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PURITY AND DECOMPOSITION THEOREMS FOR STAGGERED SHEAVES

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Abstract Two major results in the theory of ℓ-adic mixed constructible sheaves are the purity theorem (every simple perverse sheaf is pure) and the decomposition theorem (every pure object in the derived category is a direct sum of shifts of simple perverse sheaves). In this paper, we prove analogues of these results for coherent sheaves. Specifically, we work with *staggered sheaves*, which form the heart of a certain t-structure on the derived category of equivariant coherent sheaves. We prove, under some reasonable hypotheses, that every simple staggered sheaf is pure, and that every pure complex of coherent sheaves is a direct sum of shifts of simple staggered sheaves.

Keywords: coherent sheaves; derived categories; purity; decomposition theorem

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1. Introduction

Let Z be a variety over a finite field $\mathbb{F}_q$, and let $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(Z)$ denote the bounded derived category of ℓ-adic mixed constructible sheaves on Z. Recall that the *weights* of an object $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(Z)$ are certain integers defined in terms of the eigenvalues of the Frobenius morphism on the stalks of F at $\mathbb{F}_{q^n}$-points of Z. An object is said to be *pure* of weight $w \in \mathbb{Z}$ if both it and its Verdier dual have weights less than or equal to w. The theory of weights and purity plays a vital role in the proof and in applications of the Weil conjectures [**6**, **9**, **10**].

Two of the most astonishing consequences of the Weil conjectures occur in the theory of perverse sheaves, developed in [**6**, Chapter 5]. They are

(i) the Purity Theorem [**6**, Théorème 5.3.5], which states that every perverse sheaf has a canonical filtration with pure subquotients (and in particular that every simple perverse sheaf is pure), and

(ii) the Decomposition Theorem [**6**, Théorème 5.4.5], which states that over $\bar{\mathbb{F}}_q$, the pushforward of a simple perverse sheaf along a proper morphism is semisimple, i.e. a direct sum of shifts of simple perverse sheaves.

The Decomposition Theorem can, in turn, be separated into two parts: the 'Weil conjecture' part (such a pushforward preserves purity, see [**10**, Théorème I]) and the 'semisimplicity' part (every pure object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(Z)$ becomes semisimple after passing to $\bar{\mathbb{F}}_q$). Together, the Purity Theorem and the Decomposition Theorem are the source of much of the power of the theory of perverse sheaves for applications in representation theory and other areas.

In this paper, we prove analogues of the Purity Theorem and the semisimplicity part of the Decomposition Theorem in the setting of derived categories of equivariant coherent sheaves. These results would be made substantially more powerful by the discovery of a functor that preserves purity in this setting, in analogy with the Weil conjecture part above. Unfortunately, no functor is currently known to have this property.

We will work in the following setting. Let X be a scheme of finite type over an arbitrary field, and let G be an affine algebraic group acting on X with finitely many orbits. Let $\mathcal{D}^{\mathrm{b}}_G(X)$ denote the bounded derived category of G-equivariant coherent sheaves on X. The category of *staggered sheaves*, introduced in [**1**], is the heart of a certain non-standard t-structure on $\mathcal{D}^{\mathrm{b}}_G(X)$. This category shares some of the key properties of perverse sheaves: for example, every object has finite length, and the simple objects arise via the '$\mathcal{IC}$ functor' from irreducible vector bundles on orbits.

In $\mathcal{D}^{\mathrm{b}}_G(X)$ there is no single best notion of weight or purity as there is in the ℓ-adic setting. Rather, there is a large number of such notions parametrized by *baric perversities*, which are certain integer-valued functions on the set of G-orbits in X. More precisely, in [**5**] we associated to each baric perversity a *baric structure* (a certain kind of filtration of a triangulated category) on $\mathcal{D}^{\mathrm{b}}_G(X)$, which we use here to simulate the formalism of weights. We call an object $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$ *pure* of baric degree w if both it and its Serre–Grothendieck dual lie in the '$\leqslant w$' part of the baric structure. (A result of Morel [**12**] essentially states that Frobenius weights give rise to a baric structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(Z)$, so the theory of ℓ-adic mixed perverse sheaves could be redeveloped using the language of baric structures as well.)

The main results of the present paper (which are Theorems 6.5, 10.2 and 11.5) come in two incarnations, a 'baric' one and a 'skew' one. In the baric version, they state that under some reasonable hypotheses, every staggered sheaf has a canonical filtration with pure subquotients, and every pure object of $\mathcal{D}^{\mathrm{b}}_G(X)$ is a direct sum of shifts of simple staggered sheaves. The skew versions consist of essentially the same statements, but with purity replaced by a new concept called *skew-purity*.

An outline of the paper is as follows. We begin in § 2 by fixing notation and recalling relevant results about baric structures and staggered sheaves. In §§ 3 and 4, we construct two t-structures on the full triangulated subcategory of pure objects of baric degree w in $\mathcal{D}^{\mathrm{b}}_G(X)$, called the *purified standard t-structure* and the *pure-perverse t-structure*. (The latter is defined in terms of the former.) We also prove that the heart of the pure-perverse t-structure is contained in that of the staggered t-structure. In § 5, we study simple objects in the pure-perverse t-structure. They, like simple staggered sheaves, are characterized by a certain uniqueness property, and this allows us to prove that every simple staggered sheaf lies in the heart of a suitable pure-perverse t-structure. This is a major step towards the baric version of the Purity Theorem, whose proof is completed in § 6.

Next, in § 7, which is essentially independent of the rest of the paper, we give a combinatorial classification of staggered t-structures on a variety consisting of a single G-orbit. This allows us to give an elementary criterion for a certain Ext^1-vanishing condition that appears as a hypothesis throughout the rest of the paper. Section 8 contains some results on vanishing of higher Ext-groups; these lay the groundwork for the definition of skew-purity in § 9. The skew version of the Purity Theorem is proved in § 10, and both versions of the Decomposition Theorem are proved together in § 11. Finally, § 12 illustrates the use of the Decomposition Theorem by means of an application to the equivariant K-theory of smooth toric varieties.

2. Preliminaries and notation

Let X be a scheme of finite type over a field $\Bbbk$. Let G be an affine algebraic group over $\Bbbk$, acting on X. Assume that G acts on X with finitely many orbits. Here, and throughout the paper, an *orbit* is a reduced, locally closed G-invariant subscheme containing no proper non-empty closed G-invariant subschemes. X itself need not be reduced. Let $\mathbb{O}(X)$ denote the set of G-orbits in X.

For each orbit $C \in \mathbb{O}(X)$, let $i_C \colon \bar{C} \hookrightarrow X$ denote the inclusion morphism of the closure of C as a reduced closed subscheme, and let $\mathcal{I}_C \subset \mathcal{O}_X$ denote the corresponding ideal sheaf.

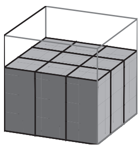
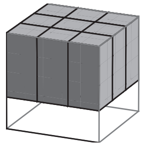
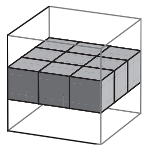
Remark 2.1. Some earlier references on staggered sheaves, including most of [**1**] and a significant part of [**5**], imposed much weaker hypotheses: the setting was a scheme of finite type over some noetherian base scheme admitting a dualizing complex, acted on by an affine group scheme over the same base, with no assumption on the number of orbits. In the present paper, only the results of §§ 3 and 4 hold in such great generality. The main results do not, and it simplifies the discussion to impose these conditions at the outset.

We uniformly adopt the convention that terms like 'open subscheme', 'closed subscheme' and 'irreducible' are always to be interpreted in a G-invariant sense. That is, 'open subscheme' should always be understood to mean 'G-invariant open subscheme', and a subscheme is 'irreducible' if it is not a union of two proper closed (G-invariant) subschemes.

Let $\mathcal{C}_G(X)$ denote the category of G-equivariant coherent sheaves on X, and let $\mathcal{D}^{\mathrm{b}}_G(X)$ (respectively $\mathcal{D}^-_G(X)$, $\mathcal{D}^+_G(X)$) denote the full subcategory of the bounded (respectively bounded above, bounded below) derived category of G-equivariant quasicoherent sheaves on X consisting of objects with coherent cohomology. It is known [**7**, Corollary 1] that $\mathcal{D}^{\mathrm{b}}_G(X)$ and $\mathcal{D}^-_G(X)$ are equivalent to bounded and bounded-above derived categories of $\mathcal{C}_G(X)$, respectively. As usual, we let $\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}$ and $\mathcal{D}^-_G(X)^{\leqslant n}$ denote the subcategories of $\mathcal{D}^{\mathrm{b}}_G(X)$ and $\mathcal{D}^-_G(X)$, respectively, consisting of objects $\mathcal{F}$ with $h^k(\mathcal{F}) = 0$ for $k > n$. $\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}$ and $\mathcal{D}^+_G(X)^{\geqslant n}$ are defined similarly. We also have the truncation functors

$$\tau^{\leqslant n} \colon \mathcal{D}^+_G(X) \to \mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n} \quad \text{and} \quad \tau^{\geqslant n} \colon \mathcal{D}^-_G(X) \to \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}.$$

Over the course of this paper, we will consider a rather large number of different kinds of subcategories of $\mathcal{D}^{\mathrm{b}}_G(X)$, all of which are denoted by decorating the symbol '$\mathcal{D}^{\mathrm{b}}_G(X)$' with various left and right superscripts and subscripts. To avoid confusion, it is helpful to visualize these subcategories as various regions in a large three-dimensional grid in which the vertical axis represents cohomological degree in $\mathcal{D}^{\mathrm{b}}_G(X)$. (See §§ 2.4 and 4 for the meanings of the other axes.) Thus, the standard t-structure and its heart may be pictured as follows:

$\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}$: 
$\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}$: 
$\mathcal{C}_G(X)$: 

2.1. Duality and codimension

By [**7**, Proposition 1], X possesses an equivariant Serre–Grothendieck dualizing complex. Choose one, once and for all, and denote it by ω_X. We denote the Serre–Grothendieck duality functor by $\mathbb{D} = \mathrm{R}\mathcal{H}\mathrm{om}(\cdot, \omega_X)$.

For each orbit $C \in \mathbb{O}(X)$, there is a unique integer

$$\operatorname{cod} C \quad \text{such that} \quad Ri^!_C\omega_X|_C \in \mathcal{D}^{\mathrm{b}}_G(C)^{\leqslant \operatorname{cod} C} \cap \mathcal{D}^{\mathrm{b}}_G(C)^{\geqslant \operatorname{cod} C}.$$

This integer differs from the ordinary Krull codimension of C by some constant depending only on ω_X (see [**11**, § V.3] and [**1**, § 6]). Thus, $\operatorname{cod} Y$ can be made to agree with the ordinary codimension by replacing ω_X by a suitable shift, but we do not assume here that any such specific normalization has been made.

2.2. s-structures and altitude

Suppose $\mathcal{C}_G(X)$ is equipped with an increasing filtration $\{\mathcal{C}_G(X)_{\leqslant w}\}_{w\in\mathbb{Z}}$ by Serre subcategories. Let

$$\mathcal{C}_G(X)_{\geqslant w} = \{\mathcal{G} \in \mathcal{C}_G(X) \mid \operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0 \text{ for all } \mathcal{F} \in \mathcal{C}_G(X)_{\leqslant w-1}\}. \tag{2.1}$$

For any sheaf $\mathcal{F} \in \mathcal{C}_G(X)$ and any integer $w \in \mathbb{Z}$, there is a unique maximal subsheaf of $\mathcal{F}$ in $\mathcal{C}_G(X)_{\leqslant w}$, denoted $\sigma_{\leqslant w}\mathcal{F}$. Conversely, the sheaf $\sigma_{\geqslant w+1}\mathcal{F} = \mathcal{F}/\sigma_{\leqslant w}\mathcal{F}$ is the unique largest quotient of $\mathcal{F}$ lying in $\mathcal{C}_G(X)_{\geqslant w+1}$.

The categories $(\{\mathcal{C}_G(X)_{\leqslant w}\}, \{\mathcal{C}_G(X)_{\geqslant w}\})_{w\in\mathbb{Z}}$ constitute an *s-structure* on X if they satisfy a rather lengthy list of axioms given in [**1**], mostly having to do with Ext-vanishing conditions on closed subschemes. We will not review the full definition in the general case here, but we will give an explicit description of a certain class of s-structures below.

If X is endowed with an s-structure, a sheaf $\mathcal{F} \in \mathcal{C}_G(X)$ is said to be *s-pure of step w* if it lies in $\mathcal{C}_G(X)_{\leqslant w} \cap \mathcal{C}_G(X)_{\geqslant w}$. (In [**1**], this property was simply called 'pure', but here we call it 's-pure' to avoid confusion with the notions of baric and skew purity, see § 2.4.) An s-structure on X induces s-structures on all locally closed subschemes of X, and in particular on all orbits.

Given an orbit $C \in \mathbb{O}(X)$, recall that $Ri^!_C\omega_X[\operatorname{cod} C]|_C$ lies in $\mathcal{C}_G(C)$ (that is, it is concentrated in cohomological degree 0). According to [**1**, § 6], there is a unique integer

$$\operatorname{alt} C \quad \text{such that} \quad Ri^!_C\omega_X[\operatorname{cod} C]|_C \in \mathcal{C}_G(C)_{\leqslant \operatorname{alt} C} \cap \mathcal{C}_G(C)_{\geqslant \operatorname{alt} C}.$$

This integer is called the *altitude* of C. Finally, the *staggered codimension* of C, denoted $\operatorname{scod} C$, is defined by

$$\operatorname{scod} C = \operatorname{alt} C + \operatorname{cod} C.$$

Let us now return to the question of how to construct an s-structure. Consider the special case where X is a reduced scheme consisting of a single G-orbit. In this case, the conditions for a collection $(\{\mathcal{C}_G(X)_{\leqslant w}\}, \{\mathcal{C}_G(X)_{\geqslant w}\})_{w\in\mathbb{Z}}$ to constitute an s-structure reduce to the following much simpler conditions.

(1) For every sheaf $\mathcal{F} \in \mathcal{C}_G(X)$, there exist integers v, w such that $\mathcal{F} \in \mathcal{C}_G(X)_{\geqslant v} \cap \mathcal{C}_G(X)_{\leqslant w}$.

(2) If $\mathcal{F} \in \mathcal{C}_G(X)_{\leqslant w}$ and $\mathcal{G} \in \mathcal{C}_G(X)_{\leqslant v}$, then $\mathcal{F} \otimes \mathcal{G} \in \mathcal{C}_G(X)_{\leqslant w+v}$.

(3) If $\mathcal{F} \in \mathcal{C}_G(X)_{\geqslant w}$ and $\mathcal{G} \in \mathcal{C}_G(X)_{\geqslant v}$, then $\mathcal{F} \otimes \mathcal{G} \in \mathcal{C}_G(X)_{\geqslant w+v}$.

(Here, the categories $\{\mathcal{C}_G(X)_{\geqslant w}\}_{w\in\mathbb{Z}}$ are still given by (2.1).) In § 7, we will give a constructive classification of all s-structures on a single orbit.

Now, suppose X contains more than one orbit, and assume that each orbit is endowed with an s-structure. Assume also that the following condition holds:

$$\text{for each orbit } C \subset X, \text{ the sheaf } i^*_C\mathcal{I}_C|_C \text{ is in } \mathcal{C}_G(C)_{\leqslant -1}. \tag{2.2}$$

(The sheaf in question is simply the conormal bundle of C.) By [**4**, Theorem 1.1], the condition (2.2) implies that there is a unique s-structure on X whose restriction to each orbit coincides with the given s-structure on that orbit. In practice, the easiest way to produce explicit examples of s-structures seems to be to specify one on each orbit and then invoke [**4**, Theorem 1.1].

Not every s-structure on X arises in this way, but every s-structure for which condition (2.2) holds does. Following [**5**], s-structures with this property are said to be *recessed*.

We assume for the remainder of the paper that X is endowed with a fixed recessed s-structure. For examples, see [**4**, **14**].

2.3. Perversities

A *perversity* (or *perversity function*) is simply a function $q\colon \mathbb{O}(X) \to \mathbb{Z}$. A perversity q is said to be *monotone* if whenever $C' \subset \bar{C}$, we have $q(C') \geqslant q(C)$.

A number of constructions in the sequel depend on the choice of a perversity. We will often refer to specific kinds of perversities, such as 'baric perversities', 'Deligne–Bezrukavnikov perversities' and 'staggered perversities'. These are not intrinsically different kinds of objects; rather, the adjectives serve merely to indicate how a particular perversity will be used (e.g. to construct a baric structure).

Given a perversity $q\colon \mathbb{O}(X) \to \mathbb{Z}$, we define three different kinds of 'dual perversity', as follows:

$$\begin{array}{ll} \textit{baric} \text{ dual:} & \hat{q}(C) = 2\operatorname{alt} C - q(C), \\ \textit{Deligne–Bezrukavnikov} \text{ dual:} & \tilde{q}(C) = \operatorname{cod} C - q(C), \\ \textit{staggered} \text{ dual:} & \bar{q}(C) = \operatorname{scod} C - q(C). \end{array}$$

A perversity is called *comonotone* if its dual is monotone. This condition is, of course, ambiguous, but the intended type of duality will be clear from context whenever this term is used.

The *middle perversity* of a given kind (baric, Deligne–Bezrukavnikov, or staggered) is the unique perversity that is equal to its own dual. Clearly, the middle baric perversity is given by

$$q(C) = \operatorname{alt} C.$$

Similarly, the middle Deligne–Bezrukavnikov and staggered perversities, when they exist, are given by the formulae

$$q(C) = \tfrac{1}{2}\operatorname{cod} C \quad \text{and} \quad q(C) = \tfrac{1}{2}\operatorname{scod} C,$$

respectively. However, these formulae make sense only when all $\operatorname{cod} C$ or all $\operatorname{scod} C$, respectively, are even.

2.4. Baric structures

Following [**5**], a *baric structure* on a triangulated category $\mathfrak{D}$ is a pair of collections of thick subcategories $(\{\mathfrak{D}_{\leqslant w}\}, \{\mathfrak{D}_{\geqslant w}\})_{w\in\mathbb{Z}}$ satisfying the following axioms.

(1) $\mathfrak{D}_{\leqslant w} \subset \mathfrak{D}_{\leqslant w+1}$ and $\mathfrak{D}_{\geqslant w} \supset \mathfrak{D}_{\geqslant w+1}$ for all w.

(2) $\operatorname{Hom}(A, B) = 0$ whenever $A \in \mathfrak{D}_{\leqslant w}$ and $B \in \mathfrak{D}_{\geqslant w+1}$.

(3) For any object $X \in D$, there is a distinguished triangle $A \to X \to B \to$ with $A \in \mathfrak{D}_{\leqslant w}$ and $B \in \mathfrak{D}_{\geqslant w+1}$.

(4) For any object $X \in D$, there exist integers v, w such that $X \in \mathfrak{D}_{\geqslant v} \cap \mathfrak{D}_{\leqslant w}$.

(The last axiom was not part of the definition of 'baric structure' in [**5**]; rather, a baric structure satisfying this extra condition was called *bounded*. In this paper, however, all baric structures will be bounded.) Given a baric structure on $\mathfrak{D}$, the inclusion functor $\mathfrak{D}_{\leqslant w} \hookrightarrow \mathfrak{D}$ admits a right adjoint, denoted $\beta_{\leqslant w}$, and the inclusion $\mathfrak{D}_{\geqslant w} \hookrightarrow \mathfrak{D}$ admits a left adjoint $\beta_{\geqslant w}$. The functors $\beta_{\leqslant w}$ and $\beta_{\geqslant w}$ are called *baric truncation functors*. For any object X and any $w \in \mathbb{Z}$, there is a distinguished triangle

$$\beta_{\leqslant w} X \to X \to \beta_{\geqslant w+1} X \to,$$

and any distinguished triangle as in Axiom (3) above is canonically isomorphic to this one.

The main result of [**5**] was the construction of a family of baric structures on $\mathcal{D}^{\mathrm{b}}_G(X)$, which we now recall. Let $q\colon \mathbb{O}(X) \to \mathbb{Z}$ be a perversity. We define a full subcategory of $\mathcal{C}_G(X)$ as follows:

$$ {}_q\mathcal{C}_G(X)_{\leqslant w} = \{\mathcal{F} \in \mathcal{C}_G(X) \mid i_C^*\mathcal{F}|_C \in \mathcal{C}_G(C)_{\leqslant \lfloor (w+q(C))/2 \rfloor} \text{ for all } C \in \mathbb{O}(X)\}. \tag{2.3}$$

Note that this does not agree with the definition in [**5**]: in [**5**], pullbacks to orbits were required to lie in $\mathcal{C}_G(C)_{\leqslant w+q(C)}$, not $\mathcal{C}_G(C)_{\leqslant \lfloor (w+q(C))/2 \rfloor}$. Thus, the relationship between the two definitions is as follows:

$$ {}_q\mathcal{C}_G(X)_{\leqslant w} \text{ as in [5]} = {}_{2q}\mathcal{C}_G(X)_{\leqslant 2w} \text{ as in the present paper.}$$

(The reason for this change will be explained below.) Next, let

$$\left.\begin{aligned} {}_q\mathcal{D}^-_G(X)_{\leqslant w} &= \{\mathcal{F} \in \mathcal{D}^-_G(X) \mid h^k(\mathcal{F}) \in {}_q\mathcal{C}_G(X)_{\leqslant w} \text{ for all } k\}, \\ {}_q\mathcal{D}^+_G(X)_{\geqslant w} &= \{\mathcal{F} \in \mathcal{D}^+_G(X) \mid \mathrm{Hom}(\mathcal{G},\mathcal{F}) = 0 \text{ for all } \mathcal{G} \in {}_q\mathcal{D}^-_G(X)_{\leqslant w-1}\}. \end{aligned}\right\} \tag{2.4}$$

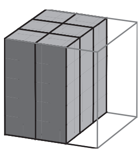
Let ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$ and ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$ denote the bounded versions of these categories, i.e. the intersections of the categories above with $\mathcal{D}^{\mathrm{b}}_G(X)$. According to [**5**, Theorem 6.4], $(\{{}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}\}, \{{}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}\})_{w\in\mathbb{Z}}$ is a baric structure on $\mathcal{D}^{\mathrm{b}}_G(X)$. We write ${}_q\beta_{\leqslant w}$ and ${}_q\beta_{\geqslant w}$ for its baric truncation functors, and we let

$$ {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]} = {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}.$$

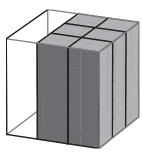
${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$ is a full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}_G(X)$. Its objects are said to be *pure of baric degree* w (with respect to the baric perversity q). Note that for a sheaf in $\mathcal{C}_G(X)$, there is no concise relationship between purity and s-purity: neither condition implies the other.

In the three-dimensional grid picture of $\mathcal{D}^{\mathrm{b}}_G(X)$, the horizontal axis represents baric degree. Thus, the various categories associated to a baric structure may be drawn as follows:

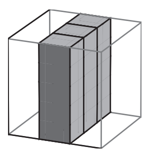
${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$:



${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$:

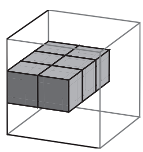


${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$:



Observe that the category ${}_q\mathcal{C}_G(X)_{\leqslant w}$ is simply $\mathcal{C}_G(X) \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. We draw it thus:

${}_q\mathcal{C}_G(X)_{\leqslant w}$:



However, it would be misleading to draw a similar picture of $\mathcal{C}_G(X) \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$, because $\mathcal{C}_G(X)$ is not, in general, generated by the subcategories ${}_q\mathcal{C}_G(X)_{\leqslant w}$ and

$\mathcal{C}_G(X) \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$. The latter category does not seem to have very good properties, and it will not make an appearance in the sequel. (See [**5**] for more information about this category.)

The following useful result states that these baric structures are both *hereditary* (well-behaved on closed subschemes) and *local* (well-behaved on open subschemes).

