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$\mathrm{H}^4(\mathrm{Co}_0;\mathrm{Z})=\mathrm{Z}/24$

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We show that the 4th integral cohomology of Conway's group Co_0 is a cyclic group of order 24, generated by the 1st fractional Pontryagin class of the 24-dimensional representation.

Let $\mathrm{Co}_0 = 2.\mathrm{Co}_1$ denote the linear isometry group of the Leech lattice, the largest of the Conway groups. By definition, it has a 24-dimensional complex representation, which we will denote by $\mathrm{Leech} \otimes \mathbf{C}$. The main result of this paper is the following:

Theorem 0.1. The group cohomology $\mathrm{H}^4(\mathrm{Co}_0;\mathbf{Z})$ is isomorphic to $\mathbf{Z}/24$. Furthermore,

(1) The Chern class $c_2(\mathrm{Leech} \otimes \mathbf{C}) \in \mathrm{H}^4(\mathrm{Co}_0;\mathbf{Z})$ generates a subgroup of index 2.

(2) There exists a subgroup $\mathrm{CSD} \subset \mathrm{Co}_0$ of order 48, for which the restriction map $\mathrm{H}^4(\mathrm{Co}_0;\mathbf{Z}) \to \mathrm{H}^4(\mathrm{CSD};\mathbf{Z})$ is injective. CSD is isomorphic to $\mathbf{Z}/3 \times 2D_8$, the product of the cyclic group of order 3 and the "binary dihedral" or "generalized quaternion" group of order 16.

As Theorem 5.3, we conclude that $\mathrm{H}^4(\mathrm{Co}_1;\mathbf{Z})$ is isomorphic to $\mathbf{Z}/12$. In Section 6, we compute the Chern classes $c_2(V)$ for each complex irreducible representation V of Co_0—they all have the form $c_2(V) = k(V)c_2(\mathrm{Leech} \otimes \mathbf{C})$ for $k(V) \in \mathbf{Z}/12$.

We note that the fact that the p-primary part of $\mathrm{H}^4(\mathrm{Co}_0;\mathbf{Z})$ vanishes for $p > 3$ is not new—the entire cohomology ring $\mathrm{H}^\bullet\left(\mathrm{Co}_0;\mathbf{Z}\left[\frac{1}{2},\frac{1}{3}\right]\right) \cong \mathrm{H}^\bullet\left(\mathrm{Co}_1;\mathbf{Z}\left[\frac{1}{2},\frac{1}{3}\right]\right)$, and the subring generated by Chern classes, was obtained by the late C. Thomas in [32, Section 3].

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These days, similar results to Theorem 0.1 are sometimes obtained by software advances—for example, [30] determined

$$\mathrm{H}^4(M_{24};\mathbf{Z}) = \mathbf{Z}/12 \qquad \text{(where } M_{24} \text{ denotes the largest Mathieu group)} \tag{0.1}$$

by developing HAP. But the Conway group is large, and we believe that there is no known chain model for $\mathrm{H}^4(\mathrm{Co}_0;\mathbf{Z})$ that can be feasibly handled by a computer.

Our argument actually appeals directly to (0.1), as well as to some GAP-assisted (more specifically, Derek Holt's "Cohomolo" package) computations of H^1 and H^2 of Mathieu groups with twisted coefficients. And we have used GAP and Sage extensively while exploring the Conway group. But formally, our proof is less direct. It is largely based on analyzing a pair of subgroups of Co_0 that contain the 2- and the 3-Sylow subgroups, and that split as semidirect products

$$2^{12} : M_{24}, \qquad 3^6 : 2M_{12}.$$

The colon and the use of 2^n, 3^m for commutative groups follows the ATLAS notation, see Section 1.1. These two subgroups are closely related to the Niemeier lattices A_1^{24} and A_2^{12}, Kneser 2- and 3-neighbors of the Leech lattice.

Our argument is also based on some good luck, as the existence of such a small subgroup CSD that detects H^4 is not a priori clear, but it's crucial for our computation. In fact we encounter the same good luck when studying the Mathieu groups M_{23} and M_{24}, leading to a less computer-intensive proof of (0.1); see Section 5. Theorem 8.1 gives yet another connection between our calculation of $\mathrm{H}^4(\mathrm{Co}_0;\mathbf{Z})$ and the group $\mathrm{H}^4(M_{24};\mathbf{Z})$ of (0.1).

For any particular finite group G, the determination of the group cohomology $\mathrm{H}^\bullet(G;\mathbf{Z})$ is a challenging problem in algebraic topology. The low-degree groups are more accessible, and have concrete group- and representation-theoretical significance with 19th century pedigrees. For example, $\mathrm{H}^2(G;\mathbf{Z})$ is the group of one-dimensional characters of G, and $\mathrm{H}^3(G;\mathbf{Z})$ classifies the twisted group algebras for G. The Pontryagin dual of $\mathrm{H}^3(G;\mathbf{Z})$, which the universal coefficient theorem identifies with $\mathrm{H}_2(G)$, is the Schur multiplier of G.

More recently, a similar role has emerged for $\mathrm{H}^4(G;\mathbf{Z})$—for instance, it classifies monoidal structures on the category of vector bundles on G that have the form of convolution [27, App. E]. Nora Ganter has proposed to call the Pontryagin dual $\mathrm{H}_3(G)$ the "categorical" Schur multiplier of G. We have named the subgroup CSD for "categorical Schur detector".

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The notion of spin structure (and of string obstruction) for a representation reveals a little more structure in Theorem 0.1—a distinguished generator for $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \cong \mathbf{Z}/24$, called the "first fractional Pontryagin class" of the defining representation, denoted $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})$. Its construction is briefly reviewed in Section 1.4. In terms of Chern classes, $2\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R}) = -c_2(\mathrm{Leech} \otimes \mathbf{C})$.

We use the explicit generator to compute the restriction map $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \to \mathrm{H}^4(G; \mathbf{Z})$ for some subgroups $G \subset \mathrm{Co}_0$, that is, we compute the fractional Pontryagin class of the G-action on $\mathbf{R}^{24}$. In particular, in Theorem 7.1 we compute the restriction to all cyclic subgroups of Co_0, where we find a peculiar relationship with Frame shapes of elements—this result generalizes the relationship discovered in [17] between cycle types of permutations in M_{24} and $\mathrm{H}^4(M_{24}; \mathbf{Z})$. In Theorems 8.1 and 8.2 we study the restrictions of $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})$ to certain "umbral" subgroups of Co_0, and relate the answers to the calculations of [9]. A general connection between $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})$ and various forms of moonshine is discussed in [18].

We begin with some preliminary remarks in Section 1, in particular recalling the standard transfer-restriction argument that allows theorems like Theorem 0.1 to be proved prime-by-prime. In Section 2 we quickly dispense with the large primes $p \geq 5$. We handle the prime $p = 3$ in Section 3. The most interesting story occurs at the prime $p = 2$, which is the subject of Section 4; that section completes the proof of Theorem 0.1. Section 5 summarizes our proof of Theorem 0.1 and also computes $\mathrm{H}^4(M_{24}; \mathbf{Z})$, $\mathrm{H}^4(M_{23}; \mathbf{Z})$, and $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z})$. Section 6 explains the computation of the map $c_2 : R(\mathrm{Co}_0) \to \mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})$—the output of the computation is summarized in a table on the last page of that section. In Section 7 we explain the computation of the restriction of $\frac{p_1}{2}$ to all cyclic subgroups of Co_0 and Section 8 discusses the restriction to umbral subgroups.

1 Preliminary Remarks

1.1 Notation

We generally follow the ATLAS [10] for notation for finite groups, and regularly refer to it (often without mention) for known facts. The cyclic group of order n is denoted variously $\mathbf{Z}/n$, $\mathbf{F}_n$ (when n is prime and we are thinking of it as a finite field), and just "n". Elementary abelian groups are denoted n^k and extra special groups n^{1+k}. An extension with normal subgroup N and cokernel G is denoted $N.G$ or occasionally NG; an extension that is known to split is written $N : G$. The conjugacy classes of elements of order n in

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a group G are named nA, nB, ..., ordered by increasing size of the class (decreasing size of the centralizer).

When G is a finite group and A is a G module we write $\mathrm{H}^\bullet(G; A)$ for the group cohomology of G with coefficients in A. But when G is a Lie group we will write $\mathrm{H}^\bullet(BG)$ to avoid confusion with the cohomology of the manifold underlying G.

1.2 The Conway group

We now recite some standard material about Co_0. By definition, Conway's largest group $\mathrm{Co}_0 = 2.\mathrm{Co}_1$ is the automorphism group of the Leech lattice. Its order factors as

$$|\mathrm{Co}_0| = 2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23. \tag{1.1}$$

The Leech lattice is the unique rank-24 self-dual even lattice with no roots. It can be constructed in many ways [12, Ch. 24]. One standard construction of the Leech lattice begins with the Golay code $\mathfrak{g}_{24} : 2^{12} \hookrightarrow 2^{24}$ [19], which is the unique Lagrangian subspace for the standard ("Euclidean") inner product on $2^{24} = \mathbf{F}_2^{24}$ containing no vectors of Hamming length 4. The subgroup of S_{24} preserving the Golay code is the largest Mathieu group M_{24}. Let $\mathrm{Nie}(A_1^{24}) \subset \mathbf{R}^{24}$ denote the Niemeier lattice with root system $A_1^{24} = (\sqrt{2}\mathbf{Z})^{24}$. It is constructed from $\mathfrak{g}_{24}$ as the pullback

$$\begin{tikzcd}
\mathrm{Nie}(A_1^{24}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \left(\frac{\sqrt{2}}{2}\mathbf{Z}\right)^{24} \arrow[d, two heads, "\mathrm{mod}\ (\sqrt{2}\mathbf{Z})^{24}"] \\
2^{12} \arrow[r, hook, "\mathfrak{g}_{24}"] & 2^{24}
\end{tikzcd}$$

By construction, the automorphism group of $\mathrm{Nie}(A_1^{24})$ contains (and in fact is equal to) the semidirect product $2^{24} : M_{24}$, where M_{24} acts by permuting the coordinates and 2^{24} acts by basic reflections. The subgroup $2^{12} : M_{24}$ (in which $2^{12} \subset 2^{24}$ via the Golay code) preserves a unique index-2 sublattice L of $\mathrm{Nie}(A_1^{24})$. This sublattice L can be extended to a self-dual lattice in exactly three ways: it can be extended back to $\mathrm{Nie}(A_1^{24})$; it can be extended to an odd lattice (the so-called "odd Leech lattice" discovered by [28]); and it can be extended to a new even lattice. The 3rd of these is by definition the *Leech lattice*.

By construction, then, $\mathrm{Co}_0 = \mathrm{Aut}(\text{Leech lattice})$ contains a subgroup of shape $2^{12} : M_{24}$. As the order of M_{24} is divisible by 2^{10}, a 2-Sylow subgroup of $2^{12} : M_{24}$ has order 2^{22} and is also a 2-Sylow subgroup of Co_0. A similar construction of the Leech lattice using the ternary Golay code $3^6 \hookrightarrow 3^{12}$ and the Niemeier lattice with root system A_2^{12} provides Co_0 with a subgroup of shape $3^6 : 2M_{12}$ containing the 3-Sylow. (It extends

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to a maximal subgroup of shape $2 \times (3^6 : 2M_{12})$.) The complete list of maximal subgroups of Co_0 was worked out in [35].

1.3 Transfer-restriction

These large subgroups that contain Sylows are very useful for computing cohomology of finite groups, because of the following standard result.

Lemma 1.1. Let G be a finite group. Then $\mathrm{H}^k(G; \mathbf{Z})$ is finite abelian for $k \geq 1$, and so splits as $\mathrm{H}^k(G; \mathbf{Z}) = \bigoplus_p \mathrm{H}^k(G; \mathbf{Z})_{(p)}$ where the sum ranges over primes p and $\mathrm{H}^k(G; \mathbf{Z})_{(p)}$ has order a power of p. Fix a prime p and suppose that $S \subset G$ is a subgroup such that p does not divide the index $|G|/|S|$, that is, such that S contains the p-Sylow of G. Then the *restriction* map $\alpha \mapsto \alpha|_S \, \mathrm{H}^k(G; \mathbf{Z})_{(p)} \to \mathrm{H}^k(S; \mathbf{Z})_{(p)}$ is an injection onto a direct summand.

Proof. Define the *transfer* map $\alpha \mapsto \mathrm{tr}_{G/S}\alpha : \mathrm{H}^k(S; \mathbf{Z}) \to \mathrm{H}^k(G; \mathbf{Z})$ by summing over the fibers of the finite covering $\mathrm{B}S \to \mathrm{B}G$ [4, Section XII.8]. The composition $\alpha \mapsto \mathrm{tr}_{G/S}(\alpha|_S)$ is multiplication by $|G|/|S|$, and so is invertible on $\mathrm{H}^k(G; \mathbf{Z})_{(p)}$. ∎

Thus, in order to understand $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})$, we may work prime by prime. The only primes that participate are those that divide $|\mathrm{Co}_0|$ (1.1). It is known [32] that $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(p)} = 0$ for $p \geq 5$ (we will also verify this directly). As already mentioned, subgroups containing the 2- and 3-Sylows are $2^{12} : M_{24}$ and $3^6 : 2M_{12}$.

1.4 Fractional Pontryagin class

Let $B\mathrm{SO}$ and $B\mathrm{Spin}$ denote the homotopy colimits $\varinjlim_n B\mathrm{SO}(n)$ and $\varinjlim_n B\mathrm{Spin}(n)$, respectively. Then $\mathrm{H}^4(B\mathrm{SO}; \mathbf{Z})$ and $\mathrm{H}^4(B\mathrm{Spin}; \mathbf{Z})$ are both isomorphic to $\mathbf{Z}$. The former group is generated by the 1st Pontryagin class p_1. The restriction map $\mathrm{H}^4(B\mathrm{SO}; \mathbf{Z}) \to \mathrm{H}^4(B\mathrm{Spin}; \mathbf{Z})$ is multiplication by 2, and so the generator of $\mathrm{H}^4(B\mathrm{Spin}; \mathbf{Z})$ is called the 1st *fractional Pontryagin class* and denoted $\frac{p_1}{2}$. The restriction maps $\mathrm{H}^4(B\mathrm{SO}; \mathbf{Z}) \to \mathrm{H}^4(B\mathrm{SO}(n); \mathbf{Z})$ and $\mathrm{H}^4(B\mathrm{Spin}; \mathbf{Z}) \to \mathrm{H}^4(B\mathrm{Spin}(n); \mathbf{Z})$ are isomorphisms for $n \geq 5$.

