Project description — Lagrangian surfaces, Legendrian knots, and the microlocal theory of sheaves

PI Treumann proposes to employ constructible sheaf techniques in the study of symplectic geometry, especially in low dimensions. The application of constructible sheaves to symplectic geometry is a major theme of the PI's recent works on mirror symmetry, and many of these themes and tools will be applied in the proposed projects. The results obtained from prior support, reviewed in §1.1, will be explained in some detail as background for the proposal. In §2, themes for a several-year program of study on Legendrian knots and Lagrangian surfaces are described, with background given in §2.1 and detailed proposals in §2.2–2.4. Broader impacts of previous support are described in §3.1, and broader impacts of the proposal are described in §3.2.

1 Results from prior NSF support

Treumann's prior NSF support from the past five years comes from the following award:

- NSF-DMS-1206520 "Mirror symmetry and the microlocal theory of sheaves" ($105,479, 09/01/12–08/31/15)

The broader impact of the prior award is discussed in §3.1. Here we discuss the intellectual merit of the prior award.

1.1 Intellectual merit of the prior award

The PI's work during the first two years of the supported period resulted in the following papers

(1) H. Ruddat, N. Sibilla, D. Treumann and E. Zaslow, "Skeleta of affine hypersurfaces," Geometry and Topology 2014.

(2) V. Shende, D. Treumann and E. Zaslow, "Legendrian knots and constructible sheaves," 2014 preprint available at arXiv:1403.0490

(3) D. Treumann and A. Venkatesh, "Functoriality, Smith theory, and the Brauer homomorphism," 2014 preprint available at arXiv:1407.2346.

The paper most relevant to the current proposal is (2), but many of the same tools, constructions, and themes arise in paper (1) and its antecedents (prior to NSF support) . We will explain some of these in some detail in §1.1.1 and §1.1.2. The content of paper (3) is briefly explained in §1.1.3, with further details omitted as not relevant to the proposal.

1.1.1 Mirror symmetry and skeleta of affine hypersurfaces

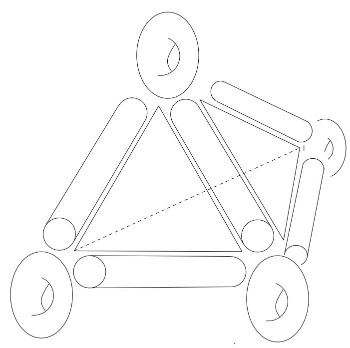
Paper (1) is the most recent paper in a series (begun prior to NSF support) by the PI et al to apply constructible sheaf techniques to **homological mirror symmetry**, especially at the **large volume/large complex structure limit**. Other papers in this series are [PI-FLTZ1, PI-FLTZ2, PI-SiTZ, PI-T3].

Paper (1) answers a very natural and concrete question in algebraic geometry, which we explain before turning to mirror symmetry. Rene Thom's classic Morse-theoretic proof of the Lefschetz hyperplane theorem [AF] shows that a smooth affine variety of complex dimension n has the homotopy type of a cell complex of real dimension n. If it arises as a deformation retract, let us call such an n-dimensional cell complex a **skeleton** of the affine variety Y_0. Thom's procedure is very general but not very explicit. The question addressed by (1) is to find such a skeleton, by a combinatorial recipe, for the simplest affine varieties: hypersurfaces in $\mathbf{C}^{n+1}$ and $(\mathbf{C}^*)^{n+1}$, cut out by a single equation $\{f = 0\}$. The PI et al gave an answer in terms of the **Newton polytope** Δ of the (Laurent) polynomial f.

Example 1.1. Let $Y_0 \subset (\mathbf{C}^*)^3$ be the affine quartic surface cut out by the equation

$$f(x, y, z) = x + y + z + \frac{1}{xyz} = 1$$

The Newton polytope (the convex hull of those vectors in $\mathbf{Z}^3$ corresponding to exponents of monomials appearing in f) is a tetrahedron Δ. Then Y_0 deformation retracts onto a space obtained by gluing together four tori, six cylinders, and 4 triangles together in the shape of $\partial\Delta$, as roughly[1] indicated in the following picture:



Let us state precisely a fairly general special case of the main result of [PI-RSTZ]:

Theorem 1.2 (RSTZ). *Let $\Delta \subset \mathbf{R}^{n+1}$ be a lattice polytope containing 0 in the interior of its convex hull. Let $\mathcal{T}$ be a coherent[2] triangulation of the boundary of Δ. If f is a sufficiently generic Laurent polynomial whose Newton polytope is Δ, then the hypersurface in $(\mathbf{C}^*)^{n+1}$ cut out by f has the homotopy type of the following subset of $\partial\Delta \times \mathrm{Hom}(\mathbf{Z}^{n+1}, S^1) \cong S^n \times (S^1)^{n+1}$:*

$$\bigcup_{\tau \in \mathcal{T}} \{(x, \phi) \mid x \in \tau \text{ and } \phi(v) = 1 \text{ whenever } v \text{ is a vertex of } \tau\} \tag{1}$$

[1] the result does not actually fit into $\mathbf{R}^3$ so well.

[2] we omit the definition, but every Δ admits at least one coherent triangulation

The skeleton (1) had appeared previously in the PI's work on the **coherent-constructible correspondence** (CCC) [PI-FLTZ1]. It is the **Legendrian boundary** (at contact infinity) of a **conic Lagrangian** in $T^*(S^1)^{n+1}$, that gives a **microlocal condition** (singular support, in the sense of Kashiwara and Schapira [KS]) that cuts out a full subcategory of the derived category of constructible sheaves on $(S^1)^{n+1}$. The coherent constructible correspondence asserts that this category of constructible sheaves is in turn equivalent to the derived category of coherent sheaves on a certain toric variety.

