Part D. Project description
Mirror symmetry and the microlocal theory of sheaves

Project description for David Treumann, Northwestern University

The PI proposes to use microlocal sheaf theory to investigate mirror symmetry for Calabi-Yau hypersurfaces in toric varieties.

Mirror symmetry is a geometric phenomenon first observed in physics: it is a relationship that can exist between the complex geometry of one manifold and the symplectic geometry of a different manifold. Kontsevich conjectured that this relationship could be formalized mathematically as an equivalence of categories. This proposal concerns these "homological mirror symmetry" (HMS) conjectures.

The microlocal theory of sheaves was developed by Saito, Kashiwara, Kawai, and others for the algebraic analysis of linear PDEs, and later saw many applications to representation theory. There have recently been some striking applications to symplectic geometry, for example Nadler's [N1] results on the nearby Lagrangian problem, and Tamarkin's new proof of Hamiltonian nondisplacability [Ta]. In [FLTZ2], the PI with Fang, Liu and Zaslow have used sheaf theory to give a proof of HMS for toric manifolds.

This proposal outlines a method to use similar sheaf-theoretic techniques to give a proof of HMS for non-toric manifolds.

1 Previous work

We discuss in this section some of the PI's recent work most directly relevant to the proposal.

1. In [FLTZ1, Tr1], PI Treumann et al proved that the derived category of constructible sheaves on $(S^1)^n$ (denoted $Sh((S^1)^n)$) contains the derived category of coherent sheaves on every compact toric manifold—this is called the *coherent constructible correspondence* (CCC).
2. In [FLTZ2], the CCC was shown to be a form of homological mirror symmetry, under the microlocalization equivalence $Sh((S^1)^n) \cong \mathrm{Fuk}(T^*(S^1)^n) \cong \mathrm{Fuk}((\mathbb{C}^*)^n)$ of [NZ, N1]. The CCC combined with the Nadler-Zaslow construction was proved to be compatible with the SYZ transform, verifying for toric varieties the thesis of [SYZ] that "mirror symmetry is T-duality."
3. The PI has investigated several miscellaneous-but-important aspects of the CCC. The CCC was extended to orbifolds in [FLTZ3], and McKay/Kawamata-style equivalences [Ka] between orbifolds and their crepant resolutions was investigated in [FLTZ4]. A dictionary relating toric vector bundles to "homologically convex" sheaves on $\mathbb{R}^n$ was established in [Tr2], and the constructible counterpart of the nefness condition for vector bundles was given.
4. The PI began to study mirror symmetry for reducible varieties with toric components in [STZ1] using the CCC. These varieties often appear at the large complex structure limit of the HMS equivalence, in which case they should be mirror to an exact symplectic manifold.

1.1 The coherent-constructible correspondence (CCC)

In [FLTZ1] PI Treumann et al proved that the equivariant derived category of a toric variety was equivalent to a category of constructible sheaves on a real vector space $M_{\mathbb{R}} = \mathbb{R}^n$. This equivalence is called the coherent-constructible correspondence or CCC.

Recall that a convex polytope $\triangle \subset M_{\mathbb{R}}$ with vertices in $M_{\mathbb{Z}} = \mathbb{Z}^n$ (i.e. a **lattice polytope**) determines a pair $(X(\triangle), \mathcal{L}(\triangle))$ where $X = X(\triangle)$ is an n-dimensional toric variety and $\mathcal{L} = \mathcal{L}(\triangle)$ is an ample line bundle on X. The curvature of $\mathcal{L}$ is a symplectic form on X, so one can recover $\triangle$ from the pair $(X, \mathcal{L})$ by taking the image of the moment map $X \to M_{\mathbb{R}}$ determined by the Hamiltonian $\mathrm{U}(1)^n$-action. Let us denote this inverse construction by $\mathcal{L} \mapsto \triangle_{\mathcal{L}}$. If one fixes X and varies $\mathcal{L}$, the lattice polytopes $\triangle_{\mathcal{L}}$ can change size and location but maintain the same shape in some sense.

The CCC can be motivated as a generalization of this dictionary between ample line bundles and lattice polytopes. Line bundles have a categorical structure—given $\mathcal{L}_1$ and $\mathcal{L}_2$ it is possible to form the Hom group $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2)$ and more generally the Ext groups $\mathrm{Ext}^i(\mathcal{L}_1, \mathcal{L}_2)$, which are homs in the derived category. To put polytopes on equal categorical footing we regard them as constructible sheaves. More precisely, given a convex subset $U \subset \mathbb{R}^n$, we introduce the **costandard sheaf** on $M_{\mathbb{R}}$ attached to U, call it $\mathrm{cost}(U)$. This is the constant sheaf of rank one on the interior[1] of U extended by zero to all of $M_{\mathbb{R}}$. One computes

$$\mathrm{Ext}^i\left(\mathcal{L}_1, \mathcal{L}_2\right) \cong \mathrm{Ext}^i\left(\mathrm{cost}\left(\triangle\left(\mathcal{L}_1\right)\right), \mathrm{cost}\left(\triangle\left(\mathcal{L}_2\right)\right)\right)$$

which is essentially equivalent to a formula of Demazure's for the weight spaces of the line bundle cohomology $H^i(\mathcal{L}_2 \otimes \mathcal{L}_1^{-1})$. The CCC of PI Treumann et al [FLTZ1] is a full embedding of dg categories

$$\kappa : \mathrm{Coh}_T(X) \hookrightarrow Sh_{cc}(M_{\mathbb{R}})$$

where on the left we have the derived category of torus-equivariant coherent[2] sheaves on X, and on the right we have the differential-graded category of constructible sheaves—the *cc* in the subscript stands for "constructible with compact support." If one works with nonequivariant sheaves, one obtains [Tr1] a wrapped-up version of this embedding

$$\overline{\kappa} : \mathrm{Coh}(X) \hookrightarrow Sh_c(M_{\mathbb{R}}/M_{\mathbb{Z}})$$

where on the right we have constructible sheaves on a compact torus.

1.1.1 The CCC and the Fukaya category

PI Treumann et al have addressed the important problem of determining **the image of the CCC**, i.e. which constructible sheaves on $\mathbb{R}^n$ or on $(S^1)^n$ come from coherent sheaves on a fixed toric variety. The solution of this problem uses the microlocal language of Kashiwara and Schapira, which makes features of constructible sheaves on M visible in the cotangent bundle T^*M. Let us explain the Kashiwara-Schapira story in terms of a fairly recent development, i.e. the Fukaya-theoretic picture of [NZ].