Lemma 2.2 (Achar and Treumann [5, Lemma 6.6]). *Let $j\colon U \hookrightarrow X$ be the inclusion of an open subscheme, and $i\colon Z \hookrightarrow X$ the inclusion of a closed subscheme. Then*

(1) *j^* takes ${}_q\mathcal{D}^-_G(X)_{\leqslant w}$ to ${}_q\mathcal{D}^-_G(U)_{\leqslant w}$ and ${}_q\mathcal{D}^+_G(X)_{\geqslant w}$ to ${}_q\mathcal{D}^+_G(U)_{\geqslant w}$;*

(2) *Li^* takes ${}_q\mathcal{D}^-_G(X)_{\leqslant w}$ to ${}_q\mathcal{D}^-_G(Z)_{\leqslant w}$;*

(3) *$Ri^!$ takes ${}_q\mathcal{D}^+_G(X)_{\geqslant w}$ to ${}_q\mathcal{D}^+_G(Z)_{\geqslant w}$;*

(4) *i_* takes ${}_q\mathcal{D}^-_G(Z)_{\leqslant w}$ to ${}_q\mathcal{D}^-_G(X)_{\leqslant w}$ and ${}_q\mathcal{D}^+_G(Z)_{\geqslant w}$ to ${}_q\mathcal{D}^+_G(X)_{\geqslant w}$.*

By applying the duality functor $\mathbb{D}$ to the categories that constitute some given baric structure on $\mathcal{D}^{\mathrm{b}}_G(X)$, one can obtain a new baric structure, said to be *dual* to the given one. It follows from the construction in [**5**, § 6] that the dual baric structure to $(\{{}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}\}, \{{}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}\})_{w\in\mathbb{Z}}$ is the baric structure associated by the above formulae to the dual baric perversity:

$$\mathbb{D}({}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}) = {}_{\hat{q}}\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant -w} \quad \text{and} \quad \mathbb{D}({}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}) = {}_{\hat{q}}\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant -w}.$$

In particular, if q is the middle baric perversity $q(C) = \operatorname{alt} C$, then the baric structure $(\{{}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}\}, \{{}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}\})_{w\in\mathbb{Z}}$ is self-dual. We adopt the convention that when the left-subscript perversity is omitted, this self-dual baric structure is meant:

$$\begin{aligned}\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w} &= {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w} \text{ with respect to } q(C) = \operatorname{alt} C,\\ \mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w} &= {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w} \text{ with respect to } q(C) = \operatorname{alt} C.\end{aligned}$$

From § 6 on, we will work almost exclusively with this self-dual baric structure.

Remark 2.3. The existence of a self-dual baric structure is why the definition of ${}_q\mathcal{C}_G(X)_{\leqslant w}$ was changed from that in [**5**]: in the notation of [**5**], the definitions (2.4) can give rise to a self-dual baric structure only if $\operatorname{alt} C$ is even for all $C \in \mathbb{O}(X)$. Here, we do not wish to impose that restriction on the s-structure, and we circumvent it by modifying the definition of ${}_q\mathcal{C}_G(X)_{\leqslant w}$.

2.5. Staggered t-structures

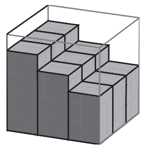
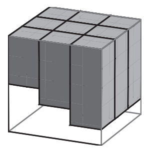
Let $q\colon \mathbb{O}(X) \to \mathbb{Z}$ be a perversity. We define full subcategories of $\mathcal{D}^-_G(X)$ and $\mathcal{D}^+_G(X)$ as follows:

$$\begin{aligned}{}^q\mathcal{D}^-_G(X)^{\leqslant n} &= \{\mathcal{F} \in \mathcal{D}^-_G(X) \mid h^k(\mathcal{F}) \in {}_{2q}\mathcal{C}_G(X)_{\leqslant n-2k} \text{ for all } k\},\\ {}^q\mathcal{D}^+_G(X)^{\geqslant n} &= \{\mathcal{F} \in \mathcal{D}^+_G(X) \mid \operatorname{Hom}(\mathcal{G}, \mathcal{F}) = 0 \text{ for all } \mathcal{G} \in {}^q\mathcal{D}^-_G(X)^{<n}\}.\end{aligned}$$

We also write ${}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}$ and ${}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}$ for the corresponding bounded categories:

$$ {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n} = {}^q\mathcal{D}^{-}_G(X)^{\leqslant n} \cap \mathcal{D}^{\mathrm{b}}_G(X), \qquad {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n} = {}^q\mathcal{D}^{+}_G(X)^{\geqslant n} \cap \mathcal{D}^{\mathrm{b}}_G(X). $$

We may draw pictures of these categories as follows:

${}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}$:  ${}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}$: 

Although these pictures are useful, care should be exercised in interpreting them. In particular, for a fixed perversity q, it does not make sense to superimpose the picture for, say, ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$ with that for ${}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}$, because in the latter, the horizontal axis represents baric degree with respect to the baric perversity $2q$, not q. Also, the picture above for ${}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}$ may be interpreted as saying that for each k, we have $\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k} \cap {}_{2q}\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant -2k} \subset {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}$. It does *not* say that $h^k(\mathcal{F}) \in {}_{2q}\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant -2k}$ for $\mathcal{F} \in {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}$; indeed, the latter condition is false in general.

According to [**5**, Theorem 8.1], $({}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}, {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0})$ is a bounded, non-degenerate t-structure on $\mathcal{D}^{\mathrm{b}}_G(X)$. Moreover, its heart

$$ {}^q\mathcal{M}(X) = {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0} \cap {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}, $$

known as the category of *staggered sheaves* (of perversity q), is a finite-length category. Its truncation functors are denoted

$$ {}^q\tau^{\leqslant n} \colon \mathcal{D}^{\mathrm{b}}_G(X) \to {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n} \quad \text{and} \quad {}^q\tau^{\geqslant n} \colon \mathcal{D}^{\mathrm{b}}_G(X) \to {}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}, $$

and the associated cohomology functors are denoted

$$ {}^q h^n \colon \mathcal{D}^{\mathrm{b}}_G(X) \to {}^q\mathcal{M}(X). $$

The simple objects in this category are parametrized by pairs $(C, \mathcal{L})$, where $C \in \mathbb{O}(X)$, and $\mathcal{L}$ is an irreducible vector bundle on C. To describe the structure of the corresponding simple object, we require the notion of the *intermediate-extension functor*. This is a fully faithful functor

$$ {}^q j^C_{!*} \colon {}^q\mathcal{M}(C) \to {}^q\mathcal{M}(\bar{C}) $$

that takes an object $\mathcal{F} \in {}^q\mathcal{M}(C)$ to the unique object of ${}^q\mathcal{M}(\bar{C})$ with the following properties:

(1) ${}^q j^C_{!*}\mathcal{F}|_C \cong \mathcal{F}$;

(2) for any smaller orbit $C' \subset \bar{C} \smallsetminus C$, we have

$$ Li^*_{C'}\,{}^q j^C_{!*}\mathcal{F} \in {}^q\mathcal{D}^{-}_G(\overline{C'})^{\leqslant -1} \quad \text{and} \quad Ri^!_{C'}\,{}^q j^C_{!*}\mathcal{F} \in {}^q\mathcal{D}^{+}_G(\overline{C'})^{\geqslant 1}. $$

Now, an irreducible vector bundle $\mathcal{L} \in \mathcal{C}_G(C)$ is necessarily s-pure; suppose it is s-pure of step v. Then $\mathcal{L}[v - q(C)]$ is an object of ${}^q\mathcal{M}(C)$, and the object

$$ {}^q\mathcal{IC}(\bar{C}, \mathcal{L}[v - q(C)]) = i_{C*} j^C_{!*} \mathcal{L}[v - q(C)], $$

known as a (*staggered*) *intersection cohomology complex*, is a simple object of ${}^q\mathcal{M}(X)$. Every simple object of ${}^q\mathcal{M}(X)$ arises in this way.

Remark 2.4. In a subsequent paper [**2**, Corollary 5.2], it is shown that ${}^q j^C_{!*}$ is an exact functor, unlike the intermediate extension functor for ordinary constructible perverse sheaves.

As with baric structures, there is an easy description of the dual t-structure to a given staggered t-structure: according to [**5**, Theorem 8.6], it is the staggered t-structure associated to the dual staggered perversity. That is,

$$ \mathbb{D}({}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}) = {}^{\bar{q}}\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant -n} \quad \text{and} \quad \mathbb{D}({}^q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}) = {}^{\bar{q}}\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant -n}. $$

In particular, if $\operatorname{scod} C$ is even for all $C \in \mathbb{O}(X)$, then the staggered t-structure associated to the middle staggered perversity $q(C) = \frac{1}{2} \operatorname{scod} C$ is self-dual.

2.6. Sheaves on non-reduced schemes

We conclude with a useful lemma comparing various categories of sheaves on a non-reduced scheme with those on its associated reduced scheme.

Lemma 2.5. *Let X_{red} denote the reduced scheme associated to X, and let $t \colon X_{\mathrm{red}} \hookrightarrow X$ be the natural map. Let $q \colon \mathbb{O}(X) \to \mathbb{Z}$ be a perversity.*

(1) *If $\mathcal{F} \in \mathcal{C}_G(X)$, we have $\mathcal{F} \in \mathcal{C}_G(X)_{\leqslant w}$ if and only if $t^*\mathcal{F} \in \mathcal{C}_G(X_{\mathrm{red}})_{\leqslant w}$.*

(2) *If $\mathcal{F} \in \mathcal{C}_G(X)$, we have $\mathcal{F} \in {}_q\mathcal{C}_G(X)_{\leqslant w}$ if and only if $t^*\mathcal{F} \in {}_q\mathcal{C}_G(X_{\mathrm{red}})_{\leqslant w}$.*

(3) *If $\mathcal{F} \in \mathcal{D}^-_G(X)$, we have $\mathcal{F} \in \mathcal{D}^-_G(X)^{\leqslant n}$ if and only if $Lt^*\mathcal{F} \in \mathcal{D}^-_G(X_{\mathrm{red}})^{\leqslant n}$.*

(4) *If $\mathcal{F} \in \mathcal{D}^-_G(X)$, we have $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)_{\leqslant w}$ if and only if $Lt^*\mathcal{F} \in {}_q\mathcal{D}^-_G(X_{\mathrm{red}})_{\leqslant w}$.*

There is a dual version of this lemma involving '$\geqslant$' categories and the $t^!$ and $Rt^!$ functors, but this statement suffices for our needs.

Proof. Part (1) is contained in [**1**, Proposition 4.1], and part (4) is contained in [**5**, Proposition 4.11]. Part (2) is obvious from the definition.

It remains to prove part (3). If $\mathcal{F} \in \mathcal{D}^-_G(X)^{\leqslant n}$, then clearly $Lt^*\mathcal{F} \in \mathcal{D}^-_G(X_{\mathrm{red}})^{\leqslant n}$, since Lt^* is right t-exact. Conversely, suppose $\mathcal{F} \notin \mathcal{D}^-_G(X)^{\leqslant n}$. Let k be the largest integer such that $h^k(\mathcal{F}) \neq 0$. Of course, we have $k > n$. By applying Lt^* to the distinguished triangle

$$ \tau^{<k}\mathcal{F} \to \mathcal{F} \to h^k(\mathcal{F})[-k] \to $$

and then forming the cohomology long exact sequence, one sees that $h^k(Lt^*\mathcal{F}) \cong t^*h^k(\mathcal{F})$. The functor t^* kills no non-zero sheaf, so $h^k(Lt^*\mathcal{F}) \neq 0$, and hence $Lt^*\mathcal{F} \notin \mathcal{D}^-_G(X_{\mathrm{red}})^{\leqslant n}$. □

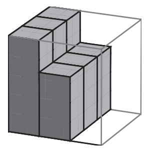
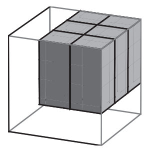
3. Pure sheaves

Let $q\colon \mathbb{O}(X) \to \mathbb{Z}$ be a baric perversity. The category of ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$ of pure objects is not stable under the standard truncation functors, so the standard t-structure on $\mathcal{D}^{\mathrm{b}}_G(X)$ does not induce a t-structure on ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$. Our goal in this section is to find an 'easy' t-structure on ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$ that resembles the standard t-structure on $\mathcal{D}^{\mathrm{b}}_G(X)$ as closely as possible.

Let us define full subcategories of $\mathcal{D}^-_G(X)$ and $\mathcal{D}^+_G(X)$ by

$$
\begin{aligned}
{}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w} &= (\mathcal{D}^-_G(X)^{\leqslant n} * {}_q\mathcal{D}^-_G(X)_{<w}) \cap {}_q\mathcal{D}^-_G(X)_{\leqslant w},\\
{}_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w} &= \mathcal{D}^+_G(X)^{\geqslant n} \cap {}_q\mathcal{D}^+_G(X)_{\geqslant w}.
\end{aligned}
$$

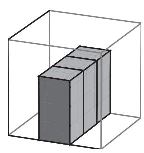
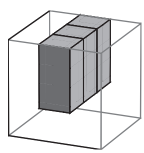
(For the '$*$' operation on triangulated categories, see [**6**, § 1.3.9].) Note that the definition of ${}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ involves the condition '$< w$'. Let ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{\leqslant w}$ and ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{\geqslant w}$ denote the bounded versions of these categories, i.e. the intersections of the above categories with $\mathcal{D}^{\mathrm{b}}_G(X)$. These categories may be pictured as follows:

${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{\leqslant w}$:  ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{\geqslant w}$: 

Finally, we denote the intersections of these categories with the category ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$ of pure objects by

$$
{}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{[w]} = {}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]} \quad \text{and} \quad {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{[w]} = {}_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]},
$$

and we draw them thus:

${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{[w]}$:  ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{[w]}$: 

The pictures suggest that $({}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{[w]}, {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}_{[w]})$ is a t-structure on ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$. The main result of this section states that this is, in fact, the case.

Lemma 3.1. *Let $j\colon U \hookrightarrow X$ be the inclusion of an open subscheme, and $i\colon Z \hookrightarrow X$ the inclusion of a closed subscheme. Then*

(1) *j^* takes ${}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ to ${}_q\mathcal{D}^-_G(U)^{\leqslant n}_{\leqslant w}$ and ${}_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w}$ to ${}_q\mathcal{D}^+_G(U)^{\geqslant n}_{\geqslant w}$;*

(2) *Li^* takes ${}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ to ${}_q\mathcal{D}^-_G(Z)^{\leqslant n}_{\leqslant w}$;*

(3) *$Ri^!$ takes ${}_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w}$ to ${}_q\mathcal{D}^+_G(Z)^{\geqslant n}_{\geqslant w}$;*

(4) *i_* takes ${}_q\mathcal{D}^-_G(Z)^{\leqslant n}_{\leqslant w}$ to ${}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ and ${}_q\mathcal{D}^+_G(Z)^{\geqslant n}_{\geqslant w}$ to ${}_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w}$.*

Proof. Immediate from Lemma 2.2 and well-known t-exactness properties of these functors with respect to the standard t-structure. $\square$

Lemma 3.2. *If* $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ *and* $\mathcal{G} \in {}_q\mathcal{D}^+_G(X)^{>n}_{\geqslant w}$, *then* $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$. *Conversely, if* $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)_{\leqslant w}$ *and* $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$ *for all* $\mathcal{G} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{>n}_{[w]}$, *then* $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$.

Proof. First, suppose $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$, and find a distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ with $\mathcal{F}' \in \mathcal{D}^-_G(X)^{\leqslant n}$ and $\mathcal{F}'' \in \mathcal{D}^-_G(X)_{<w}$. If $\mathcal{G} \in {}_q\mathcal{D}^+_G(X)^{>n}_{\geqslant w}$, then $\operatorname{Hom}(\mathcal{F}', \mathcal{G}) = \operatorname{Hom}(\mathcal{F}'', \mathcal{G}) = 0$, so we see that $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$ as well.

On the other hand, given $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)_{\leqslant w}$ such that $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{>n}_{[w]}$, form the distinguished triangle

$$\tau^{\leqslant n}\mathcal{F} \to \mathcal{F} \to \tau^{>n}\mathcal{F} \to \ .$$

To show that $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$, it suffices to show that $\tau^{>n}\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{<w}$. Suppose this is not the case, and let $\mathcal{G} = {}_q\beta_{\geqslant w}\tau^{>n}\mathcal{F}$. Since ${}_q\beta_{\geqslant w}$ is left t-exact, we see that $\mathcal{G} \in {}_q\mathcal{D}^+_G(X)^{>n}_{\geqslant w}$. Clearly, $\operatorname{Hom}((\tau^{\leqslant n}\mathcal{F})[1], \mathcal{G}) = 0$, so the fact that $\operatorname{Hom}(\tau^{>n}\mathcal{F}, \mathcal{G}) \neq 0$ implies that $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \neq 0$, a contradiction. $\square$

Lemma 3.3. *For any* $\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$, *there is a distinguished triangle* $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ *with* $\mathcal{F}' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{\leqslant w}$ *and* $\mathcal{F}'' \in \mathcal{D}^{\mathrm{b}}_G(X)^{>n}_{[w]}$.

Proof. Let $\mathcal{F}'' = {}_q\beta_{\geqslant w}\tau^{>n}\mathcal{F}$. Since $\tau^{>n}$ is right baryexact in the sense of [**5**, Definition 2.4], we have $\tau^{>n}\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)^{>n} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. Then, the left t-exactness of ${}_q\beta_{\geqslant w}$ (with respect to the standard t-structure) implies that

$$\mathcal{F}'' = {}_q\beta_{\geqslant w}\tau^{>n}\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)^{>n} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w} = {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{[w]}.$$

We also have a natural morphism $\mathcal{F} \to \mathcal{F}''$, obtained by composing $\mathcal{F} \to \tau^{>n}\mathcal{F}$ and $\tau^{>n}\mathcal{F} \to {}_q\beta_{\geqslant w}\tau^{>n}\mathcal{F}$. Let $\mathcal{F}'$ be the cocone of this morphism. We already know that $\mathcal{F}' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. The octahedral diagram below shows that $\mathcal{F}' \in \mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n} * {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{<w}$, so $\mathcal{F}' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{\leqslant w}$, as desired:

$$\begin{tikzcd}
 & & & \tau^{>n}\mathcal{F} \arrow[dddl] \arrow[dr, "+1"] & & & \\
 & & & & \tau^{\leqslant n}\mathcal{F} \arrow[dllll] \arrow[dddl] & & \\
\mathcal{F} \arrow[uurrr] \arrow[drr] & & & & & & {}_q\beta_{<w}\tau^{>n}\mathcal{F} \arrow[uulll] \arrow[ull, "+1"'] \\
 & & \mathcal{F}'' \arrow[urrrr, "+1"'] \arrow[dr, "+1"] & & & & \\
 & & & \mathcal{F}' \arrow[uulll] \arrow[uurrr] & & &
\end{tikzcd}$$

$\square$

Proposition 3.4. $({}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{[w]}, {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}_{[w]})$ *is a non-degenerate, bounded t-structure on* ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$.

Proof. It is clear that ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{\leqslant w} \subset {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 1}_{\leqslant w}$ and ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}_{\geqslant w} \supset {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 1}_{\geqslant w}$. Next, given $\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$, form a distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ as in Lemma 3.3. According to that lemma, $\mathcal{F}''$ necessarily lies in ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$, so it follows that $\mathcal{F}'$ does as well. From Lemma 3.2, we see that $({}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{[w]}, {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}_{[w]})$ is indeed a t-structure on ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$.

It is clear that no non-zero object can belong to ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{[w]}$, or even ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{\geqslant w}$, for all n. On the other hand, the only objects that belong to ${}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{\leqslant w}$ for all n are those in ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{<w}$, and only the zero object lies in ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{<w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$. Thus, this t-structure is non-degenerate. Its boundedness then follows from the boundedness of the standard t-structure on $\mathcal{D}^{\mathrm{b}}_G(X)$. □

Definition 3.5. The t-structure of Proposition 3.4 is called the *purified standard t-structure*, or simply the *purified t-structure*, on ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$. Its truncation functors are denoted

$$ {}_q\tau^{\leqslant n}_{[w]} : {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]} \to {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{[w]} \quad \text{and} \quad {}_q\tau^{\geqslant n}_{[w]} : {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]} \to {}_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{[w]}. $$

4. Pure-perverse coherent sheaves

Let $q\colon \mathbb{O}(X) \to \mathbb{Z}$ be a function. In this section, we construct a new t-structure on the category ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$ of pure objects, called the *pure-perverse t-structure*. It is related to the purified standard t-structure in the same way the perverse coherent t-structure of [**7**] is related to the standard t-structure on $\mathcal{D}^{\mathrm{b}}_G(X)$. We then prove that the heart of the pure-perverse t-structure is contained in the heart of a suitable staggered t-structure $({}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}, {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0})$. This is an important step towards the Purity Theorem, as it will enable us to prove in the next section that a certain operation in the heart of the staggered t-structure can be replaced by one in the heart of the pure-perverse t-structure.

The construction of the pure-perverse t-structure closely follows the construction of the perverse coherent t-structure in [**7**]. As in [**7**], the pure-perverse t-structure depends on the choice of a monotone and comonotone Deligne–Bezrukavnikov perversity, i.e. a function $p\colon \mathbb{O}(X) \to \mathbb{Z}$ satisfying

$$ 0 \leqslant p(C') - p(C) \leqslant \operatorname{cod} C' - \operatorname{cod} C $$

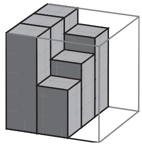
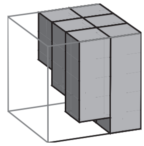
whenever $C' \subset \bar{C}$.

Fix a monotone and comonotone Deligne–Bezrukavnikov perversity $p\colon \mathbb{O}(X) \to \mathbb{Z}$. Define full subcategories of ${}_q\mathcal{D}^{-}_G(X)_{\leqslant w}$ and ${}_q\mathcal{D}^{+}_G(X)_{\geqslant w}$ as follows:

$$ {}^p_q\mathcal{D}^{-}_G(X)^{\leqslant n}_{\leqslant w} = \{\mathcal{F} \mid Li^*_C\mathcal{F}|_C \in {}_q\mathcal{D}^{-}_G(C)^{\leqslant n+p(C)}_{\leqslant w} \text{ for all } C \in \mathbb{O}(X)\}, $$
$$ {}^p_q\mathcal{D}^{+}_G(X)^{\geqslant n}_{\geqslant w} = \mathbb{D}({}^{\tilde{p}}_{\hat{q}}\mathcal{D}^{-}_G(X)^{\leqslant -n}_{\leqslant -w}). $$

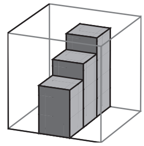
It follows from the gluing theorem for baric structures [**5**, Theorem 4.12] and induction on the number of orbits that ${}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w} \subset {}_q\mathcal{D}^-_G(X)_{\leqslant w}$, and hence that ${}^p_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w} \subset {}_q\mathcal{D}^+_G(X)_{\geqslant w}$.

The set $\mathbb{O}(X)$ is, of course, partially ordered by inclusion. Suppose for a moment that this partial order is, in fact, a total order. In this case, we can draw pictures of the above categories similar to our pictures of other subcategories of $\mathcal{D}^{\mathrm{b}}_G(X)$, by regarding the third axis of the grid as representing orbits in $\mathbb{O}(X)$, with larger orbits closer to the reader, and smaller orbits father away. Since p takes larger values on smaller orbits, we may draw the bounded versions of ${}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ and ${}^p_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w}$ thus:

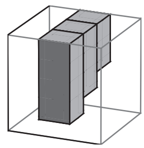
${}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{\leqslant w}$:  ${}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{\geqslant w}$: 

(The picture of ${}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{\geqslant w}$ has been drawn from an unusual perspective to make its structure visible.) We will also work with the intersections of these categories with the pure category ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$:

$${}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{[w]} = {}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{\leqslant w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}:$$



$${}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{[w]} = {}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{\geqslant w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}:$$



These pictures do not make much sense if $\mathbb{O}(X)$ is not totally ordered, but they may nevertheless be a helpful source of intuition.

Lemma 4.1. *Let $j\colon U \hookrightarrow X$ be the inclusion of an open subscheme, and $i\colon Z \hookrightarrow X$ the inclusion of a closed subscheme. Then*

- *j^* takes ${}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ to ${}^p_q\mathcal{D}^-_G(U)^{\leqslant n}_{\leqslant w}$ and ${}^p_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w}$ to ${}^p_q\mathcal{D}^+_G(U)^{\geqslant n}_{\geqslant w}$;*
- *Li^* takes ${}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ to ${}^p_q\mathcal{D}^-_G(Z)^{\leqslant n}_{\leqslant w}$;*
- *$Ri^!$ takes ${}^p_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w}$ to ${}^p_q\mathcal{D}^+_G(Z)^{\geqslant n}_{\geqslant w}$;*
- *i_* takes ${}^p_q\mathcal{D}^-_G(Z)^{\leqslant n}_{\leqslant w}$ to ${}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ and ${}^p_q\mathcal{D}^+_G(Z)^{\geqslant n}_{\geqslant w}$ to ${}^p_q\mathcal{D}^+_G(X)^{\geqslant n}_{\geqslant w}$.*

Proof. Parts (1) and (2) are immediate from the definition of ${}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$, and part (3) follows by duality. Similarly, because i_* commutes with $\mathbb{D}$, the second part of part (4) follows from the first part.

It remains to show that if $\mathcal{F} \in {}^p_q\mathcal{D}^-_G(Z)^{\leqslant n}_{\leqslant w}$, then $i_*\mathcal{F} \in {}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$. We must show that for any orbit $C \in \mathbb{O}(X)$, $Li^*_C i_*\mathcal{F}|_C \in {}_q\mathcal{D}^-_G(C)^{\leqslant n+p(C)}_{\leqslant w}$. In fact, it suffices to consider the case where C is a closed orbit contained in Z: if $C \not\subset Z$, then $Li^*_C i_*\mathcal{F}|_C = 0$, and if C is not closed, the operation $Li^*_C(\cdot)|_C$ factors as restriction to the open subscheme $V = X \smallsetminus (\bar{C} \smallsetminus C)$ followed by pullback to the closed subscheme $C \subset V$, and we already know by part (1) that restriction to V takes ${}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ to ${}^p_q\mathcal{D}^-_G(V)^{\leqslant n}_{\leqslant w}$.

Assume, therefore, that C is a closed orbit contained in Z. If

$$\mathcal{F} \in {}^p_q\mathcal{D}^-_G(Z)^{\leqslant n}_{\leqslant w}$$

but

$$Li^*_C i_*\mathcal{F} \notin {}_q\mathcal{D}^-_G(C)^{\leqslant n+p(C)}_{\leqslant w},$$

then, by Lemma 3.2, there exists an object $\mathcal{G} \in {}_q\mathcal{D}^{\mathrm{b}}_G(C)^{>n+p(C)}_{[w]}$ such that

$$\mathrm{Hom}(Li^*_C i_*\mathcal{F}, \mathcal{G}) \neq 0.$$

By adjunction, it follows that

$$\mathrm{Hom}(\mathcal{F}, Ri^! i_{C*}\mathcal{G}) \neq 0,$$

and by Lemma 3.1, we have $Ri^! i_{C*}\mathcal{G} \in {}_q\mathcal{D}^+_G(Z)^{>n+p(C)}_{\geqslant w}$. Now, let $W = Z \smallsetminus C$, and consider the exact sequence

$$\varinjlim_{Z'} \mathrm{Hom}(Li^*_{Z'}\mathcal{F}, Ri^!_{Z'} Ri^! i_{C*}\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, Ri^! i_{C*}\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}|_W, Ri^! i_{C*}\mathcal{G}|_W),$$

where $i_{Z'}\colon Z' \hookrightarrow Z$ ranges over all closed subscheme structures on $C \subset Z$. (For an explanation of this exact sequence, see, for instance, the proof of [**7**, Proposition 2]. Similar sequences will be used in Lemmas 4.3 and 4.6 and in Proposition 9.3.) The last term vanishes since $Ri^! i_{C*}\mathcal{G}|_W = 0$. Moreover, we have

$$Li^*_{Z'}\mathcal{F} \in {}_q\mathcal{D}^-_G(Z')^{\leqslant n+p(C)}_{\leqslant w}$$

for any subscheme structure, by Lemma 2.5. On the other hand,

$$Ri^!_{Z'} Ri^! i_{C*}\mathcal{G} \in {}_q\mathcal{D}^+_G(Z')^{>n+p(C)}_{\geqslant w}$$

by Lemma 3.1, so the first term above vanishes by Lemma 3.2. Thus, the middle term vanishes as well, a contradiction. Therefore, $i_*\mathcal{F} \in {}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$. □

Lemma 4.2. *Let d be the minimum value of* $\mathrm{cod}\, C$ *over all $C \in \mathbb{O}(X)$. If $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ and $\mathcal{G} \in {}_{\hat{q}}\mathcal{D}^-_G(X)^{\leqslant d-n}_{\leqslant -w}$, then* $\mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathcal{G}) = 0$.

Proof. We know, by the definition of ${}_{\hat{q}}\mathcal{D}^-_G(X)^{\leqslant d-n}_{\leqslant -w}$, that there is a distinguished triangle $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to$ with $\mathcal{G}' \in \mathcal{D}^-_G(X)^{<d-n}$ and $\mathcal{G}'' \in {}_{\hat{q}}\mathcal{D}^-_G(X)_{<-w}$. The fact that $\mathcal{G} \in {}_{\hat{q}}\mathcal{D}^-_G(X)_{\leqslant -w}$ implies that $\mathcal{G}' \in {}_{\hat{q}}\mathcal{D}^-_G(X)_{\leqslant -w}$ as well. Applying $\mathbb{D}$, we obtain a distinguished triangle

$$\mathbb{D}\mathcal{G}'' \to \mathbb{D}\mathcal{G} \to \mathbb{D}\mathcal{G}' \to .$$

Note that $\mathbb{D}\mathcal{G}'' \in {}_q\mathcal{D}^+_G(X)_{>w}$. Since $\mathcal{F} \in {}_q\mathcal{D}^-_G(X)_{\leqslant w}$, we see that $\mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathcal{G}'') = \mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathcal{G}''[1]) = 0$, so $\mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathcal{G}')$. Now, $\mathcal{F}$ arises in some distinguished triangle

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$$

with $\mathcal{F}' \in \mathcal{D}^-_G(X)^{\leqslant n}$ and $\mathcal{F}'' \in {}_q\mathcal{D}^-_G(X)_{<w}$. Note that the definition of d is such that $\mathbb{D}(\mathcal{D}^-_G(X)^{<d-n}) \subset \mathcal{D}^+_G(X)^{>n}$. Therefore, we see that $\mathbb{D}\mathcal{G}' \in {}_q\mathcal{D}^+_G(X)_{\geqslant w} \cap \mathcal{D}^+_G(X)^{>n}$. It follows that $\mathrm{Hom}(\mathcal{F}', \mathbb{D}\mathcal{G}') = 0$ and $\mathrm{Hom}(\mathcal{F}'', \mathbb{D}\mathcal{G}') = 0$. We conclude that $\mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathcal{G}') = 0$, and hence that $\mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathcal{G}) = 0$, as desired. □

Lemma 4.3. *If* $\mathcal{F} \in {}^p_q\mathcal{D}^-_G(X)^{\leqslant n}_{\leqslant w}$ *and* $\mathcal{G} \in {}^p_q\mathcal{D}^+_G(X)^{>n}_{\geqslant w}$, *then* $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

Proof. We proceed by noetherian induction, and assume the statement is known on all proper closed subschemes of X. Let $\mathcal{G}' = \mathbb{D}\mathcal{G} \in {}^{\tilde{p}}_{\hat{q}}\mathcal{D}^-_G(X)^{<-n}_{\leqslant -w}$. Choose an open orbit $C \in \mathbb{O}(X)$, and let $U \subset X$ be the corresponding open subscheme. By Lemma 2.5, $\mathcal{F}|_U \in {}_q\mathcal{D}^-_G(U)^{\leqslant n+p(C)}_{\leqslant w}$ and $\mathcal{G}'|_U \in {}_{\hat{q}}\mathcal{D}^-_G(U)^{<-n+\tilde{p}(C)}_{\leqslant -w}$. Of course, $-n + \tilde{p}(C) = \operatorname{cod} C - (n + p(C))$, so by Lemma 4.2, $\mathrm{Hom}(\mathcal{F}|_U, \mathbb{D}\mathcal{G}'|_U) = 0$. Now, let Z be the complementary closed subspace to U, and consider the exact sequence

$$\varinjlim_{Z'} \mathrm{Hom}(Li^*_{Z'}\mathcal{F}, Ri^!_{Z'}\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $i_{Z'}\colon Z' \hookrightarrow X$ ranges over all closed subscheme structures on Z. We have just seen that the last term vanishes. Since $Li^*_{Z'}\mathcal{F} \in {}^p_q\mathcal{D}^-_G(Z')^{\leqslant n}_{\leqslant w}$ and $Ri^!_{Z'}\mathcal{G} \in {}^p_q\mathcal{D}^+_G(Z')^{>n}_{\geqslant w}$, the first term vanishes by induction. So $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$, as desired. □

Proposition 4.4. $({}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{[w]}, {}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}_{[w]})$ *is a non-degenerate, bounded t-structure on* ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$.

Definition 4.5. The t-structure of Proposition 4.4 is called the *pure-perverse t-structure.* Its truncation functors are denoted

$${}^p_q\tau^{\leqslant n}_{[w]}\colon {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]} \to {}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant n}_{[w]} \quad \text{and} \quad {}^p_q\tau^{\geqslant n}_{[w]}\colon {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]} \to {}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant n}_{[w]},$$

and its heart, denoted ${}^p_q\mathcal{P}(X)_{[w]}$, is called the category of *pure-perverse coherent sheaves.*

Proof. In view of Lemma 4.3, to show that these categories form a t-structure, it remains only to show that for any $\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$, there is a distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ with $\mathcal{F}' \in {}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{[w]}$ and $\mathcal{F}'' \in {}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{>0}_{[w]}$. Our argument closely follows the proof of [**7**, Theorem 1]. Choose an open orbit $C \in \mathbb{O}(X)$ on which p achieves its minimum value, and let $U \subset X$ be the corresponding open subscheme. (The monotonicity of p guarantees that its minimum value is achieved on an open orbit.) Let $\mathcal{F}_1 = {}_q\tau^{\leqslant p(C)}_{[w]}\mathcal{F}$. By Lemma 3.1 and the monotonicity of p, we have that $\mathcal{F}_1 \in {}^p_q\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{[w]}$. Form the distinguished triangle

$$\mathcal{F}_1 \to \mathcal{F} \to \mathcal{G}_1 \to,$$

where $\mathcal{G}_1 = {}_q\tau_{[w]}^{>p(C)}\mathcal{F}$. It is clear that $\mathbb{D}\mathcal{F} \in {}_{\hat{q}}\mathcal{D}_G^{\mathrm{b}}(X)_{\leqslant -w}$, and it follows from [**1**, Lemmas 6.1 and 6.6] that $\mathbb{D}\mathcal{F}|_U \in \mathcal{D}_G^{\mathrm{b}}(X)^{\leqslant \operatorname{cod} C - n}$, so that we have $\mathbb{D}(\mathcal{G}_1)|_U \in {}_{\hat{q}}\mathcal{D}_G^{\mathrm{b}}(X)_{[-w]}^{<\tilde{p}(C)}$. Therefore, in the distinguished triangle

$$ {}_{\hat{q}}\tau_{[-w]}^{<\tilde{p}(C)}(\mathbb{D}\mathcal{G}_1) \to \mathbb{D}\mathcal{G}_1 \to {}_{\hat{q}}\tau_{[-w]}^{\geqslant\tilde{p}(C)}\mathbb{D}\mathcal{G}_1 \to, $$

the support of the last term is contained in the complement of U. Let

$$ \mathcal{G} = \mathbb{D}({}_{\hat{q}}\tau_{[-w]}^{\geqslant\tilde{p}(C)}\mathbb{D}\mathcal{G}_1) \quad \text{and} \quad \mathcal{F}_2 = \mathbb{D}({}_{\hat{q}}\tau_{[-w]}^{<\tilde{p}(C)}(\mathbb{D}\mathcal{G}_1)). $$

Since $\tilde{p}$ is monotone, ${}_{\hat{q}}\tau_{[-w]}^{<\tilde{p}(C)}(\mathbb{D}\mathcal{G}_1) \in {}^{\tilde{p}}_{\hat{q}}\mathcal{D}_G^{\mathrm{b}}(X)_{[-w]}^{<0}$, and therefore $\mathcal{F}_2 \in {}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{>0}$. We now have

$$ \mathcal{F} \in \{\mathcal{F}_1\} * \{\mathcal{G}\} * \{\mathcal{F}_2\}, $$

with

$$ \mathcal{F}_1 \in {}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\leqslant 0}, \qquad \mathcal{F}_2 \in {}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{>0}, $$

and $\mathcal{G}$ supported on a proper closed subscheme. It follows by noetherian induction that $({}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\leqslant 0}, {}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\geqslant 0})$ is a t-structure. (See the proof of [**7**, Theorem 1] for more details on this argument.)

Next, let d be the minimum value of p on X, and let e be its maximum value. We then have

$$ {}_q\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\leqslant d} \subset {}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\leqslant 0} \subset {}_q\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\leqslant e}. $$

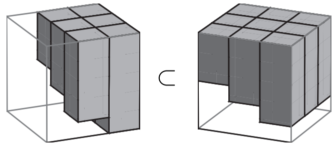
Then the non-degeneracy and boundedness of the purified standard t-structure imply that no non-zero object belongs to all ${}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\leqslant n}$, and every object belongs to some ${}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\leqslant n}$. By duality, corresponding statements hold for ${}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\geqslant n}$ as well, so the t-structure $({}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\leqslant 0}, {}^{p}_{q}\mathcal{D}_G^{\mathrm{b}}(X)_{[w]}^{\geqslant 0})$ is non-degenerate and bounded. □

Under suitable conditions on the perversity function, it is possible to define an 'intermediate-extension' functor for pure-perverse coherent sheaves, following the pattern of [**7**, Theorem 2]. Simple objects in this category arise in this way (cf. [**7**, Corollary 4]). In the next section (see Proposition 5.2), we will carry out a slight generalization of this construction.

The remainder of the section is devoted to establishing a relationship between pure-perverse coherent sheaves and staggered sheaves.

Lemma 4.6. *Suppose that $\mathcal{F} \in {}^{p}_{q}\mathcal{D}_G^{+}(X)_{\geqslant w}^{\geqslant 0}$. Let $r\colon \mathbb{O}(X) \to \mathbb{Z}$ be the function $r(C) = p(C) + \lceil (q(C)+w)/2 \rceil$. Then $\mathcal{F} \in {}^{r}\mathcal{D}_G^{+}(X)^{\geqslant 0}$.*

This statement can be thought of as saying that under a suitable change of coordinates, we have



The 'change of coordinates' is the change in the notion of baric degree between the two pictures: the left-hand picture shows baric degree with respect to q, and the right-hand picture shows baric degree with respect to $2r \approx 2p + q + w$.

Proof. It suffices to show that $\mathrm{Hom}(\mathcal{G},\mathcal{F}) = 0$ for all $\mathcal{G} \in {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant -1}$. By induction on the number of non-zero cohomology sheaves of $\mathcal{G}$, we may assume without loss of generality that $\mathcal{G}$ is concentrated in a single degree: suppose $\mathcal{G} \cong \mathcal{G}'[n+1]$ for some sheaf $\mathcal{G}' \in {}_{2r}\mathcal{C}_G(X)_{\leqslant 2n}$.

Choose an open orbit $C \in \mathbb{O}(X)$, and let $U \subset X$ be the corresponding open subscheme. Then $\mathcal{G}'|_U \in \mathcal{C}_G(U)_{\leqslant r(C)+n}$. By Lemma 4.1, we have

$$\mathcal{F}|_U \in {}^p_q\mathcal{D}^+_G(U)^{\geqslant 0}_{\geqslant w} = {}_q\mathcal{D}^+_G(U)^{\geqslant p(C)}_{\geqslant w}.$$

$\mathcal{G}$ is concentrated in degree $-n-1$, so if $-n-1 < p(C)$, we clearly have $\mathrm{Hom}(\mathcal{G}|_U, \mathcal{F}|_U) = 0$. Now, assume $-n-1 \geqslant p(C)$. It follows that

$$r(C) + n = p(C) + \left\lceil \frac{q(C)+w}{2} \right\rceil + n \leqslant \left\lceil \frac{q(C)+w}{2} \right\rceil - 1 = \left\lfloor \frac{q(C)+w-1}{2} \right\rfloor.$$

It follows that $\mathcal{G}'|_U \in \mathcal{C}_G(U)_{\leqslant \lfloor (q(C)+w-1)/2 \rfloor} = {}_q\mathcal{C}_G(U)_{\leqslant w-1}$. Thus, in this case, $\mathcal{G}|_U \in {}_q\mathcal{D}^{\mathrm{b}}_G(U)_{\leqslant w-1}$, and we see once again that $\mathrm{Hom}(\mathcal{G}|_U, \mathcal{F}|_U) = 0$. The result then follows by noetherian induction from the exact sequence

$$\varinjlim_{Z'} \mathrm{Hom}(Li^*_{Z'}\mathcal{G}, Ri^!_{Z'}\mathcal{F}) \to \mathrm{Hom}(\mathcal{G},\mathcal{F}) \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

□

Proposition 4.7. *Let $r\colon \mathbb{O}(X) \to \mathbb{Z}$ be such that $p(C) + \lfloor (q(C)+w)/2 \rfloor \leqslant r(C) \leqslant p(C) + \lfloor (q(C)+w)/2 \rfloor$. Then ${}^p_q\mathcal{P}(X)_{[w]} \subset {}^r\mathcal{M}(X)$.*

Proof. Suppose $\mathcal{F} \in {}^p_q\mathcal{P}(X)_{[w]}$. Let $r_1(C) = p(C) + \lfloor (q(C)+w)/2 \rfloor$. The preceding lemma tells us that $\mathcal{F} \in {}^{r_1}\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}$. On the other hand, $\mathbb{D}\mathcal{F} \in {}^{\tilde{p}}_{\hat{q}}\mathcal{P}(X)_{[-w]}$, and invoking the preceding lemma again tells us that $\mathbb{D}\mathcal{F} \in {}^{r_2}\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}$, where

$$\begin{aligned} r_2(C) &= \tilde{p}(C) + \left\lceil \frac{\hat{q}(C)-w}{2} \right\rceil \\ &= \operatorname{cod} C - p(C) + \left\lceil \operatorname{alt} C - \frac{q(C)+w}{2} \right\rceil \\ &= \operatorname{scod} C - \left(p(C) + \left\lfloor \frac{q(C)+w}{2} \right\rfloor \right). \end{aligned}$$

By duality, we have $\mathcal{F} \in {}^{r_3}\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}$, where $r_3(C) = p(C) + \lfloor (q(C)+w)/2 \rfloor$. Thus, for any $r\colon \mathbb{O}(X) \to \mathbb{Z}$ with $r_3(C) \leqslant r(C) \leqslant r_1(C)$, we have $\mathcal{F} \in {}^r\mathcal{M}(X)$. □

5. Intermediate-extension functors

In the previous section, we proved that every pure-perverse coherent sheaf is a staggered sheaf with respect to a suitable staggered perversity. In this section, we will prove a kind of converse to this: we will show that every simple staggered sheaf is pure-perverse with respect to suitable Deligne–Bezrukavnikov and baric perversities.

Fix an orbit C_0, and let $j\colon C_0 \hookrightarrow \bar{C}_0$ denote the inclusion. We define a staggered perversity ${}^\flat r\colon \mathbb{O}(X) \to \mathbb{Z}$ by

$$ {}^\flat r(C) = \begin{cases} r(C) - 1 & \text{if } \bar{C} \subsetneq \bar{C}_0, \\ r(C) & \text{otherwise.} \end{cases} $$

Next, we define an open subscheme $\tilde{C}_0 \subset \bar{C}_0$ by

$$ \tilde{C}_0 = \bar{C}_0 \smallsetminus \bigcup_{\{C \subset \bar{C}_0 \mid \operatorname{cod} C - \operatorname{cod} C_0 \geqslant 2\}} \bar{C}. $$

Let $p\colon \mathbb{O}(\bar{C}_0) \to \mathbb{Z}$ be a Deligne–Bezrukavnikov perversity such that

$$ 0 < p(C) - p(C_0) < \operatorname{cod} C - \operatorname{cod} C_0 \quad \text{for all } C \subset \bar{C}_0 \smallsetminus \tilde{C}_0. \tag{5.1} $$

Define two functions ${}^\flat p, {}^\sharp p\colon \mathbb{O}(\bar{C}) \to \mathbb{Z}$ as follows:

$$ {}^\flat p(C) = \begin{cases} p(C_0) & \text{if } C \subset \tilde{C}_0, \\ p(C) - 1 & \text{if } C \subset \bar{C}_0 \smallsetminus \tilde{C}_0, \end{cases} $$

$$ {}^\sharp p(C) = \begin{cases} p(C_0) & \text{if } C = C_0, \\ p(C_0) + 1 & \text{if } C \subset \tilde{C}_0 \smallsetminus C_0, \\ p(C) + 1 & \text{if } C \subset \bar{C}_0 \smallsetminus \tilde{C}_0. \end{cases} $$

It is easy to verify that ${}^\flat p$ and ${}^\sharp p$ are themselves monotone and comonotone Deligne–Bezrukavnikov perversities, so they give rise to additional pure-perverse t-structures on ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$. Note also that ${}^\flat p(C) \leqslant p(C) \leqslant {}^\sharp p(C)$ for all $C \subset \bar{C}$. (For $C \subset \tilde{C}_0 \smallsetminus C_0$, this follows from the fact that $0 \leqslant p(C) - p(C_0) \leqslant \operatorname{cod} C - \operatorname{cod} C_0 = 1$.) Therefore, for any baric perversity q, we have

$$ {}^{{}^\flat p}_{q}\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{[w]} \subset {}^{p}_{q}\mathcal{D}^{\mathrm{b}}_G(X)^{\leqslant 0}_{[w]} \quad \text{and} \quad {}^{{}^\sharp p}_{q}\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}_{[w]} \subset {}^{p}_{q}\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant 0}_{[w]}. $$

Define full subcategories of ${}^p_q\mathcal{P}(\tilde{C}_0)$ and of ${}^p_q\mathcal{P}(\bar{C}_0)$ as follows:

$$ {}^p_q\mathcal{P}^\natural(\tilde{C}_0)_{[w]} = {}^{{}^\flat p}_{q}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant 0}_{[w]} \cap {}^{{}^\sharp p}_{q}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\geqslant 0}_{[w]}, $$
$$ {}^p_q\mathcal{P}^\natural(\bar{C}_0)_{[w]} = {}^{{}^\flat p}_{q}\mathcal{D}^{\mathrm{b}}_G(\bar{C}_0)^{\leqslant 0}_{[w]} \cap {}^{{}^\sharp p}_{q}\mathcal{D}^{\mathrm{b}}_G(\bar{C}_0)^{\geqslant 0}_{[w]}. $$

Lemma 5.1. *Let $\mathcal{L}$ be a sheaf in $\mathcal{C}_G(C_0)$ that is s-pure of step $v \in \mathbb{Z}$. Define a Deligne–Bezrukavnikov perversity $p\colon \mathbb{O}(\tilde{C}_0) \to \mathbb{Z}$ and a baric perversity $q\colon \mathbb{O}(\tilde{C}_0) \to \mathbb{Z}$ by*

$$ p(C) = r(C_0) - v \quad \text{and} \quad q(C) = \operatorname{alt} C_0 + 2\,{}^\flat r(C) - 2r(C_0). $$

Let $w = 2v - \operatorname{alt} C_0$. Then ${}^r j_{!*}\mathcal{L}[v - r(C_0)]|_{\tilde{C}_0} \in {}^p_q\mathcal{P}^\natural(\tilde{C}_0)_{[w]}$.

Proof. Let $\mathcal{F} = {}^r j_{!*}(\mathcal{L}[v - r(C_0)])|_{\tilde{C}_0}$. We know that $\mathcal{F} \in {}^{\flat r}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant 0}$, so $\tau^{<r(C_0)-v}\mathcal{F}$ belongs to ${}^{\flat r}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant 0}$ as well. Since $\mathcal{F}|_{C_0} \cong \mathcal{L}[v - r(C_0)]$, we see that $\tau^{<r(C_0)-v}\mathcal{F}$ is supported on $\tilde{C}_0 \smallsetminus C_0$, so in fact $\tau^{<r(C_0)-v}\mathcal{F} \in {}^r\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant -1}$. But there can be no non-zero morphism from an object of ${}^r\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant -1}$ to one in ${}^r\mathcal{M}(\tilde{C}_0)$, so $\tau^{<r(C_0)-v}\mathcal{F} = 0$, and $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\geqslant r(C_0)-v}$.

Next, we have

$$h^k(\mathcal{F}) \in {}_{2\flat r}\mathcal{C}_G(\tilde{C}_0)_{\leqslant -2k} = {}_q\mathcal{C}_G(\tilde{C}_0)_{\leqslant -2k+2r(C_0)-\operatorname{alt} C_0}.$$

We have just seen that $h^k(\mathcal{F}) = 0$ for $k < r(C_0) - v$. When $k \geqslant r(C_0) - v$, we have

$$-2k + 2r(C_0) - \operatorname{alt} C_0 \leqslant 2v - \operatorname{alt} C_0 = w,$$

and the inequality is strict when $k > r(C_0)-v$. Thus, $\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)_{\leqslant w}$, and $\tau^{>r(C_0)-v}\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)_{<w}$. The distinguished triangle

$$\tau^{\leqslant r(C_0)-v}\mathcal{F} \to \mathcal{F} \to \tau^{>r(C_0)-v}\mathcal{F} \to$$

then shows that

$$\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant r(C_0)-v}_{\leqslant w} = {}^{\flat p}_{q}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant 0}_{\leqslant w}.$$

It remains to show that $\mathcal{F} \in {}^{\sharp p}_{q}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\geqslant 0}_{\geqslant w}$. Let $\mathcal{G} = \mathbb{D}\mathcal{F}$. Then $\mathcal{G}$ also arises as an intermediate-extension. Specifically, let $\mathcal{L}' = (\mathbb{D}\mathcal{L})[\operatorname{cod} C_0]$; then $\mathcal{L}'$ is a sheaf in $\mathcal{C}_G(C_0)$ that is pure of step $v' = \operatorname{alt} C_0 - v$. We have

$$\mathcal{G} = {}^{\bar{r}} j_{!*}(\mathcal{L}'[v' - \bar{r}(C_0)])|_{\tilde{C}_0}.$$

By the arguments above, we know that $\mathcal{G} \in {}_{q'}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant \bar{r}(C_0)-v'}_{\leqslant w'}$, where

$$q'(C) = \operatorname{alt} C_0 + 2{}^\flat\bar{r}(C) - 2\bar{r}(C_0) \quad \text{and} \quad w' = 2v' - \operatorname{alt} C_0.$$

Observe that

$$w' = 2(\operatorname{alt} C_0 - v) - \operatorname{alt} C_0 = \operatorname{alt} C_0 - 2v = -w.$$

Next, note that $\operatorname{cod} C - \operatorname{cod} C_0 = {}^\sharp p(C) - {}^\flat p(C)$ for all $C \subset \tilde{C}_0$, so

$$\begin{aligned}
{}^\sharp p(C) &= \operatorname{cod} C - \operatorname{cod} C_0 + {}^\flat p(C)\\
&= \operatorname{cod} C - \operatorname{cod} C_0 + (r(C_0) - v)\\
&= \operatorname{cod} C - \operatorname{cod} C_0 + (\operatorname{alt} C_0 + \operatorname{cod} C_0 - \bar{r}(C_0) - (\operatorname{alt} C_0 - v'))\\
&= \operatorname{cod} C - (\bar{r}(C_0) - v').
\end{aligned}$$

It follows that

$$\mathbb{D}({}_{q'}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\leqslant \bar{r}(C_0)-v'}_{\leqslant -w}) = {}^{\sharp p}_{\hat{q}'}\mathcal{D}^{\mathrm{b}}_G(\tilde{C}_0)^{\geqslant 0}_{\geqslant w}.$$

From the formula

$$\hat{q}'(C) = 2\operatorname{alt} C - (\operatorname{alt} C_0 + 2{}^\flat\bar{r}(C) - 2\bar{r}(C_0)),$$

we see that $\hat{q}'(C_0) = \operatorname{alt} C_0 = q(C_0)$, and that for $C \subset \tilde{C}_0 \smallsetminus C_0$, we have

$$\begin{aligned}\hat{q}'(C) &= 2 \operatorname{alt} C - \operatorname{alt} C_0 - 2(\operatorname{scod} C - r(C) - 1) + 2(\operatorname{scod} C_0 - r(C_0)) \\ &= \operatorname{alt} C_0 - 2(\operatorname{cod} C - \operatorname{cod} C_0) + 2 + 2r(C) - 2r(C_0) > q(C).\end{aligned}$$

Thus, $\hat{q}'(C) \geqslant q(C)$ for all C, so $\mathcal{F} \cong \mathbb{D}\mathcal{G} \in {}^{\sharp}{}^{p}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\tilde{C}_0)^{\geqslant 0}_{\geqslant w}$, as desired. □

Proposition 5.2. *Let $\tilde{\jmath}\colon \tilde{C}_0 \hookrightarrow \bar{C}_0$ denote the inclusion. Assume that $p\colon \mathbb{O}(X) \to \mathbb{Z}$ satisfies condition (5.1). Then $\tilde{\jmath}^*$ induces an equivalence of categories ${}^{p}_{q}\mathcal{P}^{\natural}(\bar{C}_0)_{[w]} \to {}^{p}_{q}\mathcal{P}^{\natural}(\tilde{C}_0)_{[w]}$.*

Proof. The proof of this proposition is copied verbatim, except for minor changes in notation, from [**3**, Proposition 2.3], which in turn is closely based on [**7**, Theorem 2]. Let

$$J_{!*}\colon {}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)_{[w]} \to {}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)_{[w]}$$

be the functor ${}^{\flat}{}^{p}_{q}\tau^{\leqslant 0} \circ {}^{\sharp}{}^{p}_{q}\tau^{\geqslant 0}$. We claim that $J_{!*}$ actually takes values in ${}^{p}_{q}\mathcal{P}^{\natural}(\bar{C}_0)_{[w]}$. Given $\mathcal{F} \in {}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)_{[w]}$, let $\mathcal{F}_1 = {}^{\sharp}{}^{p}_{q}\tau^{\geqslant 0}\mathcal{F}$. Then we have a distinguished triangle

$$({}^{\flat}{}^{p}_{q}\tau^{\geqslant 1}\mathcal{F}_1)[-1] \to J_{!*}(\mathcal{F}) \to \mathcal{F}_1 \to \ .$$

Note that

$$({}^{\flat}{}^{p}_{q}\tau^{\geqslant 1}\mathcal{F}_1)[-1] \in {}^{\flat}{}^{p}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)^{\geqslant 2}_{[w]}.$$

Now, ${}^{\sharp}p(C) - {}^{\flat}p(C) \leqslant 2$ for all $C \subset \bar{C}_0$, and this implies that

$${}^{\flat}{}^{p}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)^{\geqslant 2}_{[w]} \subset {}^{\sharp}{}^{p}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)^{\geqslant 0}_{[w]}.$$

Clearly,

$$\mathcal{F}_1 \in {}^{\sharp}{}^{p}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)^{\geqslant 0}_{[w]},$$

so it follows that $J_{!*}\mathcal{F} \in {}^{\sharp}{}^{p}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)^{\geqslant 0}_{[w]}$. Since $J_{!*}$ obviously takes values in ${}^{\flat}{}^{p}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)^{\leqslant 0}_{[w]}$, we have $J_{!*}\mathcal{F} \in {}^{p}_{q}\mathcal{P}^{\natural}(\bar{C}_0)_{[w]}$.

Next, note that if $\mathcal{F} \in {}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)_{[w]}$ is such that $\mathcal{F}|_{\tilde{C}_0} \in {}^{p}_{q}\mathcal{P}^{\natural}(\tilde{C}_0)_{[w]}$, then both $({}^{\sharp}{}^{p}_{q}\tau^{\geqslant 0}\mathcal{F})|_{\tilde{C}_0}$ and $({}^{\flat}{}^{p}_{q}\tau^{\leqslant 0}\mathcal{F})|_{\tilde{C}_0}$, and hence $(J_{!*}\mathcal{F})|_{\tilde{C}_0}$, are isomorphic to $\mathcal{F}|_{\tilde{C}_0}$. In particular, we can see now that $\tilde{\jmath}^*$ is essentially surjective. Given $\mathcal{F} \in {}^{p}_{q}\mathcal{P}^{\natural}(\tilde{C}_0)_{[w]}$, let $\tilde{\mathcal{F}}$ be any object in $\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)$ such that $\tilde{\jmath}^*\tilde{\mathcal{F}} \cong \mathcal{F}$. (Such an object exists by [**7**, Corollary 2].) Replacing $\tilde{\mathcal{F}}$ by ${}_{q}\beta_{\leqslant w}{}_{q}\beta_{\geqslant w}\tilde{\mathcal{F}}$, we may assume that $\tilde{\mathcal{F}} \in {}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)_{[w]}$. Then $\mathcal{F}' = J_{!*}(\tilde{\mathcal{F}})$ is an object of ${}^{p}_{q}\mathcal{P}^{\natural}(\bar{C}_0)_{[w]}$ such that $\tilde{\jmath}^*\mathcal{F}' \cong \mathcal{F}$.

Now, if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism in ${}^{p}_{q}\mathcal{P}^{\natural}(\tilde{C}_0)_{[w]}$, then by [**7**, Corollary 2], we can find objects $\mathcal{F}'$ and $\mathcal{G}'$ in $\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)$ and a morphism $\phi'\colon \mathcal{F}' \to \mathcal{G}'$ such that $\tilde{\jmath}^*\mathcal{F}' \cong \mathcal{F}$, $\tilde{\jmath}^*\mathcal{G}' \cong \mathcal{G}$, and $\tilde{\jmath}^*\phi' \cong \phi$. By applying ${}_{q}\beta_{\leqslant w} \circ {}_{q}\beta_{\geqslant w}$ and then $J_{!*}$, we may assume that $\mathcal{F}'$, $\mathcal{G}'$, and ϕ' actually belong to ${}^{p}_{q}\mathcal{P}^{\natural}(\bar{C}_0)_{[w]}$. This shows that $\tilde{\jmath}^*$ is full.

To show that $\tilde{\jmath}^*$ is faithful, it suffices to show that if ϕ is an isomorphism, then ϕ' must be as well. Since $\phi'|_{\tilde{C}_0}$ is an isomorphism, the kernel and cokernel of ϕ' must be supported on $\bar{C}_0 \smallsetminus \tilde{C}_0$. Thus, the proof of the proposition will be complete once we prove that an object of ${}^p_q\mathcal{P}^\natural(\bar{C}_0)_{[w]}$ has no non-zero subobjects or quotients in ${}^p_q\mathcal{P}(\bar{C}_0)_{[w]}$ that are supported on $\bar{C}_0 \smallsetminus \tilde{C}_0$.

Let $\mathcal{F} \in {}^p_q\mathcal{P}^\natural(\bar{C}_0)_{[w]}$, and let $\mathcal{G} \in {}^p_q\mathcal{P}(\bar{C}_0)_{[w]}$ be a non-zero object supported on $\bar{C}_0 \smallsetminus \tilde{C}_0$. We will actually show that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{G}, \mathcal{F}) = 0$. There exists some closed subscheme structure $i\colon Z \hookrightarrow \bar{C}_0$ on $\bar{C}_0 \smallsetminus \tilde{C}_0$ and some object $\mathcal{G}' \in {}^p_q\mathcal{P}(Z)_{[w]}$ such that $\mathcal{G} \cong i_*\mathcal{G}'$. Then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(Li^*\mathcal{F}, \mathcal{G}')$. By Lemma 4.1, $Li^*\mathcal{F} \in {}^{\flat p}_{\ q}\mathcal{D}^-_G(Z)^{\leqslant 0}_{\leqslant w}$. Clearly, ${}^{\flat p}_{\ q}\mathcal{D}^-_G(Z)^{\leqslant 0}_{\leqslant w} = {}^p_q\mathcal{D}^-_G(Z)^{\leqslant -1}_{\leqslant w}$, and since $\mathcal{G}' \in {}^p_q\mathcal{D}^{\mathrm{b}}_G(Z)^{\geqslant 0}_{\geqslant w}$, we see that $\mathrm{Hom}(Li^*\mathcal{F}, \mathcal{G}') = 0$. Similarly, $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(\mathcal{G}', Ri^!\mathcal{F}) = 0$ because $Ri^!\mathcal{F} \in {}^{\sharp p}_{\ q}\mathcal{D}^+_G(Z)^{\geqslant 0}_{\geqslant w} = {}^p_q\mathcal{D}^+_G(Z)^{\geqslant 1}_{\geqslant w}$. □

Proposition 5.3. *Let $\mathcal{L} \in \mathcal{C}_G(C_0)$ be a coherent sheaf, s-pure of step v. Define a Deligne–Bezrukavnikov perversity $p\colon \mathbb{O}(\bar{C}_0) \to \mathbb{Z}$ and a baric perversity $q\colon \mathbb{O}(\bar{C}_0) \to \mathbb{Z}$ by*

$$p(C) = \begin{cases} r(C_0) - v & \text{if } C \subset \tilde{C}_0, \\ r(C_0) - v + \operatorname{cod} C - \operatorname{cod} C_0 - 1 & \text{if } C \subset \bar{C}_0 \smallsetminus \tilde{C}_0, \end{cases}$$

$$q(C) = \begin{cases} \operatorname{alt} C_0 + 2r(C) - 2r(C_0) - 2\operatorname{cod} C + 2\operatorname{cod} C_0 & \text{if } C \subset \tilde{C}_0, \\ \operatorname{alt} C_0 + 2r(C) - 2r(C_0) - 2\operatorname{cod} C + 2\operatorname{cod} C_0 + 1 & \text{if } C \subset \bar{C}_0 \smallsetminus \tilde{C}_0. \end{cases}$$

Let $w = 2v - \operatorname{alt} C_0$. Then ${}^r j_{!*}(\mathcal{L}[v - r(C_0)]) \in {}^p_q\mathcal{P}^\natural(\bar{C}_0)_{[w]}$.

Proof. We first prove that p is a monotone and comonotone Deligne–Bezrukavnikov perversity. Suppose $C' \subset \bar{C}$. It is easy to check that

$$p(C') - p(C) = \begin{cases} 0 & \text{if } C, C' \subset \tilde{C}_0, \\ \operatorname{cod} C' - \operatorname{cod} C_0 - 1 & \text{if } C \subset \tilde{C}_0 \text{ but } C' \not\subset \tilde{C}_0, \\ \operatorname{cod} C' - \operatorname{cod} C & \text{if } C, C' \subset \bar{C}_0 \smallsetminus \tilde{C}_0. \end{cases}$$

In all cases, it follows that

$$0 \leqslant p(C') - p(C) \leqslant \operatorname{cod} C' - \operatorname{cod} C.$$

Moreover, in the case where $C = C_0$ and $C' \subset \bar{C}_0 \smallsetminus \tilde{C}_0$, we know that $\operatorname{cod} C' - \operatorname{cod} C_0 \geqslant 2$, and it follows that condition (5.1) holds.

Note that for $C \subset \tilde{C}_0 \smallsetminus C_0$, we have $\operatorname{cod} C - \operatorname{cod} C_0 = 1$, so the restrictions to $\mathbb{O}(\tilde{C}_0)$ of the functions p and q defined here agree with those defined in Lemma 5.1. Let $\mathcal{F} = {}^r j_{!*}(\mathcal{L}[v - r(C_0)])$. By Lemma 5.1, $\mathcal{F}|_{\tilde{C}_0} \in {}^p_q\mathcal{P}^\natural(\tilde{C}_0)_{[w]}$. Then, because the inequalities (5.1) hold, we may invoke Proposition 5.2, which gives us a unique object $\mathcal{G} \in {}^p_q\mathcal{P}^\natural(\bar{C}_0)_{[w]}$ such that $\tilde{\jmath}^*\mathcal{G} \cong \mathcal{F}|_{\tilde{C}_0}$. We must show that $\mathcal{G} \cong \mathcal{F}$.

A straightforward calculation shows that

$${}^\flat p(C) + \frac{q(C) + w}{2} = \begin{cases} r(C_0) & \text{if } C = C_0, \\ r(C) - 1 & \text{if } C \subset \tilde{C}_0 \smallsetminus C_0, \\ r(C) - \frac{3}{2} & \text{if } C \subset \bar{C}_0 \smallsetminus \tilde{C}_0. \end{cases}$$

Thus, ${}^{\flat}p(C) + \lceil (q(C)+w)/2 \rceil = {}^{\flat}r(C)$. Since $\mathcal{G} \in {}^{\flat p}_{\;q}\mathcal{P}(\bar{C}_0)_{[w]}$, Proposition 4.7 tells us that $\mathcal{G} \in {}^{\flat r}\mathcal{M}(\bar{C}_0)$. Similarly, we have

$$
{}^{\sharp}p(C) + \frac{q(C)+w}{2} = \begin{cases} r(C_0) & \text{if } C = C_0, \\ r(C) & \text{if } C \subset \tilde{C}_0 \smallsetminus C_0, \\ r(C) + \frac{1}{2} & \text{if } C \subset \bar{C} \smallsetminus \tilde{C}_0. \end{cases}
$$

Let $s(C) = {}^{\sharp}p(C) + \lceil (q(C)+w)/2 \rceil$. Then, as before, Proposition 4.7 tells us that $\mathcal{G} \in {}^{s}\mathcal{M}(\bar{C}_0)$. But s and ${}^{\sharp}r$ agree on $\mathbb{O}(\bar{C}_0) \smallsetminus \mathbb{O}(\tilde{C}_0)$, and we already know that $\mathcal{G}|_{\tilde{C}_0} \cong \mathcal{F}|_{\tilde{C}0} \in {}^{\sharp r}\mathcal{M}(\tilde{C}_0)$, so we may conclude that $\mathcal{G} \in {}^{\sharp r}\mathcal{M}(\bar{C}_0)$.

Since $\mathcal{F}$ is, up to isomorphism, the unique object in ${}^{\flat r}\mathcal{M}(\bar{C}_0) \cap {}^{\sharp r}\mathcal{M}(\bar{C}_0)$ with the property that $\mathcal{F}|_{C_0} \cong \mathcal{L}[v - r(C_0)]$, we conclude that $\mathcal{G} \cong \mathcal{F}$, as desired. $\square$

The formulae for the perversities used in Proposition 5.3 are carefully chosen so as to ensure that, after calculating ${}^{\flat}p(C) + (q(C)+w)/2$ and ${}^{\sharp}p(C) + (q(C)+w)/2$, we are able to invoke Proposition 4.7. Unfortunately, those calculations have the aesthetically unpleasant property of not being integer-valued. We could perhaps improve the aesthetics by modifying the definition of q.

Let us briefly study how this would change the subsequent calculations. We retain all the notation used in the proof of Proposition 5.3, including the definition of q. We have proved that $\mathcal{F} \in {}_{q}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)_{\leqslant w}$, or, equivalently, that

$$
i_C^* h^k(\mathcal{F})|_C \in \mathcal{C}_G(C)_{\leqslant \lfloor (w+q(C))/2 \rfloor} \tag{5.2}
$$

for all k. Note that $w \equiv \operatorname{alt} C_0 \pmod 2$. From the definition of q, we see that

$$
q(C) + w \equiv \begin{cases} 0 \pmod 2 & \text{if } C \subset \tilde{C}_0, \\ 1 \pmod 2 & \text{if } C \subset \bar{C}_0 \smallsetminus \tilde{C}_0. \end{cases}
$$

For $n \equiv 1 \pmod 2$, we have $\lfloor n/2 \rfloor = (n-1)/2$, so we can refine (5.2) by defining $q' \colon \mathbb{O}(\bar{C}_0) \to \mathbb{Z}$ by

$$
\begin{aligned}
q'(C) &= \operatorname{alt} C_0 + 2r(C) - 2r(C_0) - 2\operatorname{cod} C + 2 \operatorname{cod} C_0 \\
&= \begin{cases} q(C) & \text{if } C \subset \tilde{C}_0, \\ q(C) - 1 & \text{if } C \subset \bar{C}_0 \smallsetminus \tilde{C}_0. \end{cases}
\end{aligned}
$$

We then have

$$
i_C^* h^k(\mathcal{F})|_C \in \mathcal{C}_G(C)_{\leqslant (w+q'(C))/2},
$$

so $\mathcal{F} \in {}_{q'}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)_{\leqslant w}$.

By replacing q by q', we have lost the two-sided nature of Proposition 5.3: it is not true in general that $\mathcal{F} \in {}_{q'}\mathcal{D}^{\mathrm{b}}_{G}(\bar{C}_0)_{\geqslant w}$. For a one-sided statement alone, however, we could further replace q' by any larger function. Pushing forward to $\mathcal{D}^{\mathrm{b}}_{G}(X)$ by $i_{C_0 *}$, we obtain the following useful result.

Corollary 5.4. *Let $\mathcal{L} \in \mathcal{C}_G(C_0)$ be a coherent sheaf, s-pure of step v. Let $q\colon \mathbb{O}(X) \to \mathbb{Z}$ be any baric perversity such that*

$$q(C) \geqslant \operatorname{alt} C_0 + 2r(C) - 2r(C_0) - 2\operatorname{cod} C + 2\operatorname{cod} C_0 \quad \text{if } C \subset \bar{C}_0,$$

and let $w = 2v - \operatorname{alt} C_0$. Then ${}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - r(C_0)]) \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$.

Note that no conditions are imposed on the values of $q(C)$ for $C \not\subset \bar{C}_0$. Since ${}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - r(C_0)])$ is supported on $\bar{C}_0$, it is clear that the values of q outside $\bar{C}_0$ have no bearing on this statement.

Recall that a simple staggered sheaf $\mathcal{F} = {}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - r(C_0)])$ is characterized by the property that $Li^*_C\mathcal{F} \in {}^r\mathcal{D}^-_G(\bar{C})^{<0}$ and $Ri^!_C\mathcal{F} \in {}^r\mathcal{D}^+_G(\bar{C})^{>0}$ for all $C \subset \bar{C}_0 \smallsetminus C_0$. The following result, which illustrates the use of Corollary 5.4, gives a baric analogue of this property in the case of the self-dual staggered perversity. (This result will not be used in the sequel.)

Proposition 5.5. *Assume that $r(C) = \frac{1}{2}\operatorname{scod} C$. Let $\mathcal{L} \in \mathcal{C}_G(C_0)$ be a coherent sheaf, s-pure of step v, and let $w = 2v - \operatorname{alt} C_0$. For any orbit $C \subset \bar{C}_0 \smallsetminus C_0$, we have*

$$Li^*_C{}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \tfrac{1}{2}\operatorname{scod} C_0]) \in \mathcal{D}^-_G(\bar{C})_{<w}$$

and

$$Ri^!_C{}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \tfrac{1}{2}\operatorname{scod} C_0]) \in \mathcal{D}^+_G(\bar{C})_{>w}.$$

Proof. Consider the baric perversity $q\colon \mathbb{O}(X) \to \mathbb{Z}$ given by

$$q(C) = \begin{cases} \operatorname{alt} C & \text{if } C \not\subset \bar{C}_0 \smallsetminus C_0, \\ \operatorname{alt} C - 1 & \text{if } C \subset \bar{C}_0 \smallsetminus C_0. \end{cases}$$

This function obeys the condition in Corollary 5.4 with respect to the middle staggered perversity $r(C) = \frac{1}{2}\operatorname{scod} C$:

$$q(C) \geqslant \operatorname{alt} C_0 + \operatorname{scod} C - \operatorname{scod} C_0 - 2\operatorname{cod} C + 2\operatorname{cod} C_0 = \operatorname{alt} C + \operatorname{cod} C_0 - \operatorname{cod} C$$

for all $C \subset \bar{C}_0$, since $\operatorname{cod} C_0 - \operatorname{cod} C \leqslant -1$ for any $C \subset \bar{C}_0 \smallsetminus C_0$. Invoking that corollary, we have $\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \frac{1}{2}\operatorname{scod} C_0]) \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. It follows that

$$Li^*_C\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \tfrac{1}{2}\operatorname{scod} C_0]) \in {}_q\mathcal{D}^-_G(\bar{C})_{\leqslant w}$$

by Lemma 2.2. Since $q(C') = \operatorname{alt} C' - 1$ for all $C' \in \mathbb{O}(\bar{C})$, it follows that

$$Li^*_C\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \tfrac{1}{2}\operatorname{scod} C_0]) \in \mathcal{D}^-_G(\bar{C})_{<w}.$$

The same argument applies to

$$\mathbb{D}\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \tfrac{1}{2}\operatorname{scod} C_0]) \cong \mathcal{IC}(\bar{C}_0, \mathbb{D}(\mathcal{L}[v - \tfrac{1}{2}\operatorname{scod} C_0])),$$

and shows that $Li^*_C\mathbb{D}\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \frac{1}{2}\operatorname{scod} C_0]) \in \mathcal{D}^-_G(\bar{C})_{<-w}$. Since

$$Ri^!_C\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \tfrac{1}{2}\operatorname{scod} C_0]) \cong \mathbb{D}(Li^*_C\mathbb{D}\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \tfrac{1}{2}\operatorname{scod} C_0])),$$

we conclude that $Ri^!_C\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - \frac{1}{2}\operatorname{scod} C_0]) \in \mathcal{D}^+_G(\bar{C})_{>w}$, as desired. □

6. The baric purity theorem

In this section, we prove the baric version of the Purity Theorem for staggered sheaves. Henceforth, unless otherwise specified, all references to baric degrees, purity, and baric truncation should be understood to be with respect to the self-dual baric structure $(\{\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}\}, \{\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}\})_{w\in\mathbb{Z}}$ corresponding to the middle baric perversity $q(C) = \operatorname{alt} C$. In particular, the left-subscript 'q' will generally be omitted.

Definition 6.1. A staggered perversity $r\colon \mathbb{O}(X) \to \mathbb{Z}$ is said to be *moderate* if for any two orbits $C, C' \subset X$ with $C' \subset \bar{C}$, the following inequalities all hold:

$$\operatorname{cod} C' - \operatorname{cod} C \leqslant r(C') - r(C) \leqslant \operatorname{alt} C' - \operatorname{alt} C, \tag{6.1}$$

$$\tfrac{1}{2}\operatorname{alt} C' - \tfrac{1}{2}\operatorname{alt} C \leqslant r(C') - r(C) \leqslant \tfrac{1}{2}\operatorname{alt} C' + \operatorname{cod} C' - \tfrac{1}{2}\operatorname{alt} C - \operatorname{cod} C. \tag{6.2}$$

Remark 6.2. Note that a necessary condition for the existence of a moderate staggered perversity is that

$$\operatorname{cod} C' - \operatorname{cod} C \leqslant \operatorname{alt} C' - \operatorname{alt} C$$

whenever $C' \subset \bar{C}$. Under these conditions, the staggered perversities $r(C) = \lfloor \frac{1}{2}\operatorname{scod} C\rfloor$ and $r(C) = \lceil \frac{1}{2}\operatorname{scod} C\rceil$ are automatically moderate.

Lemma 6.3. *Let $\mathcal{L} \in \mathcal{C}_G(C_0)$ be a coherent sheaf, s-pure of step v. If r is a moderate staggered perversity, ${}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - r(C_0)])$ is pure of baric degree $w = 2v - \operatorname{alt} C_0$.*

Proof. Let $\mathcal{F} = {}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - r(C_0)])$. It follows from the inequalities (6.2) that

$$\operatorname{alt} C \geqslant \operatorname{alt} C_0 + 2r(C) - 2r(C_0) - 2\operatorname{cod} C + 2\operatorname{cod} C_0$$

for all $C \subset \bar{C}_0$. Then Corollary 5.4 tells us that $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. Note that the dual of a moderate perversity is also moderate, so we may apply the same argument to $\mathbb{D}\mathcal{F} \in {}^{\bar{r}}\mathcal{M}(X)$. We find that $\mathbb{D}\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant -w}$, so $\mathcal{F}$ is pure of baric degree w. $\square$

Proposition 6.4. *Let $r\colon \mathbb{O}(X) \to \mathbb{Z}$ be a moderate staggered perversity. Then the category of staggered sheaves ${}^r\mathcal{M}(X)$ is stable under the baric truncation functors $\beta_{\leqslant w}$ and $\beta_{\geqslant w}$ with respect to the middle baric perversity.*

Proof. Since every staggered sheaf has finite length, we may proceed by induction on the length of $\mathcal{F}$. If $\mathcal{F}$ is simple, Lemma 6.3 tells us that $\mathcal{F}$ is pure. In particular, every baric truncation functor takes $\mathcal{F}$ either to itself or to 0.

Now, suppose $\mathcal{F}$ is not simple. Let $\mathcal{F}' \subset \mathcal{F}$ be a simple subobject, and form a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

For any $w \in \mathbb{Z}$, we obtain a distinguished triangle

$$\beta_{\leqslant w}\mathcal{F}' \to \beta_{\leqslant w}\mathcal{F} \to \beta_{\leqslant w}\mathcal{F}'' \to .$$

The first term is in ${}^r\mathcal{M}(X)$ because $\mathcal{F}'$ is simple, and the last term is in ${}^r\mathcal{M}(X)$ by induction. Therefore, $\beta_{\leqslant w}\mathcal{F} \in {}^r\mathcal{M}(X)$ as well. The same argument shows that ${}^r\mathcal{M}(X)$ is stable under $\beta_{\geqslant w}$ as well. $\square$

Below is the first major theorem of the paper. The parts of this theorem correspond to Proposition 5.3.1, Corollaire 5.3.4, Théorème 5.3.5 and Théorème 5.4.1 in [**6**], respectively.

Theorem 6.5 (baric purity). *Suppose X is endowed with a recessed s-structure. Let $r\colon \mathbb{O}(X) \to \mathbb{Z}$ be a moderate staggered perversity.*

(1) *Let $\mathcal{F}$ be a staggered sheaf. If $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$, then every subquotient of $\mathcal{F}$ is in $\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. If $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$, then every subquotient of $\mathcal{F}$ is in $\mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$.*

(2) *Every simple staggered sheaf is pure.*

(3) *Every staggered sheaf $\mathcal{F}$ admits a unique finite filtration*

$$\cdots \subset \mathcal{F}_{\leqslant w-1} \subset \mathcal{F}_{\leqslant w} \subset \mathcal{F}_{\leqslant w+1} \subset \cdots$$

such that $\mathcal{F}_{\leqslant w}/\mathcal{F}_{\leqslant w-1} \in \mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$.

(4) *Let $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$. Then $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$ if and only if ${}^r h^i(\mathcal{F}) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$ for all i, and $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$ if and only if ${}^r h^i(\mathcal{F}) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$ for all i.*

Proof. (1) Suppose we have a short exact sequence of staggered sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, with $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. Applying the functor $\beta_{>w}$ to this sequence yields a new short exact sequence in ${}^r\mathcal{M}(X)$ with middle term 0. Therefore, $\beta_{>w}\mathcal{F}' = \beta_{>w}\mathcal{F}'' = 0$ as well. The proof for $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\geqslant w}$ is similar.

(2) This was proved in Lemma 6.3.

(3) The desired filtration is given by $\mathcal{F}_w = \beta_{\leqslant w}\mathcal{F}$.

(4) If all ${}^r h^i(\mathcal{F}) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$, the fact that $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$ follows (by induction on the number of non-zero cohomology objects) from the fact that $\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$ is stable under extensions. Conversely, suppose $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. We proceed by induction on the number of non-zero cohomology objects. If $\mathcal{F}$ has only one non-zero cohomology object, there is nothing to prove. Otherwise, choose some k such that ${}^r\tau^{\leqslant k}\mathcal{F}$ and ${}^r\tau^{\geqslant k+1}\mathcal{F}$ are both non-zero. By Proposition 6.4, $\beta_{>w}{}^r\tau^{\geqslant k+1}\mathcal{F} \in {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k+1}$, so

$$\operatorname{Hom}({}^r\tau^{\geqslant k+1}\mathcal{F}, \beta_{>w}{}^r\tau^{\geqslant k+1}\mathcal{F}) \cong \operatorname{Hom}(\mathcal{F}, \beta_{>w}{}^r\tau^{\geqslant k+1}\mathcal{F}) = 0,$$

where the last equality holds because $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. It follows that $\beta_{>w}{}^r\tau^{\geqslant k+1}\mathcal{F} = 0$, so ${}^r\tau^{\geqslant k+1}\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$, and hence ${}^r\tau^{\leqslant k}\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$ as well. By induction, we know that all cohomology objects of ${}^r\tau^{\geqslant k+1}\mathcal{F}$ and of ${}^r\tau^{\leqslant k}\mathcal{F}$ lie in $\mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$, so all ${}^r h^i(\mathcal{F}) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\leqslant w}$. □

7. s-structures on a G-orbit

In this section only, we assume that the ground field $\Bbbk$ is algebraically closed.

Let $C \subset X$ be a G-orbit. Our goal in this section is to classify s-structures on C in terms of the representation theory of a certain algebraic torus T_C, defined as follows. Choose a closed point $x \in C$, and let $H \subset G$ be the stabilizer of x. We assume throughout this section that H is connected. Let $R \subset H$ be the radical of H, and let $U \subset H$ be the unipotent radical of H. Let T_C be a maximal torus of R. A routine argument, using the fact that T_C is equal to its normalizer in R, shows that T_C is canonical. That is, making different choices above would lead to a torus canonically isomorphic to T_C.

Next, let $\mathcal{O}_H$ and $\mathcal{O}_U$ denote the $\Bbbk$-algebras of regular functions on H and U respectively. We will regard them as H-modules and in particular as T_C-modules via the action

$$g \cdot f \colon h \mapsto f(g^{-1}hg).$$

Let $X(T_C)$ and $Y(T_C)$ denote the character and cocharacter lattices of T_C, respectively. Let $\mathfrak{u}$ denote the Lie algebra of U, and define a subset Υ_C by

$$\Upsilon_C = \{\upsilon \in X(T_C) \mid \upsilon \text{ occurs in the adjoint action of } T_C \text{ on } \mathfrak{u}\}.$$

Let $S(\mathfrak{u}^*)$ denote the symmetric algebra on the dual vector space to $\mathfrak{u}$. In other words, $S(\mathfrak{u}^*)$ is the ring of regular functions $\mathfrak{u} \to \Bbbk$. Next, let $-\mathbb{N}\Upsilon_C$ denote the set of all non-positive integer linear combinations of elements of Υ_C. Clearly, the set of T_C-weights on $\mathfrak{u}^*$ is $-\Upsilon_C$, and the set of T_C-weights on $S(\mathfrak{u}^*)$ is $-\mathbb{N}\Upsilon_C$. We will see later that $-\mathbb{N}\Upsilon_C$ is also the set of T_C-weights on $\mathcal{O}_H$ and on $\mathcal{O}_U$.

Recall that the category $\mathcal{C}_G(C)$ of G-equivariant coherent sheaves on C is equivalent to the category $\mathcal{R}(H)$ of finite-dimensional algebraic representations of H. Moreover, this equivalence respects tensor products and internal Hom. We will freely use this equivalence below. In particular, an 's-structure' can be understood to mean a collection of full subcategories $(\{\mathcal{R}(H)_{\leqslant w}\}, \{\mathcal{R}(H)_{\geqslant w}\})_{w\in\mathbb{Z}}$, subject to various axioms.

Let $\mathrm{Irr}(H)$ denote the set of isomorphism classes of irreducible H-representations. By Schur's Lemma, T_C must act on any irreducible representation by a character, so we get a natural map

$$\mathrm{Irr}(H) \to X(T_C) \quad \text{denoted by} \quad V \mapsto \chi_V.$$

Recall that in any s-structure, any simple object must be pure of some step. Since an orbit C contains no non-trivial open subschemes, any s-structure on $\mathcal{C}_G(H) \cong \mathcal{R}(H)$ is additive on tensor products: if $V, W \in \mathcal{R}(H)$ are both pure, then so is $V \otimes W$, and $\operatorname{step} V \otimes W = \operatorname{step} V + \operatorname{step} W$.

Lemma 7.1. *Suppose $\mathcal{C}_G(C) \cong \mathcal{R}(H)$ is endowed with an s-structure. If $V, V' \in \mathrm{Irr}(H)$ with $\chi_V = \chi_{V'}$, then $\operatorname{step} V = \operatorname{step} V'$.*

Proof. Consider the reductive group $M = H/U$. We identify T_C with its image in M, namely, the identity component of the centre of M. U acts trivially in any irreducible representation of H, so the simple objects in $\mathcal{R}(H)$ can be identified with the irreducible representations of M. The top exterior power $\bigwedge^{\dim V} V$ is a one-dimensional M-representation contained in $\bigotimes^{\dim V} V$, so we have $\operatorname{step} \bigwedge^{\dim V} V = (\dim V) \operatorname{step} V$. Finally, we can take a further tensor power, and conclude that

$$\operatorname{step}\left(\bigotimes^{\dim V'} \bigwedge^{\dim V} V\right) = (\dim V')(\dim V) \operatorname{step} V.$$

Similar reasoning shows that

$$\operatorname{step}\left(\bigotimes^{\dim V}\bigwedge^{\dim V'} V'\right) = (\dim V')(\dim V)\operatorname{step} V'.$$

Because the group $M/T_C \cong H/R$ is semisimple, it has a unique one-dimensional representation (the trivial one), so two one-dimensional representations of M are isomorphic if and only if T_C acts on them by the same character. In our setting, T_C clearly acts on both $\operatorname{step}\bigotimes^{\dim V'}\bigwedge^{\dim V} V$ and $\operatorname{step}\bigotimes^{\dim V}\bigwedge^{\dim V'} V'$ by the same character, namely, $(\dim V')(\dim V)\chi_V$. These representations are isomorphic, and the lemma follows from the step calculations above. □

Every character of T_C occurs as χ_V for some $V \in \operatorname{Irr}(H)$. Identifying $Y(T_C)$ with $\operatorname{Hom}(X(T_C), \mathbb{Z})$, we can now define a map

$$\Psi_C \colon \{s\text{-structures on } C\} \to Y(T_C)$$

by the following formula, which is well-defined by the preceding lemma:

$$\Psi_C(\text{a given } s\text{-structure})(\lambda) = \operatorname{step}(\text{any } V \in \operatorname{Irr}(H) \text{ with } \chi_V = \lambda).$$

Since any s-structure is determined by the steps of simple objects, it is clear from Lemma 7.1 that distinct s-structures give rise to distinct cocharacters in $Y(T_C)$. In other words, Ψ_C is injective. We can describe the image of Ψ_C quite precisely.

Definition 7.2. A cocharacter $\phi \in Y(T_C)$ is said to be *semifocused* if $\phi(\upsilon) \leqslant 0$ for all $\upsilon \in \Upsilon_C$. It is *focused* if $\phi(\upsilon) < 0$ for all $\upsilon \in \Upsilon_C$.

Theorem 7.3. *If X and G are schemes over an algebraically closed field, then a cocharacter $\phi \in Y(T_C)$ is in the image of Ψ_C if and only if it is semifocused.*

Before proving this theorem, we need the following basic result.

Lemma 7.4. *Let $K \subset H$ be a T_C-stable subgroup containing U. Then there is a T_C-equivariant isomorphism of varieties $K \cong K/U \times \mathfrak{u}$.*

Note that in characteristic 0, this lemma is straightforward: K admits a Levi decomposition $K \cong K/U \ltimes U$, and the exponential map provides a T_C-equivariant isomorphism of varieties $\mathfrak{u} \to U$. Neither Levi decompositions nor the exponential map necessarily exist in positive characteristic, however.

Proof. The structure theory of unipotent groups provides a filtration

$$1 = U_0 \subset U_1 \subset \cdots \subset U_n = U \tag{7.1}$$

with the following properties:

(1) each U_i is a normal subgroup of H, and therefore of K and of U_{i+1};

(2) each U_i is stable under the action of T_C; and

(3) each subquotient U_{i+1}/U_i is isomorphic to $\mathbb{G}_a$.

Note that as a consequence of (1), each of the schemes K/U_i is affine.

Let us show that each projection $K/U_{i-1} \to K/U_i$ admits a T_C-equivariant section. It is convenient to use the language of algebraic stacks: put $X_i = K/U_i$ and let $[X_i/T_C]$ denote the quotient stack. The map $[X_{i-1}/T_C] \to [X_i/T_C]$ is a $\mathbb{G}_a \cong U_i/U_{i-1}$-torsor over $[X_i/T_C]$ in the flat topology. To show that it has a section it suffices to show that $H^1_{\text{flat}}([X_i/T_C]; \mathbb{G}_a) = 0$. Note that because $\mathbb{G}_a$ is commutative, we have access to higher cohomology groups and the machinery of spectral sequences. In particular, associated to the composition of maps

$$[X_i/T_C] \to [\mathrm{pt}/T_C] \to \mathrm{pt},$$

there is the Leray spectral sequence

$$E_2^{pq} = H^p([\mathrm{pt}/T_C]; H^q_{\text{flat}}(X_i; \mathbb{G}_a)) \implies H^{p+q}_{\text{flat}}([X_i/T_C]; \mathbb{G}_a).$$

We have $H^q_{\text{flat}}(X_i; \mathbb{G}_a) \cong H^q_{\text{Zar}}(X_i; \mathcal{O}_{X_i})$, which vanishes for $q > 0$ because X_i is affine. Moreover, because the category of T_C-representations is semisimple, the cohomology groups $H^p([\mathrm{pt}/T_C]; \mathcal{F})$ vanish for $p > 0$ and any coherent sheaf $\mathcal{F}$ on the classifying stack $[\mathrm{pt}/T_C]$. Thus we have the required vanishing of $H^1_{\text{flat}}([X_i/T_C]; \mathbb{G}_a)$, and every map $K/U_{i-1} \to K/U_i$ has a T_C-equivariant section.

It follows that there is a T_C-equivariant isomorphism $K/U_{i-1} \cong K/U_i \times U_i/U_{i-1}$. Now, consider the Lie algebra version of the filtration (7.1):

$$0 = \mathfrak{u}_0 \subset \mathfrak{u}_1 \subset \cdots \subset \mathfrak{u}_n = \mathfrak{u}.$$

Each quotient $\mathfrak{u}_i/\mathfrak{u}_{i-1}$ may be identified with the Lie algebra of U_i/U_{i-1}. The exponential map makes sense for $\mathbb{G}_a$ in arbitrary characteristic, and provides a T_C-equivariant isomorphism $\mathfrak{u}_i/\mathfrak{u}_{i-1} \to U_i/U_{i-1}$. Combining the isomorphisms $K/U_{i-1} \cong K/U_i \times \mathfrak{u}_i/\mathfrak{u}_{i-1}$ for all i, we obtain

$$K \cong K/U \times \mathfrak{u}_1 \times \mathfrak{u}_2/\mathfrak{u}_1 \times \cdots \times \mathfrak{u}/\mathfrak{u}_{n-1}.$$

Again using the fact that T_C-representations are semisimple, we see that there is a T_C-equivariant isomorphism $\mathfrak{u}_1 \times \mathfrak{u}_2/\mathfrak{u}_1 \times \cdots \times \mathfrak{u}/\mathfrak{u}_{n-1} \cong \mathfrak{u}$, as desired. □

Applying this lemma in the special cases $K = U$ and $K = H$, we obtain the following result.

Corollary 7.5. *We have isomorphisms of T_C-representations $S(\mathfrak{u}^*) \cong \mathcal{O}_U$ and $\mathcal{O}_H \cong \mathcal{O}_{H/U} \otimes S(\mathfrak{u}^*)$.*

Since T_C acts trivially on $\mathcal{O}_{H/U}$, the last part of this corollary implies that T_C acts with the same set of weights on $\mathcal{O}_H$ and on $S(\mathfrak{u}^*)$.

Proof of Theorem 7.3. Let M be an H-module. Note that the comodule structure map $\gamma_M \colon M \to M \otimes \mathcal{O}_H$ is H-equivariant. (This is easiest to see by identifying $M \otimes \mathcal{O}_H$ with the vector space of regular functions $H \to M$.) In particular, this map is T_C-equivariant and preserves weights.

Define $\sigma_{\leqslant w} M$ to be the vector subspace of M spanned by those weight vectors whose weight χ satisfies $\phi(\chi) \leqslant w$. To say that ϕ defines an s-structure is equivalent to saying

$\sigma_{\leqslant w}M$ is an H-submodule of M. If $m \in M$ has weight χ, then we may write $\gamma_M(m)$ as $\sum m_i \otimes f_i$ where f_i has weight $-\upsilon_i$ for some $\upsilon_i \in \Upsilon$ and m_i has weight $\chi + \upsilon_i$. Thus, $\gamma_M(\sigma_{\leqslant w}M) \subset \sigma_{\leqslant w}M \otimes \mathcal{O}_H$ if and only if $\phi(\upsilon) \leqslant 0$ for all $\upsilon \in \Upsilon$. □

We conclude this section with an Ext-vanishing result for certain s-structures.

Theorem 7.6. *Suppose that H has a Levi factor M and that the category of M-representations is semisimple. Let ϕ be a semifocused cocharacter, and consider the corresponding s-structure $(\{\mathcal{C}_G(C)_{\leqslant w}\}, \{\mathcal{C}_G(C)_{\geqslant w}\})_{w\in\mathbb{Z}}$. The following conditions are equivalent.*

(1) *The cocharacter ϕ is focused.*

(2) *For any two simple objects $\mathcal{F}, \mathcal{G} \in \mathcal{C}_G(C)$ that are both s-pure of step w, we have $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$.*

Note that the conditions on H always hold in characteristic 0, and they always hold in arbitrary characteristic when H is solvable.

Proof. Suppose $\mathcal{R}(H)$ carries an s-structure such that the corresponding cocharacter ϕ is focused. Let $V_1, V_2 \in \mathcal{R}(H)$ be simple objects that are both s-pure of step w. Suppose we have a short exact sequence

$$0 \to V_2 \to V \to V_1 \to 0. \tag{7.2}$$

As a sequence of M-representations, this sequence splits by assumption, and we can find a subspace $V_1' \subset V$ that is isomorphic as an M-representation to V_1. Let us show that V_1' is an H-submodule of V. It suffices to show that V_1' is stable under multiplication by U; as in the proof of Theorem 7.3, this follows from the fact that the comodule structure map $\gamma_V : V \to V \otimes \mathcal{O}_U$ preserves weights. Indeed, if $v \in V_1'$, then write $\gamma_V(v)$ as $\sum v_i \otimes u_i$, where $v_i \in V$ and $u_i \in \mathcal{O}_U$ are weight vectors for T_C of weights χ_i and υ_i, respectively. Since ϕ is focused, and $\upsilon_i \in -\mathbb{N}\Upsilon_C$, we must have $\phi(\chi_i) < w$ unless $\upsilon_i = 0$. The latter condition only holds when u_i is a constant; thus v_i cannot lie in V_2. Thus the sequence (7.2) splits.

Conversely, suppose ϕ is semifocused but not focused. Then $\sigma_{\geqslant 0}\mathfrak{u} \neq 0$. By a slight abuse of notation, let us denote by $\sigma_{\leqslant 0}\mathcal{O}_H$ and $\sigma_{\leqslant 0}S(\mathfrak{u}^*)$ the subspaces of $\mathcal{O}_H$ and $S(\mathfrak{u}^*)$, respectively, spanned by all T_C-weight spaces whose weight χ satisfies $\phi(\chi) \leqslant 0$. (This notation is an abuse because $\mathcal{O}_H$ and $S(\mathfrak{u}^*)$ are infinite-dimensional and therefore not objects of $\mathcal{R}(H)$.) It follows from Corollary 7.5 that

$$\sigma_{\leqslant 0}\mathcal{O}_H \cong \mathcal{O}_{H/U} \otimes \sigma_{\leqslant 0}S(\mathfrak{u}^*).$$

Now, identify the dual space $(\sigma_{\geqslant 0}\mathfrak{u})^*$ with a subspace of $\mathfrak{u}^*$. Since the weights occurring in $S(\mathfrak{u}^*)$ are linear combinations with non-positive coefficients of the weights in Υ_C, it is easy to see that $\sigma_{\leqslant 0}S(\mathfrak{u}^*) \cong S((\sigma_{\geqslant 0}\mathfrak{u})^*)$.

We claim that $\sigma_{\leqslant 0}\mathcal{O}_H$ is a Hopf subalgebra of $\mathcal{O}_H$. Indeed, the multiplication map $\mathcal{O}_H \otimes \mathcal{O}_H \to \mathcal{O}_H$, the comultiplication map $\mathcal{O}_H \to \mathcal{O}_H \otimes \mathcal{O}_H$, and the antipode (inverse) map

$\mathcal{O}_H \to \mathcal{O}_H$ are all H- and therefore T_C-equivariant, so the restrictions of these maps to $\sigma_{\leqslant 0}\mathcal{O}_H$ endow that space with the structure of a Hopf algebra. Thus, $H' = \operatorname{Spec} \sigma_{\leqslant 0}\mathcal{O}_H$ is an affine algebraic group over $\Bbbk$, and the inclusion $\sigma_{\leqslant 0}\mathcal{O}_H \hookrightarrow \mathcal{O}_H$ corresponds to a surjective group homomorphism $H \to H'$. Note that H' cannot be reductive: the largest reductive quotient of H is H/U, but since $\mathcal{O}_{H/U}$ can be identified with a subalgebra of $\sigma_{\leqslant 0}\mathcal{O}_H$, the group H/U is a non-trivial quotient of H'.

Let U' be the unipotent radical of H'. Because the quotient map $H \cong M \ltimes U \to H/U \cong M$ factors through H', the latter group inherits a Levi decomposition with the same Levi factor: we have $H' \cong M \ltimes U'$. Now, find a faithful representation of H' on some vector space V. Such a representation is not semisimple: the space $V^{U'}$ of U'-fixed vectors (which is not all of V because $U' \neq 1$) is an H'-invariant subspace with no H'-invariant complement. $V^{U'}$ does, of course, admit an M-invariant complement; let V_1 be an irreducible M-representation in that complement, and suppose it is s-pure of step w. Let V' be the smallest H-stable subspace containing V_1, and find a filtration

$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = V'$$

such that W_i/W_{i-1} is simple for each i. Since V_1 is not contained in any proper submodule of V', we must have $W_n/W_{n-1} \cong V_1$. Moreover, since $V' \neq V_1$, we know that $n \geqslant 2$. Let $W = V'/W_{n-2}$, and let $W' = W_{n-1}/W_{n-2} \subset W$. We then have a short exact sequence

$$0 \to W' \to W \to V_1 \to 0.$$

This sequence cannot split: if W contained an H-stable subspace isomorphic to V_1, its preimage in V' would be a proper H-stable subspace of V' containing V_1. Thus, $\operatorname{Ext}^1(V_1, W') \neq 0$. To finish the proof of the theorem, it remains only to show that $\operatorname{step} W' = w$. As usual, there is an M-stable subspace $V_1' \subset W$ that is isomorphic to V_1 as an M-representation. Moreover, there is some vector $v \in V_1'$ whose image under the comodule map $\gamma_W \colon W \to W \otimes \mathcal{O}_{H'}$ is not contained in $V_1' \otimes \mathcal{O}_{H'}$. That is, if we write $\gamma_W(v)$ in the form $\sum v_i \otimes u_i$, where all the $v_i \in W$ and all the $u_i \in \mathcal{O}_{H'}$ are weight vectors, say of weights χ_i and υ_i, respectively, there is at least one non-zero term with $v_i \notin V_1$, and therefore $v_i \in W'$. Now, $\phi(\upsilon_i) = 0$ by the construction of H', so it follows that $\phi(v_i) = w$, and hence that $\operatorname{step} W' = w$. Thus, we have exhibited a pair of simple objects $V_1, W' \in \mathcal{R}(H)$, both s-pure of step w, such that $\operatorname{Ext}^1(V_1, W') \neq 0$. $\square$

8. Higher Ext-vanishing over a closed orbit

Consider the following condition on an s-structure.

Definition 8.1. An s-structure is *split* if for every orbit $C \in \mathbb{O}(X)$, and any two simple objects $\mathcal{F}, \mathcal{G} \in \mathcal{C}_G(C)$ that are both s-pure of step v, we have $\operatorname{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$.

For the remainder of the paper, we assume that the fixed s-structure on X is both recessed and split. Theorem 7.6 gives a useful criterion for an s-structure to be split.

For a closed subspace $Z \subset X$, define

$$\mathcal{C}_G^{\mathrm{supp}}(X, Z)_{\geqslant w} = \{\mathcal{F} \in \mathcal{C}_G(X)_{\geqslant w} \mid \mathcal{F} \text{ is supported set-theoretically on } Z\}.$$

The main result of this section is the following Ext-vanishing result, which will be an important tool in the proofs of both decomposition theorems.

Proposition 8.2. *Let $C \subset X$ be a closed orbit, and let $\mathcal{F} \in \mathcal{C}_G(X)$ be such that $i_C^*\mathcal{F} \in \mathcal{C}_G(C)_{\leqslant w}$. For any sheaf $\mathcal{G} \in \mathcal{C}_G^{\mathrm{supp}}(X, C)_{\geqslant v}$, we have $\mathrm{Ext}^k(\mathcal{F}, \mathcal{G}) = 0$ for all $k > w - v$.*

We begin by proving a very special case of this result.

Lemma 8.3. *Let $C \subset X$ be a closed orbit, and suppose $\mathcal{F} \in \mathcal{C}_G(C)$ is simple and s-pure of step w. For any sheaf $\mathcal{G} \in \mathcal{C}_G^{\mathrm{supp}}(X, C)_{\geqslant w}$, we have $\mathrm{Ext}^1(i_{C*}\mathcal{F}, \mathcal{G}) = 0$.*

Proof. For brevity, let us write $i = i_C$. Consider the exact sequence

$$\mathrm{Hom}(i_*\mathcal{F}, \sigma_{\geqslant w+1}\mathcal{G}) \to \mathrm{Ext}^1(i_*\mathcal{F}, \sigma_{\leqslant w}\mathcal{G}) \to \mathrm{Ext}^1(i_*\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^1(i_*\mathcal{F}, \sigma_{\geqslant w+1}\mathcal{G}).$$

The first term clearly vanishes, and the last term vanishes by Axiom (S10) in the definition of an s-structure [**1**]. Thus, the middle two terms are isomorphic. To prove that $\mathrm{Ext}^1(i_*\mathcal{F}, \mathcal{G}) = 0$, we may replace $\mathcal{G}$ by $\sigma_{\leqslant w}\mathcal{G}$, and assume without loss of generality that $\mathcal{G}$ is s-pure of step w.

Now, to every sheaf $\mathcal{G} \in \mathcal{C}_G^{\mathrm{supp}}(X, C)_{\geqslant w}$, we associate an invariant $\ell(\mathcal{G})$, defined to be the smallest integer n such that $\mathcal{I}_C^n\mathcal{G} = 0$. (See the proof of [**1**, Proposition 4.1] for details.) In the exact sequence

$$0 \to \mathcal{I}_C\mathcal{G} \to \mathcal{G} \to \mathcal{G}/\mathcal{I}_C\mathcal{G} \to 0,$$

we have $\ell(\mathcal{I}_C\mathcal{G}) = \ell(\mathcal{G}) - 1$ and $\ell(\mathcal{G}/\mathcal{I}\mathcal{G}) = 1$. Now, $\mathcal{C}_G(X)_{\leqslant w}$ is a Serre subcategory of $\mathcal{C}_G(X)$, and because C is a single closed orbit, $\mathcal{C}_G^{\mathrm{supp}}(X, C)_{\geqslant w}$ is as well, as shown in the proof of [**1**, Proposition 10.1]. Thus, $\mathcal{I}\mathcal{G}$ and $\mathcal{G}/\mathcal{I}\mathcal{G}$ are both also objects of $\mathcal{C}_G^{\mathrm{supp}}(X, C)_{\geqslant w}$ that are s-pure of step w.

Consider the exact sequence

$$\mathrm{Ext}^1(i_*\mathcal{F}, \mathcal{I}_C\mathcal{G}) \to \mathrm{Ext}^1(i_*\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^1(i_*\mathcal{F}, \mathcal{G}/\mathcal{I}\mathcal{G}).$$

If the first and last terms are known to vanish, the middle one must vanish as well. Thus, by induction on $\ell(\mathcal{G})$, we can reduce to the case where $\ell(\mathcal{G}) = 1$, i.e. $\mathcal{I}_C\mathcal{G} = 0$. In that case, there must be a sheaf $\mathcal{G}' \in \mathcal{C}_G(C)$, s-pure of step w, such that $\mathcal{G} \cong i_*\mathcal{G}'$.

If $\mathrm{Ext}^1(i_*\mathcal{F}, i_*\mathcal{G}') \neq 0$, then there is a non-split short exact sequence

$$0 \to i_*\mathcal{G}' \to \mathcal{H} \to i_*\mathcal{F} \to 0. \tag{8.1}$$

Note that $\mathcal{H}$ is necessarily also s-pure of step w. If $\ell(\mathcal{H}) = 1$, then $\mathcal{H} \cong i_*\mathcal{H}'$ for some $\mathcal{H}' \in \mathcal{C}_G(C)$, and the entire short exact sequence is the push-forward of the short exact sequence

$$0 \to \mathcal{G}' \to \mathcal{H}' \to \mathcal{F} \to 0$$

in $\mathcal{C}_G(C)$. But $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}') = 0$ because the s-structure is split, so this sequence splits, as does the one in (8.1). Thus, $\mathrm{Ext}^1(i_*\mathcal{F}, i_*\mathcal{G}') = 0$.

On the other hand, if $\ell(\mathcal{H}) > 1$, then $\mathcal{I}_C\mathcal{H} \neq 0$. Since $\mathcal{I}_C(i_*\mathcal{F}) = 0$, $\mathcal{I}_C\mathcal{H}$ must be contained in the kernel of the map $\mathcal{H} \to i_*\mathcal{F}$, so $\mathcal{I}_C\mathcal{H}$ can be identified with a subsheaf of $i_*\mathcal{G}'$. That also implies that $i_*i^*\mathcal{I}_C\mathcal{H} \cong \mathcal{I}_C\mathcal{H}$. Now, $i^*\mathcal{I}_C\mathcal{H}$ is a quotient of $i^*\mathcal{I}_C \otimes i^*\mathcal{H}$. Since $i^*\mathcal{I}_C \in \mathcal{C}_G(C)_{\leqslant -1}$ by assumption, and $i^*\mathcal{H} \in \mathcal{C}_G(C)_{\leqslant w}$, we conclude that $\mathcal{I}_C\mathcal{H} \in \mathcal{C}_G(X)_{\leqslant w-1}$. But that is a contradiction: $i_*\mathcal{G}'$ is s-pure of step w and contains no non-zero subsheaf in $\mathcal{C}_G(X)_{\leqslant w-1}$. $\square$

To prove Proposition 8.2, we will carry out an Ext-group calculation using certain injective resolutions in the category of quasicoherent sheaves. Let $\mathcal{Q}_G(X)$ denote the category of G-equivariant quasicoherent sheaves, and for any closed set $Z \subset X$, let

$$\mathcal{Q}_G^{\mathrm{supp}}(X,Z)_{\geqslant w} = \left\{ \mathcal{F} \in \mathcal{Q}_G(X) \;\middle|\; \begin{array}{c}\text{every coherent subsheaf of}\\ \mathcal{F} \text{ is in } \mathcal{C}_G^{\mathrm{supp}}(X,Z)_{\geqslant w}\end{array} \right\}.$$

Proposition 8.4. *Let $C \subset X$ be a closed orbit. Every sheaf $\mathcal{F} \in \mathcal{Q}_G^{\mathrm{supp}}(X,C)_{\geqslant w}$ admits an injective resolution*

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

with $\mathcal{I}^k \in \mathcal{Q}_G^{\mathrm{supp}}(X,C)_{\geqslant w+k}$.

Proof. For brevity of notation, it will be convenient to set $\mathcal{I}^{-1} = \mathcal{F}$. We also continue to write $i = i_C$, as in the proof of Lemma 8.3. According to the proof of [**1**, Proposition 10.1], every sheaf in $\mathcal{Q}_G^{\mathrm{supp}}(X,C)_{\geqslant w}$ has an injective hull in $\mathcal{Q}_G^{\mathrm{supp}}(X,C)_{\geqslant w}$. Let $\mathcal{I}^0$ be such an injective hull of $\mathcal{F}$, and let $\partial^{-1}\colon \mathcal{F} \to \mathcal{I}^0$ be the inclusion map. For subsequent terms of the injective resolution, we proceed by induction. Suppose that the terms $\mathcal{I}^{-1}, \mathcal{I}^0, \ldots, \mathcal{I}^n$ have already been constructed, together with morphisms $\partial^k\colon \mathcal{I}^k \to \mathcal{I}^{k+1}$ for $k = -1, \ldots, n-1$. We will show below that the cokernel of ∂^{n-1} lies in $\mathcal{Q}_G^{\mathrm{supp}}(X,C)_{\geqslant w+n+1}$. Then, using the result from [**1**, Proposition 10.1] again, we may take $\mathcal{I}^{n+1}$ to be an injective hull of $\operatorname{cok}\partial^{n-1}$ that also lies in $\mathcal{Q}_G^{\mathrm{supp}}(X,C)_{\geqslant w+n+1}$.

Suppose $\operatorname{cok}\partial^{n-1} \notin \mathcal{Q}_G^{\mathrm{supp}}(X,C)_{\geqslant w+n+1}$. Then there is some non-zero coherent subsheaf $\mathcal{G} \subset \operatorname{cok}\partial^{n-1}$ that does not lie in $\mathcal{C}_G^{\mathrm{supp}}(X,C)_{\geqslant w+n+1}$. Replacing $\mathcal{G}$ by its subsheaf $\sigma_{\leqslant w+n}\mathcal{G}$, we may assume that $\mathcal{G} \in \mathcal{C}_G(X)_{\leqslant w+n}$. Because $\mathcal{G}$ is set-theoretically supported on the orbit C, it necessarily contains a non-zero subsheaf of sections with scheme-theoretic support on C. Indeed, by replacing $\mathcal{G}$ its subsheaf $i_*i^!\mathcal{G}$, we may assume that $\mathcal{G}$ itself is supported scheme-theoretically on C. That is, $\mathcal{G} \cong i_*\mathcal{G}'$ for some $\mathcal{G}' \in \mathcal{C}_G(C)_{\leqslant w+n}$. Finally, recall that $\mathcal{C}_G(C)$ is a finite-length category, so we may replace $\mathcal{G}'$ by a simple subobject. To summarize: we have a coherent sheaf $\mathcal{G} \subset \operatorname{cok}\partial^{n-1}$ such that $\mathcal{G} \cong i_*\mathcal{G}'$ for some simple object $\mathcal{G}' \in \mathcal{C}_G(C)_{\leqslant w+n}$.

Now, consider the preimage $\tilde{\mathcal{G}}$ of $\mathcal{G}$ in $\mathcal{I}^n$. Let $\mathcal{H} \subset \tilde{\mathcal{G}}$ be any coherent subsheaf not contained in $\operatorname{im}\partial^{n-1}$. (Since $\tilde{\mathcal{G}}$ is the union of all its coherent subsheaves, such a sheaf $\mathcal{H}$ exists.) The map $\mathcal{H} \to \mathcal{G}$ is surjective, because it is non-zero and $\mathcal{G}$ is simple. We thus have a short exact sequence

$$0 \to \mathcal{H} \cap \operatorname{im}\partial^{n-1} \to \mathcal{H} \to \mathcal{G} \to 0.$$

Now, by assumption, $\mathcal{I}^n$ is the injective hull of $\operatorname{im}\partial^{n-1}$, so $\mathcal{H}$ cannot contain a direct summand complementary to $\mathcal{H} \cap \operatorname{im}\partial^{n-1}$. In other words, the exact sequence above cannot split. We thus have $\mathrm{Ext}^1(i_*\mathcal{G}', \mathcal{H} \cap \operatorname{im} \partial^{n-1}) \neq 0$. But since $\mathcal{H} \cap \operatorname{im} \partial^{n-1} \in \mathcal{C}_G^{\mathrm{supp}}(X, C)_{\geqslant x+n}$, this contradicts Lemma 8.3. $\square$

We are now ready to prove the main result of this section.

Proof of Proposition 8.2. Let $\mathcal{I}^*$ be an injective resolution of $\mathcal{G}$ as constructed in the previous proposition, with $\mathcal{I}^k \in \mathcal{Q}_G^{\mathrm{supp}}(X, C)_{\geqslant v+k}$. In particular, if $k > w - v$, there are no non-zero morphisms $\mathcal{F} \to \mathcal{I}^k$: the image of such a morphism, a certain coherent subsheaf of $\mathcal{I}^k$, belongs to $\mathcal{C}_G(X)_{\leqslant w}$ and therefore does not belong to $\mathcal{C}_G^{\mathrm{supp}}(X, C)_{\geqslant v+k}$ unless it is 0. But any non-zero element of $\mathrm{Ext}^k(\mathcal{F}, \mathcal{G})$ can be represented by a suitable non-zero morphism $\mathcal{F} \to \mathcal{I}^k$. $\square$

We conclude with an application of this Ext-vanishing result. The following technical lemma will be used in § 9.

Lemma 8.5. *Let $i\colon Z \hookrightarrow X$ be the inclusion of a closed subscheme, and let $t\colon C \hookrightarrow X$ be a closed orbit contained in Z, so that $i_C = i \circ t$. Let $\mathcal{F} \in \mathcal{C}_G(X)$ be such that $i_C^*\mathcal{F} \in \mathcal{C}_G(C)_{\leqslant w}$. Then $t^*h^{-r}(Li^*\mathcal{F}) \in \mathcal{C}_G(C)_{\leqslant w-r}$ for all $r \geqslant 0$.*

Proof. We proceed by induction on r. If $r = 0$, we have $t^*h^0(Li^*\mathcal{F}) \cong t^*i^*\mathcal{F} \cong i_C^*\mathcal{F}$, and that lies in $\mathcal{C}_G(C)_{\leqslant w}$ by assumption. Now, assume that $t^*h^{-k}(Li^*\mathcal{F}) \in \mathcal{C}_G(C)_{\leqslant w-k}$ for all $k < r$. If $t^*h^{-r}(Li^*\mathcal{F}) \notin \mathcal{C}_G(C)_{\leqslant w-r}$, then there is some object $\mathcal{G} \in \mathcal{C}_G(C)_{\geqslant w-r+1}$ such that $\mathrm{Hom}(t^*h^{-r}(Li^*\mathcal{F}), \mathcal{G}) \neq 0$, or, equivalently, $\mathrm{Hom}(h^{-r}(Li^*\mathcal{F}), t_*\mathcal{G}) \neq 0$.

Note that for $k < r$, the fact that $t^*h^{-k}(Li^*\mathcal{F}) \in \mathcal{C}_G(C)_{\leqslant w-k}$ implies, by Proposition 8.2, that $\mathrm{Hom}(h^{-k}(Li^*\mathcal{F}), t_*\mathcal{G}[n]) = 0$ whenever $n \geqslant r - k$. Equivalently, we have $\mathrm{Hom}(h^{-k}(Li^*\mathcal{F})[k], t_*\mathcal{G}[n]) = 0$ for all $n \geqslant r$. Since the object $\tau^{\geqslant -r+1}Li^*\mathcal{F} \in \mathcal{D}_G^{\mathrm{b}}(Z)$ is built up by extensions from the objects $h^{-k}(Li^*\mathcal{F})[k]$ with $k = 0, \ldots, r-1$, it follows that $\mathrm{Hom}(\tau^{\geqslant -r+1}Li^*\mathcal{F}, t_*\mathcal{G}[n]) = 0$ whenever $n \geqslant r$.

Next, from the distinguished triangle

$$h^{-r}\mathcal{F}[r] \to \tau^{\geqslant -r}\mathcal{F} \to \tau^{\geqslant -r+1}\mathcal{F} \to$$

we obtain the long exact sequence

$$\begin{aligned}\mathrm{Hom}(\tau^{\geqslant -r+1}Li^*\mathcal{F}, t_*\mathcal{G}[r]) &\to \mathrm{Hom}(\tau^{\geqslant -r}Li^*\mathcal{F}, t_*\mathcal{G}[r]) \\ &\to \mathrm{Hom}(h^{-r}(Li^*\mathcal{F})[r], t_*\mathcal{G}[r]) \to \mathrm{Hom}(\tau^{\geqslant -r+1}Li^*\mathcal{F}, t_*\mathcal{G}[r+1]) \to \ .\end{aligned}$$

The first and last terms vanish by the preceding paragraph. We saw earlier that the third term is non-zero, so the second term is as well. The chain of isomorphisms

$$\mathrm{Hom}(\mathcal{F}, i_{C*}\mathcal{G}[r]) \cong \mathrm{Hom}(Li^*\mathcal{F}, t_*\mathcal{G}[r]) \cong \mathrm{Hom}(\tau^{\geqslant -r}Li^*\mathcal{F}, t_*\mathcal{G}[r])$$

shows then that $\mathrm{Hom}(\mathcal{F}, i_{C*}\mathcal{G}[r]) \neq 0$. But this is a contradiction: since $i_C^*\mathcal{F} \in \mathcal{C}_G(C)_{\leqslant w}$ and $\mathcal{G} \in \mathcal{C}_G(C)_{\geqslant w-r+1}$, we have $\mathrm{Hom}(\mathcal{F}, i_{C*}\mathcal{G}[r]) = 0$ by Proposition 8.2. $\square$

9. The skew co-t-structure

Co-t-structures on triangulated categories have appeared in the work of Bondarko [**8**] and Pauksztello [**13**]. In this section, we construct a certain family of co-t-structures on $\mathcal{D}^{\mathrm{b}}_G(X)$, and we use them to define the notion of *skew-purity*.

We begin by recalling the definition. Given a triangulated category $\mathfrak{D}$ and a pair of full subcategories $(\mathfrak{D}_{\sqsubseteq 0}, \mathfrak{D}_{\sqsupseteq 0})$, let us set $\mathfrak{D}_{\sqsubseteq n} = \mathfrak{D}_{\sqsubseteq 0}[n]$ and $\mathfrak{D}_{\sqsupseteq n} = \mathfrak{D}_{\sqsupseteq 0}[n]$. Note that this is the opposite of the usual convention with t-structures. The pair $(\mathfrak{D}_{\sqsubseteq 0}, \mathfrak{D}_{\sqsupseteq 0})$ is called a *co-t-structure* if the following four conditions hold.

(0) $\mathfrak{D}_{\sqsubseteq 0}$ and $\mathfrak{D}_{\sqsupseteq 0}$ are closed under direct summands.

(1) $\mathfrak{D}_{\sqsubseteq 0} \subset \mathfrak{D}_{\sqsubseteq 1}$ and $\mathfrak{D}_{\sqsupseteq 0} \supset \mathfrak{D}_{\sqsupseteq 1}$.

(2) $\mathrm{Hom}(A, B) = 0$ whenever $A \in \mathfrak{D}_{\sqsubseteq 0}$ and $B \in \mathfrak{D}_{\sqsupseteq 1}$.

(3) For any object $X \in \mathfrak{D}$, there is a distinguished triangle $A \to X \to B \to$ with $A \in \mathfrak{D}_{\sqsubseteq 0}$ and $B \in \mathfrak{D}_{\sqsupseteq 1}$.

Note that for a co-t-structure, the distinguished triangle in Axiom (3) is not functorial. (The usual proof fails because $A \in \mathfrak{D}_{\sqsubseteq 0}$ does not imply $A[1] \in \mathfrak{D}_{\sqsubseteq 0}$.) The properties of being *bounded* or *non-degenerate* are defined for co-t-structures in the same way as for t-structures. The reader is referred to [**8**] or [**13**] for further properties of co-t-structures.

Now, let $q\colon \mathbb{O}(X) \to \mathbb{Z}$ be a function, to be known as a *skew perversity*. Define a full subcategory of $\mathcal{D}^{-}_G(X)$ by

$$ {}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w} = \{\mathcal{F} \in \mathcal{D}^{-}_G(X) \mid h^k(X) \in {}_{2q}\mathcal{C}_G(X)_{\leqslant 2w+2k} \text{ for all } k\}. $$

Next, define a new function $\check{q}\colon \mathbb{O}(X) \to \mathbb{Z}$, called the *skew dual* of q, by

$$ \check{q}(C) = \operatorname{alt} C - \operatorname{cod} C - q(C). $$

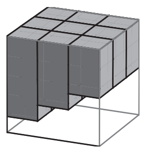
We then define a full subcategory of $\mathcal{D}^{+}_G(X)$ by

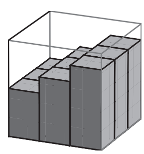
$$ {}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w} = \mathbb{D}({}_{\check{q}}\mathcal{D}^{-}_G(X)_{\sqsubseteq -w}). $$

As usual, we put

$$ {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w} = {}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w} \cap \mathcal{D}^{\mathrm{b}}_G(X) \quad \text{and} \quad {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w} = {}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w} \cap \mathcal{D}^{\mathrm{b}}_G(X). $$

The pictures of these categories resemble ‘upside-down’ versions of the categories that constitute the staggered t-structure:

${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$: 

${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$: 

Finally, we define a full subcategory of $\mathcal{D}^{\mathrm{b}}_G(X)$ as follows:

$$ {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\langle w\rangle} = {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}. $$

Objects of ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\langle w\rangle}$ are said to be *skew-pure* of *skew-degree* w.

The following lemma collects some basic properties of these categories. The proofs are routine and will be omitted.

Lemma 9.1.

(1) *${}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$ and ${}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w}$ are closed under extensions and direct summands.*

(2) *${}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$ is stable under all standard truncation functors $\tau^{\leqslant n}$ and $\tau^{\geqslant n}$.*

(3) *${}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}[1] = {}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w+1}$ and ${}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w}[1] = {}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w+1}$.*

(4) *For every $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$, there exist integers v, w such that $\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq v} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$.*

Lemma 9.2. *Let $j\colon U \hookrightarrow X$ be the inclusion of an open subscheme, and $i\colon Z \hookrightarrow X$ the inclusion of a closed subscheme. Then*

(1) *j^* takes ${}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$ to ${}_q\mathcal{D}^{-}_G(U)_{\sqsubseteq w}$ and ${}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w}$ to ${}_q\mathcal{D}^{+}_G(U)_{\sqsupseteq w}$;*

(2) *Li^* takes ${}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$ to ${}_q\mathcal{D}^{-}_G(Z)_{\sqsubseteq w}$;*

(3) *$Ri^!$ takes ${}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w}$ to ${}_q\mathcal{D}^{+}_G(Z)_{\sqsupseteq w}$;*

(4) *i_* takes ${}_q\mathcal{D}^{-}_G(Z)_{\sqsubseteq w}$ to ${}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$ and ${}_q\mathcal{D}^{+}_G(Z)_{\sqsupseteq w}$ to ${}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w}$.*

Proof. For parts (1) and (4), the statements about ${}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$ follow from the fact that j^* and i_* are exact, baryexact functors, and the statements about ${}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w}$ then follow from the fact that j^* and i_* commute with $\mathbb{D}$. Part (3) follows by duality from part (2).

It remains only to prove part (2). Let $\mathcal{F} \in {}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$. We first consider the special case where $\mathcal{F}$ is concentrated in a single degree, say degree n. Thus, $\mathcal{F}[n]$ is an object in ${}_{2q}\mathcal{C}_G(X)_{\leqslant 2w+2n}$. Let $i_C\colon C \hookrightarrow X$ be an orbit contained in Z, and let $j\colon U \hookrightarrow X$ be the inclusion of the open subscheme $U = X \smallsetminus (\bar{C} \smallsetminus C)$. Thus, C is a closed orbit in U. Let $t\colon C \hookrightarrow Z \cap U$ be the inclusion of C into $Z \cap U$. By assumption, $i_C^*\mathcal{F}[n]|_C \in \mathcal{C}_G(C)_{\leqslant w+q(C)+n}$, so by Lemma 8.5,

$$ t^*h^{-r}(Li^*\mathcal{F}[n]|_U) \in \mathcal{C}_G(C)_{\leqslant w+q(C)+n-r} \quad \text{for all } r \geqslant 0. $$

Clearly, $t^*h^{-r}(Li^*\mathcal{F}[n]|_U) \cong i_C^*h^{n-r}(Li^*\mathcal{F})|_C$, so we have just shown that $h^k(Li^*\mathcal{F}) \in {}_{2q}\mathcal{C}_G(Z)_{\leqslant 2w+2k}$. Thus, $Li^*\mathcal{F} \in {}_q\mathcal{D}^{-}_G(Z)_{\sqsubseteq w}$.

Since ${}_q\mathcal{D}^{-}_G(Z)_{\sqsubseteq w}$ is stable under extensions, an induction argument on the number of non-zero cohomology sheaves shows that for all $\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$, we have $Li^*\mathcal{F} \in {}_q\mathcal{D}^{-}_G(Z)_{\sqsubseteq w}$. Finally, consider a general object $\mathcal{F} \in {}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$. For any k, we can form a distinguished triangle

$$ Li^*(\tau^{\leqslant k-1}\mathcal{F}) \to Li^*\mathcal{F} \to Li^*(\tau^{\geqslant k}\mathcal{F}) \to \ . $$

Clearly, $\tau^{\geqslant k}\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$, so we already know that $Li^*(\tau^{\geqslant k}\mathcal{F}) \in {}_q\mathcal{D}^{-}_G(Z)_{\sqsubseteq w}$. Moreover, the long exact sequence associated to the distinguished triangle above shows that $h^k(Li^*\mathcal{F}) \cong h^k(Li^*\tau^{\geqslant k}\mathcal{F})$, and hence that $h^k(Li^*\mathcal{F}) \in {}_{2q}\mathcal{C}_G(Z)_{\leqslant 2w+2k}$. Since this holds for all k, we conclude that $Li^*\mathcal{F} \in {}_q\mathcal{D}^{-}_G(Z)_{\sqsubseteq w}$, as desired. □

Proposition 9.3. *If* $\mathcal{F} \in {}_q\mathcal{D}^{-}_G(X)_{\sqsubseteq w}$ *and* $\mathcal{G} \in {}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w+1}$, *then* $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0$.

Proof. We proceed by noetherian induction, and assume the result is already known for all proper closed subschemes of X. Let a be an integer such that $\mathcal{G} \in \mathcal{D}^{+}_G(X)^{\geqslant a}$. Then $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\tau^{\geqslant a}\mathcal{F},\mathcal{G})$. Moreover, we have $\tau^{\geqslant a}\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. Thus, we may reduce to the case where $\mathcal{F}$ actually belongs to ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$, by replacing $\mathcal{F}$ by $\tau^{\geqslant a}\mathcal{F}$ if necessary. Next, recall that $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathbb{D}\mathcal{G},\mathbb{D}\mathcal{F})$, and suppose $\mathbb{D}\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant b}$. We may similarly reduce to the case where $\mathcal{G} \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$ by replacing $\mathcal{G}$ by $\mathbb{D}\tau^{\geqslant b}\mathbb{D}\mathcal{G}$ if necessary.

Once we have reduced to the case where both $\mathcal{F}$ and $\mathcal{G}$ are bounded, we may, by induction on the number of non-zero cohomology sheaves, further reduce to the case where $\mathcal{F}$ and $\mathbb{D}\mathcal{G}$ are each concentrated in a single degree. Suppose that $\mathcal{F}$ is concentrated in degree k, and $\mathbb{D}\mathcal{G}$ in degree m. That is,

$$\mathcal{F}[k] \in {}_{2q}\mathcal{C}_G(X)_{\leqslant 2w+2k} \quad \text{and} \quad (\mathbb{D}\mathcal{G})[m] \in {}_{2\check{q}}\mathcal{C}_G(X)_{\leqslant -2w-2+2m}.$$

Let $C \subset X$ be an open orbit, and let $U \subset X$ be the corresponding (possibly non-reduced) subscheme. Consider the usual exact sequence

$$\varinjlim_{Z'} \mathrm{Hom}(Li^*_{Z'}\mathcal{F}, Ri^!_{Z'}\mathcal{G}) \to \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $i_{Z'} \colon Z' \hookrightarrow X$ ranges over all closed subscheme structures on $X \smallsetminus U$. Since $Li^*_{Z'}\mathcal{F} \in {}_q\mathcal{D}^{-}_G(Z')_{\sqsubseteq w}$ and $Ri^!_{Z'}\mathcal{G} \in {}_q\mathcal{D}^{+}_G(X)_{\sqsupseteq w+1}$, the first term vanishes by assumption. To finish the proof, then, it suffices to show that the third term vanishes.

Since the associated reduced scheme of U is the single orbit C, U has no non-empty (G-invariant) proper open subschemes. The fact that $\mathbb{D}\mathcal{G}|_U$ is concentrated in degree m then implies, by [**1**, Lemma 6.6], that $\mathcal{G}|_U$ is concentrated in degree $\operatorname{cod} C - m$. Since

$$\mathbb{D}\mathcal{G}[m]|_U \in {}_{2\check{q}}\mathcal{C}_G(U)_{\leqslant -2w-2+2m} = \mathcal{C}_G(U)_{\leqslant \operatorname{alt} C - \operatorname{cod} C - q(C) - w - 1 + m},$$

we know that $\mathcal{G}[\operatorname{cod} C - m]|_U \in \mathcal{C}_G(U)_{\geqslant \operatorname{cod} C + q(C) + w + 1 - m}$, and hence that $\mathcal{G}[\operatorname{cod} C - m]|_U \in \mathcal{C}^{\mathrm{supp}}_G(U, C)_{\geqslant \operatorname{cod} C + q(C) + w + 1 - m}$. Similarly,

$$\mathcal{F}[k]|_U \in {}_{2q}\mathcal{C}_G(U)_{\leqslant 2w+2k} = \mathcal{C}_G(U)_{\leqslant q(C)+w+k},$$

and therefore $i^*_C\mathcal{F}[k]|_C \in \mathcal{C}_G(C)_{\leqslant q(C)+w+k}$. Proposition 8.2 tells us that

$$\mathrm{Hom}(\mathcal{F}[k]|_U, \mathcal{G}[\operatorname{cod} C - m + n]|_U) = 0$$

whenever $n > k + m - \operatorname{cod} C - 1$. In particular, taking $n = k + m - \operatorname{cod} C$, we find that $\mathrm{Hom}(\mathcal{F}[k]|_U, \mathcal{G}[k]|_U) \cong \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) = 0$, as desired. □

Proposition 9.4. *For any $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$, there is a distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ with $\mathcal{F}' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ and $\mathcal{F}'' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$. Moreover, if $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$, then $\mathcal{F}'$ and $\mathcal{F}''$ may be chosen to lie in $\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$ as well.*

Proof. We proceed by noetherian induction, in a manner similar to the proof of Proposition 4.4. Let us first treat the special case where $\mathcal{F}$ is concentrated in a single degree, say $\mathcal{F} \cong h^k(\mathcal{F})[-k]$. Choose an open orbit $C \in \mathbb{O}(X)$ on which $\operatorname{cod} C$ achieves its minimum value, and let $U \subset X$ be the corresponding open subscheme. Consider the sheaf $\sigma_{\leqslant q(C)+k}(\mathcal{F}[k])|_U \in \mathcal{C}_G(U)_{\leqslant q(C)+k} = {}_{2q}\mathcal{C}_G(U)_{\leqslant 2k}$. By [**5**, Lemma 6.3], there exists a subsheaf of $\mathcal{F}[k]$ in ${}_{2q}\mathcal{C}_G(X)_{\leqslant 2k}$ whose restriction to U is $\sigma_{\leqslant q(C)+k}(\mathcal{F}[k])|_U$. Denote this subsheaf by $\mathcal{F}_1[k]$. That is, we denote by $\mathcal{F}_1$ an object of $\mathcal{D}^{\mathrm{b}}_G(X)$ concentrated in degree k such that $\mathcal{F}_1[k]$ is the subsheaf of $\mathcal{F}[k]$ obtained by invoking [**5**, Lemma 6.3]. Clearly, $\mathcal{F}_1$ lies in ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq 0}$ and is concentrated in degree k.

Next, let $\mathcal{F}'$ be the cone of the obvious morphism $\mathcal{F}_1 \to \mathcal{F}$. Clearly, $\mathcal{F}'$ is also concentrated in degree k, and $\mathcal{F}'[k]|_U \cong \sigma_{\geqslant q(C)+k+1}(\mathcal{F}[k]|_U)$. Because C was chosen to minimize $\operatorname{cod} C$, we have $\mathbb{D}\mathcal{F}' \in \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant \operatorname{cod} C - k}$. Moreover, by [**1**, Proposition 6.8], $\mathbb{D}\mathcal{F}'|_U$ is concentrated in degree $\operatorname{cod} C - k$, and

$$\begin{aligned}(\mathbb{D}\mathcal{F}')[k - \operatorname{cod} C]|_U \in \mathcal{C}_G(U)_{\leqslant \operatorname{alt} C - q(C) - k - 1} &= \mathcal{C}_G(U)_{\leqslant \check{q}(C) + \operatorname{cod} C - k - 1} \\ &= {}_{2\check{q}}\mathcal{C}_G(U)_{\leqslant 2(\operatorname{cod} C - k) - 2}.\end{aligned}$$

By invoking [**5**, Lemma 6.3] again, we can find an object $\mathcal{G}_1 \in \mathcal{D}^{\mathrm{b}}_G(X)$, concentrated in degree $\operatorname{cod} C - k$, such that $\mathcal{G}_1[k - \operatorname{cod} C]$ is a subsheaf of $h^{\operatorname{cod} C - k}(\mathbb{D}\mathcal{F}')$ lying in ${}_{2\check{q}}\mathcal{C}_G(X)_{\leqslant 2(\operatorname{cod} C - k) - 2}$, and such that $\mathcal{G}_1|_U \cong \mathbb{D}\mathcal{F}'|_U$. By construction, $\mathcal{G}_1 \in {}_{\check{q}}\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq -1}$, and we have a natural map $\mathcal{G}_1 \to \mathbb{D}\mathcal{F}'$.

Let $\mathcal{F}_2 = \mathbb{D}\mathcal{G}_1$, and let $\mathcal{G}$ denote the cocone of the morphism $\mathcal{F}' \to \mathcal{F}_2$. Then $\mathcal{F}_2 \in \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$, and it follows that $\mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$ as well. We have

$$\mathcal{F} \in \{\mathcal{F}_1\} * \{\mathcal{G}\} * \{\mathcal{F}_2\},$$

with $\mathcal{F}_1 \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq 0}$ and $\mathcal{F}_2 \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq 1}$. Moreover, since $\mathcal{F}'|_U \cong \mathcal{F}_2|_U$, we see that $\mathcal{G}$ is supported on a proper closed subscheme. It follows by noetherian induction that $\mathcal{F}$ sits in a suitable distinguished triangle. Since $\mathcal{F}_1$, $\mathcal{G}$, and $\mathcal{F}_2$ all lie in $\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$, the last assertion in the lemma holds by induction as well.

Now, for general $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$, we proceed by induction on the number of non-zero cohomology sheaves. Choose some k such that $\tau^{\leqslant n}\mathcal{F}$ and $\tau^{\geqslant n+1}\mathcal{F}$ are both non-zero, and thus have fewer non-zero cohomology sheaves than $\mathcal{F}$. Find distinguished triangles

$$\mathcal{F}_1' \to \tau^{\leqslant n}\mathcal{F} \to \mathcal{F}_1'' \to \quad \text{and} \quad \mathcal{F}_2' \to \tau^{\geqslant n+1}\mathcal{F} \to \mathcal{F}_2'' \to$$

with $\mathcal{F}_1', \mathcal{F}_2' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq 0}$ and $\mathcal{F}_1'', \mathcal{F}_2'' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq 1}$. Consider the composition $\mathcal{F}_2'[-1] \to \tau^{\geqslant n+1}[-1]\mathcal{F} \to \tau^{\leqslant n}\mathcal{F}$, which we denote by f. Now, $\operatorname{Hom}(\mathcal{F}_2'[-1], \mathcal{F}_1'') = 0$ (because $\mathcal{F}_2'[-1] \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq -1}$), so $f \in \operatorname{Hom}(\mathcal{F}_2'[-1], \tau^{\leqslant n}\mathcal{F})$ factors through $\mathcal{F}_1'$. We thus obtain a commutative square

$$\begin{tikzcd}
\mathcal{F}_2'[-1] \arrow[r] \arrow[d] & \tau^{\geqslant n+1}\mathcal{F}[-1] \arrow[d] \\
\mathcal{F}_1' \arrow[r] & \tau^{\leqslant n}\mathcal{F}
\end{tikzcd}$$

Let us complete this diagram using the 9-lemma [**6**, Proposition 1.1.11], and then rotate:

$$\begin{tikzcd}
\mathcal{F}_1' \arrow[r] \arrow[d] & \tau^{\leqslant n}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_1'' \arrow[r] \arrow[d] & {} \\
\mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & {} \\
\mathcal{F}_2' \arrow[r] \arrow[d] & \tau^{\geqslant n+1}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_2'' \arrow[r] \arrow[d] & {} \\
{} & {} & {} &
\end{tikzcd}$$

We see that $\mathcal{F}' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq 0}$ and $\mathcal{F}'' \in {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq 1}$ because those categories are stable under extensions. Moreover, if $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$, then the same holds for $\tau^{\leqslant n}\mathcal{F}$ and $\tau^{\geqslant n+1}\mathcal{F}$, and therefore for $\mathcal{F}_1'$, $\mathcal{F}_1''$, $\mathcal{F}_2'$, and $\mathcal{F}_2''$ by induction. It follows that $\mathcal{F}'$ and $\mathcal{F}''$ lie in $\mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$, as desired. □

Combining the results above, we see that have established the following fact.

Theorem 9.5. *$({}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq 0}, {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq 0})$ is a non-degenerate, bounded co-t-structure on $\mathcal{D}^{\mathrm{b}}_G(X)$.*

We conclude this section with an alternate characterization of $\mathcal{D}^-_G(X)_{\sqsubseteq w}$ and $\mathcal{D}^+_G(X)_{\sqsupseteq w}$ that will be useful in the sequel.

Proposition 9.6. *For $\mathcal{F} \in \mathcal{D}^-_G(X)$, the following conditions are equivalent.*

(1) $\mathcal{F} \in \mathcal{D}^-_G(X)_{\sqsubseteq w}$.

(2) $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$ *for all* $\mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$.

(3) $Li^*_C\mathcal{F}|_C \in \mathcal{D}^-_G(C)_{\sqsubseteq w}$ *for all orbits* $C \subset X$.

Similarly, for $\mathcal{F} \in \mathcal{D}^+_G(X)$, the following are equivalent.

(1) $\mathcal{F} \in \mathcal{D}^+_G(X)_{\sqsupseteq w}$.

(2) $\mathrm{Hom}(\mathcal{G}, \mathcal{G}) = 0$ *for all* $\mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w-1}$.

(3) $Ri^!_C\mathcal{F}|_C \in \mathcal{D}^+_G(C)_{\sqsupseteq w}$ *for all orbits* $C \subset X$.

Proof. The second set of equivalences follows from the first by applying $\mathbb{D}$, so we will only prove the first. The fact that (1) implies (2) and (3) was shown in Proposition 9.3 and Lemma 9.2, respectively. Next, we show that (2) implies (1). If $\mathcal{F} \notin \mathcal{D}^-_G(X)_{\sqsubseteq w}$, then there is some k such that $h^k(\mathcal{F}) \notin {}_{2\llcorner r \lrcorner}\mathcal{C}_G(X)_{\leqslant 2w+2k}$. It follows that $\tau^{\geqslant k}\mathcal{F} \notin \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. Using Proposition 9.4, form a distinguished triangle $\mathcal{F}' \to \tau^{\geqslant k}\mathcal{F} \to \mathcal{F}'' \to$ with $\mathcal{F}' \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w} \cap \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$ and $\mathcal{F}'' \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1} \cap \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$. The morphism $\tau^{\geqslant k}\mathcal{F} \to \mathcal{F}''$ is necessarily non-zero, and because $\mathcal{F}'' \in \mathcal{D}^{\mathrm{b}}_G(X)^{\geqslant k}$, the composition $\mathcal{F} \to \tau^{\geqslant k}\mathcal{F} \to \mathcal{F}''$ is non-zero as well.

Finally, we show that (3) implies (1). We proceed by induction on the number of orbits in X. Let $C_0 \subset X$ be an open orbit, and let Z be the complementary closed subset. For $\mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$, consider the exact sequence

$$\varinjlim_{Z'} \operatorname{Hom}(Li^*_{Z'}\mathcal{F}, Ri^!_{Z'}\mathcal{G}) \to \operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}(\mathcal{F}|_{C_0}, \mathcal{G}|_{C_0}),$$

where $i_{Z'}\colon Z' \hookrightarrow X$ ranges over all closed subscheme structures on Z. We assume that $Li^*_C\mathcal{F}|_C \in \mathcal{D}^-_G(C)_{\sqsubseteq w}$ for all orbits C. In particular, this holds for all orbits in such closed subscheme Z'. Since each Z' has fewer orbits than X, we know inductively that $Li^*_{Z'}\mathcal{F} \in \mathcal{D}^-_G(Z')_{\sqsubseteq w}$. We also know that $\mathcal{F}|_{C_0} \in \mathcal{D}^-_G(C_0)_{\sqsubseteq w}$, and, by Lemma 9.2 again, that $Ri^!_{Z'}\mathcal{G} \in \mathcal{D}^+_G(Z')_{\sqsupseteq w+1}$ and $\mathcal{G}|_{C_0} \in \mathcal{D}^{\mathrm{b}}_G(C_0)_{\sqsupseteq w+1}$. Therefore, by Proposition 9.3, the first and last terms above vanish. It follows that $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$, as desired. $\square$

10. The skew purity theorem

We prove the skew version of the Purity Theorem in this section. Of course, we must specify a skew perversity with respect to which skew-purity statements are to be understood. Given a moderate staggered perversity $r\colon \mathbb{O}(X) \to \mathbb{Z}$, we associate to it a skew perversity, denoted $\llcorner r \lrcorner\colon \mathbb{O}(X) \to \mathbb{Z}$, as follows:

$$\llcorner r \lrcorner(C) = r(C) - \operatorname{cod} C.$$

Note that this operation transforms staggered duals into skew duals:

$$\llcorner \bar{r} \lrcorner(C) = (\operatorname{scod} C - r(C)) - \operatorname{cod} C = \operatorname{alt} C - \operatorname{cod} C - (r(C) - \operatorname{cod} C) = (\llcorner r \lrcorner)\check{}(C).$$

Henceforth, we will generally omit the perversity from the notation for skew categories. Unless otherwise specified, the categories $\mathcal{D}^-_G(X)_{\sqsubseteq w}$ and $\mathcal{D}^+_G(X)_{\sqsupseteq w}$ should be understood to be defined with respect to $\llcorner r \lrcorner(C)$.

Lemma 10.1. *Let $\mathcal{L} \in \mathcal{C}_G(C_0)$ be a coherent sheaf, s-pure of step v. For any staggered perversity r, the object ${}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - r(C_0)])$ is skew-pure of skew degree $w = 2v - 2r(C_0) + \operatorname{cod} C_0$.*

As a special case, this lemma implies that for any $n \in \mathbb{Z}$, $\mathcal{L}[n]$ is a skew-pure object of $\mathcal{D}^{\mathrm{b}}_G(C_0)$ of skew degree $v + n - r(C_0) + \operatorname{cod} C_0$.

Proof. Let $j\colon C_0 \hookrightarrow \bar{C}_0$ be the inclusion, and let $\mathcal{F} = {}^r j_{!*}(\mathcal{L}[v - r(C_0)])$. Of course, ${}^r\mathcal{IC}(\bar{C}_0, \mathcal{L}[v - r(C_0)]) \cong i_{C_0*}\mathcal{F}$, so it suffices to show that $\mathcal{F}$ is skew-pure of skew degree w.

We saw in the proof of Lemma 5.1 that $h^k(\mathcal{F}) = 0$ for $k < r(C_0) - v$. Next, let $u = 2v - \operatorname{alt} C_0$, and consider the function $q\colon \mathbb{O}(\bar{C}_0) \to \mathbb{Z}$ given by

$$q(C) = 2\llcorner r \lrcorner(C) + 2w + 2(r(C_0) - v) - u.$$

Direct calculation shows that $q(C)$ satisfies the condition of Corollary 5.4. That statement tells us that $\mathcal{F} \in {}_q\mathcal{D}^{\mathrm{b}}_G(\bar{C}_0)_{\leqslant u}$, or, equivalently, that

$$\mathcal{F} \in {}_{2\llcorner r \lrcorner}\mathcal{D}^{\mathrm{b}}_G(\bar{C}_0)_{\leqslant 2w+2(r(C_0)-v)}.$$

In other words, for all $k \geqslant r(C_0) - v$, we have

$$h^k(\mathcal{F}) \in {}_{2\llcorner r \lrcorner}\mathcal{C}_G(\bar{C}_0)_{\leqslant 2w+2(r(C_0)-v)} \subset {}_{2\llcorner r \lrcorner}\mathcal{C}_G(\bar{C}_0)_{\leqslant 2w+2k}.$$

Thus, $\mathcal{F} \in {}_{\llcorner r \lrcorner}\mathcal{D}^{\mathrm{b}}_G(\bar{C}_0)_{\sqsubseteq w}$. The same argument shows that $\mathbb{D}\mathcal{F} \cong {}^{\bar{r}}j_{!*}(\mathbb{D}(\mathcal{L}[v - r(C_0)]))$ belongs to ${}_{\llcorner \bar{r} \lrcorner}\mathcal{D}^{\mathrm{b}}_G(\bar{C}_0)_{\sqsubseteq w'}$, where $w' = 2(\operatorname{alt} C_0 - v) - 2\bar{r}(C_0) + \operatorname{cod} C_0 = -w$. Thus, $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(\bar{C}_0)_{\langle w \rangle}$, as desired. □

Theorem 10.2 (skew purity). *Suppose X is endowed with a recessed, split s-structure. Let $r\colon \mathbb{O}(X) \to \mathbb{Z}$ be a staggered perversity.*

(1) *Let $\mathcal{F}$ be a staggered sheaf. If $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$, then every subquotient of $\mathcal{F}$ is in $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. If $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$, then every subquotient of $\mathcal{F}$ is in $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$.*

(2) *Every simple staggered sheaf is skew-pure.*

(3) *Every staggered sheaf $\mathcal{F}$ admits a unique finite filtration*

$$\cdots \subset \mathcal{F}_{\sqsubseteq w-1} \subset \mathcal{F}_{\sqsubseteq w} \subset \mathcal{F}_{\sqsubseteq w+1} \subset \cdots$$

such that $\mathcal{F}_{\sqsubseteq w}/\mathcal{F}_{\sqsubseteq w-1}$ is skew-pure of skew degree w.

(4) *Let $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$. Then $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ if and only if ${}^r h^i(\mathcal{F}) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w+i}$ for all i, and $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$ if and only if ${}^r h^i(\mathcal{F}) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+i}$ for all i.*

Proof. (1) We will prove the statement for $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$; the statement for $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$ then follows by duality. Note that any subquotient of $\mathcal{F}$ arises by extensions among the composition factors of $\mathcal{F}$, so it suffices to prove that every composition factor of $\mathcal{F}$ is in $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. If $\mathcal{F}$ is simple, then it is skew-pure by Lemma 10.1, and there is nothing to prove. Otherwise, let $\mathcal{F}_1$ be a simple quotient of $\mathcal{F}$. Since $\operatorname{Hom}(\mathcal{F}, \mathcal{F}_1) \neq 0$, we know by Proposition 9.3 that $\mathcal{F}_1 \notin \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$. Since $\mathcal{F}_1$ is skew-pure, it must lie in $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. Therefore, $\mathcal{F}_1[-1] \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ as well. Let $\mathcal{F}_2 \subset \mathcal{F}$ be the kernel of the morphism $\mathcal{F} \to \mathcal{F}_1$. From the distinguished triangle $\mathcal{F}_1[-1] \to \mathcal{F}_2 \to \mathcal{F} \to$, we see that $\mathcal{F}_2 \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. Since $\mathcal{F}_2$ has shorter length than $\mathcal{F}$, we know that all it composition factors lie in $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. Thus, all composition factors of $\mathcal{F}$ lie in $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$, as desired.

(2) This was proved in Lemma 10.1.

(3) We follow the proof of [**6**, Théorème 5.3.5]. Given an integer w, let $\mathcal{S}^+$ (respectively $\mathcal{S}^-$) denote the set of isomorphism classes of simple staggered sheaves of skew-degree greater than w (respectively less than or equal to w). Clearly, if $\mathcal{G} \in \mathcal{S}^-$ and $\mathcal{G}' \in \mathcal{S}^+$, then $\mathcal{G}'[1] \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$ as well, so $\mathrm{Hom}(\mathcal{G}, \mathcal{G}'[1]) = 0$ by Proposition 9.3. The sets $\mathcal{S}^+$ and $\mathcal{S}^-$ thus satisfy the hypotheses of [**6**, Lemme 5.3.6], which then tells us that every staggered sheaf $\mathcal{F}$ admits a unique subobject $\mathcal{F}_{\sqsubseteq w}$ belonging to $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ such that the quotient $\mathcal{F}/\mathcal{F}_{\sqsubseteq w}$ belongs to $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$. The functoriality of this assignment guarantees that $\mathcal{F}_{\sqsubseteq w-1} \subset \mathcal{F}_{\sqsubseteq w}$ (so that we do indeed obtain a filtration) and that $\mathcal{F}_{\sqsubseteq w}/\mathcal{F}_{\sqsubseteq w-1}$ is skew-pure of skew degree w. Finally, the uniqueness of this filtration follows from part (1).

(4) Again, we will prove only the statement about $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. First, suppose ${}^r h^i(\mathcal{F}) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w+i}$ for all i. Then ${}^r h^i(\mathcal{F})[-i] \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. Using the fact that $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ is stable under extensions, it follows by induction on the number of non-zero ${}^r h^i(\mathcal{F})$ that $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ as well. Conversely, suppose $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$. By a minor abuse of terminology, we define the *total length* of $\mathcal{F}$ to be the sum of lengths of all ${}^r h^i(\mathcal{F})$. We proceed by induction on total length. Let k be the largest integer such that ${}^r h^k(\mathcal{F}) \neq 0$, and let $\mathcal{F}_1$ be a simple quotient of ${}^r h^k(\mathcal{F})$. Note that ${}^r\tau^{\geqslant k}\mathcal{F} \cong {}^r h^k(\mathcal{F})[-k]$. From the adjunction $\mathrm{Hom}(\mathcal{F}, \mathcal{F}_1[-k]) \cong \mathrm{Hom}({}^r\tau^{\geqslant k}\mathcal{F}, \mathcal{F}_1[-k])$, we see that there is a natural non-zero morphism $\mathcal{F} \to \mathcal{F}_1[-k]$. $\mathcal{F}_1[-k]$ is skew-pure and not in $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$, by Proposition 9.3, so $\mathcal{F}_1[-k] \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$, or $\mathcal{F}_1 \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w+k}$. Next, let $\mathcal{F}_2$ be the cocone of the morphism $\mathcal{F} \to \mathcal{F}_1[-k]$. From the distinguished triangle

$$\mathcal{F}_1[-k-1] \to \mathcal{F}_2 \to \mathcal{F} \to$$

and the fact that $\mathcal{F}_1[-k-1] \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$, we see that $\mathcal{F}_2 \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ as well. It has shorter total length, so by assumption, ${}^r h^i(\mathcal{F}_2) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w+i}$ for all i. Now, consider the cohomology long exact sequence associated to the distinguished triangle above. We see that ${}^r h^i(\mathcal{F}) \cong {}^r h^i(\mathcal{F}_2)$ for $i < k$, whereas for $i = k$, we have a short exact sequence $0 \to {}^r h^k(\mathcal{F}_2) \to {}^r h^k(\mathcal{F}) \to \mathcal{F}_1 \to 0$. It follows that ${}^r h^i(\mathcal{F}) \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w+i}$ for all i, as desired. □

11. The decomposition theorems

In this section, we prove the two versions of the Decomposition Theorem. In contrast with the two Purity Theorems, whose proofs involved different arguments, the two Decomposition Theorems have essentially identical proofs, and we will prove them simultaneously.

We retain the assumption that X is endowed with a recessed, split s-structure. Let $r\colon \mathbb{O}(X) \to \mathbb{Z}$ be a fixed staggered perversity.

Proposition 11.1. *Let $\mathcal{F} \in {}^r\mathcal{M}(X)$. The following conditions are equivalent.*

(1) *$\mathcal{F}$ is simple and skew-pure of skew degree w.*

(2) *$\mathcal{F} \cong \mathcal{IC}(\bar{C}, \mathcal{L}[(w - \operatorname{cod} C)/2])$, where $\mathcal{L} \in \mathcal{C}_G(C)$ is an irreducible vector bundle that is s-pure of step $(w - \operatorname{cod} C)/2 + r(C)$.*

If furthermore r is moderate, these conditions are equivalent to

(3) *$\mathcal{F}$ is simple and pure of baric degree $w + r(c) - \bar{r}(C)$.*

In particular, in the case where $r(C) = \frac{1}{2}\operatorname{scod} C$, the baric and skew degrees of a simple staggered sheaf coincide.

Proof. We know that every simple staggered sheaf is of the form ${}^{r}\mathcal{IC}(\bar{C}, \mathcal{L}[v - r(C)])$ for some irreducible vector bundle $\mathcal{L}$. From Lemma 10.1, we have $v - r(C) = (w - \operatorname{cod} C)/2$, and this establishes the equivalence of parts (1) and (2). Next, in case r is moderate, Lemma 6.3 tells us that the baric degree of $\mathcal{F}$ is

$$\begin{aligned} 2v - \operatorname{alt} C &= w - \operatorname{cod} C + 2r(C) - \operatorname{alt} C \\ &= w + r(C) + (r(C) - \operatorname{scod} C) \\ &= w + r(C) - \bar{r}(C). \end{aligned}$$

This establishes the equivalence of part (3) with the other two. □

Note that in the special case of the self-dual staggered perversity $r(C) = \frac{1}{2}\operatorname{scod} C$, the baric degree and skew degree of a simple staggered sheaf coincide.

Proposition 11.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be staggered sheaves.*

(1) *If $\mathcal{F}$ is skew-pure of skew degree w and $\mathcal{G}$ is skew-pure of skew degree v, then $\operatorname{Hom}(\mathcal{F}, \mathcal{G}[k]) = 0$ for all $k > w - v$.*

(2) *Assume that $r(C) = \frac{1}{2}\operatorname{scod} C$ and r is moderate. If $\mathcal{F}$ is pure of baric degree w and $\mathcal{G}$ is pure of baric degree v, then $\operatorname{Hom}(\mathcal{F}, \mathcal{G}[k]) = 0$ for all $k > w - v$.*

Proof. Part (1) is an immediate consequence of Proposition 9.3 and the fact that $\mathcal{G}[k] \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq v+k}$, and part (2) then follows using Proposition 11.1. □

Proposition 11.3 (cf. [6, Théorème 5.3.8]).

(1) *Every skew-pure staggered sheaf is semisimple.*

(2) *Assume $r(C) = \frac{1}{2}\operatorname{scod} C$ and r is moderate. Then every pure staggered sheaf is semisimple.*

Proof. The proofs of the two parts are identical, and we prove them simultaneously. Let $\mathcal{F}$ be a (skew-)pure staggered sheaf, and let $\mathcal{F}' \subset \mathcal{F}$ be the sum of all simple subobjects of $\mathcal{F}$. $\mathcal{F}'$ is the largest semisimple subobject of $\mathcal{F}$. We must show that $\mathcal{F}' = \mathcal{F}$. Form a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

By Theorem 6.5 or 10.2, $\mathcal{F}'$ and $\mathcal{F}''$ are also (skew-)pure of degree w, and then by Proposition 11.2, $\operatorname{Hom}(\mathcal{F}'', \mathcal{F}'[1]) = 0$. It follows that this short exact sequence splits, and that $\mathcal{F} \cong \mathcal{F}' \oplus \mathcal{F}''$. If $\mathcal{F}'' \neq 0$, then any simple subobject of $\mathcal{F}''$ would also be a simple subobject of $\mathcal{F}$ not contained in $\mathcal{F}'$, a contradiction. □

Proposition 11.4 (cf. [6, Théorème 5.4.5]). *Let* $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$.

(1) *If* $\mathcal{F}$ *is skew-pure, then* $\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} {}^r h^i(\mathcal{F})[-i]$.

(2) *Assume* $r(C) = \frac{1}{2} \operatorname{scod} C$ *and that* r *is moderate. If* $\mathcal{F}$ *is pure, then*

$$\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} {}^r h^i(\mathcal{F})[-i].$$

Proof. Again, we prove the two parts simultaneously. We proceed by induction on the number of non-zero staggered cohomology objects of $\mathcal{F}$. If $\mathcal{F}$ has at most one non-zero cohomology object, then there is nothing to prove. Otherwise, let k be the largest integer such that ${}^r h^k(\mathcal{F}) \neq 0$, and form the distinguished triangle

$${}^r\tau^{\leqslant k-1}\mathcal{F} \to \mathcal{F} \to {}^r h^k(\mathcal{F})[-k] \to .$$

It follows from Theorem 6.5 or 10.2 that the staggered truncation functor ${}^r\tau^{\leqslant k-1}$ preserves (skew-)purity. Since ${}^r\tau^{\leqslant k-1}\mathcal{F}$ has fewer non-zero cohomology objects than $\mathcal{F}$, we have

$${}^r\tau^{\leqslant k-1}\mathcal{F} \cong \bigoplus_{i \leqslant k-1} {}^r h^i(\mathcal{F})[-i]$$

by assumption. Then

$$\begin{aligned} \operatorname{Hom}({}^r h^k(\mathcal{F})[-k], ({}^r\tau^{\leqslant k-1}\mathcal{F})[1]) &\cong \bigoplus_{i \leqslant k-1} \operatorname{Hom}({}^r h^k(\mathcal{F})[-k], {}^r h^i(\mathcal{F})[-i+1]) \\ &\cong \bigoplus_{i \leqslant k-1} \operatorname{Hom}({}^r h^k(\mathcal{F}), {}^r h^i(\mathcal{F})[k+1-i]). \end{aligned}$$

We claim that $\operatorname{Hom}({}^r h^k(\mathcal{F}), {}^r h^i(\mathcal{F})[k+1-i]) = 0$ for all i. In the setting of skew-purity, Theorem 10.2 tells us that ${}^r h^k(\mathcal{F})$ is skew-pure of skew degree $w+k$, and that each ${}^r h^i(\mathcal{F})$ is skew-pure of skew degree $w+i$. In the setting of baric purity, Theorem 6.5 tells us that ${}^r h^k(\mathcal{F})$ and all the ${}^r h^i(\mathcal{F})$ are pure of baric degree w. Since $k+1-i > (w+k)-(w+i)$ and $k+1-i > 0$, Proposition 11.2 tells us in both cases that $\operatorname{Hom}({}^r h^k(\mathcal{F}), {}^r h^i(\mathcal{F})[k+1-i]) = 0$. Thus, $\operatorname{Hom}({}^r h^k(\mathcal{F})[-k], ({}^r\tau^{\leqslant k-1}\mathcal{F})[1]) = 0$, so in the distinguished triangle above, we find that

$$\mathcal{F} \cong {}^r\tau^{\leqslant k-1}\mathcal{F} \oplus {}^r h^k(\mathcal{F})[-k] \cong \bigoplus_{i \in \mathbb{Z}} {}^r h^i(\mathcal{F})[-i],$$

as desired. □

Combining the preceding two propositions with the formulae in Proposition 11.1 relating step, baric degree, and skew degree, we obtain the following theorem.

Theorem 11.5 (decomposition). *Assume that X is endowed with a recessed, split s-structure.*

(1) *Every skew-pure complex $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\langle w\rangle}$ admits a decomposition*

$$\mathcal{F} \cong \bigoplus_{i=1}^{n} {}^{r}\mathcal{IC}(\bar{C}_i, \mathcal{L}_i[(w - k_i - \operatorname{cod} C)/2])[k_i],$$

where each $\mathcal{L}_i \in \mathcal{C}_G(C_i)$ is an irreducible vector bundle that is s-pure of step $(w - k_i - \operatorname{cod} C)/2 + r(C_i)$.

(2) *Assume $r(C) = \frac{1}{2} \operatorname{scod} C$ and that r is moderate. Every pure complex $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$ admits a decomposition*

$$\mathcal{F} \cong \bigoplus_{i=1}^{n} \mathcal{IC}(\bar{C}_i, \mathcal{L}_i[(w - \operatorname{cod} C_i)/2])[k_i],$$

where each C_i is an orbit such that $w \equiv \operatorname{cod} C_i \pmod 2$, and each $\mathcal{L}_i \in \mathcal{C}_G(C_i)$ is an irreducible vector bundle that is s-pure of step $(w + \operatorname{alt} C_i)/2$.

Remark 11.6. Most of the work for this result lies in the proofs of the purity theorems. Indeed, suppose $\mathfrak{D}$ is a triangulated category equipped with t- and co-t-structures. Suppose furthermore that objects in the heart of the t-structure have finite length, and that the simple objects are 'skew-pure' for the given co-t-structure. Then an analogue of Theorem 11.5 holds in that category. The proof is a straightforward adaptation of the arguments in this section, which have themselves been adapted from arguments in [**6**, Chapter 5].

12. Pointwise purity and tensor products on smooth toric varieties

In this section, we impose the following assumption on the scheme X:

$$\begin{aligned} &X \text{ is smooth and is endowed with a dualizing complex and} \\ &\qquad \text{an } s\text{-structure such that } \operatorname{alt} C = \operatorname{cod} C \text{ for all } C \in \mathbb{O}(X). \end{aligned} \tag{12.1}$$

Later, we will see that smooth toric varieties admit dualizing complexes and s-structures satisfying (12.1). We will work exclusively with the self-dual staggered perversity

$$r(C) = \tfrac{1}{2} \operatorname{scod} C = \operatorname{cod} C = \operatorname{alt} C,$$

and we will drop the superscript 'r' from most notation related to the staggered t-structure. Note that the smoothness assumption implies that functors like $\otimes^L$, $\mathrm{R}\mathcal{H}\mathrm{om}$, and Li^* and $Ri^!$ (for $i\colon Z \to X$ a closed subscheme) actually take values in the bounded derived category. Our goal in this section is to illustrate some uses of Theorem 11.5: we first show that certain derived tensor products in $\mathcal{D}^{\mathrm{b}}_G(X)$ are automatically semisimple, and then we apply this to obtain a positivity statement about the equivariant K-theory of smooth toric varieties.

12.1. Pointwise purity

To gain some control over the behaviour of derived tensor products, we must impose a condition called 'pointwise purity' that somewhat resembles the notion of pointwise purity for mixed ℓ-adic sheaves. Pointwise purity implies skew-purity, but the converse is not true in general. However, we will see later that on a smooth toric variety, every simple staggered sheaf is pointwise pure. (For comparison, recall that simple ℓ-adic perverse sheaves are not pointwise pure in general, but they are pointwise pure on flag varieties of reductive algebraic groups.)

Definition 12.1. Assume that X is smooth. An object $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$ is said to be *pointwise pure* of weight w if for each orbit $C \subset X$, we have that $Li^*_C\mathcal{F}|_C$ and $Ri^!_C\mathcal{F}|_C$ are both skew-pure of skew-degree w.

Proposition 12.2. *If $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$ is pointwise pure of weight w, and $\mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)$ is pointwise pure of weight v, then $\mathcal{F} \otimes^L \mathcal{G}$ is pointwise pure of weight $w + v$.*

Proof. Recall that $Li^*_C(\mathcal{F} \otimes^L \mathcal{G}) \cong Li^*_C\mathcal{F} \otimes^L Li^*_C\mathcal{G}$ and that $Ri^!_C(\mathcal{F} \otimes^L \mathcal{G}) \cong Ri^!_C\mathcal{F} \otimes^L Li^*_C\mathcal{G}$ (see [**11**, Proposition III.8.8]). Thus, it suffices to show that on a single orbit, a derived tensor product of skew-pure objects is again skew-pure.

Since skew-pure objects are semisimple by Theorem 11.5, we may further reduce to considering the derived tensor product of (shifts of) two simple staggered sheaves on a single orbit. Let $\mathcal{L}, \mathcal{L}' \in \mathcal{C}_G(C)$ be two irreducible vector bundles, and let $n, m \in \mathbb{Z}$. Then the objects $\mathcal{L}[n]$ and $\mathcal{L}'[m]$ are both skew-pure; assume they have weights w and v, respectively. By the remark following Lemma 10.1, we have $w = n + \operatorname{step}\mathcal{L}$ and $v = m + \operatorname{step}\mathcal{L}'$. We have

$$\mathcal{L}[n] \otimes^L \mathcal{L}'[m] \cong (\mathcal{L} \otimes \mathcal{L}')[n + m].$$

Since $\mathcal{L} \otimes \mathcal{L}' \in \mathcal{C}_G(C)$ is s-pure, with $\operatorname{step}\mathcal{L} \otimes \mathcal{L}' = \operatorname{step}\mathcal{L} + \operatorname{step}\mathcal{L}'$, the result follows by another application of Lemma 10.1. $\square$

It follows from Proposition 9.6 that a pointwise pure object of weight w is automatically skew-pure of skew degree w. The next statement then follows from Theorem 11.5.

Proposition 12.3. *If $\mathcal{F}, \mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)$ are pointwise pure, then $\mathcal{F} \otimes^L \mathcal{G}$ is semisimple.*

12.2. Smooth toric varieties

Staggered t-structures on toric varieties have been studied by the second author [**14**]. In particular, [**14**] explains how to describe s-structures, staggered perversity functions, etc., in the language of cones and fans in lattices. We will use that language to show that smooth toric varieties provide a setting where the results of § 12.1 apply.

Let Λ be a finitely generated free abelian group of rank N, and let $\Sigma \subset \Lambda$ be a fan corresponding to a smooth toric variety X. Let G denote the canonical torus acting on X.

Each one-dimensional cone $\kappa \subset \Sigma$ is the set of non-negative integer multiples of a single element. Let A_κ denote the negative of that element, so that $\kappa = \mathbb{N} \cdot (-A_\kappa)$. More generally, for cones $\kappa \subset \Sigma$ of dimension greater than 1, let

$$A_\kappa = \sum_{\{\kappa' \subset \kappa \mid \dim \kappa' = 1\}} A_{\kappa'}.$$

As in [**14**, Theorem 3.5], the collection $\boldsymbol{A} = \{A_\kappa\}$ determines an s-structure on X.

To describe this s-structure explicitly, let us consider line bundles on X. Recall that any piecewise-linear function $\underline{\psi}\colon \Sigma \to \mathbb{Z}$ determines a G-equivariant line bundle, denoted $L(\underline{\psi})$. (Here, 'piecewise-linear' simply means 'additive on cones.') It follows from [**14**, § 3.2] that if $C \subset X$ is the orbit corresponding to κ, then

$$\operatorname{step} i_C^* L(\underline{\psi})|_C = -\underline{\psi}(-A_\kappa). \tag{12.2}$$

In [**14**], dualizing complexes are attached to piecewise-linear functions by a slightly different convention: given $\underline{\chi}\colon \Sigma \to \mathbb{Z}$, we set $\omega_X = \omega_X(\underline{\chi}) = \Omega_X^{\dim X} \otimes L(\underline{\chi})$, where $\Omega_X^{\dim X}$ is the canonical bundle. According to [**14**, Corollary 4.8], there is a self-dual perversity with respect to the dualizing complex attached to the function $\underline{\chi}$ determined by setting $\underline{\chi}(-A_\kappa) = -1$ for all one-dimensional cones $\kappa \subset \Sigma$. This function clearly satisfies

$$\underline{\chi}(-A_\kappa) = -\dim \kappa \quad \text{for all cones } \kappa \subset \Sigma.$$

From [**14**, Proposition 3.6 and Corollary 4.8], we see that the associated self-dual perversity is given by $r(C) = \dim X - \dim C = \operatorname{alt} C$. In particular, condition (12.1) holds. In our case, it may be checked that $L(\underline{\chi})$ is in fact the inverse of the canonical bundle $\Omega_X^{\dim X}$, so we simply have $\omega_X \cong \mathcal{O}_X$.

Proposition 12.4. *Every simple staggered sheaf on a smooth toric variety is pointwise pure.*

Proof. Since pointwise purity is a local property, it suffices to prove this statement on an affine toric variety. Furthermore, every smooth affine toric variety is isomorphic (as a G-variety) to $\mathbb{A}^k \times (\mathbb{G}_m)^{n-k}$, and this variety admits an equivariant embedding into $\mathbb{A}^n$, so let us assume that $X = \mathbb{A}^n$. We identify Λ with $\mathbb{Z}^n$ in such a way that $\Sigma = \{(a_1, \ldots, a_n) \mid a_i \geqslant 0 \text{ for all } i\}$. For $I \subset \{1, \ldots, n\}$, let $\kappa_I \subset \Sigma$ consist of those $(a_1, \ldots, a_n) \in \Sigma$ with $a_i = 0$ for $i \notin I$, and let $C_I \subset X$ denote the corresponding G-orbit. For brevity, we will write A_I instead of A_{κ_I}.

A piecewise-linear function $\underline{\psi}\colon \Sigma \to \mathbb{Z}$ (or the corresponding line bundle $L(\underline{\psi})$) is said to be *I-steady* if $\underline{\psi}(-A_{\{i\}}) = 0$ for all $i \notin I$. Note that the restriction of a line bundle $L(\underline{\psi})$ to the orbit C_I is determined by $\underline{\psi}|_{\kappa_I}$, so any line bundle $L(\underline{\psi})$ can be replaced by an I-steady one without changing $i_C^* L(\underline{\psi})$. Every equivariant line bundle on C_I arises as the restriction of an equivariant line bundle on X, and therefore of an I-steady equivariant line bundle on X.

Consider coherent sheaves of the form

$$\mathcal{L}_{I,\underline{\psi}} = \mathcal{O}_{\bar{C}_I} \otimes L(\underline{\psi}) \quad \text{where } \underline{\psi} \text{ is } I\text{-steady}. \tag{12.3}$$

We will show below that the $\mathcal{L}_{I,\underline{\psi}}$ are pointwise pure. Admitting this claim for a moment, we have by Proposition 12.3 that they are semisimple. But the $\mathcal{L}_{I,\underline{\psi}}$ are clearly indecomposable, so they must in fact be shifts of simple staggered sheaves. From the preceding paragraph, we know that every line bundle on C_I arises as some $i^*_{C_I}\mathcal{L}_{I,\underline{\psi}}|_{C_I}$, so it follows that every simple staggered sheaf is a shift of some $\mathcal{L}_{I,\underline{\psi}}$. Thus, the proposition will follow from the pointwise purity of the $L_{I,\underline{\psi}}$.

Before proceeding, we require some additional notation. