black/white vertex in the center of each black/white triangle. It also has n white vertices along each edge of the original triangle, one in the middle of each outward-facing edge of a white triangle. There is an edge between any black vertex and each of its three white neighbors, as well as between each white vertex on the boundary and the white vertex in the center of the white triangle whose boundary it lies on.

We associate a bicolored graph $\Gamma \subset \Sigma$ to an ideal triangulation of Σ' by embedding Γ_n into each triangle. This is done so that the white vertices on the boundaries of adjacent triangles coincide. We

can choose the alterating Legendrian of Γ so that the σ_i lie in null regions of its front projections, and so that it is Legendrian isotopic to β inside $T^\infty(\Sigma \smallsetminus \sigma)$.

5.2. **The square move.** There is a local operation on quadrilateral faces of bicolored graphs — the so-called *square move* — that induces isotopies between alternating Legendrians. The corresponding conjugate Lagrangians each determine an abelian chart on the moduli space; we will show here that these charts are related by a cluster $\mathcal{X}$-transformation.

Geometrically, the associated conjugate Lagrangians differ by a certain Lagrangian surgery (see Figure 13); one could imagine using this fact directly to compare, Floer theoretically, the spaces of local systems supported on each. Instead, we use the results of Section 4.3, which capture all the relevant Floer theoretic data in the categories of alternating sheaves associated to the two alternating Legendrians. The comparison between these categories is computed using local calculations of the [GKS] equivalence.

The local model for the square move is the Legendrian isotopy $\Lambda \to \Lambda'$ pictured in Figure 12. Let $\mathcal{F} \in Sh_\Lambda(D^2)$ be an alternating sheaf and N, W, S, E its nonzero stalks near the boundary of the picture. We will compute the image of $\mathcal{F}$ under $Sh_\Lambda \xrightarrow{\sim} Sh_{\Lambda'}$ in terms of the positive face coordinates of Definition 4.17. Here we must consider these not just for closed faces of the graph, but also for the four regions on the boundary. The associated coordinates are more properly isomorphisms

$$X_{NE} : N \xrightarrow{\sim} E, \quad X_{ES} : E \xrightarrow{\sim} S, \quad X_{SW} : S \xrightarrow{\sim} W, \quad X_{WN} : W \xrightarrow{\sim} N,$$

which together with the positive coordinate of the middle region satisfy

$$X_M = -(X_{NE} X_{ES} X_{SW} X_{WN})^{-1}.$$

For example, X_{NE} is explicitly the composition of

(1) the isomorphism of N with a microstalk of $\mathcal{F}$ on the component of Λ passing immediately below the Northern region
(2) parallel transport in $\mathcal{F}_\Lambda$ to a microstalk of $\mathcal{F}$ at the far right of the picture
(3) the isomorphism between this microstalk and E.

The remaining isomorphisms X_{ES}, X_{SW}, X_{WN} can be described symmetrically. Note that the existence of the isomorphisms in (1) and (3) depends on the vanishing of $\mathcal{F}$ in the middle and Northeast null regions. Given an alternating sheaf microsupported on Λ', we similarly denote its positive coordinates by $Y_{NE}, Y_{ES}, Y_{SW}, Y_{WN}$, and Y_M.

Proposition 5.8. *Let $\Lambda, \Lambda' \subset T^\infty D^2$ be the alternating Legendrians related by the square move of Figure 12. Let $\mathcal{F} \in Sh_\Lambda(D^2)$ be an alternating sheaf and $\mathcal{F}' \in Sh_{\Lambda'}(D^2)$ its image under the isotopy equivalence $Sh_\Lambda \xrightarrow{\sim} Sh_{\Lambda'}$. Then $\mathcal{F}$ is an alternating sheaf if and only if $X_M \neq -1$, and its*
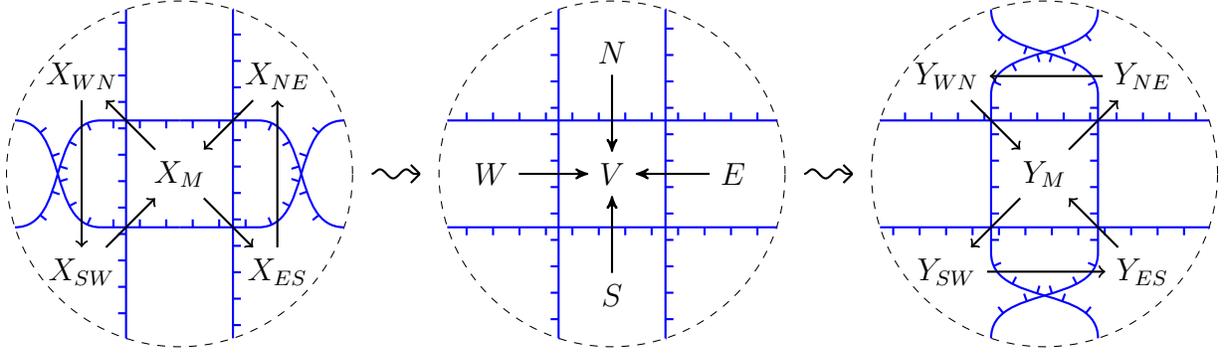
FIGURE 12. The alternating Legendrian Λ' on the right is obtained from Λ on the left by a square move. The isotopy between them can be chosen to pass through $\Lambda_\#$ pictured in the middle. If $\mathcal{F} \in Sh_\Lambda(D^2)$ is alternating, its image $\mathcal{F}_\#$ in $Sh_{\Lambda_\#}(D^2)$ is described by the stalks and generization maps pictured. The dual quivers of the bicolored graphs of Λ, Λ' have vertices labeling the positive face coordinates on their spaces of alternating sheaves.

positive face coordinates are related to those of $\mathcal{F}$ by

$$X_{NE} = Y_{NE}(1 + Y_M), \quad X_{ES} = Y_{ES}(1 + Y_M^{-1})^{-1}, \quad X_{SW} = Y_{SW}(1 + Y_M),$$

$$X_{WN} = Y_{WN}(1 + Y_M^{-1})^{-1}, \quad X_M = Y_M^{-1}.$$

Proof. Denote by $\mathcal{F}_\#$ the image of $\mathcal{F}$ under $Sh_\Lambda \xrightarrow{\sim} Sh_{\Lambda_\#}$, where $\Lambda_\#$ is as in the middle of Figure 12. $\mathcal{F}_\#$ is determined by the data of a generic stalk V in the middle region, along with the four generization maps from S, W, N, and E.

We claim that in terms of $\mathcal{F}_\#$, X_{NE} is the composition of

(1) the generization map from N to V
(2) the map from V to the microstalk of $\mathcal{F}_\#$ on the component of $\Lambda_\#$ passing immediately below the middle region
(3) parallel transport in $(\mathcal{F}_\#)_{\Lambda_\#}$ to a microstalk of $\mathcal{F}_\#$ at the far right of the picture
(4) the isomorphism between this microstalk and E.

Note that under the isotopy $\Lambda \to \Lambda_\#$ the component of $\Lambda_\#$ passing below the middle region corresponds to the component of Λ passing below the Northern region. We obtain $\Lambda_\#$ from Λ by performing Reidemesiter III moves at the black vertices of its bipartite graph followed by a Reidemeister II. The above description of X_{NE} follows from Lemma 2.9, which asserts the invariance of microlocal parallel transport under Reidemeister III, and the following observation about Reidemeister II. In both sides of Figure 5 there is a map from A to $Cone(B \to C)$: on the left the composition $A \to C \to Cone(B \to C)$ and on the right $A \to Cone(C' \to A) \to Cone(B \to C)$. Here $Cone(C' \to A) \to Cone(B \to C)$ is the canonical isomorphism coming from $C' \cong Cone(A \oplus B \to C)[-1]$. These two maps agree by elementary homological algebra, and are exactly the maps being compared at the beginning of the two descriptions of X_{NE}.

More explicitly, the above description identifies X_{NE} with the natural composition $N \xrightarrow{\sim} V/S \xrightarrow{\sim} E$. Note that $N \to V/S$ being invertible is exactly the condition that $\mathcal{F}_\#$ arose from an alternating sheaf on Λ. It also follows that N, E, S, W are all isomorphic to a single invertible $\Bbbk$-module I (if $\mathcal{F}$ were not alternating $\mathcal{F}_\#$ could have $N \cong S \cong I_1$ and $W \cong E \cong I_2$ for nonisomorphic invertible modules I_1, I_2). Fixing identifications of each with I and of V with I^2, we encode the generization maps as a 2×4 matrix with columns labeled S, W, N, E and entries in $\Bbbk = \mathrm{End}_\Bbbk I$. We write Δ_{NE}, etc., for the minors of this matrix, keeping track of orders of indices so, for example, $\Delta_{NE} = -\Delta_{EN}$. The crossing conditions imply that minors of cyclically consecutive columns are invertible, and in terms of minors we can rewrite the above calculation as $X_{NE} = \Delta_{NS}/\Delta_{ES}$. By symmetry, we also have:

$$X_{NE} = \frac{\Delta_{SN}}{\Delta_{SE}}, \quad X_{ES} = \frac{\Delta_{NE}}{\Delta_{NS}}, \quad X_{SW} = \frac{\Delta_{NS}}{\Delta_{NW}}, \quad X_{WN} = \frac{\Delta_{SW}}{\Delta_{SN}}$$

$$Y_{NE} = \frac{\Delta_{WN}}{\Delta_{WE}}, \quad Y_{ES} = \frac{\Delta_{WE}}{\Delta_{WS}}, \quad Y_{SW} = \frac{\Delta_{ES}}{\Delta_{EW}}, \quad Y_{WN} = \frac{\Delta_{EW}}{\Delta_{EN}}$$

The remaining holonomy X_M is determined by the relation $X_M^{-1} = -X_{WN}X_{NE}X_{ES}X_{SW}$, likewise for Y_M.

Recall the 2-term Plücker relation

$$\Delta_{SN}\Delta_{EW} = \Delta_{SE}\Delta_{NW} + \Delta_{SW}\Delta_{EN}$$

Dividing by $\Delta_{ES}\Delta_{WE}$ and reordering indices, we obtain the desired relation

$$X_{NE} = \frac{\Delta_{SN}}{\Delta_{SE}} = \frac{\Delta_{WN}}{\Delta_{WE}}\left(1 - \frac{\Delta_{WS}\Delta_{EN}}{\Delta_{ES}\Delta_{WN}}\right) = Y_{NE}(1 + Y_M).$$

The remaining relations follow from a symmetric calculation. $\qquad\square$

By locality, the preceding result also determines how alternating sheaves microsupported on more complex alternating Legendrians transform under square moves. The more general transformation rules are naturally expressed in the language of cluster algebra [Fom, Lec, Kel2], which we briefly review following the notation of [FG2, GHK]. Below we write $[a]_+$ for $\max(a, 0)$.

Definition 5.9. A **seed** $s = (N, \{e_i\})$ is the data of a lattice N with skew-symmetric integral form $\{,\}$ and a finite collection $\{e_i\}_{i \in I} \subset N$ of distinct primitive elements indexed by a set I. The mutation of s at $k \in I$ is the seed $\mu_k s = (N, \{\mu_k e_i\})$, where

(5.2.1)
$$\mu_k e_i = \begin{cases} e_i + [\{e_i, e_k\}]_+ e_k & i \neq k \\ -e_k & i = k. \end{cases}$$

To a seed we associate a quiver without oriented 2-cycles and with vertex set $\{v_i\}_{i \in I}$. The number of arrows from v_i to v_j is $[\{e_i, e_j\}]_+$, and if the e_i are a basis the seed is determined up to isomorphism by the quiver. Conversely, given such a quiver Q we have a seed given by $\mathbb{Z}Q_0$ with its natural basis and skew-symmetric form; in the literature one often only considers seeds of this form. One can also consider seeds related to skew-symmetrizable matrices, but these do not arise in

our setting. We also suppress a discussion of frozen indices, obviated in our case by allowing the e_i to fail to generate N.

Given a seed $s = (N, \{e_i\})$, we write $M = \operatorname{Hom}(N, \mathbb{Z})$ and consider the dual algebraic tori

$$\mathcal{X}_s = \operatorname{Spec} \mathbb{Z}N, \quad \mathcal{A}_s = \operatorname{Spec} \mathbb{Z}M,$$

We let $z^n \in \mathbb{Z}N$ denote the monomial associated to $n \in N$, likewise $z^m \in \mathbb{Z}M$ for $m \in M$.

Definition 5.10. For $k \in I$, the **cluster $\mathcal{X}$- and $\mathcal{A}$-transformations** $\mu_k : \mathcal{X}_s \dashrightarrow \mathcal{X}_{\mu_k s}$, $\mu_k : \mathcal{A}_s \dashrightarrow \mathcal{A}_{\mu_k s}$ are the rational maps defined by

$$(5.2.2) \qquad \mu_k^* z^n = z^n (1 + z^{e_k})^{\{e_k, n\}}, \quad \mu_k^* z^m = z^m (1 + z^{\{e_k, -\}})^{-\langle e_k, m \rangle},$$

where $\langle e_k, m \rangle$ denotes the evaluation pairing. We use the term **signed cluster transformations** to refer to the counterparts of these maps where the plus signs are replaced by minus signs.

Let T be an infinite $|I|$-ary tree with edges labeled by I so that the edges incident to a given vertex have distinct labels. Fix a root $t_0 \in T_0$ and label it by the seed s. Label the remaining $t \in T_0$ by seeds s_t such that if t and t' are connected by an edge labeled k, and t' is farther from t_0 than t, then $s_{t'} = \mu_k s_t$.

Definition 5.11. A **cluster $\mathcal{X}$-structure** on Y is a collection $\{\mathcal{X}_{s_t} \hookrightarrow Y\}_{t \in T_0}$ of open maps such that the images of $\mathcal{X}_{s_t}$ and $\mathcal{X}_{\mu_k s_t}$ are related by a cluster $\mathcal{X}$-transformation for all t, k. A partial cluster $\mathcal{X}$-structure is the same but with maps only for a subset of T_0, a cluster $\mathcal{A}$-structure the same but with $\mathcal{A}$-tori and $\mathcal{A}$-transformations, and a signed cluster structure the same but with signed cluster transformations.

When the e_i are linearly independent, the notions of signed and ordinary cluster $\mathcal{X}$-structure coincide: given a homomorphism $\sigma : N \to \{\pm 1\}$ such that $\sigma(e_i) = -1$ for all i, the automorphism $z^n \mapsto \sigma(n) z^n$ intertwines the signed and ordinary cluster transformations.

There is a canonical seed associated to an embedded bicolored graph $\Gamma \subset \Sigma$ and a collection of marked points $\sigma \subset \Sigma$. Let $\{\partial F_i\} \subset H_1(\Gamma; \mathbb{Z})$ be the set of boundaries of faces F_i not meeting σ, where by faces we mean the contractible regions of $\Sigma \smallsetminus \Gamma$. If L is the conjugate Lagrangian of Γ, we have $H_1(\Gamma; \mathbb{Z}) \cong H_1(L; \mathbb{Z})$, so the intersection pairing makes $(H_1(L; \mathbb{Z}), \{\partial F_i\})$ a seed.

The quiver of $(H_1(L; \mathbb{Z}), \{\partial F_i\})$ has vertices labeled by $\{\partial F_i\}$ and $\langle e_i, e_j \rangle_+$ arrows from e_i to e_j. It can be drawn on Σ as follows: the vertex labeled by ∂F_i is drawn in F_i, and an edge of Γ with distinctly colored endpoints and separating two faces is crossed by an arrow with the white endpoint on its right. This is pictured in Figure 12. More precisely, the drawn quiver may have oriented 2-cycles, but removing these one obtains the quiver of $(H_1(L; \mathbb{Z}), \{\partial F_i\})$. Comparing the left picture of **??** and its rotation gives:

Proposition 5.12. [GK, Sec. 4.1] *Let $\Gamma \subset \Sigma$ be an embedded bicolored graph and Γ' the result of performing a square move at a face F_k. There is a homeomorphism of their conjugate Lagrangians L, L' which identifies the seed $(H_1(L'; \mathbb{Z}), \{\partial F_i'\})$ with the one obtained from $(H_1(L; \mathbb{Z}), \{\partial F_i\})$ by mutation at ∂F_k.*

With this in hand we can state:

Theorem 5.13. *Let L be the conjugate Lagrangian of an alternating Legendrian $\Lambda \subset T^\infty \Sigma$, and L', Λ' their counterparts upon performing a square move at a face F_k not meeting σ. We identify the underlying topological spaces of L and L' as in Proposition 5.12, and identify the spaces of alternating sheaves with $Loc_1(L)$ as in Definition 4.17. Under the isomorphism of moduli spaces induced by the Legendrian isotopy $\Lambda \to \Lambda'$, the inclusions*

$$Loc_1(L) \hookrightarrow \mathcal{M}_1(\Lambda, \sigma) \simeq \mathcal{M}_1(\Lambda', \sigma) \hookleftarrow Loc_1(L)$$

are related by the signed cluster $\mathcal{X}$-transformation associated to the mutation of $(H_1(L; \mathbb{Z}), \{\partial F_i\})$ at ∂F_k.

Proof. Since the isotopy $\Lambda \to \Lambda'$ is stationary outside a neighborhood of the face F, the equivalence $Sh_\Lambda(\Sigma) \cong Sh_{\Lambda'}(\Sigma)$ restricts to the identity outside such a neighborhood. On the other hand, in a neighborhood of F_k it restricts to the equivalence explicitly computed in Proposition 5.8. Since positive and standard face coordinates differ by a sign, the formulas computed there are exactly those expressing the signed cluster $\mathcal{X}$-transformation associated to mutation of $(H_1(L; \mathbb{Z}), \{\partial F_i\})$ at ∂F_k. $\square$

The result extends to framed moduli spaces in an obvious way. Strictly speaking, in the unframed case the points of $Loc_1(L)$ have $\mathbb{G}_m$ stabilizers, but we use the usual cluster terminology regardless.

Corollary 5.14. *Suppose $\Lambda \in T^\infty \Sigma$ is an alternating Legendrian. Then $\mathcal{M}_1(\Lambda, \sigma)$ has a partial, signed cluster $\mathcal{X}$-structure with charts labeled by alternating Legendrians obtained by some series of square moves from Λ. If σ is nonempty it has an ordinary partial cluster $\mathcal{X}$-structure.*

Proof. The first part of the corollary is immediate, the second follows from the remark after Definition 5.11 since the ∂F_k are independent in homology when σ is nonempty. $\square$

Given an exact Lagrangian L with a nodal singularity, one can produce two exact Lagrangians L_+, L_- which coincide with L outside a neighborhood of the singularity. These are said to differ by Lagrangian surgery [LS, Pol]. Both L_+ and L_- have degenerations to L through smooth exact Lagrangians but are not themselves related by a Hamiltonian isotopy. The degenerations to L are accompanied by the collapse of a sphere, referred to as the vanishing cycle of the surgery. Surgery of exact Lagrangians is directly related to wall-crossing phenomena in Floer cohomology [FOOO, Chap. 10], and thus to the appearance of cluster transformations in symplectic geometry in the guise of wall-crossing transformations [KoS, KoS2].

In the present setting, Lagrangian surgery on conjugate Lagrangians provides the symplectic interpretation of the square move on bipartite graphs:

Proposition 5.15. *Let L, L' be conjugate Lagrangians related by performing a square move at $\partial F_k \in H_1(L; \mathbb{Z})$. Then L and L' are related by a Lagrangian surgery whose vanishing cycle is ∂F_k.*
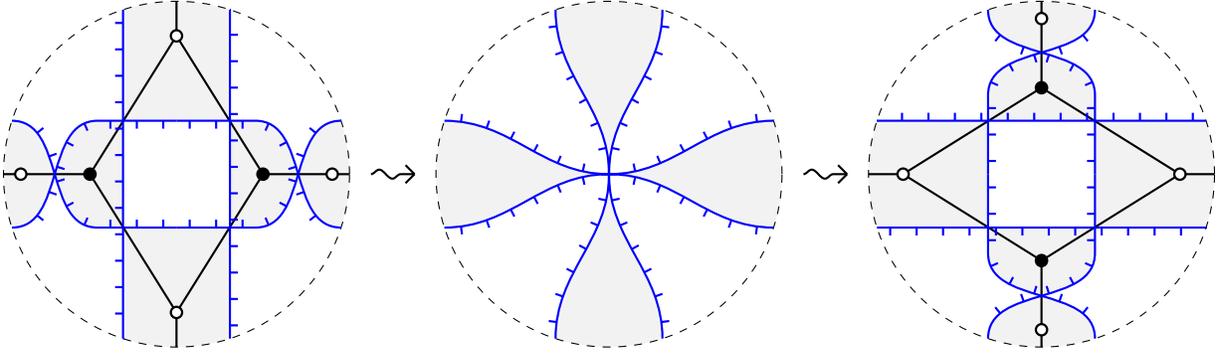
FIGURE 13. The square move as Lagrangian surgery. The left and right frames show the front projections of alternating Legendrians Λ, Λ' related by a square move of their associated bipartite graphs. The shaded regions indicate the projections of a family of exact Lagrangian fillings, which become singular in the middle frame.

Proof. Consider the Legendrian isotopy between the alternating Legendrians Λ, Λ' of L, L' pictured in Figure 13. There is exactly one moment in the isotopy, pictured in the middle frame, where the associated Legendrian is not itself alternating. Thus at every other time we can associate a family of conjugate Lagrangians filling the Legendrians which appear in the isotopy. In the middle frame these degenerate to a nodal Lagrangian, whose intersection with the cotangent space of the singular point of the front projection is the union of the horizontal and vertical conormal lines. That the vanishing cycle is as claimed is evident. That L and L' are not Hamiltonian isotopic, but honestly differ by a surgery, follows from Theorem 5.13: L and L' define different subcategories of alternating sheaves in $Sh_\Lambda \cong Sh_{\Lambda'}$, hence the subcategories of $Fuk_\Lambda(T^*\Sigma) \cong Fuk_{\Lambda'}(T^*\Sigma)$ consisting of Lagrangian branes supported on L and L' are distinct.   □

5.3. **The boundary measurement map.** In Section 5.1 we saw that essentially all Legendrian braid satellites of cocircles have alternating Legendrian isotopy representatives. This leads to the question of describing the associated cluster charts in terms of natural coordinates on spaces of β-filtered local systems. We treat here the fundamental case of positroid strata, showing that our study of Legendrian isotopy recovers the boundary measurement map of Postnikov. This says in particular that, in terms of Plücker coordinates, the cluster charts produced by Hamiltonian isotopy of conjugate Lagrangians can be described as sums over perfect matchings on bipartite graphs.

Recall from Section 5.1 that a reduced plabic graph in D^2 is one whose alternating Legendrian satisfies Definition 5.1. These are exactly the Legendrians which, up to isotopy, arise from positroids:

Proposition 5.16. *Given a reduced plabic graph Γ, there exists a unique cyclic rank matrix r such that Λ_Γ and Λ_r are Legendrian isotopic.*

Proof. First note that the components of Λ_r determine a bijection between incoming and outgoing intersections of its front projection with the boundary of D^2, and this bijection determines r. Thus

given Γ we define the associated cyclic rank matrix as the one corresponding to the boundary matching of Λ_Γ (in the terminology of [Pos] the restriction that all boundary-adjacent vertices of Γ are white means we need only consider matchings or permutations rather than decorated permutations). On the other hand, Λ_r is clearly reduced and by Proposition 5.3 any reduced Legendrians with the same boundary matching are Legendrian isotopic. $\square$

Moreover, Proposition 5.3 tells us that the Legendrian Λ_Γ admits a *contractible* space of isotopies, fixed at the boundary of the disk, to the positroid Legendrian Λ_r. Thus by Proposition 2.7, there is a canonical isomorphism of framed moduli spaces $\mathcal{M}_1^{fr}(\Lambda_\Gamma) \cong \mathcal{M}_1^{fr}(\Lambda_r)$. On the other hand, in Theorem 3.9 we gave a canonical identification $\mathcal{M}_1^{fr}(\Lambda_r) \cong \Pi_r$ of the positroid stratum with the framed moduli space of the positroid Legendrian. If L is the conjugate Lagrangian of Λ_Γ, we can compose this identification with the family Floer theoretic morphism $Loc_1^{fr}(L) \hookrightarrow \mathcal{M}_1^{fr}(\Lambda_\Gamma)$ to obtain a toric chart on the positroid stratum. Finally, recalling that L retracts to the graph Γ, we have the composition

$$\mathbb{F}_\Gamma : Loc_1^{fr}(\Gamma) \cong Loc_1^{fr}(L) \hookrightarrow \mathcal{M}_1^{fr}(\Lambda_\Gamma) \cong \mathcal{M}_1^{fr}(\Lambda_r) \cong \Pi_r.$$

Here, the framing on L or on Γ is again a trivialization of each connected component of the boundary. That is, $Loc_1^{fr}(\Gamma)$ is the algebraic torus $H^1(\Gamma, \partial\Gamma; \mathbb{G}_m)$, where $\partial\Gamma = \Gamma \cap \partial\mathcal{D}$. We also implicitly use the standard trivialization of Definition 4.17 to define $\mathbb{F}_\Gamma$.

On the other hand, the motivation for considering reduced plabic graphs in [Pos] is that each gives rise to a **boundary measurement map**

$$\mathbb{B}_\Gamma : Loc_1^{fr}(\Gamma) \hookrightarrow \Pi_r.$$

We recall the definition of $\mathbb{B}_\Gamma$ as reformulated by the main result of [Tal]. Fix a cyclically-ordered labeling of the boundary vertices of Γ by $\{1, \ldots, n\}$. Orient Γ so that every white vertex has exactly one incoming edge and every black vertex has exactly one outgoing edge; following [Pos] this is called a perfect orientation of Γ. Let $I \subset [1, n]$ be the subset of incoming boundary vertices of Γ, which necessarily has k elements. If $J \subset [1, n]$ is any other k-element subset, we say a flow from I to J is a collection of disjoint self-avoiding oriented cycles in Γ, relative to $\partial\Gamma$, such that each nonclosed cycle connects a boundary vertex in I to a boundary vertex in J.

Each flow F gives rise to a function on $Loc_1^{fr}(\Gamma)$, which by a slight abuse we also denote by F. Then $\mathbb{B}_\Gamma$ is defined by the condition that the pullback of the Jth Plücker coordinate to $Loc_1^{fr}(\Gamma)$ is

$$(5.3.1) \qquad\qquad \mathbb{B}_\Gamma^* \Delta_J = \sum_{F:I \to J} F,$$

where the sum is over all flows from I to J. This definition turns out to be independent of the choice of perfect orientation (of course, it only *ratios* of Plücker coordinates that are meaningful, and changing the orientation may rescale all of them by a common factor). Note also that the labels of the boundary vertices are used to determine the sign of $\mathbb{B}_\Gamma^* \Delta_J$.

We want to compare $\mathbb{F}_\Gamma$ and $\mathbb{B}_\Gamma$, but we can see already that we can only expect them to agree up to certain signs. For example, the definition of $\mathbb{B}_\Gamma$ implicitly depends on how the boundary vertices

are labeled by $1, \ldots, n$, since this ordering is needed to fix the signs of Plücker coordinates in Equation (5.3.1). The definition of $\mathbb{F}_\Gamma$, on the other hand, is manifestly independent of the boundary labels. This is related to the fact that $\mathbb{F}_\Gamma$ transforms by a *signed* cluster transformation under square moves (it is defined using standard face coordinates, so Theorem 5.13 applies). The maps $\mathbb{F}_\Gamma$ do not naturally define a positive locus in $\mathrm{Gr}(k, n)$, and we cannot expect them to given that they are cyclically invariant: when k is even the usual positive part of $\mathrm{Gr}(k, n)$ is itself not cyclically invariant.

Theorem 5.17. *Let Γ be a reduced plabic graph and Π_r the associated positroid stratum. The maps $\mathbb{B}_\Gamma$ and $\mathbb{F}_\Gamma$ coincide up to signs of Plücker coordinates.*

Proof. The main idea is that as Γ ranges over the set of all reduced plabic graphs of all positroid strata, $\mathbb{B}_\Gamma$ is determined by certain "recursion relations" with respect to direct sum and projections [ABCGPT, Section 4.4]. Thus it suffices to show that $\mathbb{F}_\Gamma$ satisfies the same relations (up to signs), and to verify the theorem by hand in the trivial cases when Π_r is the open stratum of $\mathrm{Gr}(1, 3)$ or $\mathrm{Gr}(2, 3)$.

For any pair of Grassmannians there is a direct sum map $\mathrm{Gr}(k_1, n_1) \times \mathrm{Gr}(k_2, n_2) \to \mathrm{Gr}(k_1 + k_2, n_1 + n_2)$ which on $\Bbbk$-points acts by taking $(\Bbbk^{k_1} \twoheadrightarrow E_1, \Bbbk^{k_2} \twoheadrightarrow E_2)$ to $\Bbbk^{k_1 + k_2} \twoheadrightarrow E_1 \oplus E_2$ (here E_1, E_2 are locally free modules of ranks n_1, n_2). Let Γ_1 and Γ_2 be two reduced plabic graphs with associated positroid strata $\Pi_{r_1} \subset \mathrm{Gr}(k_1, n_1)$, $\Pi_{r_2} \subset \mathrm{Gr}(k_2, n_2)$. Let Γ_3 denote the reduced plabic graph which is the disjoint union of Γ_1 and Γ_2, with boundary vertices labeled so that those from Γ_1 retain their original labels while those from Γ_2 have n_1 added to their labels. Let $\Pi_{r_3} \subset \mathrm{Gr}(k_1 + k_2, n_1 + n_2)$ be the positroid stratum associated with Γ_3; it is the image of $\Pi_{r_1} \times \Pi_{r_2}$ under the direct sum map. There is an obvious isomorphism $Loc_1^{fr}(\Gamma_1) \times Loc_1^{fr}(\Gamma_2) \cong Loc_1^{fr}(\Gamma_3)$. The boundary measurement map $\mathbb{B}_{\Gamma_3}$ is determined by $\mathbb{B}_{\Gamma_1}$, $\mathbb{B}_{\Gamma_2}$ in the sense that the following diagram commutes (note that the indexing prescription on Γ_3 fixes the signs of Plücker coordinates in the bottom map):

$$
\begin{array}{ccc}
Loc_1^{fr}(\Gamma_1) \times Loc_1^{fr}(\Gamma_2) & \xrightarrow{\mathbb{B}_{\Gamma_1} \times \mathbb{B}_{\Gamma_2}} & \Pi_{r_1} \times \Pi_{r_2} \\
\downarrow & & \downarrow \\
Loc_1^{fr}(\Gamma_3) & \xrightarrow{\mathbb{B}_{\Gamma_3}} & \Pi_{r_3}
\end{array}
$$

On the other hand, if we replace the boundary measurement maps above by their counterparts $\mathbb{F}_{\Gamma_1}$, $\mathbb{F}_{\Gamma_2}$, $\mathbb{F}_{\Gamma_3}$, then the above diagram still commutes. It suffices to show that the direct sum map corresponds to the isotopy isomorphism $\mathcal{M}_1^{fr}(\Lambda_{r_1}) \times \mathcal{M}_1^{fr}(\Lambda_{r_2}) \xrightarrow{\sim} \mathcal{M}_1^{fr}(\Lambda_{r_3})$ under the identification of Theorem 3.9 (by Proposition 5.3 there is a unique such isomorphism, and this uniqueness forces the diagram to commute). This follows from the appearance of direct sums in the Reidemeister-II move (see Figure 5): the isotopy relates the maximal-rank stalk of a sheaf in $\mathcal{M}_1^{fr}(\Lambda_{r_3})$ to the
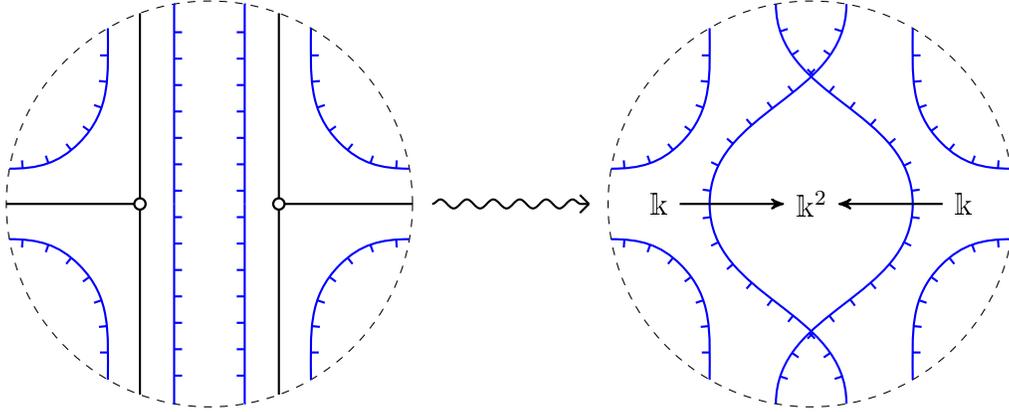
FIGURE 14. The isotopy corresponding to the direct sum map when Γ_1 and Γ_2 have a single trivalent white vertex, so $\Pi_{r_1} = \Pi_{r_2}$ is the big positroid stratum in $\mathrm{Gr}(1,3)$. The framed moduli space of the left picture is manifestly isomorphic $\Pi_{r_1} \times \Pi_{r_2}$, applying the construction of Theorem 3.9 separately to its left and right halves. The framed moduli space of the right picture is manifestly isomorphic to a positroid stratum Π_{r_3} in $\mathrm{Gr}(2,6)$ where the first three (and last three) columns of any matrix representative are pairwise linearly dependent. The crossing conditions on the right assert that the stalk of a sheaf in the middle region is canonically identified with the direct sum of a stalk from the left region and the right region.

maximal rank stalks of sheaves in $\mathcal{M}_1^{fr}(\Lambda_{r_1})$, $\mathcal{M}_1^{fr}(\Lambda_{r_2})$ by a sequence of Reidemeister-II's. This determines the map of positroid strata since by the construction of Theorem 3.9 these stalks and the maps they receive from the boundary stalks determine the maps $\mathbb{F}_{\Gamma_1}$, $\mathbb{F}_{\Gamma_2}$, $\mathbb{F}_{\Gamma_3}$. See Figure 14 for an example.

Now let r be a cyclic rank matrix of type (k,n) such that $r_{12} = 2$, and let Γ be a reduced plabic graph for r. Assume that the bicolored graph Γ' obtained by gluing boundary vertices 1 and 2 together is again a reduced plabic graph, and let r' be its associated cyclic rank matrix. We have a projection map $\Pi_r \to \Pi_{r'} \subset \mathrm{Gr}(k-1, n-2)$ which on $\Bbbk$-points take $\Bbbk^n \twoheadrightarrow E$ to $\Bbbk^{n-2} \twoheadrightarrow E/\langle v_1 - v_2 \rangle$, where v_1, v_2 are the images of 1 in the first two factors of $\Bbbk^n$. There is a natural map $Loc_1^{fr}(\Gamma) \twoheadrightarrow Loc_1^{fr}(\Gamma')$, since the framings let us identify the stalks at boundary vertices 1 and 2 of a framed local system on Γ. The boundary measurement map $\mathbb{B}_{\Gamma'}$ is determined by $\mathbb{B}_\Gamma$ in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
Loc_1^{fr}(\Gamma) & \xrightarrow{\;\;\mathbb{B}_\Gamma\;\;} & \Pi_r \\
\downarrow & & \downarrow \\
Loc_1^{fr}(\Gamma') & \xrightarrow[\;\;\mathbb{B}_{\Gamma'}\;\;]{} & \Pi_{r'}
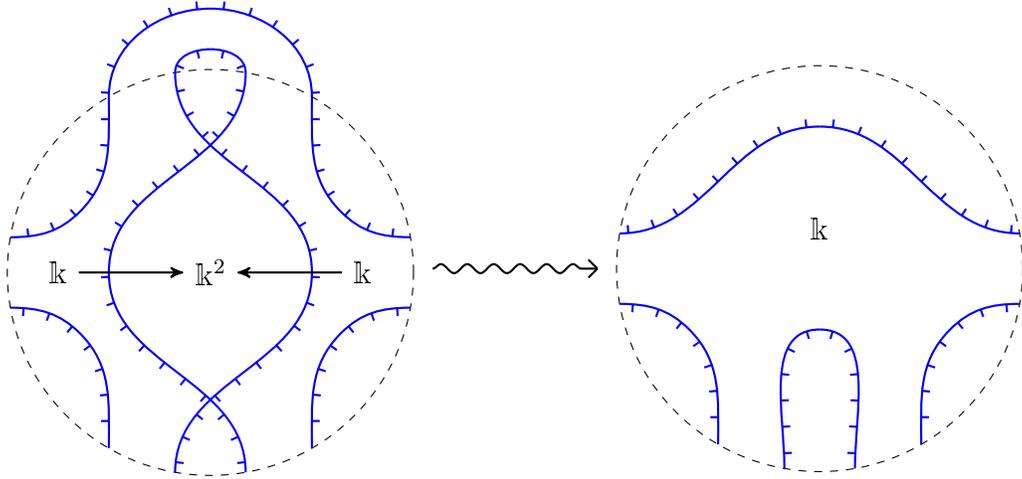\end{array}
$$

FIGURE 15. The isotopy corresponding to the projection map when Π_r is the positroid stratum from the right picture of Figure 14. The starting point is to cap off Λ_r to obtain the Legendrian $\widehat{\Lambda}_r$ whose front projection is on the left; there is a canonical map from $\mathcal{M}_1^{fr}(\Lambda_r)$ to $\mathcal{M}_1^{fr}(\widehat{\Lambda}_r)$. When we isotope to the right hand side the rank two region in the middle is replaced by a rank one region where the boundary stalks from the left and right sides of the picture are identified up to a scalar. Under the correspondence of Theorem 3.9 this is exactly the projection map from Π_r to $\Pi_{r'}$

As above, we claim the diagram still commutes up to signs of Plücker coordinates after replacing the boundary measurement maps by $\mathbb{F}_\Gamma$, $\mathbb{F}_{\Gamma'}$. In terms of alternating Legendrians, gluing vertices 1 and 2 of Γ together corresponds to "capping off" the front projection of Λ_G with two strands going outside the disk, then pulling the cap back inside the disk. Let $\widehat{\Lambda}_r$ be the Legendrian obtained from Λ_r by capping off its front projection in the same way. There is a natural map $\mathcal{M}_1^{fr}(\Lambda_r) \to \mathcal{M}_1^{fr}(\widehat{\Lambda}_r)$ constructed the same way as the gluing map $Loc_1^{fr}(\Gamma) \twoheadrightarrow Loc_1^{fr}(\Gamma')$. It suffices to show that the projection map corresponds to the composition of this with the isotopy isomorphism $\mathcal{M}_1^{fr}(\widehat{\Lambda}_r) \to \mathcal{M}_1^{fr}(\Lambda_{r'})$ under the identification of Theorem 3.9 (as in the direct sum case, by Proposition 5.3 there is a unique such isomorphism, and this uniqueness forces the diagram to commute).

Adding the cap to Λ_r, however, exactly imposes the relation that the boundary stalks of a sheaf at vertices 1 and 2 are identified under generization maps into the disk. Since Γ' is reduced and boundary stalks F_1, F_2 have distinct images in F_x, the innermost strands of Λ_r that are glued together in $\widehat{\Lambda}_r$ are distinct and cross twice. The isotopy between $\widehat{\Lambda}_r$ and $\Lambda_{r'}$ pulls the inner part of the cap through the picture, a series of Reidemeister-III moves, and then pulls it apart by a Reidemeister-II. The resulting isotopy isomorphism acts on maximal-rank stalks exactly by the projection map; see Figure 15 for a simple example.

Finally, we consider the base cases of $\mathrm{Gr}(1,3)$ and $\mathrm{Gr}(2,3)$. For the former there is essentially nothing to show, so we only explicitly discuss the latter. There is only one reduced plabic graph, a trivalent black vertex connected to three white vertices numbered 1, 2, and 3 along the boundary. Let X_{12}, X_{23}, X_{31} denote the face holonomies on $Loc_1^{fr}(\Gamma)$, i.e. X_{12} is the parallel transport from the trivialized stalk at vertex 1 to the one at vertex 2. With the perfect orientation such that 1 is a sink, 2 and 3 sources, we have

$$\mathbb{B}_\Gamma^* \Delta_{12} = X_{31}, \quad \mathbb{B}_\Gamma^* \Delta_{13} = X_{12}^{-1}, \quad \mathbb{B}_\Gamma^* \Delta_{23} = 1.$$

On the other hand, $\mathbb{F}_\Gamma$ is determined by the invariance of microlocalization under Legendrian isotopy, i.e. Lemma 2.9. In standard face coordinates we compute that

$$\mathbb{F}_\Gamma^* \Delta_{12} = X_{31}, \quad \mathbb{F}_\Gamma^* \Delta_{13} = -X_{12}^{-1}, \quad \mathbb{F}_\Gamma^* \Delta_{23} = 1.$$

This agrees with $\mathbb{B}_\Gamma$ up to signs, completing the proof. $\qquad\square$

6. DISTINGUISHING FILLINGS

A standard way of distinguishing Hamiltonian isotopy classes of Lagrangians in a symplectic manifold is to compute Floer homology. These calculations are in general nontrivial, and it is also nontrivial to organize the information extracted from them in a meaningful way. In this section, we observe that our results thus far allow us to package a great deal of Floer theoretic information into structures of cluster algebra. We apply this towards the task of distinguishing Lagrangian fillings of Legendrian knots. We also explain how results about alternating Legendrians in $T^\infty \mathbb{R}^2$ (which necessarily have nontrivial winding number around the fibers) lead to results about Legendrians in $\mathbb{R}^3$. This provides, for example, new combinatorial constructions of inequivalent exact fillings of Legendrian torus links in $\mathbb{R}^3$, as well as information about how these fillings are related by surgery.

Fix some Legendrian Λ, and let Λ_α be a collection of alternating Legendrians equipped with Legendrian isotopies to Λ. Let L_α be the exact filling of Λ obtained by Hamiltonian isotopy from the conjugate Lagrangian filling of Λ_α. Assume in addition that the various Λ_α can be isotoped to each other via square moves. Then it follows from our results that the comparison of charts amongst the $Loc_1(L_\alpha)$ are governed by cluster transformation rules computable from the dual quiver to the bicolored graph determining any one of the Λ_α. These rules, and in particular the question of whether two such charts are the same, have received extensive study in the combinatorial literature. The following is immediate from our Floer-theoretic interpretation of the comparison between the $Loc_1(L_\alpha)$:

Proposition 6.1. *In the above setting, if L_α is Hamiltonian isotopic (fixing the boundary) to L_β, then the induced rational morphism $Loc_1(L_\alpha) \dashrightarrow Loc_1(L_\beta)$ is a regular isomorphism.*

We can apply the above notion to any class of links which have alternating representatives. For example, let $\bigcirc \subset T^\infty \mathbb{R}^2$ be a co-circle, β a positive braid, and Δ the half-twist. Construction 5.5 asserts that every word for β gives rise to an alternating representative of $\beta\Delta^2 \hookrightarrow \bigcirc$, but there are

generally more. If $\beta = T_{k,n}$ is the (k, n) torus braid, then $T_{k,n}\Delta^2 = T_{k,n+k}$ and $T_{k,n+k} \hookrightarrow \bigcirc$ is the braid corresponding to the big positroid stratum of $Gr(k, n + k)$. The enumeration of inequivalent reduced plabic graphs for a fixed positroid was studied in [OPS]. In the case of the big stratum of $Gr(k, n+k)$, they are in bijection with maximal collections of pairwise weakly separated k-element subsets of $[1, n]$. One says two k-element subsets $I, J \subset [1, n]$ are weakly separated if they can be cyclically shifted so that every element of $I \smallsetminus (I \cap J)$ is less than every element of $J \smallsetminus (I \cap J)$.

Proposition 6.2. *The link $T_{k,n+k} \hookrightarrow \bigcirc$ admits a collection of exact Lagrangian fillings into $T^*\mathbb{R}^2$ labeled by maximal pairwise weakly separated k-element subsets of $[1, k + n]$. No two are Hamiltonian isotopic. In particular, if $k = 2$ the number of distinct exact Lagrangian fillings is at least the Catalan number C_n.*

Proof. That distinct maximal weakly separated collections correspond to reduced plabic graphs whose boundary measurement maps have distinct images follows from the results of [MuSp]. Loc. cited shows the image of the boundary measurement map is defined (up to a fixed global automorphism, the so-called twist [MaSc]) by the nonvanishing of a collection of Plücker coordinates associated to the graph as in [Sco]. Distinct weakly separated collections correspond to distinct collections of Plücker coordinates, hence their nonvanishing loci are distinct. The main statement then follows from Proposition 6.1, and the Catalan numbers of the $k = 2$ case appear since in this case reduced plabic graphs are in correspondence with triangulations of an n-gon [FZ].   □

There is analogous notion of weakly separated collection in a more general positroid, and using this Proposition 6.2 generalizes to any positive annular braid arising from a positroid stratum. We refer to [OPS] for the relevant definitions and results. Except for the open positroid stratum of $\mathrm{Gr}(2, n)$, we do not know of a closed formula for the number of maximal weakly separated collections.

A Legendrian of the form $\beta \hookrightarrow \bigcirc$ lives in $T^\infty\mathbb{R}^2$; the above statement concerns its fillings in $T^*\mathbb{R}^2$. However, it is more common to consider Legendrians in the standard contact $\mathbb{R}^3$ and their fillings in its symplectization $\mathbb{R}^4$. Of course, we can view $\mathbb{R}^3 = J^1(\mathbb{R})$ as half the co-circle bundle of $\mathbb{R}^2$, and correspondingly view $\mathbb{R}^4$ as $T^*\mathbb{R}^2$. However, now the front projections of the knots will have cusps, a phenomenon we have avoided throughout this paper.

Nonetheless, our techniques have implications for this setting. The basic point is the existence of the following symplectomorphism, which was explained to us by Emmy Murphy.

Proposition 6.3. *Let $\mathbf{H}^2_{x,y}$ be the lower half plane $(y < 0)$, and let $T^-\mathbf{R}^2_{a,b}$ be the cotangent vectors evaluating negatively on ∂_b. There is an exact symplectomorphism $T^*\mathbf{H}_{x,y<0} \cong T^-\mathbf{R}^2_{a,b}$, inducing a contactomorphism $T^\infty\mathbf{H}^2_{x,y} \cong T^{\infty,-}\mathbf{R}^2_{a,b}$, carrying a co-circle $\bigcirc \subset T^\infty\mathbf{H}^2_{x,y}$ to an eye unknot $\eye \subset T^{\infty,-}\mathbf{R}^2_{a,b}$.*

Proof. The essence of the proof is the observation that "Fourier transform" provides a symplecto-morphism $T^*\mathbf{R}^2_{x,y} \cong T^*\mathbf{R}^2_{a,b}$: in coordinates $(x, y; \xi, \eta)$ and $(a, b; \alpha, \beta)$, we set $a = \xi, b = \eta, \alpha = -x, \beta = -y$. Under the coordinate identification of these spaces with $\mathbf{R}^4$, the symplectomorphism

preserves the radial vector field and therefore induces a contactomorphism between any pairs of hypersurfaces related by a combination of this symplectomorphism and a variable dilation.

Let us review the relevant symplectic geometry of this statement about variable dilations. Suppose $(M, \omega = d\theta)$ is an exact symplectic manifold, with V satisfying $\iota_V \omega = \theta$ the associated Liouville vector field. Now suppose $i : C \to M$ is a hypersurface transverse to V. Then $i^*\theta$ is a coorientation on C and defines a contact structure. Now suppose $f : C \to \mathbf{R}$ is a function. Extend fV to M and let ϕ_t represent the flow by this vector field. Then we claim $\phi_t|_C : C \to \phi_t(C)$ is a contactomorphism for all t. Proof: $\frac{d}{dt}(\phi_t^*\theta) = \phi_t^*(\mathcal{L}_{fV}\theta) = \phi_t^*(\iota_{fV} d\theta) = \phi_t^*(f\iota_V d\theta) = (\phi_t^* f)\phi_t^*\theta$; therefore $\phi_t^*\theta = \theta$ at $t = 0$ and only changes by scalar multiples at each point, leaving the contact structure (but not its coorientation) unchanged.

Let's be explicit about how these constructions look in coordinates.

Contact manifold	$T^\infty \mathbf{R}^{2,-}_{x,y<0}$	$T^- \mathbf{R}^2_{a,b}\|_{S^1}$ or $T^{\infty,-}(\mathbf{R}^2_{a,b} \setminus 0)$
Symplectization	$T' \mathbf{R}^{2,-}_{x,y<0}$	$T^-(\mathbf{R}^2_{a,b} \setminus 0)$
Coordinates	$x, y < 0$	$a = -\xi,\ b = -\eta$
	ξ, η	$\alpha = x,\ \beta = y < 0$
Symplectic form ω	$\omega = dxd\xi + dyd\eta$	$\omega = dad\alpha + dbd\beta$
Liouville field V	$\frac{1}{2}(x\partial_x + \xi\partial_\xi + y\partial_y + \eta\partial_\eta)$	$\frac{1}{2}(a\partial_a + \alpha\partial_\alpha + b\partial_b + \beta\partial_\beta)$
Primitive $\psi = \iota_V \omega$	$\frac{1}{2}(xd\xi - \xi dx + yd\eta - \eta dy)$	$\frac{1}{2}(ad\alpha - \alpha da + bd\beta - \beta db)$
Symplectic Filling	$T^* \mathbf{R}^{2,-}_{x,y}$	$T^- \mathbf{R}^2_{a,b}$

In the right column, we see that there are two contact manifolds to which we can associate the same symplectization. This is because a point in $T^- \mathbf{R}^2_{a,b}|_{S^1}$ has $a^2 + b^2 = 1$ and $\beta < 0$, so is related by dialtion to a unique point in $T^{\infty,-}(\mathbf{R}^2_{a,b} \setminus 0)$, i.e. with $\alpha^2 + \beta^2 = 1$ and $\beta < 0$ — see Figure 16
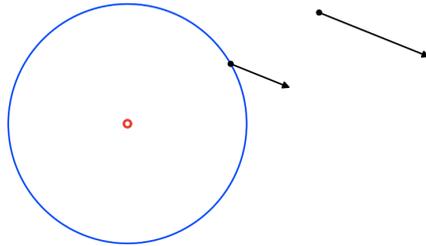

FIGURE 16. Dilation of all four coordinates scales a downward vector at a point on the unit circle to a downward, unit vector at a nonzero point. That is, $T^- \mathbf{R}^2_{a,b}|_{S^1} \cong T^{\infty,-}(\mathbf{R}^2_{a,b} \setminus 0)$.

So these two hypersurfaces are contactomorphic, and therefore their symplectizations are symplectomorphic. The later one $T^{\infty,-}(\mathbf{R}_{a,b}^2 \setminus 0)$ can also be expressed as the hypersurface $\beta = -1$.

Let us be explicit about radial projection, combined with the coordinate change of Fourier transform, gives a contactomorphism of the hypersurface $\{\beta = -1\} \subset T^-(\mathbf{R}_{a,b}^2)$ with $\{\xi^2 + \eta^2 = 1\} \subset T^*\mathbf{R}_{x,y<0}^{2,]}$. The point $(a, b; \alpha, \beta = -1)$ radially projects to

(6.0.1) $$\left(x = -\frac{\alpha}{\sqrt{a^2 + b^2}}, \quad y = -\frac{1}{\sqrt{a^2 + b^2}}, \quad \xi = \frac{a}{\sqrt{a^2 + b^2}}, \quad \eta = \frac{b}{\sqrt{a^2 + b^2}} \right)$$

Let us define the angular coordinate θ so that $\xi = \cos(\theta)$ and $\eta = \sin(\theta)$, and pull back the one-form $\psi|_{\beta=-1}$ to the hypersurface $\{\xi^2 + \eta^2 = 1\}$ coordinatized by (x, y, θ). Putting $c := \cos(\theta)$ and $s := \sin(\theta)$, we get

$$\frac{1}{2y^2}\left(-cdx - sdy + (-xs + yc)d\theta \right)$$

which since it is conformal to $\psi|_{\{\xi^2 + \eta^2 = 1\}}$ establishes that the map is a contactomorphism.

The bottom line of the table therefore shows that the coordinate change of Fourier transform exhibits an exact symplectomorphism between symplectic fillings extending the symplectizations of $T^\infty \mathbf{R}_{x,y<0}^{2,-}$ and $T^{\infty,-}(\mathbf{R}_{a,b}^2 \setminus 0)$ and inducing a contactomorphism of these hypersurfaces.

Now we want to show that the Legendrian unknot fiber of a point maps under the coordinate changes to a Legendrian unknot with standard eye-shaped front diagram. This allows us to conclude that the satellite constructions are related by the contactomorphism as claimed.

To see the second point, let us first write the contact hypersurface as $\mathbf{R}_{std}^3$. So set $\beta = -1$ in the right column. Then the one-form $\psi := \iota_V \omega$ restricts to $\psi = \frac{1}{2}(ad\alpha - \alpha da - db)$. Put

$$k = \frac{1}{2}(-b + a\alpha), \qquad j = \alpha, \qquad i = a.$$

Then $\psi = dk - jdi$. The transformation is easily inverted: $a = i$, $b = ij - 2k$, $\alpha = j$. Also note that $a = b = 0 \iff i = k = 0$, so for $(a, b) \neq (0, 0)$ we have the standard contact structure on $\mathbf{R}_{i,j,k}^3 \setminus \{j\text{-axis}\}$. Next we set $\xi^2 + \eta^2 = 1$ in the first column. Using polar coordinates to write $(\xi, \eta) = (\rho c, \rho s,)$ we coordinatize the hypersurface $\rho = 1$ with (x, y, θ). Then the one-form ψ restricts to $-\frac{1}{2}(cdx + sdy) - \frac{1}{2}(xs - yc)d\theta$. This is not the contact structure $-cdx' - sdy'$ induced from the tautological one-form, in which the circle fiber is Legendrian. However, we can use Moser's trick to define an explicit contactomorphism. Set $x' = \frac{1+s^2}{4}x - \frac{sc}{4}y$, $y' = \frac{1+c^2}{4}y - \frac{sc}{4}x$, $\theta' = \theta$, which inverts to $x = \frac{1+c^2}{2}x' + \frac{sc}{2}y'$, $y = \frac{1+s^2}{2}y' + \frac{sc}{2}x'$, $\theta = \theta'$. Then the one-form $-cdx' - sdy'$ pulls back to $-\frac{1}{4}(cdx + sdy + (xs - yc)d\theta)$ defining the contactomorphism.

Now we can investigate the Legendrian circle fiber over the point $x' = 0$, $y' = -1$, parametrized by θ.[1] In coordinates x, y, θ we have $x = -\frac{sc}{2}$, $y = -\frac{1+s^2}{2}$. What does this look like in the standard

---

[1] In fact, a circle fiber projects to a point in the plane, so we should properly Reeb flow it a bit first (taking $x' = ac$, $y' = -1 + ac$ for a small) and then apply the contactomorphism to the cocircle bundle of the spirograph plane. The resulting unknot eye turns out a bit droopy, but with no modifications of any substance.

contact structure on $\mathbf{R}^3_{i,j,k}$? Using Equation (6.0.1) and the contactomorphisms between $\mathbf{R}^3_{i,j,k}$ and $T^*\mathbf{R}^2_{a,b}|_{S^1}$ and $T^\infty \mathbf{R}^2_{x,y}$, and putting $\xi = c = \cos(\theta), \eta = s = \sin(\theta)$, we must solve

$$c = i/\Delta, \qquad s = (ij - 2k)/\Delta, \qquad x = -j/\Delta, \qquad y = -1/\Delta,$$

where $\Delta := \sqrt{a^2 + b^2} = \sqrt{i^2 + (ij - 2k)^2}$. We find

$$i = -\frac{c}{y}, \qquad j = \frac{x}{y}, \qquad k = \frac{1}{2y^2}(-cx + sy).$$

With x and y as above we find

$$i = \frac{2c}{1 + s^2}, \qquad j = \frac{sc}{1 + s^2}, \qquad k = \frac{-2s^3}{(1 + s^2)^2}.$$

This Legendrian knot has a "standard" eye-shaped front projection i-k plane, as shown below:


$\square$

Corollary 6.4. *Let $\Lambda \subset J^1(S^1)$ be a Legendrian, and let $\bigcirc \subset T^\infty \mathbf{H}^2_{x,y}$ and $\varominus \subset T^{\infty,-}\mathbf{R}^2_{a,b}$ be as above. Then exact Lagrangian fillings of the satellite $\Lambda \hookrightarrow \bigcirc$ inside $T^*\mathbf{H}^2_{x,y}$ are in bijective correspondence with those of $\Lambda \hookrightarrow \varominus$ inside $T^-\mathbf{R}^2_{a,b}$, respecting Hamiltonian isotopy.*

A higher-dimensional example of sheaves corresponding to a singular filling of a two-torus braided around a two-sphere was recently considered in [Nad4].

Remark 6.5. Since fillings are compact relative to the boundary, there is no difference between discussing fillings in $T^*\mathbf{H}^2$ and in $T^*\mathbf{R}^2$.

Corollary 6.4 allows us to translate our results to statements about Legendrians in the usual contact $\mathbb{R}^3$. To make this explicit, let us describe more explicitly the links of the form $\Lambda \hookrightarrow \varominus$.

For a positive braid β, we write $\beta^\vdash$ for the Legendrian with the following front diagram:
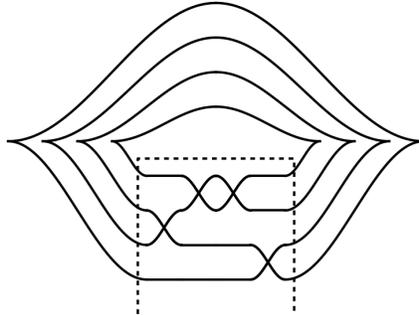
Here the braid is $\beta = s_2 s_3^2 s_1$. In general, we can place any positive braid in the the interior of the dashed region. We called this the "rainbow closure" in [STZ, Sec. 6.2.]. This Legendrian is a maximal Thurston-Benniquin representative of the braid closure of β [Tan], related to the above satellite construction as follows:

Proposition 6.6. *Let β be a positive braid. Then $\Delta\beta\Delta \leftrightarrow \frown$ and $\beta^{\succ}$ are Legendrian isotopic.*

Proof. According to [Ng], the isotopy of Figure 17 relates $\beta^{\succ}$. to $\Delta\beta\Delta \leftrightarrow \frown$. $\square$
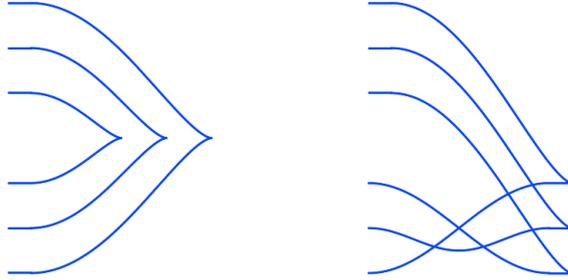

FIGURE 17.   Equivalence of rainbow closure (left) and satellite with half twist at cusps (right).

By [EH], $T_{k,n}^{\succ}$ is the unique Legendrian (k, n) torus knot of maximal Thurston-Bennequin number.

Corollary 6.7. *The Legendrian (k, n) torus link of maximal Thurston-Bennequin number admits a collection of exact Lagrangian fillings labeled by maximal pairwise weakly separated k-element subsets of $[1, k + n]$. No two are Hamiltonian isotopic. In particular, if $k = 2$ the number of distinct exact Lagrangian fillings is at least the Catalan number C_n.*

We leave it to the reader to formulate the analogous statement related to more general positroids using the notion of weakly separated collections in a positroid [OPS].

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