The **unwrapped Fukaya category** of the cotangent bundle T^*Y of a real analytic manifold Y is introduced in [NZ], and a quasi-equivalence (called **microlocalization**) between the A_∞-category $\mathrm{Fuk}(T^*Y)$ and the dg category $Sh_c(Y)$ was established in [NZ, N1]. Crucially, objects of $\mathrm{Fuk}(T^*Y)$ are allowed to be noncompact, so long as they are under some control at infinity. The basic objects of $\mathrm{Fuk}(T^*Y)$ are the **standard** branes: if f is a nonnegative smooth function

[1] The name "standard sheaf" is reserved for the Verdier dual, which is constant on both the interior and on the boundary of U

[2] If X has singularities, one should replace this with the smaller category of perfect complexes [FLTZ1]

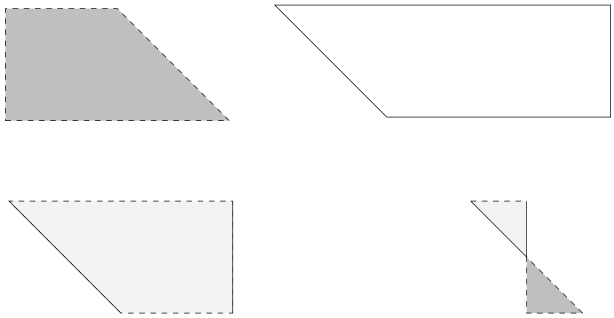


Figure 1: Constructible sheaves associated to several equivariant line bundles on the Hirzebruch surface F_1 (i.e., on the blowup of $\mathbb{P}^2$ at a T-fixed point. The top row shows sheaves associated to ample and anti-ample line bundles. The bottom row shows line bundles that are neither ample nor anti-ample. The darkly shaded (resp. lightly shaded, resp. ushaded) region is in homological degree -2 (resp. -1, resp. 0).

on Y that vanishes outside of an open set $U \subset Y$, then the graph of $d\log(f)$ defines an object of $\mathrm{Fuk}(T^*Y)$, which under microlocalization goes to the constant sheaf on the closure of U, also called the **standard** sheaf attached to U. These standard objects generate the respective categories—in $Sh_c(Y)$, this is true almost by definition, and in $\mathrm{Fuk}(T^*Y)$ is a theorem of Nadler's [N1].

In microlocal sheaf theory, a fundamental role is played by **conic Lagrangians**, these are singular subsets of T^*Y that are Lagrangian along their smooth locus, and stable by positive dilations in the cotangent fibers. Given a conic Lagrangian $\Lambda \subset T^*Y$, Kashiwara and Schapira construct a triangulated subcategory of $Sh_c(Y)$ given by sheaves whose **singular support** belongs to Λ. Roughly speaking, for each covector ξ there is a Morse-theoretic test to see if F changes in the direction of ξ, and $SS(F)$ is defined to be the set of those covectors which do detect a change. If a Lagrangian is asymptotic to Λ, then the corresponding constructible sheaf will have singular support in Λ.

The paper [FLTZ1] introduced a conic Lagrangian $\Lambda_\Sigma \subset T^*M_{\mathbb{R}}$ for each rational polyhedral fan Σ in the dual vector space $N_{\mathbb{R}} = M_{\mathbb{R}}^*$, namely

$$\Lambda_\Sigma = \bigcup_{\sigma \in \Sigma} (\sigma^\perp + \mathbb{Z}^n) \times -\sigma$$

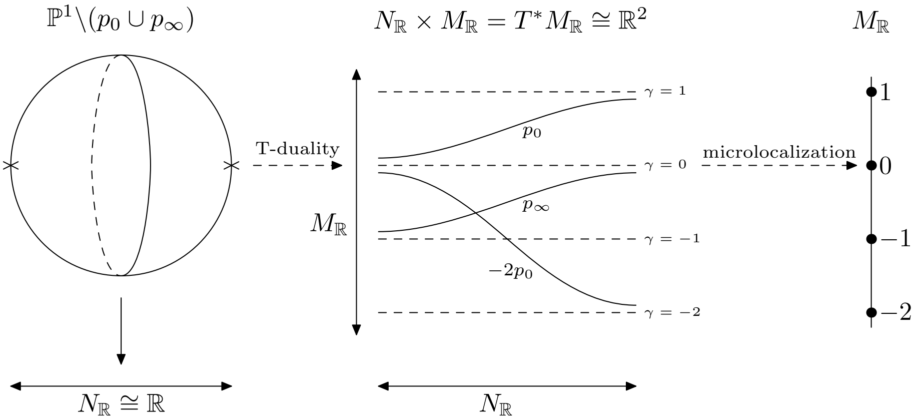
The image of Λ_Σ under the covering map $T^*\mathbb{R}^n \to T^*(S^1)^n$ is another conic Lagrangian, call it $\overline{\Lambda}_\Sigma$. The fan Σ also leads to a toric variety X_Σ, and the CCC carries coherent sheaves on X_Σ to constructible sheaves with singular support in $\overline{\Lambda}_\Sigma$ (or Λ_Σ if one starts with an equivariant coherent sheaf).

$$\kappa : \mathrm{Coh}_T(X) \xrightarrow{\sim} Sh_{cc}(\mathbb{R}^n; \Lambda_\Sigma)$$
$$\overline{\kappa} : \mathrm{Coh}(X) \to Sh_c((S^1)^n; \overline{\Lambda}_\Sigma)$$

The top row is an equivalence—this was established in [FLTZ1] by a delicate induction argument. This is still unknown for the bottom row, but it can be established in some important cases, e.g. for $X_\Sigma = \mathbb{P}^n$, and the PI conjectures that it is true in general.

1.2 The SYZ transform and mirror symmetry

The PI's results of 1.1 and 1.1.1 were inspired by mirror symmetry, especially by the T-duality thesis of [SYZ]. In [FLTZ2] the PI et al proved a basic compatibility between the constructions of [FLTZ1] and of [SYZ], extending Abouzaid's results [A2, A3] on HMS for toric varieties.



The fibers of the moment map $X \to \triangle$ are all isotropic tori; over the interior of $\triangle$ they are Lagrangian tori[3]. The interior of $\triangle$ can be (nonlinearly) identified with $N_\mathbb{R}$, and each of the Lagrangian fibers with $N_\mathbb{R}/N_\mathbb{Z}$, the T-dual to $M_\mathbb{R}/M_\mathbb{Z}$. Now given a line bundle $\mathcal{L}$ on X (related to $\triangle$ or not) we may perform the SYZ construction as follows. Pick a $\mathrm{U}(1)^n$-invariant connection $A_\mathcal{L}$ on $\mathcal{L}$, it will be flat over each fiber of $X \to \triangle$ and so determine a map from π_1 to $\mathrm{U}(1)$, or equivalently a point of the dual torus $M_\mathbb{R}/M_\mathbb{Z}$. In other words we have constructed a map from $N_\mathbb{R}$ to $M_\mathbb{R}/M_\mathbb{Z}$. If $A_\mathcal{L}$ is compatible with the complex structure on X then the graph of this map a Lagrangian submanifold $N_\mathbb{R} \times M_\mathbb{R}/M_\mathbb{Z} = T^*(M_\mathbb{R}/M_\mathbb{Z})$, called the **T-dual** or **SYZ Lagrangian**. This construction is consistent with the CCC is the following ways

1. As one gets closer to the boundary of $\triangle$, at least one of the radii of the Lagrangian fibers shrinks to zero. Correspondingly, the holonomy of $A_\mathcal{L}$ around that radius shrinks to 1, yielding an asymptotic condition on the T-dual Lagrangian.
2. An equivariant structure on $\mathcal{L}$ gives a canonical lift of this Lagrangian to the universal cover $T^*M_\mathbb{R}$. When $\mathcal{L}$ is ample, this Lagrangian is the graph of the $d\log(f)$ for some positive function f on the interior of the moment polytope associated to $\mathcal{L}$—i.e. it is a standard Lagrangian.

1.3 Homological mirror symmetry and ribbon graphs

The PI's work has established constructible sheaf methods as important in understanding **mirror symmetry at the large volume/large complex structure limit**. PI Treumann et al's paper [STZ1] treats these ideas in the one-dimensional case, and project proposals 2.1, 2.2, 2.3 below are closely related to higher-dimensional investigations. The large volume limit is very often an exact symplectic manifold, and constructions (old and new) in microlocal sheaf theory are well-suited to investigate such manifolds.

[3] This is true regardless for any of the symplectic forms on X that arise in the manner of 1.1, a fortunate consequence of the commutativity of tori

Let X be a punctured Riemann surface and let $\Gamma \subset X$ be a one-dimensional deformation retract of X. The orientation on X induces a cyclic order at each vertex of Γ, i.e. Γ is a **ribbon graph**. Without loss of generality we may assume that Γ arises from a subharmonic function f on X by taking the union of stable manifolds for f. Kontsevich has conjectured that in this regime the Fukaya category of X can be computed locally on Γ.

In [STZ1] the PI defined, starting from a ribbon graph Γ, a category $\mathrm{CPM}(\Gamma)$ ("constructible plumbing model") defined only using the language of constructible sheaves. $\mathrm{CPM}(\Gamma)$ serves as a standin for the Fukaya category $\mathrm{Fuk}(X)$. The category is defined when each vertex of Γ has degree 4 or less, and requires a chordal structure or grading. (The latter amounts to giving a grading on the symplectic manifold X in the sense of [S3]). The recipe is roughly as follows:

- Each four-valent vertex of Γ is modeled by the category of sheaves on $\mathbb{R}$ that are locally constant away from the origin. These are precisely the sheaves on $\mathbb{R}$ whose singular support is contained in the union of the axes in $T^*\mathbb{R} \cong \mathbb{R}^2$—denote this conic Lagrangian by $\Lambda_4 \subset T^*\mathbb{R}$, and the category by $Sh(\mathbb{R}, \Lambda_4)$.
- Each edge of Γ determines a functor from $Sh(\mathbb{R}; \Lambda_4)$ to the derived category of vector spaces: the rays in the base directions give us stalk functors, and the rays in the fiber directions give us "microlocal stalk" functors. We may think of each of these functors as extracting from a the vertex category a local system on the edge.
- One extracts a global category by taking a homotopy limit of the diagram of constructible sheaf categories and stalk/microstalk functors.

The PI considered a combinatorial version of a torus fibration for ribbon graphs: a map to a cycle graph, with circle fibers over the vertices. From such a datum he extracted a "mirror curve" $X^\vee$, a one-dimensional Deligne-Mumford stack with nodal singularities and smooth toric components. The main result of [STZ1] is an equivalence

$$\mathcal{P}\mathrm{erf}(X^\vee) \cong \mathrm{CPM}(\Gamma)$$

Moreover, it is conjectured that $\mathrm{CPM}(\Gamma) \cong \mathrm{Fuk}(X)$. These together would yield a form of HMS in one dimension.

Remark 1.1. There has been other recent work on HMS in one dimension. Katzarkov's ideas on what mirror symmetry should look like beyond the Calabi-Yau and Fano cases have been influential here [S4, AAEKO]. In locs. cit. the mirror is constructed as a 3d Landau-Ginzburg B-model (a category of matrix factorizations). Another approach [LP] was motivated by the appearance of Fukaya categories of open surfaces in Seiberg-Witten-style 3-manifold invariants. Lekili and Perutz treat the once-punctured genus one case and their mirror is a nodal elliptic curve, which is more directly analogous to the CPM construction. Experts expect that matrix factorization categories and more standard categories of coherent sheaves can often be compared by the results of Orlov [O].

2 Research projects

2.1 Topology of affine hypersurfaces

To an affine hypersurface $Z \subset (\mathbb{C}^*)^{n+1}$, PI Treumann will construct an n-dimensional cell complex and show that **the hypersurface is homotopy equivalent to the cell complex**. This ad-

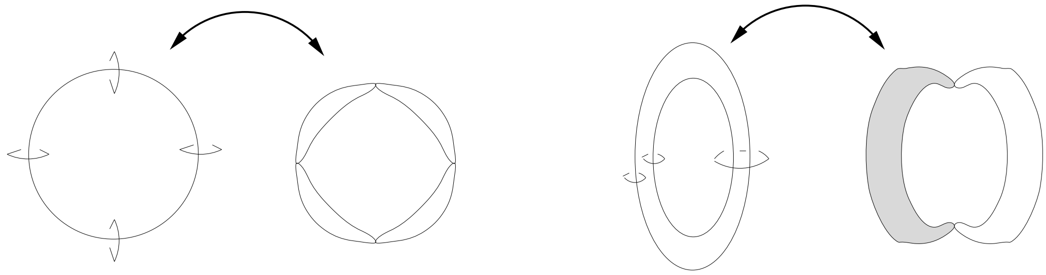


Figure 2: Some ribbon graphs and their mirror curves. The ribbon graph on the right has genus two, and the corresponding curve carries a $\mathbb{Z}/2$-gerbe over the shaded component.

dresses one of the basic problems of algebraic geometry: to decribe the topology of the zero set of a polynomial equation $f(x_0, \ldots, x_n)$. Rene Thom's classic Morse-theoretic proof of the Lefschetz hyperplane theorem [AF] showed that every affine variety of n complex dimensions has the homotopy type of an n-dimensional cell complex, but the combinatorial structure of these cell complexes is not well-understood for $n > 1$. This problem has acquired a new relevance in mirror symmetry in light of the ideas of Ruan and Kontsevich discussed elsewhere in this proposal (Section 2.3).

Let us describe the problem in more detail. Each monomial function $x_0^{a_0} \cdot \ldots \cdot x_n^{a_n}$ on $(\mathbb{C}^*)^{n+1}$ determines a lattice point $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$, and a polynomial $f : (\mathbb{C}^*)^{n+1} \to \mathbb{C}$ determines a **Newton polytope** by taking the convex hull of the monomials that appear in f. Let us write $\mathsf{Newton}(f)$ for the Newton polytope of a polynomial f. For a generic choice of coefficients, the topology of the zero set $Z(f) = \{x \in (\mathbb{C}^*)^{n+1} \mid f(x) = 0\}$ is known to depend only on $\mathsf{Newton}(f)$. Much is known about how the combinatorics of the Newton polytope influences the homology of f, for instance the work of Danilov-Khovanskii computes the Betti and mixed Hodge numbers. But the goal is to produce an explicit cell complex.

The proposed deformation retract is best described as a **torus cell complex**. A torus cell is a copy of $(S^1)^k \times \mathbb{R}^\ell$, and a torus cell complex is a space built inductively by attaching torus cells to a previously constructed torus cell complex. The torus cell complexes that appear in the proposal are **regular**: each attaching map from $(S^1)^k \to S^{\ell-1}$ is a homeomorphism onto its image, so it is easy if necessary to refine them to triangulations. We will now describe a recipe (first proposed in [TZ]) to construct a torus cell complex $\mathrm{TC}(P)$ out of a lattice polytope P.

Recipe: Suppose $P \subset \mathbb{R}^n$ has vertices at lattice points and contains the origin. If x is a point on the boundary of P, write $\sigma(x)$ for the smallest face of P that contains x. The torus complex $\mathrm{TC}(P)$ is the subspace of $\partial P \times \mathrm{Hom}(\mathbb{Z}^{n+1}, \mathbb{R}/\mathbb{Z}) \cong S^n \times (S^1)^{n+1}$ obtained by putting $(x, \phi) \in \mathrm{TC}(P)$ if ϕ carries each vertex of $\sigma(x)$ to $0 \in \mathbb{R}/\mathbb{Z}$.

Example 2.1. Let P be the polytope in $\mathbb{R}^3$ with vertices at $(1,0,0), (0,1,0), (0,0,1)$, and $(-1,-1,-1)$. Part of the torus complex $\mathrm{TC}(P)$ is displayed in Figure 2.1.

Suppose that the faces of P are all lattice points. Suppose furthermore that P is **reflexive**, i.e. that the polar dual of P is again a lattice polytope. Then the PI conjectures that $\mathrm{TC}(P)$ can be realized as a deformation retract of a generic hypersurface with Newton polytope P. This case is motivated by mirror symmetry, but actually the conjecture seems to be true in much greater generality.

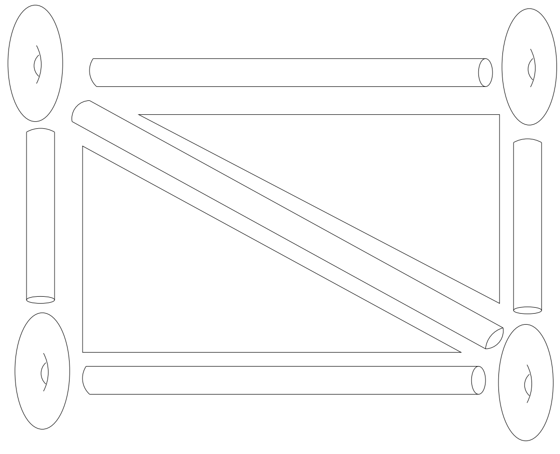


Figure 3: Part of the conjectural skeleton of the surface $Ax + By + Cz + \frac{D}{xyz} + E = 0$, for generic complex parameters A, B, C, D, E. Each tube is attached to a torus along a different circle in $S^1 \times S^1$, the resulting figure does not embed in $\mathbb{R}^3$. There is a sixth tube and two additional triangles "behind" the diagram, they are to be glued together in the shape of a tetrahedron.

Remark 2.2. (Connections to Danilov-Khovanskii) A weaker conjecture states that $\mathrm{TC}(P)$ has the same homology as the hypersurface. The PI has written a computer program in SAGE that can verify this weaker conjecture case by case, by comparing to the work of Danilov-Khovanskii. This weaker conjecture has been verified in hundreds of examples, including all 194 facet-simplicial reflexive polyhedra (3d polytopes), many four-dimensional examples, and many non-reflexive examples.

Remark 2.3. (Connections to tropical geometry) The torus complex $\mathrm{TC}(P) \subset S^n \times (S^1)^{n+1}$ projects naturally onto S^n and $(S^1)^{n+1}$. This second projection is closely related to the **coamoeba** of the associated hypersurface, i.e. its image under the argument map $(\mathbb{C}^*)^{n+1} \to (S^1)^{n+1}$—indeed the "spine" of this coamoeba coincides with the image of $\mathrm{TC}(P) \to (S^1)^{n+1}$. The other projection $\mathrm{TC}(P) \to S^n$ seems to be less closely related to the amoeba or tropicalization of the hypersurface. To define the spine of an amoeba requires an auxiliary choice (a triangulation of P) that $\mathrm{TC}(P)$ does not depend on, and the fibers of the map from the hypersurface to such a spine do not usually resemble the fibers of the map from $\mathrm{TC}(P) \to S^n$—in particular the latter fibers are more often disconnected.

Remark 2.4. (Connections to the CCC) The torus complex $\mathrm{TC}(P)$ is the Legendrian boundary of a conic Lagrangian Λ_Σ, for a certain (stacky) fan Σ. Thus it has already in some sense been intensively investigated by the PI et al. This aspect of the torus complex $\mathrm{TC}(P)$ will be discussed elsewhere in the proposal.

2.1.1 Outline

We now outline a project to study $\mathrm{TC}(P)$ in greater detail, and to prove the conjecture. In the following P denotes a lattice polytope, f a Laurent polynomial with $\mathsf{Newton}(f) = P$, and $Z \subset (\mathbb{C}^*)^{n+1}$ the hypersurface cut out by f.

1. (Hodge theory from $\mathrm{TC}(P)$). An open smooth algebraic variety carries a **mixed Hodge structure** on its cohomology [Del]. For hypersurfaces in $(\mathbb{C}^*)^{n+1}$ these have been determined by Danilov-Khovanskii. The PI will develop tools to see these structures in the cohomology of $\mathrm{TC}(P)$. There are two approaches here. (i) $\mathrm{TC}(P)$ admits a map to $S^n \cong \partial(P)$ with torus[4] fibers—if this map is analogous to an SYZ fibration then standard heuristics from mirror symmetry suggest a way to extract Hodge numbers from the Leray filtration (or spectral sequence) of this map. (ii) $\mathrm{TC}(P) \subset S^n \times (S^1)^{n+1}$ is stable under the map that raises each coordinate of the $(S^1)^{n+1}$ factor to the pth power (p a natural number). These are analogous to the Frobenius morphisms in arithmetic, and the PI expects standard arithmetic analogies between Frobenius eigenvalues and Hodge structures to apply in this combinatorial setting. Both of these approaches are amenable to computer calculations.
2. (Real points) In addition to the Frobenius-like endomorphisms of (1), the space $\mathrm{TC}(P)$ carries an involution similar to complex conjugation on Z (if the coefficients of Z were real). It is obtained by raising the torus-coordinates to the power of -1. There is more than one real structure on Z but this virtual real structure on $\mathrm{TC}(P)$ is canonically specified. The PI will show that one of the real structures on Z has the same number of connected components as the fixed points of the involution on $\mathrm{TC}(P)$. This method can be used to compare $\mathbb{R}$-points of Z to fixed points of the involution, and guides the choice of coefficients in (3).
3. (Boundary of $(\mathbb{C}^*)^{n+1}$). The complex algebraic torus $(\mathbb{C}^*)^{n+1} \cong \mathbb{R}^{n+1} \times (S^1)^{n+1}$ has a natural compactification whose boundary is $S^n \times (S^1)^{n+1}$. This puts $Z \subset (\mathbb{C}^*)^{n+1}$ and $\mathrm{TC}(P) \subset S^n \times (S^1)^{n+1}$ on the same footing, placing them in the same ambient space. The symplectic geometry of $(\mathbb{C}^*)^{n+1}$ suggests another source of flexibility. In the coordinates (y, θ) on $\mathbb{R}^{n+1} \times (S^1)^{n+1}$, we have a symplectic form $dy \wedge d\theta$, and Z can be identified with a "Hamiltonian reduction"[5] of $(\mathbb{C}^*)^{n+1}$ by the function $\mathrm{Re}(f)$. Put another way, there is a homeomorphism $Z \times \mathbb{R} \cong \mathrm{Re}(f)^{-1}(x)$. The PI will show that, for a suitably chosen f (the appropriate coefficients will be guided by (2), here) and for $x \gg 0$, the skeleton $\mathrm{TC}(P)$ embeds into $\mathrm{Re}(f)^{-1}(x)$.
4. (Refined conjecture) The conjecture is definitely false when the faces of P are not simplices. A closely related problem in mirror symmetry was solved by Batyrev and Borisov a long time ago [Ba, Bo]: the singularities of an incorrect mirror may be resolved to produce a correct mirror. Similar corrections may be made to $\mathrm{TC}(P)$, which the PI will pin down in the early stages of the project, since a better understanding of the ultimate scope of the conjecture should give at least some indication of what methods can be used to prove it.

2.2 The Kashiwara-Schapira sheaf of categories

The PI will use microlocal sheaf theory to construct a **sheaf of dg categories** on a large class of **singular Lagrangians**, and on **singular Legendrians** in cooriented contact manifolds. In the Lagrangian case the motivation for this project comes from some unpublished ideas of Ruan and Kontsevich. The *Lagrangian skeleton* of an exact symplectic manifold $(M, \omega = d\alpha)$ is the vanishing locus of its primitive α. Under standard tameness assumptions, the Liouville flow exhibits this skeleton as a deformation retract of M. Wei Dong Ruan [R] suggested that the Fukaya category of

[4]The fibers are possibly disconnected. More precisely, the fibers are of the form (torus) $\times$ (finite group)

[5]Here the quotation marks indicate that the Hamiltonian vector field $\mathrm{grad}(\mathrm{Im}(f))$ does not generate a compact group

M could be computed from a Lagrangian skeleton, and Kontsevich [K2] went further and suggested that the Fukaya category should have a local nature on Λ. In other words there should be a **sheaf of Fukaya categories** on Λ whose category of global objects is $\mathrm{Fuk}(M,\omega)$. Though the sheaf of categories envisioned by Kontsevich has not yet been rigorously constructed[6], it is natural to expect it to be equivalent to PI's proposed constructible sheaf version.

Let Y be a manifold, and let $Sh_c(Y)$ denote the dg triangulated category of constructible sheaves on Y. (Technically, we must endow Y with a real analytic structure, and consider sheaves that are constructible with respect to some subanalytic stratification of Y.) To each object $F \in Sh_c(Y)$ we can attach $SS(F)$, the singular support of F. It is a Lagrangian subset, usually singular, of T^*Y that is stable under positive dilations of the cotangent fibers, i.e. it is a **conic Lagrangian**. We may fix a conic Lagrangian Λ and consider the full subcategory of $Sh(Y)$ whose objects have $SS(F) \subset \Lambda$, call it $Sh_c(Y;\Lambda)$. These notions $SS(F)$ and $Sh_c(Y;\Lambda)$ are also discussed in Section 1.1.1.

$Sh_c(Y;\Lambda)$ is a triangulated dg category equivalent (by [NZ, N1]) to a Fukaya category of Lagrangian branes $L \subset T^*Y$ that are asymptotic to Λ. (If Λ contains the zero section, we may rephrase this as say that $L^\infty \subset \Lambda^\infty$ at contact infinity.) After the Ruan-Kontsevich proposal, we should expect $\mathrm{Fuk}(T^*Y;\Lambda)$ to be the global sections of a sheaf of Fukaya categories on T^*Y, supported on Λ. The PI will construct a sheaf of categories on Λ by different means, i.e. constructible sheaf theory.

The construction is a variant of the theory of [KS, Chapter VI], which gives a presheaf of categories. If $\Omega \subset T^*Y$ is an open subset let Ω^c denote the closed complement. Kashiwara and Schapira define the category of "microlocal sheaves on Ω" to be the quotient $Sh(Y)/Sh(Y;\Omega^c)$, i.e. the localization of $Sh(Y)$ with respect to sheaves whose singular support does not meet Ω, let us denote this category by $MSh(\Omega)$. We may regard the assignment $\Omega \mapsto MSh(\Omega)$ as a contravariant functor from the poset of open subsets of T^*Y to the ∞-category of dg categories.

For $\Lambda \subset T^*Y$ a conic Lagrangian we can define a smaller variant of MSh, called MSh_Λ:

$$MSh_\Lambda(\Omega) := Sh(Y;\Lambda \cup \Omega^c)/Sh(Y;\Omega^c)$$

This again is a presheaf of dg categories on T^*Y. It is supported on Λ in the sense that $MSh_\Lambda(\Omega)$ is the zero category whenever Ω does not meet Λ.

Example 2.5. If $Y = \mathbb{R}$ and $\Omega \subset T^*\mathbb{R} \cong \mathbb{R}^2$ is the upper half-plane, then $MSh_\Lambda(\Omega)$ is equivalent to the category of representations of the quiver

$$\bullet \xrightarrow{h} \bullet \xrightarrow{h} \cdots \xrightarrow{h} \bullet$$

with as many nodes as there are components of $\Lambda \cap \Omega$. The objects are represented in $Sh(Y;\Lambda\cup\Omega^c)$ by constant sheaves on half-infinite rays $(-\infty,t)$.

Definition 2.6. The **Kashiwara-Schapira sheaf** attached to a conic Lagrangian $\Lambda \subset T^*Y$ is the sheafification of MSh_Λ, we will denote it by $MSh^\dagger_\Lambda$.

Example 2.7. In the example 2.5, the category $MSh^\dagger_\Lambda(\Omega)$ is just a semisimple category with as many simple generators as there are components of $\Lambda\cap\Omega$. In general it is sometimes useful to think

[6] Actually, there is rapid progress. For instance see the announcement [N2].

of $MSh_\Lambda(\Omega)$ as a deformation (which is not local over Ω) of $MSh^\dagger_\Lambda(\Omega)$, so that in this example the parameter of the deformation acts by h. A similar idea has been advocated by Tamarkin [Ta, GKS]. See the PI's project 2.3.1 for more on this point of view.

Remark 2.8. A certain amount of homotopical care has to be taken to construct this object correctly. For instance, $Sh(Y)$ needs to be defined as a dg triangulated category, not a classical triangulated category, and constructing the localization $MSh(\Omega) = Sh(Y)/Sh(Y;\Omega^c)$ is slightly more complicated than classical Verdier localization. The definition of a sheaf of dg categories, and the construction of one via sheafification, requires a certain amount of higher category theory. All of these issues are easily addressed since the publication of [L], but it is possible that the lack of a similar reference in the 90s accounts for why the sheafification of MSh_Λ has not been considered before.

2.2.1 Outline.

We now outline the PI's program to study $MSh^\dagger_\Lambda$. Parts (1) and (2) indicate way to effectively work with and compute this abstractly-defined sheaf of categories. Part (3) indicates a variant of $MSh^\dagger$ which exists on general **singular Legendrian** subsets of cooriented contact manifolds.

1. (Microlocal monodromy and brane obstructions) The PI will show that the sheaf of categories $MSh^\dagger_\Lambda$ is locally constant along the smooth locus $\Lambda^{\mathrm{sm}} \subset \Lambda$, using the **microlocal stalk** functors of Kashiwara and Schapira. This local system can be trivial or not. If there is a constructible sheaf F with $SS(F) = \Lambda$, then $MSh^\dagger_\Lambda$ is trivial along Λ^{sm}, and F determines a section of $MSh^\dagger$ over Λ^{sm} which encodes the **microlocal monodromy** attached to F. This is a familiar construction in microlocal sheaf theory, especially common in the theory of perverse sheaves. The nontriviality of $MSh^\dagger_\Lambda$ over Λ^{sm} is closely related to grading and Pin obstrcutions, i.e. **brane obstructions** in Fukaya theory [S3], the obstructions that must vanish for a smooth Lagrangian to define an object of the Fukaya category.
2. (Conormal extension) The PI will develop a technique for understanding $MSh^\dagger_\Lambda$ inductively, in case that some of the components of $\Lambda \subset T^*Y$ look like they "come from a submanifold." Let $Y' \subset Y$ be a closed submanifold, and let $i : Y' \hookrightarrow Y$ denote the inclusion. There is a short exact sequence of vector bundles on Y'

   $$0 \to T^*_{Y'}Y \to T^*Y|_Y \to T^*Y' \to 0$$

   The middle term of this sequence as a closed subset of T^*Y, and if Λ is a conic Lagrangian in T^*Y' then $\pi^{-1}(\Lambda)$ is a conic Lagrangian in T^*Y, call it the **conormal extension**. The functor $i_* : Sh(Y') \to Sh(Y)$ is an equivalence of $Sh(Y';\Lambda)$ onto $Sh(Y;\pi^{-1}(\Lambda))$. The PI will show that i_* induces an equivalence of Kashiwara-Schapira sheaves

   $$\pi^{-1} MSh^\dagger_\Lambda \xrightarrow{i_*} MSh_{\pi^{-1}(\Lambda)}$$

   If σ° is a bundle of open convex cones in $T^*_{Y'}Y$, then $\sigma^\circ \cap \pi^{-1}(\Lambda)$ is a "partial conormal extension," and the equivalence above restricts to an identification

   $$MSh^\dagger_{\pi^{-1}(\Lambda)}(\Omega) \cong Sh(Y';\Lambda)$$

3. (Legendrian version) The cotangent bundle T^*Y has an associated cooriented contact manifold $T^\infty Y = (T^*Y - Y)/\mathbb{R}_{>0}$, let us call it the **cosphere bundle**. A conic Lagrangian $\Lambda \subset T^*Y$ determines a Legendrian subset $\Lambda^\infty \subset T^\infty Y$, and conversely an arbitrary (subanalytic) Legendrian subset of T^*Y determines a conic Lagrangian in T^*Y by taking the cone. The sheaf $MSh_\Lambda^\dagger$ is constant along the $\mathbb{R}_{>0}$-orbits, and so it induces a sheaf on Λ^∞, which we we denote by $MSh^\dagger_{\Lambda^\infty}$.

   Suppose M is a cooriented contact manifold and $\lambda \subset M$ is a Legendrian subset. Then M looks locally like an open subset of T^*Y and λ looks locally like Λ^∞. The PI will show that when M is equipped with a contact grading (a contact version of Seidel's notion of a graded symplectic manifold [S3]), the sheaves $MSh^\dagger_{\Lambda^\infty}$ glue together to form a sheaf of dg categories $MSh^\dagger_\lambda$ supported on λ.

2.3 Mirror symmetry at the limit

Using the Kashiwara-Schapira sheaf, the PI will construct a sheaf of categories on $\mathrm{TC}(P)$, the conjectural skeleton of an affine hypersurface in $(\mathbb{C}^*)^{n+1}$. This category should play the role of the predicted sheaf of Fukaya categories, in particular its global category should be equivalent to the Fukaya category of the hypersurface. This affine hypersurface has a mirror, and the PI will show that **the global category is equivalent to a category of coherent sheaves on the mirror**.

Let X be a toric variety, for simplicity we will assume X is smooth. Let D denote the toric anticanonical divisor of X, this is a reduced subscheme of X whose irreducible components are indexed by the faces of the moment polytope of X. D is an important space in mirror symmetry, for instance if the moment polytope of X is Fano then D is the **large complex structure limit point** of a family of Calabi-Yau hypersurfaces in X. If the moment polytope P attached to D is moreover reflexive, then the mirror partner to D is the **large volume limit** of the mirror family, which in this case can be realized as an affine hypersurface in $(\mathbb{C}^*)^n$—precisely the hypersurface whose Newton polytope is P. Let us denote this hypersurface by Z.

A deformation retract of Z is proposed in Section 2.1, the space $\mathrm{TC}(P)$. It is reasonable to furthermore conjecture that $\mathrm{TC}(P)$ can be realized as a Lagrangian skeleton of Z—the attack we have outlined in Section 2.1(3) would imply this. According to Kontsevich, it should therefore carry a sheaf of Fukaya categories. The PI will show that this sheaf can be constructed and studied in advance: though it is difficult to realize $\mathrm{TC}(P)$ as a Lagrangian subset of Z, **it is easy to realize it as a Legendrian subset** of $S^n \times (S^1)^{n+1}$ with its natural contact form.

Specifically, under the identification $S^n \times (S^1)^{n+1} \cong T^\infty(S^1)^{n+1}$, the set $\mathrm{TC}(P)$ coincides with the Legendrian boundary of one of the conic Lagrangians Λ_Σ appearing in the CCC. We may endow Λ_Σ and $\Lambda_\Sigma^\infty \cong \mathrm{TC}(P)$ with the Kashiwara-Schapira sheaf of Sections 2.2 and 2.2.1(3). Project 2.3.2(1) below describes the counterpart of this sheaf structure in terms of the CCC, i.e. how the CCC localizes over Λ_Σ.

2.3.1 Mirror symmetry near the limit

Let us recall Seidel's strategy for establishing homological mirror symmetry, developed in his proof of HMS for the quartic surface, and in Sheridan's thesis on HMS for more general projective hypersurfaces. Smooth and projective Calabi-Yau hypersurfaces arise as deformations D_t of D inside X, and there is a sense in which the derived category of coherent sheaves on D_t is a deformation of the category of coherent sheaves (or perfect complexes) on D. On the other hand a symplectic compactification of Z gives rise to a formal deformation of $\mathrm{Fuk}(Z)$ over the Novikov ring—morally the Fukaya category of the compactification has the same objects and morphisms as $\mathrm{Fuk}(Z)$, but the composition law has been deformed by contributions of holomorphic disks that extend to infinity. Once one has established mirror symmetry "at the limit" (i.e. that $\mathrm{Fuk}(Z) \cong \mathcal{P}\mathrm{erf}(D)$) one can look for arguments that the deformations coincide.

Somewhat surprisingly, this deformation is visible in terms of microlocal sheaf theory and the CCC, though the role of holomorphic disks is unclear. Let us write $\mathrm{Def}(D) \to \mathbb{C}$ for the total space of the deformation D, and $\widehat{\mathrm{Def}}(D) \to \mathrm{Spec}(\mathbb{C}[[t]])$ for the formal completion of it along the special fiber D. Then there is a map $\mathrm{Def}(D) \to X$ that carries $\widehat{\mathrm{Def}}(D)$ to the formal completion of D in X. That latter map is not an isomorphism but it is close—$\mathrm{Def}(D)$ can be realized as a blowup of the formal neighborhood along a codimension 2 locus in X. In other words, $\mathrm{Def}(D)$ is obtained by blowing up the base locus in a pencil of divisors on X. The PI will show that the formal neighborhood of D in X is directly visible in the CCC, as sections of the Kashiwara-Schapira *pre*sheaf over Λ_Σ^∞. By inverting the parameter t in $\mathbb{C}[[t]]$, we obtain an equivalence with the category of coherent sheaves on a smooth projective Calabi-Yau variety over the field of Laurent series.

2.3.2 Outline.

We now outline the project in more detail. Let X be a toric variety with fan Σ. Let D be the anticanonical divisor. Below we assume X is smooth and for simplicity non-stacky, though it will be important to develop the project for smooth toric orbifolds.

1. (Understanding the Kashiwara-Schapira sheaf on Λ_Σ). The PI will make a detailed study of the Kashiwara-Schapira sheaf (Section 2.2) on the conic Lagrangians Λ_Σ that occur in the CCC (Section 1.1.1). Suppose Λ is a conic Lagrangian in a cotangent bundle. For each open subset $\Omega \subset \Lambda$, the Kashiwara-Schapira sheaf provides a category $MSh_\Lambda^\dagger(\Omega)$, and this category is the target of a "microlocal restriction" functor from the global category of sheaves with singular support in Λ. The PI will show that, when $\Lambda = \Lambda_\Sigma \subset T^*((S^1)^{n+1})$ is the conic Lagrangian attached to a fan Σ as in Section 1.1.1, some of these restriction maps have a coherent counterpart under the CCC equivalence $\mathcal{P}\mathrm{erf}(X_\Sigma) \cong Sh_c((S^1)^{n+1};\Lambda_\Sigma)$. Each cone $\sigma \subset \Sigma$ determines an open subset $\Omega(\sigma) \subset \Lambda_\Sigma$ by setting
$$\Omega(\sigma) = \bigcup_{\tau \supset \sigma} \tau^\perp \times (-\tau^\circ)$$
This is an instance of "conormal extension" from Section 2.2(2). The PI will show that the category of sections of $MSh_\Lambda^\dagger$ over $\Omega(\sigma)$ is equivalent to the category of perfect complexes on the toric subvariety of X corresponding to σ. (In [Tr2], the PI established a special case of this (for top-dimensional σ) for the purpose of identifying the constructible counterparts of vector bundles and nef vector bundles under the CCC).
2. (Gluing the categories) The PI will compute sections of the Kashiwara-Schapira sheaf on the Legendrian Λ_Σ^∞, by appealing to an open cover where all the charts and their intersections are of the form $\Omega(\sigma)^\infty$. Since $MSh^\dagger(\Omega(\sigma)^\infty) \cong \mathcal{P}\mathrm{erf}(X_\sigma)$, we have
$$MSh^\dagger_{\Lambda_\Sigma^\infty} \cong \varprojlim \mathcal{P}\mathrm{erf}(X_\sigma)$$

where the limit is taken over the partially ordered set of nonzero cones in Σ. The same poset indexes components of the divisor D and their intersections, so the universal property gives a functor $\mathcal{P}\mathrm{erf}(D) \to \varprojlim \mathcal{P}\mathrm{erf}(X_\sigma)$. The PI will show that, since the components of D are smooth and meet each other in normal crossings, this functor is an equivalence. A similar result holds for general "TCNC" varieties, i.e. reducible varieties with toric components that meet in normal crossings. Granting that $MSh^\dagger_\Lambda$ is a good model for the Fukaya category, **this will establish a form of HMS at the limit** for the Batyrev-Borisov hypersurfaces.

3. (The formal neighborhood of D) If $\Lambda \subset T^*Y$ is a conic Lagrangian, the quotient category

$$Sh_c(Y;\Lambda)/\{\text{locally constant sheaves}\}$$

is much-studied in microlocal sheaf theory. It is exactly the category of sections of the Kashiwara-Schapira *presheaf* over the complement of the zero section in Λ, or equivalently on the Legendrian Λ^∞. Formal properties of sheafification produce a functor

$$\varsigma : \Gamma(MSh_{\Lambda^\infty}) \to \Gamma(MSh^\dagger_{\Lambda^\infty})$$

In case $\Lambda = \Lambda_\Sigma$, the PI has discussed in (2) an approach to identifying the latter category with $\mathcal{P}\mathrm{erf}(D)$. The PI will prove that **this functor is mirror to the restriction functor** $i^* : \mathrm{Coh}(\widehat{X}_D) \to \mathcal{P}\mathrm{erf}(D)$. The identification of $MSh_{\Lambda^\infty_\Sigma}$ with $\mathrm{Coh}(\widehat{X}_D)$ will follow from results of [FLTZ1]: by definition, this category is the quotient of $Sh_c((S^1)^n;\Lambda_\Sigma)$ by locally constant sheaves, and the CCC carries the latter category to skyscraper coherent sheaves supported on the open orbit in T (a coherent such sheaf must have zero-dimensional support).

It will follow that the functor ς has a left adjoint. The composition of these is a **comonad** on $\Gamma(MSh^\dagger_{\Lambda^\infty_\Sigma})$. The PI will show that all such categories $\Gamma(MSh^\dagger_\Lambda)$ admit a similar comonad, and that coalgebras over this comonad recover the unsheafified version $\Gamma(MSh_\Lambda)$ by Lurie's Barr-Beck theorem [L2].

4. (From the formal neighborhood to the deformation) A standard result on derived categories of coherent sheaves [BO] is that pullback along a proper birational map induces a full embedding (and moreover a semiorthogonal decomposition of the target category). These results apply in the formal scheme setting for the map $\widehat{\mathrm{Def}}(D) \to \widehat{X}_D$. The projects discussed so far would indicate that **a large part of the category of coherent sheaves on $\widehat{X}_D$ can be recovered from microlocal sheaf techniques**, in particular we have a full embedding

$$MSh_{\Lambda^\infty_\Sigma} \hookrightarrow \widehat{\mathrm{Def}}(D)$$

If F and G are coherent sheaves on $\widehat{\mathrm{Def}}(D)$, then $\mathrm{Hom}(F,G)$ has a $\mathbb{C}[[t]]$-module structure, in particular $MSh_{\Lambda^\infty_\Sigma}$ carries a similar $\mathbb{C}[[t]]$-linear structure. The PI will show that by extending coefficients to $\mathbb{C}((t))$, the category of coherent sheaves on a Calabi-Yau variety defined over $\mathbb{C}((t))$ is recovered.

3 Broader impact

The PI's research is disseminated to a wide audience by publication in respected journals and by giving talks in seminars and at conferences.

The proposal draws on many microlocal themes which have been influential in symplectic topology recently, such as Nadler's results on Arnold's nearby Lagrangian problem [N1], Tamarkin's proof of Arnold's nondisplacability conjecture [Ta, GKS], and the PI's proof with Fang, Liu, and Zaslow of homological mirror symmetry for toric varieties [FLTZ1, FLTZ2].

The PI is a young researcher at the end of his postdoctoral career; the PI and the grant will start at a new institution in 2012. The PI has a demonstrated facility for mentoring and collaborating with graduate students. His coauthor Bohan Fang, now a Ritt assistant professor at Columbia University, was a student of Eric Zaslow's when the papers [FLTZ1, FLTZ2] were conceived, and his coathour Nicoló Sibilla [STZ1] is a current Zaslow student. In Fall of 2010 he had many discussions with Nadler's student current Ted Stadnik about his research, which led to the PI writing a letter of recommendation for Stadnik's NSF postdoctoral fellowship application, and he is currently working closely with Nadler's student Sam Gunningham on spin TQFTs. He gave a series of lectures on graphs and their eigenvalues at Northwestern University's first *undergraduate* research conference of 2010, an experimental format for summer undergraduate research programs. The PI will continue to foster graduate and undergraduate research at his new institution.