Suppose that G is a finite group and $V : G \to \mathrm{Spin}(n)$ is a spin representation. The fractional Pontryagin class of V, denoted $\frac{p_1}{2}(V) \in \mathrm{H}^4(G; \mathbf{Z})$, is the pullback of $\frac{p_1}{2}$ along V. This class is also called the "String obstruction" because of its relation to the question of lifting a homomorphism $V : G \to \mathrm{Spin}(n)$ to a loop space map $G \to \mathrm{String}(n)$, where, for $n \geq 5$, $\mathrm{String}(n)$ is the 3-connected cover of $\mathrm{Spin}(n)$. (According to a first-hand account by Chris Douglas, the name "String(n)" for this topological group is due to

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Thomas Goodwillie. For discussion of String(n), see [29].) This is analogous to the role that the 2nd Stiefel–Whitney class $w_2(V) \in \mathrm{H}^2(G; \mathbf{Z}/2)$ plays in measuring whether an oriented representation $V : G \to \mathrm{SO}(n)$ lifts to $\mathrm{Spin}(n)$.

If $V : G \to \mathrm{O}(n)$ is merely a real representation, then $\frac{p_1}{2}(V)$ need not be defined; it is easy to come up with examples where $p_1(V)$ is odd. Suppose that V admits a lift to Spin(n), but that such a lift has not been chosen. The recipe for defining $\frac{p_1}{2}(V)$ above makes it seem that its value might depend on the choice of lift. In fact, $\frac{p_1}{2}(V)$ is well defined for real representations admitting a lift to Spin(n)—it does not depend on the choice of spin structure. Moreover, if V_1 and V_2 are two spin representations, then $\frac{p_1}{2}(V_1 \oplus V_2) = \frac{p_1}{2}(V_1) + \frac{p_1}{2}(V_2)$. One can prove these claims by studying the problem of lifting directly from O(n) to String(n) and showing that the obstruction lives in a certain generalized cohomology theory named "supercohomology" by [20, 33].

Since Co_0 is the Schur cover of a simple group, both $\mathrm{H}^2(\mathrm{Co}_0; \mathbf{Z})$ and $\mathrm{H}^3(\mathrm{Co}_0; \mathbf{Z})$ vanish, and so every real representation $V : \mathrm{Co}_0 \to \mathrm{O}(n)$ lifts uniquely to a spin representation $V : \mathrm{Co}_0 \to \mathrm{Spin}(n)$.

2 The Large Primes $p \geq 5$

We now check that $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(p)} = 0$ for $p \geq 5$, confirming the calculation from [32]. It is equivalent to show that $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z})_{(p)} = 0$ for $p \geq 5$, since the pullback map $\mathrm{H}^\bullet(\mathrm{Co}_1) \to \mathrm{H}^\bullet(\mathrm{Co}_0)$ is an isomorphism on odd parts.

Lemma 2.1. $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(p)} = 0$ for $p \geq 7$.

Proof. There is one conjugacy class each in Co_1 of order 11 and 13, and two of order 23; the p-Sylow subgroups for $p = 11$, 13, and 23 are cyclic. It follows (Pigeonhole) that, for $g \in \mathrm{Co}_1$ of order p, there exists $a \neq \pm 1 \in \mathbf{Z}/p^\times$ such that g is conjugate to g^a. The automorphism $g \mapsto g^a$ acts on $\mathrm{H}^4(\langle g \rangle; \mathbf{Z}) \cong \mathbf{Z}/p$ by multiplication by $a^2 \neq 1$, and so has no fixed points. By Lemma 1.1, $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z})_{(p)}$ injects into the conjugation-in-Co_1-fixed subgroup of $\mathrm{H}^4(\langle g \rangle; \mathbf{Z})$, which is trivial.

A similar argument handles the prime $p = 7$. Indeed, the 7-Sylow in Co_1 is a copy of $(\mathbf{Z}/7)^2$ and is contained in a subgroup isomorphic to $L_2(7)^2$ (following the ATLAS [10], $L_2(7)$ denotes the simple group $\mathrm{PSL}_2(\mathbf{F}_7)$). This is in turn contained in a maximal subgroup of shape $(L_2(7) \times A_7) : 2$. But $L_2(7)$ has a unique conjugacy class of order 7, and so, just as above, $\mathrm{H}^\bullet(L_2(7); \mathbf{Z})_{(7)}$ vanishes in degrees $\bullet \leq 4$. An application of Künneth's formula shows that $\mathrm{H}^4(L_2(7)^2; \mathbf{Z})_{(7)}$ vanishes, but $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z})_{(7)} \to \mathrm{H}^4(L_2(7)^2; \mathbf{Z})_{(7)}$ is an injection, since $L_2(7)^2$ contains the 7-Sylow. ■

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For the prime 5, we need slightly stronger technology. Suppose $G = N.J$ is an extension of finite groups. The *Lyndon–Hochschild–Serre (LHS) spectral sequence* is a spectral sequence converging to $\mathrm{H}^\bullet(G;\mathbf{Z})$ with E_2-page $\mathrm{H}^\bullet(J;\mathrm{H}^\bullet(N;\mathbf{Z}))$. This E_2 page gives an upper bound on the cohomology of G, and often this upper bound suffices.

Lemma 2.2. $\mathrm{H}^4(\mathrm{Co}_0;\mathbf{Z})_{(5)} = 0$.

Proof. Co_1 has a maximal subgroup of shape $G = 5^3 : (4 \times A_5).2$, which contains the 5-Sylow. We work out the LHS spectral sequence for this G. The center $\mathbf{Z}/4$ of $4 \times A_5$ acts with nontrivial central character on $\mathrm{H}^2(5^3;\mathbf{Z}) = 5^3$, $\mathrm{H}^3(5^3;\mathbf{Z}) = \mathrm{Alt}^2(5^3)$, and $\mathrm{H}^4(5^3;\mathbf{Z}) \cong \mathrm{Sym}^2(5^3) \oplus \mathrm{Alt}^3(5^3)$. It follows that the cohomology groups $\mathrm{H}^i(4\times A_5;\mathrm{H}^j(5^3;\mathbf{Z}))$ vanish for $j \in \{1,2,3,4\}$, and so the restriction map $\mathrm{H}^4(5^3 : (4 \times A_5);\mathbf{Z}) \to \mathrm{H}^4(4 \times A_5;\mathbf{Z})$ is an isomorphism. Choosing an element $g \in A_5$ of order 5, the restriction map $\mathrm{H}^4(4 \times A_5;\mathbf{Z})_{(5)} \to \mathrm{H}^4(\langle g\rangle;\mathbf{Z}) = \mathbf{Z}/5$ is an isomorphism.

But Co_1 has only three conjugacy classes of order 5, distinguished by their centralizers, forcing g and g^2 to be conjugate in Co_1, where g is the chosen element of order 5 in A_5. Now we may proceed as in Lemma 2.1: $g \mapsto g^2$ acts as multiplication by -1 on $\mathrm{H}^4(\langle g\rangle;\mathbf{Z})$, while the restriction map $\mathrm{H}^4(\mathrm{Co}_1;\mathbf{Z})_{(5)} \to \mathrm{H}^4(5^3 : (4 \times A_5);\mathbf{Z})_{(5)} \cong \mathrm{H}^4(\langle g\rangle;\mathbf{Z})$ is an injection into the conjugation-invariant classes in the latter. Thus, $\mathrm{H}^4(\mathrm{Co}_1;\mathbf{Z})_{(5)} = 0$. ■

3 The Prime $p = 3$

Each of Co_0 and Co_1 has four conjugacy classes of order 3, which following the ATLAS we call 3A, 3B, 3C, and 3D. They are distinguished by their traces on the Leech lattice:

$$\mathrm{trace}(3\mathrm{A}, \mathrm{Leech}) = -12, \quad \mathrm{trace}(3\mathrm{B}, \mathrm{Leech}) = 6,$$
$$\mathrm{trace}(3\mathrm{C}, \mathrm{Leech}) = -3, \quad \mathrm{trace}(3\mathrm{D}, \mathrm{Leech}) = 0.$$

In this section we will show that the restriction map

$$\mathrm{H}^4(\mathrm{Co}_0;\mathbf{Z})_{(3)} \to \mathrm{H}^4(\langle 3\mathrm{D}\rangle;\mathbf{Z}) \cong \mathbf{Z}/3 \tag{3.1}$$

is an isomorphism, where $\langle 3\mathrm{D}\rangle$ denotes any cyclic subgroup of Co_0 generated by an element in 3D. A generator for $\mathrm{H}^4(\langle 3\mathrm{D}\rangle;\mathbf{Z}) \cong \mathbf{Z}/3$ is given by $c_2(L \oplus \bar{L})$, where L and $\bar{L}$ are the two nontrivial one-dimensional representations of $\langle 3\mathrm{D}\rangle$.

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Lemma 3.1. The map (3.1) is a surjection.

Proof. Since $\mathrm{trace}(3D, \mathrm{Leech}) = 0$, $\mathrm{Leech} \otimes \mathbf{C}$ splits over $\langle 3D \rangle$ as eight copies of each of the three one-dimensional irreps of $\mathbf{Z}/3$. From this one computes

$$\begin{aligned} c_2(\mathrm{Leech} \otimes \mathbf{C}|_{3D}) &= c_2(8 \oplus 8L \oplus 8\bar{L}) \\ &= 8c_2(L \oplus \bar{L}) \\ &= -c_2(L \oplus \bar{L}) \bmod 3, \end{aligned}$$

which is nonzero in $\mathrm{H}^4(\langle 3D \rangle; \mathbf{Z})$. ■

It remains to prove that (3.1) is an injection. By the transfer-restriction Lemma 1.1, $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(3)}$ injects into $\mathrm{H}^4(3^6 : 2M_{12}; \mathbf{Z})_{(3)}$, since $3^6 : 2M_{12} \subset \mathrm{Co}_0$ contains the 3-Sylow. We first show in Lemma 3.2 that $\mathrm{H}^4(3^6 : 2M_{12}; \mathbf{Z})_{(3)} \cong \mathbf{Z}/3 \oplus \mathbf{Z}/3$. Then in Lemma 3.3 we show that the map $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(3)} \to \mathrm{H}^4(3^6 : 2M_{12}; \mathbf{Z})_{(3)}$ is not a surjection. This completes the proof that (3.1) is an isomorphism.

Lemma 3.2. $\mathrm{H}^4(3^6 : 2M_{12}; \mathbf{Z})_{(3)} \cong \mathbf{Z}/3 \oplus \mathbf{Z}/3$.

In the proof, we will see that the splitting is pretty canonical—the homomorphisms $2M_{12} \to 3^6 : 2M_{12} \to 2M_{12}$ realize $\mathrm{H}^4(2M_{12}; \mathbf{Z})_{(3)}$ as a summand of $\mathrm{H}^4(3^6 : 2M_{12}; \mathbf{Z})_{(3)}$, and one knows

$$\mathrm{H}^4(2M_{12}; \mathbf{Z})_{(3)} \cong \mathrm{H}^4(M_{12}; \mathbf{Z}) \cong \mathbf{Z}/3. \tag{3.2}$$

Equation (3.2) is an easy task for HAP [30], though it was previously computed by hand by [23] and perhaps by others.

Proof. $2M_{12}$ has two six-dimensional nontrivial modules over $\mathbf{F}_3$, the ternary Golay code and its dual, the *cocode*. These codes are discussed by Golay himself [19], and as representations of $2M_{12}$ by Coxeter in [13]. In the subgroup of Co_0 of shape $3^6 : 2M_{12}$, the module 3^6 is naturally the code module, and

$$(3^6)^\vee = \mathrm{H}^2(3^6; \mathbf{Z}) \tag{3.3}$$

is isomorphic to the cocode. We will write "E" for the cocode as a $2M_{12}$-module. The modules E and $E^\vee$ are interchanged by the outer automorphism of $2M_{12}$, and so the distinction between E and $E^\vee$ is not particularly important.

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We study the LHS spectral sequence $\mathrm{H}^\bullet(2M_{12}; \mathrm{H}^\bullet(3^6; \mathbf{Z})) \Rightarrow \mathrm{H}^\bullet(3^6 : 2M_{12}; \mathbf{Z})$. As $2M_{12}$-modules, we have

$$\mathrm{H}^2(3^6; \mathbf{Z}) \cong E, \quad \mathrm{H}^3(3^6; \mathbf{Z}) \cong \mathrm{Alt}^2(E)$$

and $\mathrm{H}^4(3^6; \mathbf{Z})$ is an extension

$$0 \to \mathrm{Sym}^2(E) \to \mathrm{H}^4(3^6; \mathbf{Z}) \to \mathrm{Alt}^3(E) \to 0. \tag{3.4}$$

The central $\mathbf{Z}/2$ in $2M_{12}$ acts trivially on $\mathrm{Sym}^2(E)$ and by the sign on $\mathrm{Alt}^3(E)$, so (3.4) is split—$\mathrm{H}^4(3^6 : \mathbf{Z}) \cong \mathrm{Sym}^2(E) \oplus \mathrm{Alt}^3(E)$ as $2M_{12}$-modules.

Moreover, since the central $\mathbf{Z}/2$ acts by the sign representation on E and $\mathrm{Alt}^3(E)$, all cohomology groups $\mathrm{H}^i(2M_{12}; E)$ and $\mathrm{H}^i(2M_{12}; \mathrm{Alt}^3(E))$ vanish. These remarks and (3.2) show that the LHS spectral sequence for $\mathrm{H}^\bullet(3^6 : 2M_{12}; \mathbf{Z})_{(3)}$ begins

$$\begin{array}{ccccc}
\mathrm{H}^0(\mathrm{Sym}^2(E)) & & & & \\
\mathrm{H}^0(\mathrm{Alt}^2(E)) & \mathrm{H}^1(\mathrm{Alt}^2(E)) & & & \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbf{Z} & 0 & 0 & 0 & \mathbf{Z}/3
\end{array}$$

To save space, in the table we have recorded just the coefficients of the cohomology group, so that $\mathrm{H}^i(V) = \mathrm{H}^i(2M_{12}; V)$.

In fact $H^0(\mathrm{Sym}^2(E))$ (and $H^0(\mathrm{Alt}^2(E))$ also) vanishes—as E is irreducible an invariant quadratic (resp. symplectic) form on E must be either zero, or nondegenerate. But such a form cannot be nondegenerate since E is not self-dual as a $2M_{12}$-representation. Since the bottom row of the spectral sequence is split off by a homomorphism $2M_{12} \to 3^6 : 2M_{12}$, it suffices to check that

$$\mathrm{H}^1\left(2M_{12}; \mathrm{Alt}^2(E)\right) = \mathbf{Z}/3.$$

We have verified this using Cohomolo. ■

Lemma 3.3. The restriction map $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(3)} \to \mathrm{H}^4(3^6 : 2M_{12}; \mathbf{Z})_{(3)}$ is not a surjection.

Proof. Consider the permutation representation $\mathrm{Perm} : M_{12} \to \mathrm{O}(12)$, and pull it back (under the same name) to $3^6 : 2M_{12}$. We will prove the Lemma by proving that the 2nd Chern class $c_2(\mathrm{Perm}) \in \mathrm{H}^4(3^6 : 2M_{12}; \mathbf{Z})_{(3)}$ does not extend to $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(3)}$.