Let us explain the microlocal study of sheaves [KS] in terms of a fairly recent development, the Fukaya-theoretic picture of [NZ]. The **unwrapped Fukaya category** of the cotangent bundle T^*M of a real analytic manifold M is introduced in [NZ], and a quasi-equivalence, (called **microlocalization**) between the A_∞-category $\mathrm{Fuk}(T^*M)$ and the dg category of constructible sheaves on M — let us call it $Sh_c(M)$ — was established in [NZ, N]. Crucially, the objects of $\mathrm{Fuk}(T^*M)$ are allowed to be noncompact, so long as they are under some control at infinity. The basic objects of $\mathrm{Fuk}(T^*M)$ are the **standard** branes: if f is a nonnegative smooth function on M that vanishes outside of an open set $U \subset M$, then the graph of $d\log(f)$ defines an object of $\mathrm{Fuk}(T^*M)$, which under microlocalization goes tot he constant sheaf on the closure of U — also called the **standard** sheaf attached to U. These standard objects generate the respective categories — in $Sh_c(M)$, this is true almost by definition, and in $\mathrm{Fuk}(T^*M)$ is a theorem of Nadler's.

In microlocal sheaf theory, a fundamental role is played by **conic Lagrangians**, these are singular subsets of T^*M that are Lagrangian along their smooth locus, and stable by positive dilations in the cotangent fibers. Given a conic Lagrangian $\Lambda \subset T^*M$, Kashiwara and Schapira construct a tirnagulted subcategory of $Sh_c(M)$ given by sheaves whose **singular support** belongs to Λ. Roughly speaking, for each covector ξ there is a Morse-theoretic test to see if a sheaf F changes in the direction of ξ, and $SS(F)$ is defined to be the set of those covectors that do detect a change. If a Lagrangian is asymptotic to Λ, then the corresponding constructible sheaf will have singular support in Λ.

A conic Lagrangian $\Lambda \subset T^*M$ is essentially determined by its "Legendrian boundary" $\Lambda^\infty \subset T^\infty M$ by taking the cone and the union with the zero section. Here $T^\infty M$, resp. Λ^∞ denotes the quotient $(T^*M - \{\text{zero section}\})/\mathbf{R}_{>0}$, resp. $(\Lambda - \{\text{zero section}\})/\mathbf{R}_{>0}$. I.e. the "contact sphere at large radius" in T^*M, and its intersection with Λ. In that case objects of $\mathrm{Fuk}(T^*M)$ are given by Lagrangian submanifolds of T^*M that "end on Λ^∞." For the class of Legendrians arising in Theorem 1.2, the CCC gives a **mirror geometry** for this symplectic problem.

1.1.2 Legendrian knots and constructible sheaves

In paper (2), the PI et al showed that microlocal sheaf theory gives a powerful Legendrian isotopy invariant of **Legendrian knots** in the standard contact $\mathbf{R}^3$ and other simple contact manifolds. If λ is a Legendrian knot, the invariant $\mathcal{C}(\lambda)$ is the category of constructible sheaves on $\mathbf{R}^2$ with singular support in the cone over λ. The knot λ lives at contact infinity, i.e. in $T^\infty \mathbf{R}^2 \cong S^1 \times \mathbf{R}^2$, which contains a contactomorphic copy of $\mathbf{R}^3$ as $(\pi, 2\pi) \times \mathbf{R}^2$. The projection of λ to $\mathbf{R}^2$ is called the **front projection** of λ, it typically looks like this:

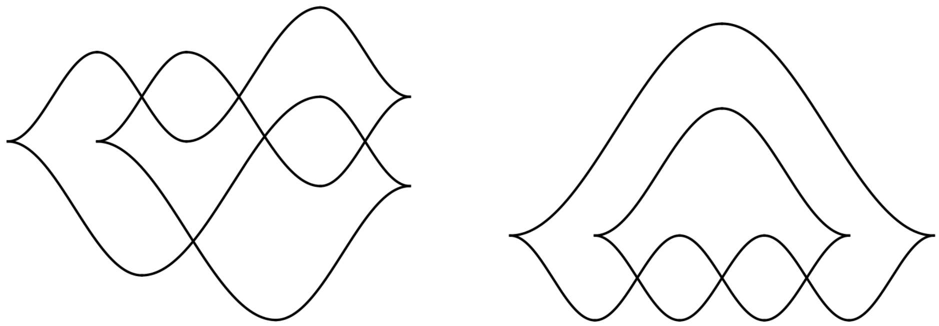


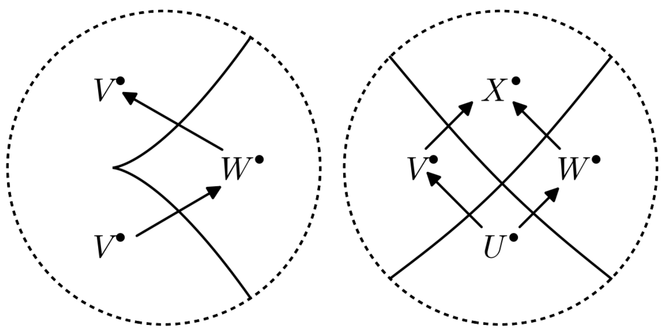
Figure 1: Left, the Legendrian figure-eight knot. Right, the Legendrian trefoil.

The front diagram $\Phi \subset \mathbf{R}^2$ determines λ, see Remark 2.2 for further discussion. The category $\mathcal{C}(\lambda)$ can be computed from Φ by fairly combinatorial means. The front diagram cuts $\mathbf{R}^2$ into regions $R \subset \mathbf{R}^2$, and an object of $\mathcal{C}(\lambda)$ is given by the following data:

(I) a chain complex of vector spaces $F^\bullet(R)$

(II) a chain map $F^\bullet(R_1) \to F^\bullet(R_2)$ whenever R_1 and R_2 are adjacent and separated by a strand of Φ, with R_1 below and R_2 above.

The maps of (II) are subject to certain local conditions near each cusp and crossing:



The condition on a cusp is that the composite $V^\bullet \to W^\bullet \to V^\bullet$ is the identity, i.e. the chain complex labeling the outside of a cusp is a summand of the chain complex labeling the inside of a cusp. The condition on a crossing is that the diagram commutes and the sequence of chain complexes $0 \to S^\bullet \to E^\bullet \oplus W^\bullet \to N^\bullet \to 0$ is exact.

We sometimes additionally impose a third **microlocal rank one** condition

(III) the cone on any map in (II) is quasi-isomorphic to vector space of dimension one placed in a single cohomological degree.

The full subcategory of $\mathcal{C}(\lambda)$ of objects obeying this condition is denoted $\mathcal{M}_1(\lambda)$.

Among the results about $\mathcal{C}(\lambda)$ and $\mathcal{M}_1(\lambda)$ obtained by the PI et al, are that it is a Legendrian isotopy invariant, that it can distinguish the famous **Chekanov pair** of Legendrian knots with the same classical invariants, and that it can be used to recover the triply-graded **HOMFLY homology** of a braid positive topological knot.

1.1.3 Geometric and arithmetic applications of Smith theory

Paper (3) is the least relevant to the current proposal, so we describe it last and very briefly. It is part of a long-term project (begun prior to NSF support) to apply "Smith theory" — a technique for relating the mod p cohomologies of spaces X and Y when Y can be realized as the fixed points of a $\mathbf{Z}/p$-action on X — to problems in modular representation theory, number theory, and low-dimensional topology. Other papers by the PI in the same family are [PI-T1, PI-T2, PI-LT].

In paper (3) above the PI and Venkatesh use Smith theory to prove many new instances of **Langlands functoriality** for mod p cohomological **automorphic forms.** For example the technique gives base change along Galois extensions of p-power order for mod p automorphic forms on any group, and an **exotic transfer** from mod 2 forms on $\mathrm{Sp}(2n)$ to $\mathrm{GL}(2n)$ (which is functoriality for an L-group homomorphism $\mathrm{SO}(2n+1) \to \mathrm{GL}(2n)$ that exists *only in characteristic two*).

2 Proposed projects

The PI proposes to study Lagrangian surfaces in (relatively simple) symplectic 4-manifolds, such as $\mathbf{R}^4$ or the cotangent bundle of a surface. The results of [PI-STZ] suggest a framework for studying them involving disparate topics — sometimes requiring new tools — in algebraic and symplectic geometry, such as nonabelian Hodge theory [Co, D, Hi, Si], irregular connections [BiqBoa, Boa], n-shifted symplectic geometry [PTTV], and cluster algebras [FZ1, FZ2, GHK].

To motivate the project, we have formulated a concrete but so far unsolved symplectic problem (on fillings of Legendrian knots, given in Conjecture 2.1 and §2.1) whose solution should be a consequence of these ideas. The proposal is presented as a plan of attack on this problem, with the applications and connections to other areas discussed alongside, and in greater detail in the Outline section §2.4.

2.0.4 Conjecture on exact Lagrangian fillings

The PI will **enumerate exact Lagrangian fillings of Legendrian knots**. The line of investigation outlined here can be carried out for general Legendrian knots, but in particular the PI will show the following:

Conjecture 2.1 (Treumann). *If $\lambda \subset S^3$ is a* **Legendrian torus knot** *of type (a, b), then*

1. *If $(a,b) = (2,n)$ or $(n,2)$, there are precisely $\binom{2n}{n}/(n+1)$ distinct exact Lagrangian fillings, up to Hamiltonian isotopy*[3].

2. *If $(a,b) = (3,4)$ or $(4,3)$, there are* **precisely 833** *distinct exact Lagrangian fillings, up to Hamiltonian isotopy. If $(a,b) = (3,5)$ or $(5,3)$, there are* **precisely 25080** *distinct exact Lagrangian fillings up to Hamiltonian isotopy.*

3. *Any other Legendrian torus knot has* **infinitely many distinct Lagrangian fillings**, *up to Hamiltonian isotopy.*

Part (1) will answer Questions 8.7 and 8.8 of [EHK], in which the authors prove that the $(2,n)$ torus knot has at least $(2^{n+1}-1)/3$ distinct exact fillings, or approximately the square root of the number conjectured in (1). To the PI's knowledge, the phenomenon predicted in part (3) is new and unexpected in Legendrian knot theory.

2.1 Background on the problem

If $\lambda \subset S^3$ is a knot, let us say that a **filling** of λ is a smoothly embedded oriented surface $\Lambda \subset D^4$, (where D^4 is a 4-dimensional ball with boundary S^3). The smallest possible genus of such a Λ is called the **4-ball genus** or **slice genus** of λ.

In symplectic geometry, it is natural to (1) replace D^4 by $\mathbf{R}^4$ with its standard symplectic structure, and S^3 by the "boundary at ∞" of $\mathbf{R}^4$ with its standard contact structure and (2) require that Λ is a **Lagrangian surface**, and λ is a **Legendrian knot**. In this case (under some additional technical hypotheses to ensure control of the the noncompact Λ) we say that Λ is a Lagrangian filling of λ.

Remark 2.2. Front diagrams are a standard source of Legendrian knots. These are planar pictures of the kind shown in Figure 1 of §1.1.2. Formally, a front diagram is a curve in $\mathbf{R}^2$ that is immersed away from a finite set of *cusps*, where the tangent direction is assumed to be well-defined and horizontal. The front diagram in $\mathbf{R}^2$ lifts to a knot in $\mathbf{R}^3$ by letting the third coordinate measure the slope of the tangent curve — it is clear that any parametrization of such a knot must obey a differential equation (as the third coordinate depends on the the derivatives of the first two coordinates), it turns out to be the condition of being perpendicular to a contact one-form.