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The group $2M_{12}$ has two conjugacy classes of elements of order 3. One acts on the permutation representation of M_{12} with cycle structure $1^3\,3^3$, and the other acts with cycle structure 3^4. Let $a \in 2M_{12}$ denote a representative of the 1st class and $b \in 2M_{12}$ a representative of the 2nd. Then

$$c_2(\mathrm{Perm})|_{\langle a\rangle} = 3c_2(L \oplus \bar{L}) = 0, \qquad c_2(\mathrm{Perm})|_{\langle b\rangle} = 4c_2(L \oplus \bar{L}) \neq 0,$$

where L and $\bar{L}$ are the two nontrivial one-dimensional representations of $\mathbf{Z}/3$.

Under the inclusion $2M_{12} \to 3^6 : 2M_{12} \to \mathrm{Co}_0$, the elements a and b have traces $\mathrm{trace}(a, \mathrm{Leech}) = 6$ and $\mathrm{trace}(b, \mathrm{Leech}) = 0$. The group $3^6 : 2M_{12}$ is small enough to completely handle on the computer—say in terms of its permutation representation of degree 729. By simply running through all elements of 3^6 one finds that there are 162 elements $x \in 3^6$ such that $xa \in 3^6 : 2M_{12}$ has order 3 and $\mathrm{trace}(xa, \mathrm{Leech}) = 0$. Choose one such x. Then $\mathrm{trace}(xa, \mathrm{Leech}) = \mathrm{trace}(b, \mathrm{Leech})$, so xa and b are conjugate in Co_0. But $c_2(\mathrm{Perm})|_{\langle xa\rangle} = c_2(\mathrm{Perm})|_{\langle a\rangle} = 0$. This shows that $c_2(\mathrm{Perm}) \in \mathrm{H}^4(3^6 : 2M_{12}; \mathbf{Z})_{(3)}$ distinguishes elements of $3^6 : 2M_{12}$ that are conjugate in Co_0, and so cannot extend to a class in $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(3)}$. ■

4 The Prime $p = 2$

To complete the proof of Theorem 0.1, we show that $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(2)} \cong \mathbf{Z}/8$. In Lemma 4.1 we find the group CSD from part (2) of the theorem, and use it to give a lower bound of 8 on the order of $\frac{p_1}{2}(\mathrm{Leech})$. Then, using this bound, we show in Lemma 4.5 that $\mathrm{H}^4(2^{12} : M_{24}; \mathbf{Z})_{(2)} \cong \mathbf{Z}/8 \oplus \mathbf{Z}/4$. Finally, in Lemma 4.6, we show that the restriction map $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(2)} \to \mathrm{H}^4(2^{12} : M_{24}; \mathbf{Z})_{(2)}$ is not a surjection. Since, by transfer-restriction Lemma 1.1, its image is a direct summand that, by Lemma 4.1, contains an element of order 8, its image must be isomorphic to $\mathbf{Z}/8$, completing the proof. Parts (1) and (2) of Theorem 0.1 follow from the isomorphism $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \cong \mathbf{Z}/24$ together with our proof of Lemma 4.1.

Lemma 4.1. The order of $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})$ is divisible by 8.

Proof. As $2\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R}) = -c_2(\mathrm{Leech} \otimes \mathbf{C})$, it suffices to show $c_2(\mathrm{Leech} \otimes \mathbf{C})$ has order divisible by 4. We will restrict $\mathrm{Leech} \otimes \mathbf{C}$ to a subgroup of Co_0 isomorphic to the binary dihedral group $2D_8$ of order 16, double covering the symmetries of the square. Of the three 2D irreducible representations of $2D_8$, two are faithful and quaternionic. The other is the 2D real defining representation of D_8. Let us call the quaternionic ones M and M'—they are exchanged by an outer automorphism of $2D_8$. Then $M : 2D_8 \hookrightarrow \mathrm{SU}(2)$ makes $2D_8$

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into one of the McKay subgroups of SU(2). Its McKay graph is

$$\begin{tikzcd} \mathrm{Triv} \arrow[dr, no head] & & & 1 \arrow[dl, no head] \\ & M \arrow[r, no head] & 2 \arrow[r, no head] & M' \\ 1 \arrow[ur, no head] & & & 1 \arrow[ul, no head] \end{tikzcd} \tag{4.1}$$

We have indicated the trivial module and the dimensions of the other irreducible modules, all of which are real and factor through D_8.

For any McKay group $M : G \hookrightarrow \mathrm{SU}(2)$, the group $\mathrm{H}^4(G; \mathbf{Z})$ is cyclically generated by $c_2(M)$, with order $|G|$. Thus, we may write $c_2(M') = kc_2(M)$ for some integer $k \in \mathbf{Z}/16$. In fact $k = 9$—to determine this, we note that $M \oplus M'$ is isomorphic to $\mathrm{Sym}^3(M)$, and (as $c_1(M) = c_1(M') = 0$), $c_2(M \oplus M') = c_2(M) + c_2(M')$. But Sym^3 of the standard representation of SU(2) on $\mathbf{C}^2$ has weights $-3, -1, 1$, and 3, and the standard representation itself has weights 1 and -1, so we compute by the splitting principle

$$c_2(\mathrm{Sym}^3(\mathbf{C}^2)) = \begin{bmatrix} c_1(-3)c_1(-1) + c_1(-3)c_1(1) + c_1(-3)c_1(3) \\ + c_1(-1)c_1(1) + c_1(-1)c_1(3) + c_1(1)c_1(3) \end{bmatrix} = 10\left[c_1(1)c_1(-1)\right] = 10c_2(\mathbf{C}^2),$$

where we have written $c_1(n) \in H^2(B\mathrm{U}(1); \mathbf{Z})$ for the Chern class of representation $\mathrm{U}(1) \to \mathrm{U}(1)$ of degree n.

We will momentarily find a copy of $2D_8$ inside Co_0 such that the central element $c \in 2D_8$ is the central element of Co_0. We may compute the restriction of any representation of Co_0 to such a subgroup, even before proving it exists. In $2D_8$, every conjugacy class of order 4 squares to c, and every conjugacy class of order 8 has 4th power equal to c. In Co_0, there is a unique conjugacy class of order 4 that squares to the central element—it is the unique conjugacy class projecting to 2B in Co_1. There is also a unique conjugacy class of order 8 whose 4th power is the central element—it is the unique conjugacy class projecting to 4E in Co_1. On $\mathrm{Leech} \otimes \mathbf{C}$, the classes of order 4 and 8 have trace 0, and the class of order 2 has trace -24, so from characters of $2D_8$ we compute

$$\mathrm{Leech} \otimes \mathbf{C}|_{2D_8} = 6M \oplus 6M'$$

so that $c_2(\mathrm{Leech} \otimes \mathbf{C})|_{2D_8} = 60c_2(M)$, and 60 has order 4 in $\mathbf{Z}/16$.

To complete the proof, it remains to construct a subgroup $2D_8 \subset \mathrm{Co}_0$ containing the central element. We originally found one by reducing it to a finite search inside of

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$2^{12} : M_{24}$, which we then implemented in Sage. Here is a simpler way that also shows that the $2D_8 \subset \mathrm{Co}_0$ and $\mathbf{Z}/3 \subset \mathrm{Co}_0$ detecting the 2- and 3-parts of cohomology can be chosen to commute with each other; the group CSD from the statement of Theorem 0.1 is simply the product $2D_8 \times \mathbf{Z}/3$ for these commuting subgroups. From [35, Section 2.2], we see that the centralizer of the class 3D in Co_1 is $3 \times A_9$, and the centralizer of its lift in Co_0 is $3 \times 2A_9$, where $2A_9$ denotes the Schur cover of the alternating group. (This group $3 \times A_9$ is the top of the "Suzuki chain" of subgroups of Co_1). The center of $2A_9$ coincides with the center of Co_0, so it suffices to find an inclusion $2D_8 \subset 2A_9$ preserving the centers. The natural inclusion $A_6 \to A_9$ lifts to an inclusion $2A_6 \to 2A_9$, and there is unique conjugacy class of subgroups of A_6 that are isomorphic to D_8. The preimage in $2A_6$ can be seen (we used GAP) to be $2D_8$. ■

Note that, assuming Theorem 0.1 has been proved, our proofs of Lemmas 3.1 and 4.1 further imply that the restriction map $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \to \mathrm{H}^4(\mathrm{CSD}; \mathbf{Z}) \cong \mathbf{Z}/48$ is an injection, verifying part (2) of the theorem. Part (1) is also an immediate consequence of our proof of Lemma 4.1.

Lemma 4.1 provides a lower bound on $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(2)}$; our next step (Lemma 4.5) in the proof of Theorem 0.1 will be to give an upper bound. We will rely on some background on the cohomology of elementary abelian 2-groups, which we now review. Let E be an elementary abelian 2-group and write $E^\vee = \hom(E, \mathbf{F}_2) \cong \hom(E, U(1))$ for the $\mathbf{F}_2$-vector space dual to E. The following is standard.

Lemma 4.2. There is an isomorphism of rings $\mathrm{H}^\bullet(E; \mathbf{Z}/2) \cong \mathrm{Sym}^\bullet(E^\vee)$. For $i \geq 1$, the reduction map $\mathrm{H}^i(E; \mathbf{Z}) \to \mathrm{H}^i(E; \mathbf{Z}/2)$ is injective. The image of $\mathrm{H}^i(E; \mathbf{Z})$ in $\mathrm{Sym}^i(E; \mathbf{Z}/2)$ is the kernel of the Bockstein, or 1st Steenrod squaring map, $\mathrm{Sq}^1 : \mathrm{H}^i(E; \mathbf{Z}/2) \to \mathrm{H}^{i+1}(E; \mathbf{Z}/2)$, which acts as a differential on $\mathrm{Sym}^\bullet(E^\vee)$. Moreover, the long sequence

$$0 \to E^\vee \xrightarrow{\mathrm{Sq}^1} \mathrm{Sym}^2(E^\vee) \xrightarrow{\mathrm{Sq}^1} \mathrm{Sym}^3(E^\vee) \xrightarrow{\mathrm{Sq}^1} \cdots$$

is exact.

Let us denote the quotient $\mathrm{Sym}^2(E^\vee)/\mathrm{Sq}^1(E^\vee)$ by $\mathrm{Alt}^2(E^\vee)$—it is the quotient of $E^\vee \otimes_{\mathbf{F}_2} E^\vee$ by the subspace generated by tensors of the form $x \otimes x$, which is the standard definition of the exterior square functor in characteristic 2 [2, Section III.7]. More generally, in characteristic 2 the alternating power $\mathrm{Alt}^k(E^\vee)$ is defined as the quotient of the tensor power $(E^\vee)^{\otimes k}$ by the subspace spanned by all tensors that contain a repeated factor—all tensors of the form $(\cdots \otimes v \otimes \cdots \otimes v \otimes \cdots)$ for $v \in E^\vee$—and can be

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identified $\mathrm{GL}(E)$-equivariantly with the quotient of $\mathrm{Sym}^k(E^\vee)$ by the subspace spanned by monomials that are not squarefree. We will also use the following:

Lemma 4.3. $\mathrm{Alt}^k(E^\vee)$ is isomorphic to a $\mathrm{GL}(E^\vee)$-stable subspace of $(E^\vee)^{\otimes k}$.

Indeed in characteristic 2, $\mathrm{Alt}^k(E^\vee)$ can be realized as the image of the norm map for the S_k-action on $(E^\vee)^{\otimes k}$:

$$(E^\vee)^{\otimes k} \to (E^\vee)^{\otimes k} \qquad w_1 \otimes \cdots \otimes w_k \mapsto \sum_{\sigma \in S_k} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}.$$

Lemma 4.2 shows that there are isomorphisms of $\mathrm{GL}(E)$-modules

$$\mathrm{H}^2(E; \mathbf{Z}) \cong E^\vee \quad \text{and} \quad \mathrm{H}^3(E; \mathbf{Z}) \cong \mathrm{Alt}^2(E^\vee). \tag{4.2}$$

A little more work gives us a useful description of $\mathrm{H}^4(E; \mathbf{Z})$ as well:

Lemma 4.4. There is a three-step filtration $F_1 \subset F_2 \subset F_3 = \mathrm{H}^4(E; \mathbf{Z})$ by $\mathrm{GL}(E)$-submodules, whose associated graded modules are

$$F_1 \cong E^\vee \qquad F_2/F_1 \cong \mathrm{Alt}^2(E^\vee) \qquad F_3/F_2 \cong \mathrm{Alt}^3(E^\vee).$$

See [26, Prop. 2.2] for another description of $\mathrm{H}^4(E; \mathbf{Z})$.

Proof. Using Lemma 4.2, it suffices to prove that the kernel K of $\mathrm{Sq}^1 : \mathrm{Sym}^4(E^\vee) \to \mathrm{Sym}^5(E^\vee)$ has such a filtration. The symmetric algebra is reduced, so the squaring maps

$$E^\vee \xrightarrow{\mathrm{Sq}^1} \mathrm{Sym}^2(E^\vee) \xrightarrow{\mathrm{Sq}^2} \mathrm{Sym}^4(E^\vee)$$

(each of which just sends f to f^2) are injective. The image of Sq^2 is contained in K—this may be seen directly, or as an instance of the Adem relation $\mathrm{Sq}^1\,\mathrm{Sq}^2 = \mathrm{Sq}^3$, where Sq^3 vanishes on $\mathrm{H}^2(-; \mathbf{F}_2)$ by definition. We have already noted the identification $\mathrm{Sym}^2(E^\vee)/\,\mathrm{Sq}^1(E^\vee) \cong \mathrm{Alt}^2(E^\vee)$. To show that $K/\,\mathrm{Sq}^2(E^\vee)$ is isomorphic to $\mathrm{Alt}^3(E^\vee)$, we may note that $K = \mathrm{Sq}^1(\mathrm{Sym}^3(E^\vee))$ and that the preimage of $\mathrm{Sq}^2(\mathrm{Sym}^2(E^\vee))$ under $\mathrm{Sq}^1 : \mathrm{Sym}^3(E^\vee) \twoheadrightarrow K$ is the subspace spanned by nonsquarefree monomials. ■

With these remarks in hand, we may now turn to our promised upper bound on $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})_{(2)}$. As discussed already in Section 1.2, the 2-Sylow in Co_0 is contained

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in a maximal subgroup of shape $2^{12} : M_{24}$, where 2^{12} denotes the extended Golay code module. This M_{24}-module is not irreducible, and it is not isomorphic to its dual module. Following [21], we will write $C_{12} = 2^{12} = 2.C_{11}$ for the extended Golay code module, and C_{11} for its simple quotient. Its dual, the *extended cocode* module, is $C_{12}^\vee = C_{11}^\vee.2$ and $C_{11}^\vee$ is its simple submodule. $C_{12}^\vee$ is the unique 12-dimensional M_{24}-module over $\mathbf{F}_2$ with no fixed points.

Lemma 4.5. $\mathrm{H}^4(C_{12} : M_{24}; \mathbf{Z})_{(2)} = \mathbf{Z}/8 \oplus \mathbf{Z}/4$.

Proof. The LHS spectral sequence for $\mathrm{H}^\bullet(C_{12} : M_{24}; \mathbf{Z})_{(2)}$ begins

$$\begin{array}{ccccc}
\mathrm{H}^0\left(M_{24}; H^4(C_{12}; \mathbf{Z})\right) & & & & \\
 & \mathrm{H}^1\left(M_{24}; \mathrm{Alt}^2\left(C_{12}^\vee\right)\right) & & & \\
 & & \mathrm{H}^2\left(M_{24}; C_{12}^\vee\right) & & \\
0 & 0 & 0 & 0 & \\
\hdashline
* & 0 & 0 & 0 & \mathbf{Z}/4.
\end{array} \tag{4.3}$$

The dashed line reminds that the bottom row splits off as a direct summand, and we have used (4.2) for the values of $\mathrm{H}^i(C_{12}; \mathbf{Z})$. The computation of $\mathrm{H}^4(M_{24}; \mathbf{Z})_{(2)} \cong \mathbf{Z}/4$ is due to [30].

C_{12} is a submodule of the permutation module $\mathbf{F}_2^{24}$. In one basis of C_{12}, we find the ATLAS generators for M_{24} act by the matrices

$$\begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&1
\end{pmatrix}
\quad
\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
1&0&1&1&0&1&1&1&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&1&0&1&1&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}. \tag{4.4}$$

The action of these matrices by right multiplication on row vectors gives $C_{12}^\vee$. This is a suitable input for Cohomolo, which verifies

$$\mathrm{H}^2\left(M_{24}; C_{12}^\vee\right) \cong \mathbf{Z}/2.$$

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The action of (4.4) on $\mathrm{Alt}^2(C_{12}^\vee)$ and $\mathrm{Alt}^3(C_{12}^\vee)$ gives larger matrix representations that are still small enough to be handled by Cohomolo:

$$\mathrm{H}^1\left(M_{24}; \mathrm{Alt}^2\left(C_{12}^\vee\right)\right) \cong \mathrm{H}^0\left(M_{24}; \mathrm{Alt}^3\left(C_{12}^\vee\right)\right) \cong \mathbf{Z}/2.$$

(The nontrivial fixed point in $\mathrm{Alt}^3(C_{12}^\vee)$ is the *triple intersection* sending Golay codewords $a, b, c \subset \{1, \ldots, 24\}$ to $|a \cap b \cap c| \mod 2$.)

Since H^0 is left exact and $\mathrm{H}^0(M_{24}; C_{12}^\vee) = \mathrm{H}^0(M_{24}; \mathrm{Alt}^2(C_{12}^\vee)) = 0$, Lemma 4.4 shows the map $\mathrm{H}^0(M_{24}; \mathrm{H}^4(C_{12}; \mathbf{Z})) \to \mathrm{H}^0(M_{24}; \mathrm{Alt}^3(C_{12}^\vee))$ is an injection. By Lemma 4.1, there is an element of order 8 in $\mathrm{H}^4(2^{12} : M_{24}; \mathbf{Z})$. From this we can conclude first that $\mathrm{H}^0(M_{24}; H^4(C_{12}; \mathbf{Z}))$ is nonzero, hence isomorphic to $\mathbf{Z}/2$, and second that all of the displayed groups in (4.3) survive to E_∞ and participate in a nontrivial extension, proving the lemma. ■

To complete the proof of Theorem 0.1, it suffices to prove the following:

Lemma 4.6. The restriction map $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \to \mathrm{H}^4(2^{12} : M_{24}; \mathbf{Z})$ is not a surjection.

Proof. Let $\mathrm{Perm} : M_{24} \to \mathrm{O}(24)$ denote the $\mathbf{R}$-linear permutation representation of M_{24}—it is isomorphic to $\mathrm{Leech} \otimes \mathbf{R}|_{M_{24}}$. As for Co_0, M_{24} is superperfect and every real representation has a unique lift to Spin—we will abuse notation and denote this lift by $\mathrm{Perm} : M_{24} \to \mathrm{Spin}(24)$ as well. Of course we have

$$\frac{p_1}{2}(\mathrm{Perm}) = \frac{p_1}{2}\left(\mathrm{Leech}|_{M_{24}}\right) \qquad \text{in } \mathrm{H}^4\left(M_{24}; \mathbf{Z}\right)$$

but we claim that the pullback of $\frac{p_1}{2}(\mathrm{Perm})$ along the projection $(C_{12} : M_{24}) \to M_{24}$ does not extend to a class on Co_0.

To see this, let $g \in M_{24}$ be an element of the M_{24}-conjugacy class 2B (acting with cycle structure 2^{12}). Then $\langle g \rangle \to \mathrm{Spin}(24)$ projects to 12 copies of the trivial representation plus 12 copies of the sign representation, and because of this

$$\frac{p_1}{2}(\mathrm{Perm})|_{\langle g \rangle} = \frac{12}{4} \neq 0 \qquad \text{in } \mathrm{H}^4(\langle g \rangle; \mathbf{Z}) \cong \mathbf{Z}/2.$$

Of course, for $x \in C_{12}$, $\mathrm{Perm}|_{\langle x \rangle}$ is the trivial representation, and so $\frac{p_1}{2}(\mathrm{Perm})|_{\langle x \rangle} = 0$.

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If $\frac{p_1}{2}(\text{Perm})$ is the restriction of a class $\lambda \in \mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})$, then its restriction $\lambda|_{\langle x\rangle}$ depends only on the conjugacy class of x in Co_0. Thus, to show that such a λ does not exist, it suffices to find an element $x \in 2^{12}$ conjugate in Co_0 to $g \in M_{24}$.

According to [35], Co_0 has four conjugacy classes of order 2: the central element $c \in \mathrm{Co}_0$; two that cover the conjugacy class 2A in Co_1; and one that covers the conjugacy class 2C in Co_1. The conjugacy class 2B in Co_1 lifts to order 4 in Co_0. These classes are distinguished by their traces:

$$\mathrm{trace}(c, \mathrm{Leech}) = -24, \quad \mathrm{trace}(\text{lifts of 2A}, \mathrm{Leech}) = \pm 8, \quad \mathrm{trace}(\text{lift of 2C}, \mathrm{Leech}) = 0.$$

The codewords in C_{12} with Hamming length 8, 12, 16, and 24, respectively, act on Leech with trace 8, 0, -8, and -24.

Thus, codewords of Hamming length 12 are conjugate in Co_0 to elements of M_{24} of M_{24}-conjugacy class 2B. But $\frac{p_1}{2}(\text{Perm}) \in \mathrm{H}^4(2^{12} : M_{24}; \mathbf{Z})$ vanishes on all codewords and does not vanish on class 2B, and so cannot extend to a cohomology class on Co_0. ∎

5 $\mathrm{H}^4(M_{24}; \mathbf{Z})$, $\mathrm{H}^4(M_{23}; \mathbf{Z})$, and $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z})$

In outline, our proof of Theorem 0.1 had the following structure, for $G = \mathrm{Co}_0$:

(1) Quickly determine $\mathrm{H}^4(G; \mathbf{Z})_{(p)} = 0$ for large primes p for which G has a very simple p-Sylow subgroup.

(2) Find a characteristic class $\alpha \in \mathrm{H}^4(G; \mathbf{Z})$ and a small subgroup $C \subset G$ such that $\alpha|_C$ has large order. This provides a lower bound on $\mathrm{H}^4(G; \mathbf{Z})$.

(3) For small primes p, find a subgroup of G containing the p-Sylow of shape $p^n : J$. Compute the E_2-page of the LHS spectral sequence for $\mathrm{H}^4(p^n : J; \mathbf{Z})$. This provides a preliminary upper bound on $\mathrm{H}^4(G; \mathbf{Z})_{(p)}$.

(4) Find a characteristic class in $\mathrm{H}^4(J; \mathbf{Z})$ whose pullback to $\mathrm{H}^4(p^n : J; \mathbf{Z})$ distinguishes elements that are conjugate in G, and so doesn't extend to G. This narrows the upper bound on $\mathrm{H}^4(G; \mathbf{Z})_{(p)}$ to agree with the lower bound, completing the proof.

In this section we will discuss, via examples, the extent to which this strategy works for other groups. We will give new proofs of the isomorphisms $\mathrm{H}^4(M_{24}; \mathbf{Z}) = \mathbf{Z}/12$ and $\mathrm{H}^4(M_{23}; \mathbf{Z}) = 0$ essentially following the steps (1)–(4). But we will see that the strategy fails for Co_1—it turns out that the bound from step (3) is insufficiently sharp. A more serious version of this obstacle is encountered when trying to compute H^4 of the

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Monster, see [24, Section 3.5] for some discussion. Nevertheless for Co_1, we are able to deduce $\mathrm{H}^4(\mathrm{Co}_1;\mathbf{Z}) = \mathbf{Z}/12$ from a simple reduction to Theorem 0.1.

We first confirm (0.1), due originally to [30]:

Theorem 5.1. The group cohomology $\mathrm{H}^4(M_{24};\mathbf{Z})$ is isomorphic to $\mathbf{Z}/12$. Let $\mathrm{Perm} : M_{24} \to S_{24}$ denote the defining permutation representation and $\mathrm{Perm}\otimes\mathbf{C}$ the corresponding complex representation. The Chern class $c_2(\mathrm{Perm}\otimes\mathbf{C})$ generates a subgroup of index 2 in $\mathrm{H}^4(M_{24};\mathbf{Z})$; since $\mathrm{H}^2(M_{24};\mathbf{Z}) = \mathrm{H}^3(M_{24};\mathbf{Z}) = 0$, the real representation $\mathrm{Perm}\otimes\mathbf{R}$ carries a unique spin structure, and $\frac{p_1}{2}(\mathrm{Perm}\otimes\mathbf{R})$ is a distinguished generator of $\mathrm{H}^4(M_{24};\mathbf{Z})$. Let $\langle 12\mathrm{B}\rangle \subset M_{24}$ denote the cyclic subgroup generated by an element of conjugacy class 12B; then the restriction map $\mathrm{H}^4(M_{24};\mathbf{Z}) \to \mathrm{H}^4(\langle 12\mathrm{B}\rangle;\mathbf{Z})$ is an isomoprhism.

Proof. For $p = 7$ and 23, the p-Sylow in M_{24} is contained in a maximal subgroup isomorphic to $L_2(p)$, giving $\mathrm{H}^4(M_{24};\mathbf{Z})_{(p)} = 0$ as in Lemma 2.1. The 3-, 5-, and 11-Sylows in M_{24} are contained in a subgroup isomorphic to M_{12}. A by-hand computation (in, e.g., [23]) gives $\mathrm{H}^4(M_{12};\mathbf{Z})_{(5)} = \mathrm{H}^4(M_{12};\mathbf{Z})_{(11)} = 0$ and $\mathrm{H}^4(M_{12};\mathbf{Z})_{(3)} = \mathbf{Z}/3$.

For reasons that will become apparent, during the proof we will denote the degree-24 permutation representation of M_{24} as Perm_{24}. Conjugacy class 12B acts with cyclic structure 12^2, from which one computes that $c_2(\mathrm{Perm}_{24}\otimes\mathbf{C})|_{\langle 12\mathrm{B}\rangle}$ has order 6. Since the permutation representation is spin, $c_2(\mathrm{Perm}_{24}\otimes\mathbf{R})$ is even. This completes the proof of the theorem for odd primes and provides the claimed upper bound for $p = 2$.

The 2-Sylow in M_{24} is contained in a maximal subgroup of shape $2^4 : A_8$; the action of A_8 on 2^4 uses the exceptional isomorphism $A_8 \cong \mathrm{GL}(4,\mathbf{F}_2)$. Using Cohomolo but not HAP, one confirms the following:

$$\mathrm{H}^2\left(A_8;\mathrm{H}^2(2^4;\mathbf{Z})\right) \cong \mathrm{H}^1\left(A_8;\mathrm{H}^3(2^4;\mathbf{Z})\right) \cong \mathbf{Z}/2 \quad \text{and} \quad \mathrm{H}^0\left(A_8;\mathrm{H}^4(2^4;\mathbf{Z})\right) = 0.$$

Furthermore, $\mathrm{H}^4(A_8;\mathbf{Z}) \cong \mathbf{Z}/12$. These provide the E_2-page of the LHS spectral sequence, from which we learn that $\mathrm{H}^4(2^4 : A_8)_{(2)} \cong \mathbf{Z}/4 \oplus X$, where the 1st summand is $\mathrm{H}^4(A_8;\mathbf{Z})_{(2)}$ and where X is one of the groups $\mathbf{Z}/2$, $(\mathbf{Z}/2)^2$, or $\mathbf{Z}/4$.

Let $\mathrm{Perm}_8\otimes\mathbf{C}$ denote the eight-dimensional complex permutation representation of A_8. Then $c_2(\mathrm{Perm}_8\otimes\mathbf{C})$ generates $\mathrm{H}^4(A_8;\mathbf{Z})$ [31]. We claim that the pullback $2c_2(\mathrm{Perm}_8\otimes\mathbf{C}) \in \mathrm{H}^4(2^4 : A_8;\mathbf{Z})$ does not extend to $\mathrm{H}^4(M_{24};\mathbf{Z})$. Indeed, let $g \in A_8$ be an element of order 4 that has a fixed point in the degree-8 permutation representation; its cycle structure is $1^2 2^1 4^1$, and so $2c_2(\mathrm{Perm}_8\otimes\mathbf{C})|_{\langle g\rangle}$ has order 2 in $\mathrm{H}^4(\langle g\rangle) \cong \mathbf{Z}/4$. Let

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$h \in A_8$ have cycle structure $1^4 2^2$; then $2c_8(\mathrm{Perm}_8 \otimes \mathbf{C}) = 0$. Choose $x \in 2^4$ such that x is not fixed by h. Then $xh \in 2^4 : A_8$ has order 4 and $2c_2(\mathrm{Perm}_8 \otimes \mathbf{C})|_{\langle xh \rangle} = 2c_2(\mathrm{Perm}_8 \otimes \mathbf{C})|_{\langle h \rangle} = 0$. But both xh and g are order-4 elements of M_{24} that fix points in the degree-24 permutation representation of M_{24}, and there is a unique conjugacy class of such elements. Since $2c_2(\mathrm{Perm}_8 \otimes \mathbf{C}) \in \mathrm{H}^4(2^4 : A_8; \mathbf{Z})$ distinguished conjugate-in-M_{24} elements, it cannot extend to a class on M_{24}.

We know that $\mathrm{H}^4(M_{24}; \mathbf{Z})_{(2)}$ contains an element of order 4, namely $\frac{p_1}{2}(\mathrm{Perm}_{24} \otimes \mathbf{R})$. If we had $X \cong \mathbf{Z}/2$ or $(\mathbf{Z}/2)^2$, then we would have $2\frac{p_1}{2}(\mathrm{Perm}_{24} \otimes \mathbf{R}) = 2c_2(\mathrm{Perm}_8 \otimes \mathbf{C}) \in \mathrm{H}^4(2^4 : A_8; \mathbf{Z})$, which is impossible since $2c_2(\mathrm{Perm}_8 \otimes \mathbf{C})$ does not extend to M_{24}. So $X \cong \mathbf{Z}/4$ and $\mathrm{H}^4(M_{24}; \mathbf{Z})_{(2)}$ is a direct summand of $(\mathbf{Z}/4)^2$ that is nonempty (since it contains $\frac{p_1}{2}(\mathrm{Perm}_{24} \otimes \mathbf{R})$) and not everything (since it does not contain $2c_2(\mathrm{Perm}_8 \otimes \mathbf{C})$). Thus, $\mathrm{H}^4(M_{24}; \mathbf{Z})_{(2)} \cong \mathbf{Z}/4$ generated by $\frac{p_1}{2}(\mathrm{Perm}_{24} \otimes \mathbf{R})$. ■

A very similar argument applies to M_{23}. The computation of $\mathrm{H}^4(M_{23}; \mathbf{Z})$ is due to Milgram [22].

Theorem 5.2. The group cohomology $\mathrm{H}^4(M_{23}; \mathbf{Z})$ vanishes.

Proof. The odd Sylow subgroups are contained in subgroups of shapes $3^2 : 8$, $5 : 4$, $7 : 3$, $11 : 5$, and $23 : 11$, where in the 1st case $8 \cong \mathbf{F}_9^\times$ acts by multiplication on $\mathbf{F}_9 \cong 3^2$; thus, $\mathrm{H}^4(M_{24}; \mathbf{Z})_{(\mathrm{odd})} = 0$ as in Lemma 2.1. The 2-Sylow is contained in a maximal subgroup isomorphic to $2^4 : A_7$. Furthermore,

$$\mathrm{H}^2\left(A_7; \mathrm{H}^2(2^4; \mathbf{Z})\right) = \mathrm{H}^1\left(A_7; \mathrm{H}^3(2^4; \mathbf{Z})\right) = \mathrm{H}^0\left(A_7; \mathrm{H}^4(2^4; \mathbf{Z})\right) = 0.$$

It follows that $\mathrm{H}^4(M_{23}; \mathbf{Z})_{(2)} \to \mathrm{H}^4(A_7; \mathbf{Z})_{(2)} \cong \mathbf{Z}/4$ is an injection onto a direct summand. But, exactly as in the proof of Theorem 5.1, $2c_2(\mathrm{Perm}_7 \otimes \mathbf{C})$ distinguishes conjugate elements in M_{23}, where Perm_7 denotes the defining permutation representation of A_7. ■

To end this section, let us show $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z}) = \mathbf{Z}/12$ by a different argument.

Theorem 5.3. $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z}) \cong \mathbf{Z}/12$.

It can be shown that the 276-dimensional representation of Co_1 is Spin. (Indeed, this is the adjoint rep of $\mathrm{PSO}(24) \supset \mathrm{Co}_1$, and the adjoint rep of $\mathrm{PSO}(2n)$ is Spin when $n = 0$ or $1 \bmod 4$.) It follows from the table at the end of Section 6 that an explicit generator for $\mathrm{H}^4(\mathrm{Co}_1)$ is $\frac{p_1}{2}(276)$.

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Proof. Consider the LHS spectral sequence for the extension $\mathrm{Co}_0 = 2.\mathrm{Co}_1$. Its E_2 page begins as follows:

$$\begin{array}{ccccc} 2 & & & & \\ 0 & 0 & 0 & & \\ 2 & 0 & \mathrm{H}^2(\mathrm{Co}_1; 2) & & \\ 0 & 0 & 0 & 0 & \\ \mathbf{Z} & 0 & 0 & \mathrm{H}^3(\mathrm{Co}_1; \mathbf{Z}) & \mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z}) \end{array} \quad = \quad \begin{array}{ccccc} 2 & & & & \\ 0 & 0 & 0 & & \\ 2 & 0 & 2 & & \\ 0 & 0 & 0 & 0 & \\ \mathbf{Z} & 0 & 0 & 2 & \mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z}). \end{array}$$

Specifically, write $E_r^{i,j}$ for the (i,j)th entry on the E_r page. On the $i+j=3$ diagonal, the groups E_2^{21}, E_2^{12}, and E_2^{03} vanish. It follows that $E_\infty^{04} = E_2^{04} = \mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z})$. On the $i+j=4$ diagonal, the groups E_2^{22} and E_2^{04} are copies of $\mathbf{Z}/2$, while E_2^{31} and E_2^{13}, hence also E_∞^{31} and E_∞^{13}, vanish.

We claim that E_∞^{04} also vanishes. Indeed, the center $\mathbf{Z}/2 \subset \mathrm{Co}_0$ acts on Leech $\otimes\, \mathbf{R}$ as 24 copies of the sign representation, so $\frac{p_1}{2}$ vanishes there (see Theorem 7.1 for a more general statement). It follows that $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \to E_2^{04}$ is zero, but $E_\infty^{04} \subset E_2^{04}$ is precisely the image of this map.

In total degree 4, the LHS filtration on $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \cong \mathbf{Z}/24$ reduces therefore to a short exact sequence

$$0 \to \mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z}) \to \mathbf{Z}/24 \to E_\infty^{22} \to 0.$$

But E_∞^{22} is a subquotient of $E_2^{22} = \mathbf{Z}/2$. We conclude that $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z})$ is either $\mathbf{Z}/12$ or $\mathbf{Z}/24$.

It remains to rule out the latter option. Equivalently, we must show that the image of $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z})$ in $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})$ does not contain an element of order 8. One can detect whether a class in $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})$ has order 8 by restricting to the binary dihedral group $2D_8 \subset \mathrm{Co}_0$. But the composition $\mathrm{H}^4(\mathrm{Co}_1; \mathbf{Z}) \to \mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \to \mathrm{H}^4(2D_8; \mathbf{Z})$ factors through $\mathrm{H}^4(D_8; \mathbf{Z}) = (\mathbf{Z}/2)^2 \oplus \mathbf{Z}/4$. ■

Theorem 5.3 is slightly surprising if one tries to repeat the strategy outlined at the beginning of this section. The maximal subgroup of Co_1 containing the 2-Sylow has shape $C_{11} : M_{24}$, where as above C_{11} denotes the irreducible Golay code module. The E_2 page of the corresponding LHS spectral sequence for $\mathrm{H}^\bullet(C_{11} : M_{24}; \mathbf{Z})_{(2)}$ can be computed

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as in the proof of Lemma 4.5:

$$\begin{array}{lllll}
2 & & & & \\
0 & 2 & 2 & & \\
0 & 2 & 2 & & \\
0 & 0 & 0 & 0 & \\
\hdashline
\mathbf{Z} & 0 & 0 & 0 & \mathbf{Z}/4.
\end{array}$$

The class in $\mathrm{H}^0(M_{24}; \mathrm{H}^4(C_{11}; \mathbf{Z}))$ is the triple intersection $(a, b, c) \mapsto |a \cap b \cap c|$—we defined it on $C_{12} = 2.C_{11}$, but it vanishes if any element is the all-1s vector—and the pullback $\mathrm{H}^0(M_{24}; \mathrm{H}^4(C_{11}; \mathbf{Z})) \to \mathrm{H}^0(M_{24}; \mathrm{H}^4(C_{12}; \mathbf{Z}))$ is an isomorphism. For comparison, the E_2 page for $\mathrm{H}^\bullet(C_{12} : M_{24}; \mathbf{Z})_{(2)}$ is

$$\begin{array}{lllll}
2 & & & & \\
0 & 2 & 2 & & \\
0 & 0 & 2 & & \\
0 & 0 & 0 & 0 & \\
\hdashline
* & 0 & 0 & 0 & \mathbf{Z}/4
\end{array}$$

and the triple intersection extends to an element with order 8. How, then, can $\mathrm{H}^4(C_{11} : M_{24}; \mathbf{Z})_{(2)}$ fail to have elements of order 8? Shouldn't the triple intersection have order 8 there?

The answer is that the triple intersection does not survive the LHS spectral sequence for $\mathrm{H}^\bullet(C_{11} : M_{24}; \mathbf{Z})_{(2)}$ but does for $\mathrm{H}^\bullet(C_{12} : M_{24}; \mathbf{Z})_{(2)}$. The extension $\mathrm{H}^3(C_{12}; \mathbf{Z}) = \mathrm{Alt}^2(C_{12}^\vee) = \mathrm{Alt}^2(C_{11}^*).C_{11}^*$ leads to a long exact sequence in M_{24}-cohomology:

$$\begin{tikzcd}
 & \mathrm{Alt}^2(C_{11}^*) & \mathrm{Alt}^2(C_{12}^\vee) & C_{11}^* \\
\mathrm{H}^0(M_{24}; -) & 0 & 0 & 0 \\
\mathrm{H}^1(M_{24}; -) & \mathbf{Z}/2 \arrow[r, "\sim"] & \mathbf{Z}/2 & \mathbf{Z}/2 \arrow[dll, "\sim"'] \\
\mathrm{H}^2(M_{24}; -) & \mathbf{Z}/2 & \mathbf{Z}/2 \arrow[r, "\sim"] & \mathbf{Z}/2
\end{tikzcd}$$

In particular, the restriction map $\mathrm{H}^2(M_{24}; \mathrm{H}^3(C_{11}; \mathbf{Z})) \to \mathrm{H}^2(M_{24}; \mathrm{H}^3(C_{12}; \mathbf{Z}))$ is 0. This provides the room needed for the d_2 differential $\mathrm{H}^0(M_{24}; \mathrm{H}^4(C_{11}; \mathbf{Z})) \to \mathrm{H}^2(M_{24}; \mathrm{H}^3(C_{11}; \mathbf{Z}))$ to be nonzero while the d_2 differential $\mathrm{H}^0(M_{24}; \mathrm{H}^4(C_{12}; \mathbf{Z})) \to \mathrm{H}^2(M_{24}; \mathrm{H}^3(C_{12}; \mathbf{Z}))$ is 0.

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6 Second Chern Classes of Representations

Let us index the representations of $2D_8$ as in (6.1):

$$\begin{tikzcd}[column sep=small, row sep=small]
V_0 \arrow[dr, no head] & & & & V_2 \arrow[dl, no head] \\
& V_6 \arrow[r, no head] & V_4 \arrow[r, no head] & V_5 & \\
V_1 \arrow[ur, no head] & & & & V_3 \arrow[ul, no head]
\end{tikzcd} \tag{6.1}$$

Thus, $M = V_6$ and $M' = V_5$ in the notation of Lemma 4.1. V_0 is the trivial representation, and V_1, V_2, V_3 are the nontrivial one-dimensional representations. The kernel of V_1 is cyclic (of order 8), while the kernels of V_2 and V_3 are quaternion groups of order 8. V_4 is the real dihedral representation into $\mathrm{O}(2)$, the symmetries of the square.

Lemma 6.1. Let V be a representation of Co_0, and suppose that $V|_{2D_8} = \bigoplus_{i=0}^{6} n_i V_i$. Then

$$c_2(V|_{2D_8}) = 4n_4 + 9n_5 + n_6 \mod 16. \tag{6.2}$$

Proof. For $i = 1, 2, 3$, put $v_i := c_1(V_i)$. Then v_1, v_2, v_3 are the three nonzero elements of $\mathrm{H}^2(2D_8;\mathbf{Z}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$, and $v_1 = v_2 + v_3$. As $\mathrm{H}^4(2D_8;\mathbf{Z})$ is cyclic, we must either have $v_2^2 = 0$ or $v_2^2 = 8$, and similarly for v_3. As v_2 and v_3 are exchanged by an outer automorphism of $2D_8$, we have $v_2^2 = v_3^2$, and therefore $v_1^2 = 0$.

(One may also see that $v_1^2 = 0$ by observing that V_1 is pulled back from a one-dimensional representation of $2D_{16}$, and that the restriction map $\mathrm{H}^4(2D_{16};\mathbf{Z}) \to \mathrm{H}^4(2D_8;\mathbf{Z})$, being a map from $\mathbf{Z}/32$ to $\mathbf{Z}/16$, must vanish on the 2-torsion subgroup of the domain. We are not sure whether or not v_2^2 and v_3^2 are zero, but to prove the lemma we will not need to know.)

In the proof of Lemma 4.1, we have already computed the total Chern classes of $V_6 = M$ and $V_5 = M'$—

$$c_t(V_6) = 1 + t^2, \qquad c_t(V_5) = 1 + 9t^2$$

—by decomposing $\mathrm{Sym}^3(V_6)$ as $V_6 \oplus V_5$. To prove the Lemma, we will appeal to the following computations:

$$c_t(V_4) = 1 + v_1 t + 4t^2, \qquad \text{and} \qquad v_2 v_3 = 0. \tag{6.3}$$

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The 1st follows from considering the decomposition of $\mathrm{Sym}^2(V_6)$ and the 2nd from considering the decomposition of $\mathrm{Sym}^4(V_6)$:

$$\mathrm{Sym}^2(V_6) = V_1 \oplus V_4, \qquad \mathrm{Sym}^4(V_6) = V_0 \oplus V_2 \oplus V_3 \oplus V_4. \tag{6.4}$$

Under the identification $\mathrm{H}^4(B\mathrm{SU}(2); \mathbf{Z}) \cong \mathbf{Z}$, we have

$$c_2(\mathrm{Sym}^n(\mathbf{C}^2)) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n, \qquad \text{i.e. } 1, 4, 10, 20, 35, \ldots$$

so (6.4) gives

$$1 + 4t^2 = (1 + v_1 t)\big(1 + v_1 t + c_2(V_4)t^2\big), \qquad 1 + 20t^2 = (1 + v_2 t)(1 + v_3 t)\big(1 + v_1 t + c_2(V_4)t^2\big),$$

which gives (6.3) upon expanding and using $v_1^2 = 0$.

Now we compute (6.2) by considering the total Chern class of the direct sum:

$$1^{n_0}(1 + v_1 t)^{n_1}(1 + v_2 t)^{n_2}(1 + v_3 t)^{n_3}\big(1 + v_1 t + 4t^2\big)^{n_4}(1 + 9t^2)^{n_5}(1 + t^2)^{n_6}.$$

In fact $n_2 = n_3$ for every representation of Co_0—this can be seen from the merging of conjugacy classes of $2D_8$ in Co_0, or just by checking each irreducible representation one by one. As

$$(1 + v_2 t)(1 + v_3 t) = 1 + (v_2 + v_3)t + v_2 v_3 t^2 = 1 + v_1 t,$$

the total Chern class of $V|_{2D_8}$ is

$$(1 + v_1 t)^{n_1 + n_2}\big(1 + v_1 t + 4t^2\big)^{n_4}(1 + 9t^2)^{n_5}(1 + t^2)^{n_6}. \tag{6.5}$$

Since $v_1^2 = 0$, the coefficient of t^2 in the expansion of (6.5) is $4n_4 + 9n_5 + n_6$. ■

Let $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_{167}$ be GAPs ordering of the conjugacy classes of Co_0, in its library of character tables. Then $\mathfrak{c}_1$ is the identity element and $\mathfrak{c}_2$ is the central element, and

(1) $\mathfrak{c}_5$ is the unique conjugacy class that squares to the central element,
(2) $\mathfrak{c}_{21}$ is the unique conjugacy class that squares to $\mathfrak{c}_5$, and
(3) $\mathfrak{c}_{13}$ is the unique conjugacy class of order 3 whose trace on Leech is zero.

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If V is any complex representation of Co_0, we have $c_2(V) = k(V)c_2(\mathrm{Leech}\otimes\mathbf{C})$ for some $k(V)\in\mathbf{Z}/12$. Theorem 0.1 implies that k depends only on

$$\mathrm{trace}(\mathfrak{c}_i, V)\text{for } i\in\{1,2,5,21,13\}. \tag{6.6}$$

The numbers $k(V_1),\ldots,k(V_{167})$, where $V_1,\ldots,V_{167}$ are the irreducible characters in the order that they appear in GAP's library, are recorded in the table below, along with the traces at (6.6). Note that $V\mapsto k(V)$ is a group homomorphism $R(\mathrm{Co}_0)\to\mathbf{Z}/12$, since $c_1(V)=0$ for every complex representation V of Co_0.

Incidentally, 153 of the 167 irreducible representations of Co_0 are real—it is easier to list the fourteen exceptions, which have GAP indices

$$17,\quad 18,\quad 27,\quad 28,\quad 121,\quad 122,\quad 125,\quad 126,\quad 128,\quad 129,\quad 135,\quad 136,\quad 142,\quad 143.$$

Any real representation of Co_0, irreducible or not, has a unique lift from $\mathrm{Co}_0\to\mathrm{O}(n)$ to $\mathrm{Co}_0\to\mathrm{Spin}(n)$, and therefore has a fractional Pontryagin class of the form $k'(V)\cdot\frac{p_1}{2}(\mathrm{Leech}\otimes\mathbf{R})$, with $k'(V) = -k(V)\bmod 12$. The discussion of Section 1.4 shows that k' is a homomorphism $RO(\mathrm{Co}_0)\cong\mathbf{Z}^{160}\to\mathbf{Z}/24$. One could compute it if one knew the "supercohomology" of $2D_8$, and the string obstruction map on $RO(2D_8)$, but we have not done the computation.

i	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_5$	$\mathfrak{c}_{21}$	$\mathfrak{c}_{13}$	$k(V_i)$	i	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_5$	$\mathfrak{c}_{21}$	$\mathfrak{c}_{13}$	$k(V_i)$
1	1	1	1	1	1	0	45	24667500	24667500	−1300	−28	105	4
2	276	276	12	0	0	10	46	24794000	24794000	560	0	−70	4
3	299	299	−13	−1	−1	2	47	25900875	25900875	3835	−41	−63	0
4	1771	1771	51	7	7	6	48	31574400	31574400	3328	0	84	8
5	8855	8855	15	−1	−7	10	49	40166280	40166280	2520	0	0	0
6	17250	17250	90	−6	0	10	50	40370176	40370176	8192	0	112	4
7	27300	27300	156	0	0	2	51	44013375	44013375	−1665	27	0	0
8	37674	37674	−78	−6	0	6	52	46621575	46621575	−6201	27	0	0
9	44275	44275	−77	7	1	8	53	51571520	51571520	0	0	56	4
10	80730	80730	−78	6	0	6	54	55255200	55255200	−4576	0	−84	0
11	94875	94875	235	−1	21	8	55	57544344	57544344	−936	0	0	0
12	313950	313950	−26	14	21	6	56	60435375	60435375	−3185	7	−105	8
13	345345	345345	377	1	−21	6	57	65270205	65270205	−3003	21	0	6
14	376740	376740	−364	0	36	0	58	66602250	66602250	4290	6	0	6
15	483000	483000	−600	0	0	4	59	77702625	77702625	−975	21	0	0
16	644644	644644	1092	28	49	4	60	83720000	83720000	0	−56	20	0
17	673750	673750	350	−14	−35	6	61	85250880	85250880	7488	0	0	0
18	673750	673750	350	−14	−35	6	62	91547820	91547820	4212	0	0	6
19	822250	822250	−910	−14	43	2	63	100725625	100725625	6825	−7	−140	4
20	871884	871884	1612	28	27	0	64	106142400	106142400	11648	56	216	0
21	1434510	1434510	246	6	0	6	65	109882500	109882500	−9100	−28	105	0
22	1450449	1450449	1001	−7	63	6	66	150732800	150732800	4096	64	104	8
23	1771000	1771000	40	0	28	4	67	163478250	163478250	−1950	−6	0	6
24	1821600	1821600	352	0	−36	0	68	184184000	184184000	4160	0	−28	0
25	2055625	2055625	1625	13	85	8	69	185912496	185912496	11088	0	162	0
26	2417415	2417415	−1001	−21	84	8	70	185955000	185955000	4200	0	0	4
27	2464749	2464749	−987	21	0	6	71	191102976	191102976	−4096	−64	216	0
28	2464749	2464749	−987	21	0	6	72	201451250	201451250	−4550	14	35	2
29	2816856	2816856	−1080	0	0	0	73	205395750	205395750	−5810	−14	189	6
30	2877875	2877875	715	−1	35	6	74	207491625	207491625	8865	−27	−162	6
31	4100096	4100096	0	0	−64	8	75	210974400	210974400	0	0	0	0
32	5494125	5494125	1365	21	0	2	76	215547904	215547904	0	0	64	0
33	5801796	5801796	−364	−28	63	0	77	219648000	219648000	0	0	0	8
34	7628985	7628985	1001	−7	−27	0	78	241741500	241741500	5460	0	0	10
35	9221850	9221850	−1598	−14	27	6	79	247235625	247235625	6825	21	0	0
36	9669660	9669660	364	−28	21	8	80	251756505	251756505	−6903	−27	0	0
37	12432420	12432420	2028	0	0	6	81	257857600	257857600	−11648	56	−56	4
38	16347825	16347825	−351	−27	0	0	82	259008750	259008750	9750	−6	0	6
39	20083140	20083140	−924	0	0	0	83	267014475	267014475	8827	7	189	0
40	21049875	21049875	−2325	15	0	6	84	280280000	280280000	0	−56	140	4
41	21528000	21528000	4160	0	36	0	85	292953024	292953024	−7488	0	0	0
42	21579129	21579129	−2223	−27	162	6	86	299710125	299710125	6565	13	−216	6
43	23244375	23244375	2975	35	105	10	87	302176875	302176875	2275	7	−105	6
44	24174150	24174150	5278	14	21	2	88	309429120	309429120	−11648	0	84	0

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i	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_5$	$\mathfrak{c}_{21}$	$\mathfrak{c}_{13}$	$k(V_i)$
89	326956500	326956500	−2340	0	0	6
90	360062976	360062976	0	0	0	8
91	387317700	387317700	−4356	0	0	6
92	402902500	402902500	−9100	28	−35	8
93	464257024	464257024	4096	−64	−56	0
94	469945476	469945476	4004	−28	−189	0
95	469945476	469945476	9828	0	0	0
96	483483000	483483000	−3640	0	−84	4
97	502078500	502078500	−1260	0	0	0
98	503513010	503513010	3354	6	0	6
99	504627200	504627200	−4096	64	56	4
100	522161640	522161640	2184	0	0	0
101	551675124	551675124	−9828	0	0	6
102	24	−24	0	0	0	1
103	2024	−2024	0	0	8	3
104	2576	−2576	0	0	8	2
105	4576	−4576	0	0	−8	8
106	40480	−40480	0	0	−8	4
107	95680	−95680	0	0	−8	0
108	170016	−170016	0	0	0	4
109	299000	−299000	0	0	8	5
110	315744	−315744	0	0	0	4
111	351624	−351624	0	0	0	11
112	388080	−388080	0	0	0	6
113	789360	−789360	0	0	48	2
114	1841840	−1841840	0	0	8	6
115	1937520	−1937520	0	0	0	6
116	4004000	−4004000	0	0	56	4
117	5051904	−5051904	0	0	−48	0
118	6446440	−6446440	0	0	112	7
119	6446440	−6446440	0	0	−56	11
120	7104240	−7104240	0	0	0	6
121	9152000	−9152000	0	0	−112	8
122	9152000	−9152000	0	0	−112	8
123	11051040	−11051040	0	0	48	0
124	13156000	−13156000	0	0	−56	0
125	15002624	−15002624	0	0	−112	4
126	15002624	−15002624	0	0	−112	4
127	15471456	−15471456	0	0	0	4
128	16170000	−16170000	0	0	0	10

i	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_5$	$\mathfrak{c}_{21}$	$\mathfrak{c}_{13}$	$k(V_i)$
129	16170000	−16170000	0	0	0	10
130	17050176	−17050176	0	0	0	0
131	17310720	−17310720	0	0	0	8
132	18987696	−18987696	0	0	0	6
133	19734000	−19734000	0	0	48	2
134	34155000	−34155000	0	0	0	9
135	40370176	−40370176	0	0	112	4
136	40370176	−40370176	0	0	112	4
137	44204160	−44204160	0	0	168	4
138	49335000	−49335000	0	0	0	1
139	50519040	−50519040	0	0	120	8
140	51571520	−51571520	0	0	−112	8
141	59153976	−59153976	0	0	0	9
142	59153976	−59153976	0	0	0	9
143	59153976	−59153976	0	0	0	9
144	62790000	−62790000	0	0	120	2
145	67358720	−67358720	0	0	−112	4
146	106260000	−106260000	0	0	−120	0
147	112519680	−112519680	0	0	−168	8
148	139243104	−139243104	0	0	0	0
149	161161000	−161161000	0	0	112	11
150	190417920	−190417920	0	0	168	0
151	210496000	−210496000	0	0	112	0
152	215547904	−215547904	0	0	64	0
153	230230000	−230230000	0	0	160	2
154	247543296	−247543296	0	0	168	0
155	282906624	−282906624	0	0	−48	4
156	287006720	−287006720	0	0	−112	0
157	313524224	−313524224	0	0	−160	4
158	342752256	−342752256	0	0	0	0
159	351624000	−351624000	0	0	168	4
160	394680000	−394680000	0	0	−120	4
161	464143680	−464143680	0	0	0	0
162	485760000	−485760000	0	0	120	0
163	517899096	−517899096	0	0	0	9
164	557865000	−557865000	0	0	0	3
165	655360000	−655360000	0	0	160	0
166	805805000	−805805000	0	0	−280	3
167	1021620600	−1021620600	0	0	0	9

7 Restrictions to Cyclic Subgroups

In this section we will give a formula for the restriction maps

$$\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \to \mathrm{H}^4(C; \mathbf{Z}), \tag{7.1}$$

where $C \subset \mathrm{Co}_0$ is any cyclic subgroup. The domain is cyclic of order 24, and has a distinguished generator $\frac{p_1}{2}$. The codomain is also cyclic, of the same order as C. It does not always have a distinguished generator but we give a naming scheme for the elements of its 24-torsion subgroup that does not require any choices—for each $k \in \mathbf{Z}$ with $24k \in |C|\mathbf{Z}$, there is well-defined class $kt^2 \in \mathrm{H}^4(C; \mathbf{Z})$. Here t^2 is the cup-square of any generator $t \in \mathrm{H}^2(C; \mathbf{Z})$. The fact that kt^2 is independent of t (when $24k = 0$ modulo the order of C) is a consequence of what Conway and Norton call the "defining property of 24" [11, Section 3]: that $a^2 = 1 \bmod 24$ whenever a is invertible mod 24.

Thus, we may report (7.1) by reporting an integer $k \in \mathbf{Z}$ such that $\frac{p_1}{2}$ is carried to kt^2. Theorem 7.1 gives a formula for k in terms of the characteristic polynomial of (any generator of) C, regarded as a 24×24 matrix. That a general formula should exist follows from the discussion in Section 1.4, but our formula will apply only to the image of $\mathrm{Co}_0 \hookrightarrow \mathrm{O}(24)$.

Actually we give the formula in terms of Frame's encoding [15] of the characteristic polynomial. Since each element $g \in \mathrm{Co}_0$ preserves a lattice, its characteristic

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polynomial $\det(g-\lambda)$ factors uniquely as $\prod_{d|o(g)}(1-\lambda^d)^{r_d}$ for some integers $r_d \in \mathbf{Z}$, and the *Frame shape* of g is the formal expression $\prod_{d|o(g)} d^{r_d}$. Frame shapes generalize cycle structures of permutations. The Frame shapes of all elements in Co_0 were computed in [25, p. 355]; the 167 conjugacy classes in Co_0 merge to only 160 different Frame shapes.

Let $\ell(g)$ denote the smallest d such that the exponent r_d of d in the Frame shape of g is nonzero. For example, $\ell(g) = 1$ if and only if $\mathrm{trace}(g, \mathbf{R}^{24}) \neq 0$. If g is a permutation matrix, then $\ell(g)$ is the length of the smallest cycle in g. Let $\epsilon(g) = \pm 1$ record the sign of the exponent $r_{\ell(g)}$. We say that g is *balanced* if there exists an N such that $r_d = \epsilon(g) r_{N/d}$ for all d. The notion of a balanced Frame shape specializes to the notion of a balanced cycle type in the sense of [11, p. 1 item (B)]; in particular every element of $M_{24} \subset \mathrm{Co}_0$ is balanced. The conjugacy class 8B in Co_0 has Frame shape $2^{-4}8^4$, and so $\ell(8\mathrm{B}) = 2$, $\epsilon(8\mathrm{B}) = -1$, and 8B are balanced. Conjugacy class 4D has Frame shape $2^{-4}4^8$ and is not balanced. The following result summarizes our calculations of $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})|_{\langle g \rangle}$:

Theorem 7.1. Suppose that $g \in \mathrm{Co}_0$, and use notation $t, o(g), \dots$ as above.

(1) If $\ell(g) = 1$, then $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})|_{\langle g \rangle} = 0$.

(2) If g is balanced, then

$$\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})|_{\langle g \rangle} = \frac{\epsilon(g) o(g)}{\ell(g)} t^2.$$

(3) If g is not balanced, then $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})|_{\langle g \rangle} = 0$.

Statement (1) is a consequence of (2) and (3), as $o(g)t^2 = 0$. We don't know any a priori reason for Theorem 7.1 to hold; all three statements (1–3) fail in general for other lattice-preserving elements of Spin(24). Our proof is case-by-case; we computed $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})$ for all 160 Frame shapes associated with the Conway group.

Specifically, we found a factorization of each $\langle g \rangle \subset \mathrm{Co}_0 \hookrightarrow \mathrm{O}(24)$ through $\mathrm{SU}(12) \to \mathrm{Spin}(24)$. Suppose more generally that $V : \mathbf{Z}/n \to \mathrm{O}(2m)$ is given. Then the $2m$ eigenvalues of $V(g)$ lie on $\mathrm{U}(1) \subset \mathbf{C}$ and come in m complex conjugate pairs. To factorize V through $\mathrm{U}(m)$ is equivalent to selecting one eigenvalue from each of these pairs. To factorize through $\mathrm{SU}(m)$ one must select them such that their product is 1. We found that for $\mathbf{Z}/n \subset \mathrm{Co}_0$, this is always possible, although the “obvious” factorization through U(12) sometimes fails. For example, for element $4\mathrm{H} \in \mathrm{Co}_0$, with Frame shape 4^6, the “obvious” factorization through U(12) uses a matrix with determinant -1; any “correct” factorization through SU(12) has spectrum that is not invariant under complex conjugation.

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Having factored $\text{Leech} \otimes \mathbf{R}|_{\langle g\rangle} = V : \mathbf{Z}/o(g) \to \mathrm{O}(2m) = \mathrm{O}(24)$ through $\mathbf{Z}/o(g) \xrightarrow{W} \mathrm{SU}(m) \hookrightarrow \mathrm{O}(2m)$, we may compute $\frac{p_1}{2}(V)$ quickly. Indeed, $\mathrm{SU}(m)$ is simply connected, and so injects into $\mathrm{Spin}(2m)$, and the restriction map $\mathrm{H}^4(B\mathrm{Spin}(2m); \mathbf{Z}) \to \mathrm{H}^4(B\mathrm{SU}(m); \mathbf{Z})$ carries $\frac{p_1}{2}$ to $-c_2$. The Cartan formula gives a recipe for the Chern classes of W in terms of the eigenvalues of $W(g)$. That is how we proved Theorem 7.1.

8 Restrictions to Umbral Subgroups

Every even unimodular lattice $L \subset \mathbf{R}^{24}$ is isometric to either Leech or to one of the 23 Niemeier lattices. If L is a Niemeier lattice, it is characterized up to isometry by its root system $\Phi_L \subset L$—the vectors of length 2 in L—and the real span of Φ_L is all of $\mathbf{R}^{24}$. Reflection through the root vectors generates a Weyl group W_L, which is normal in the full isometry group $\mathrm{Aut}(L)$. Let $U_L := \mathrm{Aut}(L)/W_L$ denote the quotient group. We will follow [7] and call U_L an "umbral group"; it is called the "glue group" $G_1.G_2$ in [12]. For instance, the Mathieu group M_{24} is an umbral group (with L of type A_1^{24}), as is the Schur cover $2M_{12}$ of the Mathieu group M_{12} (L of type A_2^{12}).

For each Niemeier lattice L there is a preferred ("holy") conjugacy class of embeddings $U_L \hookrightarrow \mathrm{Co}_0$. Two of them, for L of types A_1^{24} and A_2^{12}, have already been mentioned in Section 1.2 and used in Sections 3 and 4. In general the theory of root systems shows that a choice of simple roots $\Delta_L \subset \Phi_L$ induces a splitting $\mathrm{Aut}_L = W_L : U_L$, where U_L acts by faithfully permuting Δ_L and preserving its graph structure (the Dynkin diagram)—see [3, Section VI.1–VI.4]. Once $\Delta_L \subset L$ is fixed, the corresponding holy construction [12, Ch. 24] [1, Section 7] outputs a distinguished U_L-stable lattice $L_0 \subset \mathbf{R}^{24}$, with $L_0 \cap L$ of finite index in L, that is isometric to Leech. Since Co_0 has no outer automorphisms the composite $U_L \hookrightarrow \mathrm{Aut}(L_0) \cong \mathrm{Co}_0$ is well defined up to conjugacy. In this section we make some comments about the restriction map

$$\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) \to \mathrm{H}^4(U_L; \mathbf{Z}). \tag{8.1}$$

The coefficients of various famous q-series are integer linear combinations of entries from the character tables of umbral groups, a phenomenon called *umbral moonshine* in [6, 7]. The umbral moonshine problem is to find a family of quantum field theories V^L, on which the umbral groups act, that would explain (by taking characters) this phenomenon. These U_L-actions would induce cohomology classes $\alpha_L \in \mathrm{H}^3(U_L; \mathrm{U}(1)) \cong \mathrm{H}^4(U_L; \mathbf{Z})$, which we will call *anomalies* based on [34]. These anomalies have largely been characterized, in [9, 17], even in advance of knowing what V^L is; in all cases the restriction of α_L to a cyclic subgroup $\langle g\rangle \subset U_L$ can be extracted from the

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modularity properties (the multiplier system) of the q-series corresponding to g—see [17, Section 3.3] and [18, Section 6]—and for all but three of the umbral groups, $\mathrm{H}^4(U_L; \mathbf{Z})$ is detected on cyclic subgroups. (The exceptions are A_2^{12}, A_3^8, and A_6^4).

In this section we check that for a number of L, α_L is in the image of (8.1), and in fact

$$\alpha_L = \epsilon(L)\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})|_{U_L} \tag{8.2}$$

for a scalar $\epsilon(L)$ that generates $\mathbf{Z}/24$—we warn that we are not sure that it is true in general, and do not propose any particular relationship between the V^L and the Conway group, but we do hope that some of our calculations will be useful for moonshine. For example, for $L = A_1^{24}$ or

$$L \in \left\{A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, D_{24}\right\}, \tag{8.3}$$

we find $\epsilon(L) = -1$. For the list (8.3), Cheng–Duncan and Duncan–O'Desky have had some qualified success in realizing V^L as a free theory—at least, for solving what Duncan calls the "meromorphic module problem." One consequence of our calculations is that there is a cohomological obstruction to solving the meromorphic module problem with a free theory, and that this obstruction is not trivial for A_3^8.

Theorem 8.1. Under the standard isomorphism $\mathrm{H}^4(G; \mathbf{Z}) \cong \mathrm{H}^3(G; \mathrm{U}(1))$ for $|G| < \infty$ given by the Bockstein for the map $x \mapsto \exp(2\pi i x)$, the restriction of $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R}) \in \mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z})$ to M_{24} is minus the anomaly $\alpha \in \mathrm{H}^3(M_{24}; \mathrm{U}(1))$ computed by [17].

Proof. Given a finite group G and $g \in G$ of order $o(g)$, consider the 3-cycle

$$\gamma_g = \sum_{i=0}^{o(g)-1} g \otimes g^i \otimes g$$

in the bar complex for G. (If we consider the G-bundle on a 3d lens space $\mathrm{SU}(2)/o(g)$, whose monodromy around the nontrivial loop is g, then the homology class of γ_g is the image under the classifying map $\mathrm{SU}(2)/o(g) \to BG$ of the fundamental class.) The anomaly $\alpha \in \mathrm{H}^3(M_{24}; \mathrm{U}(1))$ of [17] is characterized by the property that for every $g \in M_{24}$, the pairing $\mathrm{H}^3(M_{24}; \mathrm{U}(1)) \otimes \mathrm{H}_3(M_{24}) \to \mathrm{U}(1)$ takes $\alpha \otimes \gamma_g$ to $\exp(-2\pi i/\ell(g))$. In [17], $\ell(g)$ is defined as the length of the shortest cycle in the degree 24 permutation representation—the notation is consistent with the $\ell(g)$ in Section 7, since the cycle type of the permutation and the Frame shape of its permutation matrix coincide.

Let $\tau \in \mathrm{H}^1(\langle g\rangle; \mathbf{R}/\mathbf{Z}) = \hom(\langle g\rangle; \mathbf{R}/\mathbf{Z})$ denote the homomorphism sending $g \mapsto 1/o(g) + \mathbf{Z}$, let $\beta : \mathrm{H}^k(G; \mathbf{R}/\mathbf{Z}) \to \mathrm{H}^{k+1}(G; \mathbf{Z})$ denote the Bockstein, and let $t = \beta(\tau) \in$

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$H^2(\langle g\rangle;\mathbf{Z})$. Then t can be represented by the cocycle

$$t\left(g^i\otimes g^j\right)=\begin{cases}0, & i+j<o(g)\\ 1, & i+j\geq o(g)\end{cases}, \qquad i,j\in\{0,\ldots,n-1\}.$$

Under the Bockstein identification $H^4(\langle g\rangle;\mathbf{Z})\cong H^3(\langle g\rangle;\mathbf{R}/\mathbf{Z})$, the cocycle t^2 is carried to $\tau\cup t$, where $\cup: H^1(\langle g\rangle;\mathbf{R}/\mathbf{Z})\otimes H^2(\langle g\rangle;\mathbf{Z})\to H^3(\langle g\rangle;\mathbf{R}/\mathbf{Z})$ denotes the cup product. We calculate the following:

$$\sum_{i=0}^{o(g)-1}[\tau\cup t]\left(g\otimes g^i\otimes g\right)=\sum_{i=0}^{o(g)-1}\tau(g)\cdot t\left(g^i\otimes g\right)=\frac{1}{o(g)}\cdot 1$$

since only the $i=o(g)-1$ term provides a nonzero value to $t\left(g^i\otimes g\right)$. Since every $g\in M_{24}$ is balanced with $\epsilon(g)=+1$, the theorem follows from part (2) of Theorem 7.1. ■

The papers [5, 14] construct super vertex algebras that explain some but not all of the umbral moonshine phenomena for the Niemeier lattices L of types A_3^8, A_4^6, A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, and D_{24}. These super vertex algebras are all of the following type. Let $U_L'=U_L$ when L is of type A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, and D_{24}, and let U_L' denote the unique (up to conjugacy) maximal subgroup of U_L isomorphic to $\mathrm{SL}_2(7)$ when $L=A_3^8$ or to $S_3\times 4$ when $L=A_4^6$. Two finite-dimensional complex representations, called in those papers $\mathfrak{b}^+$ and $\mathfrak{a}^+$, of U_L' are selected. Specifically, they take the following:

L	U_L'	$\mathfrak{b}^+$	$\mathfrak{a}^+$
A_3^8	$\mathrm{SL}_2(7)$	$\mathbf{C}^4$	$\mathbf{C}^3$
A_4^6	$S_3\times(\mathbf{Z}/4)$	$(\mathbf{R}^2\boxtimes -i)\oplus(\mathrm{sign}\boxtimes i)$	$\mathrm{triv}\boxtimes(-1\oplus i)$
A_6^4	$2A_4$	$\mathbf{C}^2$	$\mathbf{C}^1$
A_{12}^2	$\mathbf{Z}/4$	$i\oplus -i$	-1
D_6^4	S_4	$\mathbf{R}^3$	$\mathbf{R}^2$
D_8^3	S_3	$\mathbf{R}^2$	sign
D_{12}^2	S_2	sign $\oplus$ sign	triv
D_{24}	triv	triv	triv

A free field theory, also called a "$\beta\gamma bc$ system" (see e.g., [16, Chapters 11 and 12]), is then built from these representations; it consists of free bosons valued in $\mathfrak{b}^+\oplus\mathfrak{b}^-$ and free fermions valued in $\mathfrak{a}^+\oplus\mathfrak{a}^-$, where $\mathfrak{b}^-=(\mathfrak{b}^+)^*$ and $\mathfrak{a}^-=(\mathfrak{a}^+)^*$. A physical argument shows that the anomaly of the G-action on such a system is $c_2(\mathfrak{b}^+)-c_2(\mathfrak{a}^+)$, provided that $c_1(\mathfrak{b}^+)=c_1(\mathfrak{a}^+)$. (The field theories of [5, 14] also have some auxiliary free fermions, valued in a vector space called $\mathfrak{c}$, on which G acts trivially. These do not affect the anomaly.)

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We briefly explain the names for representations in the table. By "sign" and "triv" we mean the sign and trivial representations of symmetric groups. The -1 and $\pm i$ in the A_4^6 and A_{12}^2 rows denote the one-dimensional representations of $\mathbf{Z}/4$ in which the generator acts with that eigenvalue. In the $\mathfrak{b}^+$ column, the representation $\mathbf{R}^{n-1}$ of S_n is the nontrivial submodule of the permutation representation. In the $\mathfrak{a}^+$ column, the representation $\mathbf{R}^2$ is the pullback of this representation of S_3 along the surjective homomorphism $S_4 \to S_3$ (the "resolvent cubic" of Galois theory). The $\mathbf{C}^n$s in the A_3^8 and A_4^6 rows are irreducible complex n-dimensional representations, which are specified up to simultaneous complex conjugation as follows. For $L = A_3^8$, these are chosen so that, if an order-7 element of U_L' acts on $\mathfrak{a}^+$ with eigenvalue λ, then it acts on $\mathfrak{b}^+$ with trace $-\bar{\lambda}$. For $L = A_6^4$, these are chosen so that, if an order-3 element of U_L acts on $\mathfrak{a}^+$ with eigenvalue λ, then it acts on $\mathfrak{b}^+$ with trace $-\bar{\lambda}$.

Theorem 8.2. For L of types A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, and D_{24}, for the U_L-representations $\mathfrak{b}^+$ and $\mathfrak{a}^+$ in [5, 14], we have $c_1(\mathfrak{b}^+) = c_1(\mathfrak{a}^+)$ and $-\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})|_{U_L} = c_2(\mathfrak{b}^+) - c_2(\mathfrak{a}^+)$.

Proof. For L of types A_{12}^2, D_6^4, D_8^3, D_{12}^2, and D_{24}, classes in $\mathrm{H}^4(U_L; \mathbf{Z})$ are determined by their restrictions to cyclic subgroups. For all umbral groups, $\mathrm{Leech} \otimes \mathbf{R}|_{U_L}$ is a permutation representation of U_L on the nodes of the Dynkin diagram for the root system of L, and the Frame shape of an element is the cycle type of this permutation. For a given U_L one can therefore check the theorem by proving that $c_2(\mathfrak{a}^+) - c_2(\mathfrak{b}^+)$ restricts to $\frac{o(g)}{\ell(g)} t^2$ for every $g \in U_L$, by Theorem 7.1.

We will describe the case $L = D_6^4$, where $U_L = S_4$. The other cases from $\{A_{12}^2, D_6^4, D_8^3, D_{12}^2, D_{24}\}$ can be handled similarly. The following table lists the nontrivial conjugacy classes of S_4 in terms of their cycle structures on the defining degree-4 permutation, their frame shapes as elements of Co_0, and their eigenvalues in $\mathfrak{b}^+$ and $\mathfrak{a}^+$. For each $g \in S_4$, it then lists the values of $\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R})|_{\langle g \rangle}$, $c_2(\mathfrak{b}^+)|_{\langle g \rangle}$, and $c_2(\mathfrak{a}^+)|_{\langle g \rangle}$ as multiples of the canonical generator t^2 of $\mathrm{H}^4(\langle g \rangle; \mathbf{Z})$; $\frac{p_1}{2}$ is computed using Theorem 7.1, and the c_2s are immediate:

S_4	Leech	$\mathfrak{b}^+$	$\mathfrak{a}^+$	$\frac{p_1}{2}$(Leech)	$c_2(\mathfrak{b}^+)$	$c_2(\mathfrak{a}^+)$
$1^2 2^1$	$1^8 2^8$	$-1 \oplus 1 \oplus 1$	$-1 \oplus 1$	0	0	0
2^2	2^{12}	$-1 \oplus -1 \oplus 1$	$1 \oplus 1$	1	1	0
$1^1 3^1$	$1^6 3^6$	$\lambda \oplus \bar{\lambda} \oplus 1$	$\lambda \oplus \bar{\lambda}$	0	-1	-1
4^1	4^6	$i \oplus -1 \oplus -i$	$-1 \oplus 1$	1	-1	0

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In the above table, λ denotes a cube root of unity. For all $g \in S_4$, we find that $-\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R}) = c_2(\mathfrak{b}^+) - c_2(\mathfrak{a}^+) \mod \mathrm{order}(g)$.

The only remaining case is $L = A_6^4$, which we study for the remainder of the proof. This is perhaps the most interesting case, since it is one of the three Niemeier lattices for which classes in $\mathrm{H}^4(U_L; \mathbf{Z})$ are not determined by their restrictions to cyclic subgroups [9].

We first verify that $c_1(\mathfrak{b}^+) = c_1(\mathfrak{a}^+)$. The characters of the two representations are the following:

	1A	2A	4A	3A	6A	3B	6B
$\mathfrak{b}^+$	2	-2	0	$-\bar{\lambda}$	λ	$-\lambda$	$\bar{\lambda}$
$\mathfrak{a}^+$	1	1	1	λ	$\bar{\lambda}$	$\bar{\lambda}$	λ

with $\lambda = \exp(2\pi i/3)$. In particular, class 4A acts on $\mathfrak{b}^+$ with eigenvalues $i \oplus -i$, hence with determinant 1; class 3A acts on $\mathfrak{a}^+$ with eigenvalue λ and on $\mathfrak{b}^+$ with eigenvalues $1 \oplus \lambda$. Also, note that $c_2(\mathfrak{a}^+) = 0$, as $\mathfrak{a}^+$ is one-dimensional—we are left with proving

$$c_2(\mathfrak{b}^+) = -\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R}|_{U_L}). \tag{8.4}$$

The group $U_L \cong 2A_4$ is the McKay correspondent of E_6, and it has a unique faithful representation into SU(2), let us denote it by V. Meanwhile $\mathfrak{b}^+$ is one of the other two 2D representations of $2A_4$. As for any finite subgroup of SU(2), $\mathrm{H}^4(U_L; \mathbf{Z})$ is generated by $c_2(V)$ with order $|U_L| = 24$.

We analyze $\mathrm{Leech} \otimes \mathbf{R}|_{U_L}$ by thinking of it as a permutation representation on the nodes of the Dynkin diagram. This action has three orbits: the nodes at the edges of the A_6-components, the nodes at distance one from the edges, and the nodes at distance two from the edges. These orbits are abstractly isomorphic as U_L-sets; we will refer to this degree-8 permutation representation as Perm_8, so that $\mathrm{Leech} \otimes \mathbf{R}|_{U_L} \cong \mathrm{Perm}_8^{\oplus 3} \otimes \mathbf{R}$. Since $\mathrm{Perm}_8 \otimes \mathbf{R}$ is a Spin representation, we have:

$$\frac{p_1}{2}(\mathrm{Leech} \otimes \mathbf{R}|_{U_L}) = 3\frac{p_1}{2}(\mathrm{Perm}_8 \otimes \mathbf{R}).$$

The restriction maps to the cyclic subgroup $\mathbf{Z}/3 \subset U_L$ of order 3 and to the quaternion subgroup $Q_8 \subset U_L$ of order 8 give an isomorphism $\mathrm{H}^4(U_L; \mathbf{Z}) \to \mathrm{H}^4(\mathbf{Z}/3; \mathbf{Z}) \oplus \mathrm{H}^4(Q_8; \mathbf{Z}) \cong (\mathbf{Z}/3) \oplus (\mathbf{Z}/8)$. We will prove (8.4) by proving

$$c_2\left(\mathfrak{b}^+|_{\mathbf{Z}/3}\right) = -3\frac{p_1}{2}(\mathrm{Perm}_8 \otimes \mathbf{R}|_{\mathbf{Z}/3}) = 0 \text{ and } c_2\left(\mathfrak{b}^+|_{Q_8}\right) = -3\frac{p_1}{2}(\mathrm{Perm}_8 \otimes \mathbf{R}|_{Q_8}). \tag{8.5}$$

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For the left equation in (8.5), note that $\mathfrak{b}^+|_{\mathbf{Z}/3}$ splits as the sum of two one-dimensional representations, one of which is trivial, and so $c_2(\mathfrak{b}^+|_{\mathbf{Z}/3}) = 0$.

We turn to the right equation in (8.5). We have an isomorphism $\mathfrak{b}^+|_{Q_8} \cong V|_{Q_8}$, since they are both irreducible 2D representations. Let W denote the underlying four-dimensional real representation of V and let X, Y, and $Z \cong X \otimes Y$ denote the three non-trivial one-dimensional real representations of Q_8. $\mathrm{Perm}_8|_{Q_8}$ is isomorphic to the regular representation $\mathbf{Z}[Q_8]$, which over $\mathbf{R}$ decomposes as $W \oplus X \oplus Y \oplus Z \oplus 1$. The real representations W and $X \oplus Y \oplus Z$ are each Spin, so $\frac{p_1}{2}(W \oplus X \oplus Y \oplus Z) = \frac{p_1}{2}(W) + \frac{p_1}{2}(X \oplus Y \oplus Z)$.

To compute $\frac{p_1}{2}(W)$ we can use the observation that the action $W : Q_8 \to \mathrm{SO}(4)$ factors through $V : Q_8 \to \mathrm{SU}(2)$, and so

$$\frac{p_1}{2}(W) = -c_2(V).$$

To compute $\frac{p_1}{2}(X \oplus Y \oplus Z)$, note that $X \oplus Y \oplus Z : Q_8 \to \mathrm{SO}(3)$ is nothing but the image of $V : Q_8 \to \mathrm{SU}(2) \cong \mathrm{Spin}(3)$ under the canonical map $\mathrm{Spin}(3) \to \mathrm{SO}(3)$. The group Spin(3) is unusual among Spin groups in the following way. The stable class $\frac{p_1}{2} \in \mathrm{H}^4(B\mathrm{Spin}(3); \mathbf{Z})$, pulled back from the generator of $\mathrm{H}^4(B\mathrm{Spin}(\infty); \mathbf{Z})$, does not generate $\mathrm{H}^4(B\mathrm{Spin}(3); \mathbf{Z}) \cong \mathbf{Z}$, whereas $\frac{p_1}{2}$ generates $\mathrm{H}^4(B\mathrm{Spin}(n); \mathbf{Z})$ for $n \geq 5$. Rather, $\frac{p_1}{2} \in \mathrm{H}^4(B\mathrm{Spin}(3); \mathbf{Z})$ is twice the generator. To see this, note that $\mathrm{H}^4(B\mathrm{Spin}(4); \mathbf{Z}) \cong \mathbf{Z}^2$ and that the restriction maps $\mathrm{H}^4(B\mathrm{Spin}(5); \mathbf{Z}) \to \mathrm{H}^4(B\mathrm{Spin}(4); \mathbf{Z}) \to \mathrm{H}^4(B\mathrm{Spin}(3); \mathbf{Z})$ are the diagonal embedding followed by addition. On the other hand, think of Spin(3) as SU(2): $\mathrm{H}^4(B\mathrm{SU}(2); \mathbf{Z})$ is generated by c_2. After checking signs, one finds that $\frac{p_1}{2} = -2c_2$ as classes in $\mathrm{H}^4(B\mathrm{Spin}(3); \mathbf{Z}) = \mathrm{H}^4(B\mathrm{SU}(2); \mathbf{Z})$. Restricting to Q_8 gives

$$\frac{p_1}{2}(X \oplus Y \oplus Z) = -2c_2(V).$$

All together, we have

$$\frac{p_1}{2}(W) + \frac{p_1}{2}(X \oplus Y \oplus Z) = -3c_2(V) \in \mathrm{H}^4(Q_8; \mathbf{Z}).$$

Multiplying both sides by -3 gives (8.5) as desired. ■

We conclude with some calculations that show (consistent with calculations in [8]) that the anomaly α_L does not agree with $-\frac{p_1}{2}$ for L of type A_3^8 or A_4^6.

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For $L = A_4^6$, we calculate the restrictions of $\frac{p_1}{2}(\text{Leech} \otimes \mathbf{R})$ and $c_2(\mathfrak{b}^+) - c_2(\mathfrak{a}^+)$ to the elements 3A and 4A in $U_L' = S_3 \times 4$:

Name	$S_3 \times 4$	Leech	$\mathfrak{b}^+$	$\mathfrak{a}^+$	$\frac{p_1}{2}$(Leech)	$c_2(\mathfrak{b}^+)$	$c_2(\mathfrak{a}^+)$
3A	$(3^1, 0)$	3^8	$\lambda \oplus \bar{\lambda} \oplus 1$	$1 \oplus 1$	1	-1	0
4A	$(1^3, 1)$	4^6	$-i \oplus -i \oplus i$	$-1 \oplus i$	1	-1	2

Here $\lambda = \exp(2\pi i/3)$. We have recorded the eigenvalues of each element in $\mathfrak{b}^+$ and $\mathfrak{a}^+$ and their Frame shapes in the permutation representation on the nodes of the A_4^6 Dynkin diagram (equivalently the Leech representation). We also indicate each element as a pair in $S_3 \times \mathbf{Z}/4$, where the class in S_3 is indicated by its Frame shape and we write $\mathbf{Z}/4$ additively. The calculation of $\frac{p_1}{2}$(Leech) is from Theorem 7.1 and the calculations of the c_2s are routine; we record the values as multiples of the square generator of $\mathrm{H}^4(\mathbf{Z}/3; \mathbf{Z})$ or $\mathrm{H}^4(\mathbf{Z}/4; \mathbf{Z})$. In particular, whereas when restricting to element 3A, we do find $-\frac{p_1}{2}(\text{Leech} \otimes \mathbf{R}) = c_2(\mathfrak{b}^+) - c_2(\mathfrak{a}^+)$ as in Theorem 8.2, when restricting to 4A we find the opposite sign. It follows that the restrictions to element $12\text{A} = (3^1, 1)$ of $\frac{p_1}{2}(\text{Leech} \otimes \mathbf{R})$ and $c_2(\mathfrak{b}^+) - c_2(\mathfrak{a}^+)$ differ by a factor of $\epsilon(L) = 5$.

The final case studied in [14] and not covered by Theorem 8.2 is $L = A_3^8$. The calculations in [8] suggest that (8.2) should hold with $\epsilon(L) = 1$, at least after restricting to cyclic subgroups. Note, however, that the Niemeier lattice L of type A_3^8 is one of the three Niemeier lattices for which classes in $\mathrm{H}^4(U_L; \mathbf{Z})$ are not determined by their restrictions to cyclic subgroups [9]; this remains true for the maximal subgroup $U_L' \cong \mathrm{SL}(2,7)$ studied in [14].

It is easy to check that $c_1(\mathfrak{a}^+) = c_1(\mathfrak{b}^+)$, so to test (8.2), we must calculate

$$\frac{p_1}{2}(\text{Leech} \otimes \mathbf{R})|_{U_L'} \text{ and } c_2(\mathfrak{b}^+) - c_2(\mathfrak{a}^+). \tag{8.6}$$

Since $\mathrm{H}^4(\mathrm{SL}(2,7); \mathbf{Z}) \cong \mathbf{Z}/48$ has only 2- and 3-primary torsion, we may compare these classes by comparing their restrictions to the 2- and 3-Sylow subgroups of $\mathrm{SL}(2,7)$. The 3-Sylow is cyclic of order 3, its generator acts on Leech $\otimes \mathbf{R}$ with Frame shape $1^6 3^6$, and on $\mathfrak{b}^+$ and $\mathfrak{a}^+$ with Frame shapes 3^1 and $1^1 3^1$, respectively; thus, the -primary parts of (8.6) both vanish. It remains to compare the 2-primary parts.

The 2-Sylow in $\mathrm{SL}(2,7)$ is isomorphic to $2D_8$, let us index its complex representations just as in Section 6. As with all umbral groups, the action of $U_L' \cong \mathrm{SL}(2,7)$ on Leech $\otimes \mathbf{R}$ is a permutation representation on the nodes of the Dynkin diagram; after restricting to the 2-Sylow $2D_8 \subset \mathrm{SL}(2,7)$, it is the sum of the regular representation of

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$2D_8$ with the regular representation of D_8. This 24-dimensional real representation is the underlying real representation of a complex representation V that splits over $2D_8$ as

$$V = V_0 \oplus (V_1 \oplus V_2 \oplus V_3) \oplus 2V_4 \oplus V_5 \oplus V_6$$

and so $\frac{p_1}{2}(\text{Leech} \otimes \mathbf{R}|_{2D_8}) = -c_2(V)$. The representations $\mathfrak{b}^+$ and $\mathfrak{a}^+$ decompose as

$$\mathfrak{b}^+|_{2D_8} = V_5 \oplus V_6, \qquad \mathfrak{a}^+|_{2D_8} = V_1 \oplus V_2 \oplus V_3.$$

As observed in the proof of Lemma 6.1, $c_2(V_1 \oplus V_2 \oplus V_3)$ has order 2 in $\mathrm{H}^4(2D_8; \mathbf{Z})$. Thus, to compare $\frac{p_1}{2}(\text{Leech} \otimes \mathbf{R}|_{2D_8})$ with $c_2(\mathfrak{b}^+) - c_2(\mathfrak{a}^+)$, it suffices to compare $\pm c_2(2V_4 \oplus V_5 \oplus V_6)$ with $c_2(V_5 \oplus V_6)$. According to Lemma 6.1,

$$c_2(2V_4 \oplus V_5 \oplus V_6) = 2c_2(V_6), \qquad c_2(V_5 \oplus V_6) = 10c_2(V_6),$$

which certainly differ, even up to sign. Rather, we find that, for $L = A_3^8$,

$$\alpha_L|_{U_L'} = 5\frac{p_1}{2}(\text{Leech} \otimes \mathbf{R})|_{U_L'},$$

suggesting that $\epsilon(A_3^8) = 5$. (The above equation holds whether $c_2(V_1 \oplus V_2 \oplus V_3) = 0$ or $c_2(V_1 \oplus V_2 \oplus V_3) = 8c_2(V_6)$.)

But restrictions to cyclic groups can only determine a class in $\mathrm{H}^4(2D_8; \mathbf{Z}) \cong \mathbf{Z}/16$ modulo 8, and so we confirm the calculation of [8] that, for $L = A_3^8$, the multipliers in umbral moonshine agree with those that would be given if the anomaly were $+\frac{p_1}{2}(\text{Leech} \otimes \mathbf{R})|_{U_L}$, that is, if we had $\epsilon(L) = 1$.

To conclude, we remark that these cohomological methods do explain why in the case $L = A_3^8$, the authors of [14] were unable to find a "free field" realization of the entire umbral group $U_L \cong 2^4 : \mathrm{GL}(3, 2)$ reproducing the umbral moonshine functions. Indeed, all Chern classes in the 2-primary part of $\mathrm{H}^4(U_L; \mathbf{Z})$ have order 4 or less, but the previous calculations show that the anomaly for the A_3^8 moonshine of [14] has order 8.

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