If $\Phi \subset \mathbf{R}^2$ denotes a front diagram and λ its associated Legendrian knot, the assignment $\Phi \mapsto \lambda$ is many-to-one, with two front diagrams giving rise to the same knot if they differ by a sequence of Legendrian Reidemeister moves.

There are many interesting Legendrian knots in S^3 but the condition that they admit a Lagrangian filling is very strong. Indeed we have the following theorem of Chantraine [Ch] (see also [BO]): if there exists even a single oriented Lagrangian filling of λ, then (1) the genus of Λ is the 4-ball genus of λ (2) the **rotation number** of λ is zero and (3) the

[3]See remark 2.6

Thurston-Bennequin number of λ is $2g-1$. (2) and (3) are standard discrete invariants of a Legendrian knot ("classical invariants") that one can compute directly from the front diagram — for instance one concludes that the figure eight knot in Figure 1 of §1.1.2 has no Lagrangian filling — its 4-ball genus is one, and its Thurston-Bennequin number is -3.

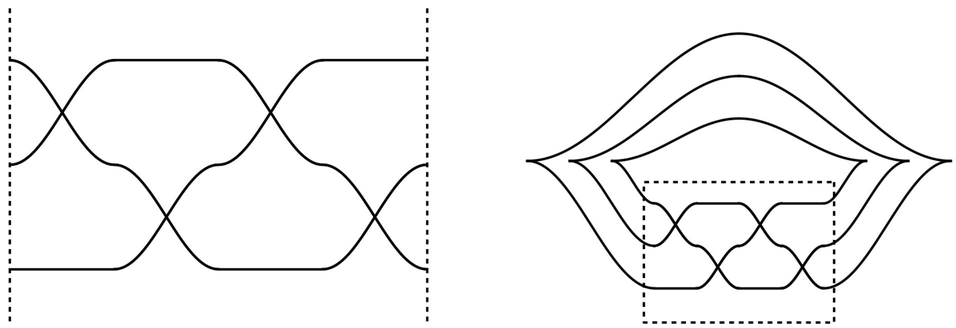


FIGURE 2: Left, a positive braid on 3 strands. Right, its rainbow closure (also, another front projection of the trefoil of figure 1)

Let us therefore restrict our attention to the following class of Legendrian knots λ. A **rainbow projection** is a front diagram obtained by closing up the cusps of a **positive braid**, as in Figure 2. The positive braid is enclosed in the dotted lines, the "rainbow" shape of the rest of the figure gives the name. A recent result [HaS] of Hayden[4] and Sabloff shows that each such knot (and more, see §2.4.5) admits a Lagrangian filling — in fact, an **exact Lagrangian filling**.

If ω denotes the standard symplectic structure on $\mathbf{R}^4$, then a Lagrangian $\Lambda \subset \mathbf{R}^4$ is called exact if for some (equivalently, for any) primitive α of ω, the restriction of α to Λ is exact. In other words, if for any α with $d\alpha = \omega$, there is an $f : \Lambda \to \mathbf{R}$ with $df = \alpha|_\Lambda$. There are many technical advantages to working with exact Lagrangians, not available for more general Lagrangians, but another, big-picture reason to study them is Arnold's **nearby Lagrangian conjecture**. Roughly speaking, this asserts that the class of exact Lagrangians up to Hamiltonian isotopy is similar to the class of general submanifolds up general smooth isotopy: there are no moduli.

More precisely (but ignoring any technical issues arising from the noncompactness of Λ and Λ'), it asserts that if Λ and Λ' are exact Lagrangians and Λ' is **nearby** Λ in the sense of being in a tubular neighborhood of Λ, then Λ and Λ' are Hamiltonian isotopic.

It is natural to ask, how many distinct (pairwise non-Hamiltonian isotopic) exact Lagrangian fillings does a knot have? The PI is aware of only the following results: (0) Many knots have no Lagrangian fillings at all, [BO, Ch] (1) the standard Legendrian unknot has a unique filling (a disk, necessarily exact) and (2) [EHK] the Legendrian $(2,n)$ knot (for n odd) has at least $(2^{n+1}-1)/4$ distinct exact fillings. The proof of (2) given in [EHK] proceeds by first constructing $\binom{2n}{n}/(n+1)$ (approximately 4^n) fillings which are not obvi-

[4] Kyle Hayden is a second-year graduate student at the PI's institution, Boston College. This result was obtained when he was an undergraduate at Haverford.

ously pairwise Hamiltonian isotopic, and showing that a symplectic-field theory invariant (the augmentation) distinguishes about 2^n of them from each other.

Part (1) of Conjecture 2.1 asserts that the list of $\binom{2n}{n}/(n+1)$ fillings is complete and has no repetitions, even if they cannot all be distinguished by the method of [EHK]. We will discuss some reasons to believe it, as well as the more startling parts (2) and (3) of the Conjecture, in the next section.

2.2 Relevance of §1.1.2

To motivate Conjecture 2.1, let us discuss Lagrangian fillings that are **not necessarily exact.** For such Lagrangians, the nearby Lagrangian conjecture fails and is replaced by the following heuristic:

> Let Λ be a Lagrangian filling of λ, not necessarily exact. The set of Lagrangian fillings Λ' close to Λ up to Hamiltonian isotopy is identified with $H^1_c(\Lambda; \mathbf{R}) \cong \mathbf{R}^{2g}$, where g is the genus[5] of Λ.

The heuristic suggests that **the moduli space of Lagrangian fillings of λ, up to Hamiltonian isotopy, is a $2g$-dimensional real manifold.** The heuristic comes from the identification of a tubular neighborhood of Λ with $T^*\Lambda$, and the notion that Λ' should be Hamiltonian isotopic to the graph of a closed one-form on Λ. As we wish to keep the knot λ at the boundary fixed, we take a one-form with compact support[6] in Λ.

Let us denote the set of equivalence classes of Lagrangian fillings of λ by $\mathcal{LF}(\lambda)$ — we are hypothesizing that it is a real manifold of dimension $2g$. We can borrow an idea from mirror symmetry[7] to **complexify** this manifold — study the moduli of pairs (Λ, ∇) where $\Lambda \in \mathcal{LF}(\lambda)$ and ∇ is a flat U(1)-connection on Λ. Denote this hypothetical moduli space by $\mathcal{M}(\lambda)$.

Remark 2.3. It is tempting to call this moduli space the **mirror of the Legendrian knot**, and indeed Conjecture 2.1 is partly inspired by a construction in mirror symmetry [GHK]. If X and Y are a mirror pair of Calabi-Yau manifolds, then homological mirror symmetry [K] (especially in light of the T-duality thesis of [SYZ]) predicts that X is a moduli space of Lagrangian tori in Y. To call $\mathcal{M}(\lambda)$ a mirror to λ stretches the philosophy a little, but it has precedents. For a precedent in physics, see [AV], which considers a "mirror curve" to a *topological* knot built as a moduli of Lagrangian cylinders (copies of $S^1 \times \mathbf{C}$) in T^*S^3 asymptotic to its conormal bundle.

The Nadler-Zaslow theory discussed in §1.1.1 identifies the **Fukaya category of a cotangent bundle** with the category of constructible sheaves on the zero section. One may visualize such a constructible sheaf as a diagram of vector spaces and linear maps drawn on the front diagram of the knot:

[5] we are assuming λ is a knot, with a single connected component, thus $H^1_c(\Lambda) \cong H^1(\Lambda)$.

[6] in general it is preferable to replace this with a suitable asymptotic condition

[7] See Remark 2.3

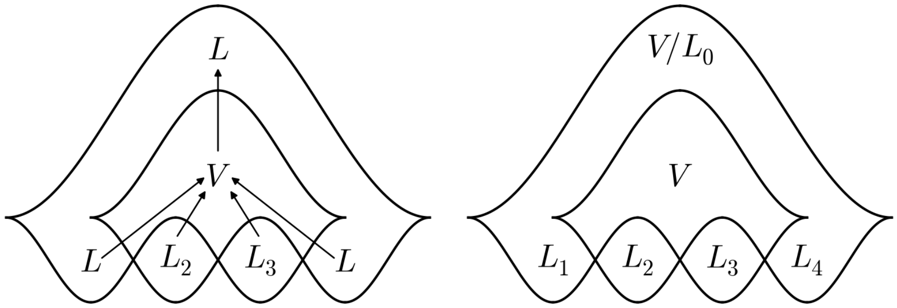


FIGURE 3: Left, a constructible sheaf on $\mathbf{R}^2$ with singular support in the Legendrian trefoil. Right, the same object with less clutter — L_1 and L_4 indicate the images of the two maps $L \to V$, and L_0 indicates the kernel of the map $V \to L$

Remark 2.4 ($\mathcal{M}(\lambda)$ and its stand-in $\mathcal{M}_1(\lambda)$)**.** There is further discussion of constructible sheaves and their microlocal study in §1.1.1, but let us emphasize something here, to explain why we regard $\mathcal{M}_1(\lambda)$ as a good stand-in for the more sketchily-defined $\mathcal{M}(\lambda)$. Given a point $(\Lambda, \nabla) \in \mathcal{M}(\lambda)$, one may attempt to define a point of $\mathcal{M}_1(\lambda)$ by taking for each vector space V_R the (unwrapped) Floer homology (twisted by ∇) of Λ against a cotangent fiber over a point in the region R. The Nadler-Zaslow theorem is essentially an elaboration on this construction; elaborate enough to produce the linear maps in Figure 3, and not just the raw vector spaces. Unfortunately the details for this procedure are only available in the literature (in [NZ]) if Λ is assumed to be exact. (In which case there is some good news to make up for it: we can allow the connection ∇ to be have $\mathbf{C}^*$-valued not just U(1)-valued holonomies. We take advantage of this in §2.3)

For many knots λ, the spaces $\mathcal{M}_1(\lambda)$ one obtains turn out to be intensively studied in different guises:

Example 2.5.

1. If λ is the Legendrian trefoil, then $\mathcal{M}_1(\lambda)$ is the moduli of 5-tuples of lines L_0, L_1, L_2, L_3, L_4 in a two-dimensional vector space V, subject to $L_0 \neq L_1 \neq L_2 \neq L_3 \neq L_4 \neq L_0$, divided by the action of GL(V). This is an affine variety, isomorphic to the complement of an anticanonical divisor (a pentagon of rational curves) in a **degree 5 del Pezzo surface**.

2. More generally, if λ is a Legendrian (p, q)-torus knot, then $\mathcal{M}_1(\lambda)$ is identified with the moduli of $p + q$ ordered points $(x_0, \ldots, x_{p+q-1})$ in $\mathbf{P}^{p-1}$, subject to a similar transversality condition (that $x_i, \ldots, x_{i+p-1}$ are in general position, where the indices $i + j$ are taken mod $p + q$), divided by the action of GL$_p$. This space is also identified with a **Grassmannian cluster variety** (or the largest **open positroid variety**) [Sc, P].

As the name in (2) suggests, these varieties are **cluster varieties**. PI Treumann will show this is true much more generally, and that it can be used to predict, and prove, the number of exact fillings of a knot.

2.3 $\mathcal{M}(\lambda)$ as a complex symplectic manifold and as a cluster variety

Recapping, it is natural to expect that the set of pairs (Λ, ∇), where Λ is a genus g Lagrangian filling and ∇ is a flat U(1)-connection on Λ, taken up to Hamiltonian isotopy, is a complex manifold of complex dimension $2g$. We denote this hypothetical manifold by $\mathcal{M}(\lambda)$, and we regard $\mathcal{M}_1(\lambda)$ constructed in [PI-STZ] to be a good and rigorous stand-in for it.

Let us now engage in a little more informal reasoning about $\mathcal{M}(\lambda)$ — more heuristics:

1. Similar to the discussion at the beginning of §2.2, a neighborhood of a point $(\Lambda, \nabla) \in \mathcal{M}(\lambda)$ is identified with a neighborhood of the origin in $H^1_c(\Lambda, \mathbf{C})$. In particular, the tangent space to (Λ, ∇) is identified with $H^1_c(\Lambda, \mathbf{C}) \cong \mathbf{C}^{2g}$, where g is the genus of Λ. As Λ has a single boundary component, $H^1_c(\Lambda, \mathbf{C})$ is naturally identified with H^1 of a smooth compactification of Λ, and carries an **intersection form**.[8] This yields a nondegenerate two-form, let us take for granted that it is closed.

2. If Λ is an *exact* filling of λ, then one may decorate Λ with a $\mathbf{C}^*$-local system, rather than just a U(1)-local system, and still obtain a point of $\mathcal{M}(\lambda)$. It is natural to expect this construction to be an **algebraic open embedding** of $H^1(\Lambda; \mathbf{C}^*) \cong (\mathbf{C}^*)^{2g}$ into $\mathcal{M}(\lambda)$, and that this inclusion is compatible with the symplectic structures.

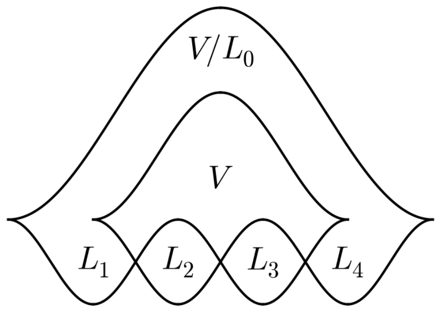
Taken together (1) and (2) indicate that there is a very rich combinatorial structure on $\mathcal{M}(\lambda)$ — it is a **cluster variety**, i.e. its coordinate algebra is a cluster algebra in the sense of [FZ1]. This is the mechanism for Conjecture 2.1: the PI will show that exact Lagrangians are in bijection with certain special open subsets of $\mathcal{M}(\lambda)$ (via (2) above), which in turn are in bijection with **clusters**, i.e. with vertices of the **mutation graph.** Cluster algebras come in two types: those with finitely many clusters and those with infinitely many clusters. The finite-type algebras have a Dynkin classification [FZ2], with known numbers of clusters (the number of vertices in a generalized associahedron). Scott [Sc] has shown that the Grassmannian cluster algebras (which are the coordinate rings of $\mathcal{M}_1$ of a torus knot) are of finite type precisely for the parameters $(2, n)$, $(3, 4)$, and $(3, 5)$ as in the Conjecture.

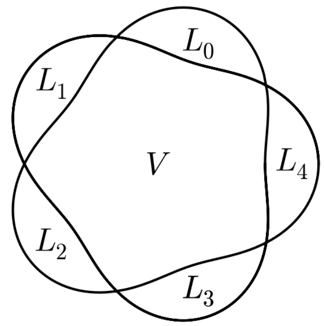
2.3.1 $\mathcal{M}_1(\lambda)$ as a hyperkahler manifold

The PI, Shende, and Zaslow have an ongoing project to identify (for certain λ, specifically algebraic knots) the spaces $\mathcal{M}_1(\lambda)$ with "wild character varieties," i.e. to the **Betti analog** of the moduli spaces of **irregular connections** considered in [BiqBoa, W]. The PI et all will show that to each of Boalch's **twisted irregular curves** [Boa, Remark 8.6] with underlying Riemann surface S, there is a Legendrian link $\lambda \subset T^\infty S$, such that the de Rham moduli of the twisted irregular curve is analytically isomorphic to the moduli of constructible sheaves on S with singular support in λ. The case where $S = \mathbf{P}^1$ with a unique singular point gives hyperkahler-rotated versions of the spaces $\mathcal{M}_1(\lambda)$.

[8] If λ is a link with many components, we expect instead $\mathcal{M}(\lambda)$ to be a Poisson manifold. See project §2.4.2

We give one example. There is a "contact Dehn twist" maneuver on $T^\infty \mathbf{R}^2$, which carries a rainbow closure to another kind of positive braid closure:





We have traded the cusps for a **full braid twist** and some vertical tangencies. One can carry the Lagrangians in $\mathbf{R}^4$ along this twist, and indeed as the diagram indicates this maneuver does not change the moduli space $\mathcal{M}_1(\lambda)$ where the arrows of the left side of Figure 3 point inward. In this case V can be taken to be the space of entire solutions to the 2nd order ODE $f'' = (z^3 + az + b)f$, and the lines $L_i \subset V$ can be taken to be **Stokes filtrations** of a rank 2 connection on $\mathbf{P}^1$ with an irregular singular point at infinity. As a and b vary, so do the lines L_i, complex analytically but not algebraically.

The PI et al are also working to show that the correspondence with irregular connections is compatible with **nonabelian Hodge theory**. Roughly speaking this asserts that for a Lagrangian surface representing a point of $\mathcal{M}_1(\lambda)$, we may take the **spectral curve of an corresponding irregular Higgs field**.

2.4 Outline

Let λ be a Legendrian knot with a rainbow front projection $\Phi \subset \mathbf{R}^2$. We regard λ as a Legendrian subset of $(\pi, 2\pi) \times \mathbf{R}^2 \subset S^1 \times \mathbf{R}^2 \cong T^\infty \mathbf{R}^2$. In [PI-STZ], the PI et al constructed a moduli space of constructible sheaves on $\mathbf{R}^2$ with singular support (at contact infinity) in λ, denoted $\mathcal{M}_1(\lambda)$. The following sub-projects to connect the theory of fillings to the geometry of $\mathcal{M}_1(\lambda)$ should lead to a proof of *most* (see remark 2.6) of Conjecture 2.1 and beyond. The approach is to replace the informal reasoning of §2.3 about $\mathcal{M}(\lambda)$ with formal reasoning about $\mathcal{M}_1(\lambda)$. Some of these projects will be carried out jointly with Shende and Zaslow.

Remark 2.6. There is one aspect of Conjecture 2.1 which is as difficult as a standard and longstanding open problem. Specifically, to prove Conjecture 2.1 one should (1) produce a list of n exact fillings of the torus knot λ (e.g. $n = 833$ or $n = \infty$), and (2) show that any other exact filling must be Hamiltonian isotopic to one from this list. But part (2) is as hard as the nearby Lagrangian conjecture (discussed in §2.1), about which unfortunately the PI has no ideas. What is actually proposed to be proven here is a variant of the Conjecture where a coarser Fukaya-theoretic equivalence relation is used in place of Hamiltonian isotopy, as in the works of Nadler and Fukaya-Seidel-Smith on the nearby Lagrangian problem [FSS, N].

2.4.1 Exact Lagrangians determine and are determined by open charts

If Λ is an exact Lagrangian filling of λ, the results of Nadler-Zaslow produce a functor from the category of local systems on Λ to the category of constructible sheaves on $\mathbf{R}^2$. If Λ ends on λ, this produces a map from the GL(1)-character variety of L to $\mathcal{M}_1(\lambda)$. The PI will show that (1) this map is an *open embedding* and that (2) if Λ and Λ' are two exact Lagrangians that determine the same chart in $\mathcal{M}_1(\lambda)$, then they are isomorphic in the Fukaya category[9]

For (1) it is very clear how to proceed, but there are details to verify and write down: the PI will show that the functor is **an algebraic map** to $\mathcal{M}_1(\lambda)$, and that moreover it is **injective and étale**. Using the functor-of-points perspective on moduli, the PI will reduce these verifications to Floer-homological statements that can be obtained by standard tools. For (2) it is likely that the techniques of [FSS, N] can be applied.

2.4.2 $\mathcal{M}_1(\lambda)$ is a complex symplectic manifold

This is already clear for torus knots, by the explicit computation of 2.5 and known results about such Grassmannian cluster varieties, but it should be true in much greater generality. The symplectic mechanism is already described in §2.3, but it is would be valuable to see it directly in the setting of constructible sheaves. It is likely there is a direct argument for Legendrian knots, but the PI will also pursue the following much more general strategy, that would also explain what goes on for **Legendrian links** and in fact **Legendrians of any dimension**.

> To prove that a complex variety $\mathcal{M}$ is symplectic, show that it is the intersection of two Lagrangians in a **1-shifted symplectic manifold**.

Roughly speaking, an n-shifted symplectic structure [PTTV] is a pairing on tangent bundles (or tangent complexes) to a homological shift (by n) of the structure sheaf. The facts relevant for this project are: (1) if X is an oriented n-manifold, then the character variety[10] of G-local systems over X has a $(2-n)$-shifted symplectic structure, and (2) if Λ_1 and Λ_2 are Lagrangians in the n-shifted symplectic manifold X, then $\Lambda_1 \times_X \Lambda_2$ is an $(n-1)$-shifted symplectic manifold and (3) if Y is an oriented $(n+1)$-manifold whose boundary is X, then the map for G-local systems on Y to G-local systems on X is Lagrangian. In [PI-STZ], the PI et al constructed a map $\mathcal{M}_1(\lambda) \to Loc_1(\lambda)$, **microlocal monodromy**, which should be analogous to (3) with $G = \mathrm{GL}_1$. When λ has a single component, this map is constant, so $\mathcal{M}_1(\lambda) = \mathcal{M}_1(\lambda) \times_{Loc_1(\lambda)} pt$ (at least set-theoretically), and $pt \to Loc_1(\lambda)$ is trivially a Lagrangian — the PI will show that **microlocal monodromy is a Lagrangian map** in the sense of shifted symplectic geometry.

[9] in fact they should even be Hamiltonian isotopic, see remark 2.6.

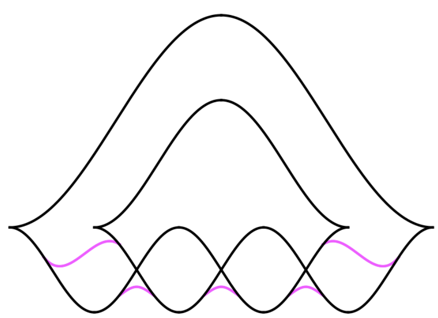
[10] in a suitably derived sense

This strategy also suggests that the analogous problem in dimension 3 (how many Lagrangians in $\mathbf{R}^6$ — or T^*S^3 or something similar — end on a given Legendrian 2-manifold at contact infinity?) leads to moduli spaces that are (-1)-shifted symplectic (in particular that have **perfect obstruction theories**), and one can study them using the tools of **Donaldson-Thomas theory.**

2.4.3 There is a log compactification of $\mathcal{M}_1(\lambda)$ with maximally degenerate boundary

Granting the complex symplectic structure of §2.4.2, the work of Gross-Hacking-Keel [GHK] shows that the existence of a cluster structure is equivalent to the existence of certain kind of algebraic compactification of $\mathcal{M}_1(\lambda)$. In fact one could then conclude that each open chart arising as in §2.4.1 was a cluster chart. What would remain of the problem of enumerating exact fillings is constructing an exact filling for each chart §2.4.4.

There is a very natural source of compactifications of $\mathcal{M}_1(\lambda)$, which the PI will construct and show to be of GHK type. These compactifications come from adding **Legendrian chords** to λ. The result is a **Legendrian ribbon graph**, say Γ. ("Ribbon" because the edges of such a graph have a natural cyclic order [OP].) The PI will define a compactification of $\mathcal{M}_1(\lambda)$, denoted $\mathcal{M}_1(\Gamma)$, by allowing the constructible sheaf to acquire singular support along these chords. The front projection of a Legendrian graph looks like this:



Omitting the purple arcs one is left with the Legendrian trefoil, whose $\mathcal{M}_1(\lambda)$ is described is §2.5. The embedding $\mathcal{M}_1(\lambda) \subset \mathcal{M}_1(\Gamma)$ is the compactification one obtains by "putting the anticanonical divisor back in." The PI will show that the boundary components of this compactification have the form $\mathcal{M}_1(\lambda')$ where λ' is a "simpler" Legendrian knot, e.g. one whose front projection has fewer crossings.

2.4.4 Each cluster chart is inhabited by an exact filling

Assuming for the moment that $\mathcal{M}_1(\lambda)$ is a cluster variety, let us say that a cluster chart is *inhabited* if there is an exact Lagrangian filling of λ which gives the cluster chart by the construction of §2.4.1. The construction of [HaS] shows that at least one of the cluster charts is inhabited, by an exact Lagrangian Λ_{HS}. The goal is now to exploit the **mutation graph structure** on clusters to prove that every cluster chart is inhabited. The mutation graph is a connected graph whose vertices are clusters — to prove each chart is inhabited it suffices to prove that a vertex adjacent to an inhabited cluster is also inhabited.

The following is a likely mechanism — which for genus one the PI learned from Seidel [Se2, p.110]. The cluster structure endows Λ_{HS} with an explicit basis $\gamma_1, \ldots, \gamma_{2g}$. For each γ_i, we may find a Lagrangian disk D_i with $D_i \cap \Lambda = \gamma_i = \partial D_i$. Then a surgery should produce a Lagrangian surface $\Lambda' = \mathbf{mutate}_i(\Lambda)$ representing the mutated cluster. The difficult part is to create the mutant Λ' so that it is exact. To achieve this, the PI will exploit the fact that $\Lambda \cup D_i$ is the skeleton of an interesting but fairly simple Weinstein manifold M (obtained by Weinstein surgery along the boundary of a fiber of $T^*\Lambda$), and reduce the creation of an exact Λ' in $\mathbf{R}^4$ to the creation of an exact Λ' in M.

2.4.5 Formulation of an analog of conjecture 2.1 for more general positive knots

This is a problem the PI plans to give to Kyle Hayden, a second-year grad student at BC. One source of cluster algebras are **seeds**, which in our setting are just $2g \times 2g$ skew-symmetric matrices J. The PI has already developed an algorithm to associate such a J to a rainbow closure, which is likely equivalent to a standard construction in the theory of positroid varieties.. The problem of doing so for more general knots would be a good entry-point for Hayden, as the main ideas (and obstacles) are already in his wheelhouse [HaS] — graded normal rulings, surgeries, intersections on surfaces.

3 Broader impacts

3.1 Broader impacts of the prior award

The PI has disseminated the results obtained so far via arxiv.org, journals, seminars and colloquia, and conferences.

The PI organized a pair of SQuaRE workshops — the first of which took place in May 2014 and the second of which will take place in summer 2015 — to bring together the authors of §1.1.2 with both senior and junior Legendrian knot experts, namely Lenhard Ng (Duke), Steven Sivek (postdoc at MIT) and Dan Rutherford (postdoc at Arkansas).

In Fall of 2014, while on leave at MSRI (for the program "Geometric Reperesentation Theory"), the PI organizes with Ivan Mirkovic the weekly members seminar, and with Kevin McGerty and Kiran Kedlaya a weekly learning seminar on D-modules with irregular singularities.

The PI worked with first-year (now second year) Boston College (BC) graduate student Kyle Hayden, who is interested in Legendrian knots and their Lagrangian fillings. (Not yet as an official advisor, though this is possible.) The PI mentored BC freshman (now sophomore) Arnav Roy and supervised an Undergraduate Research Fellowship ($2400 from BC, 06/01/2014–08/01/2014) on Smith theory and elementary number theory.

With an eye toward increasing the number of international applicants for BC's brand new Ph.D. program, and more generally encouraging international cooperation and exchange of scholars, the PI requested that the department sign a Memorandum of Understanding with the Mathematical Sciences Center (MSC) and Department of Mathematical Sciences at Tsinghua University in Beijing, China. The department and administration approved, and the PI represented BC at the May 2013 signing ceremony on Internation Cooperation, Innovation, and Collaboration at Tsinghua.

For 2012-2014 (but on leave at MSRI Fall of 2014), the PI has been a liaison to the BC Math Society, which organizes career information seminars, block parties, π-day celebrations and similar events for math majors.

3.2 Broader impacts of the proposal

The PI will continue the activities of §3.1, including working with the MSC and Tsinghua University, organizing the second SQuaRE meeting, mentoring undergraduates and grad students, and disseminating results via talks, online preprints, and journals. Additionally, the PI will experiment with a new (to him) form of dissemination of results: in September the PI began a private and many-author blog on matters related to Floer theory, microlocalization, wild ramification, nonabelian Hodge theory, and spectral networks that had become unwieldy in e-mail discussions. Participants so far are the PI, Shende, Zaslow, Xin Jin (a student of Nadler's at Berkeley) and Harold Williams (a 2014 Ph.D. and new postdoc at Texas), with Andy Neitzke in the audience. At least some of its present and future content will be made public in or before 2015.

The PI will continue to mentor Kyle Hayden and Arnav Roy. A project for Hayden is described in §2.4.5, it is possible this could lead to a thesis problem. BC has two programs for funding undergraduate research, the Undergraduate Research Fellowship and the Advanced Study Grant. The PI will nominate Roy for both of them in 2015. Each summer the PI will run a working-and-learning seminar for Boston-area grad students and postdocs, on topics related to the proposal.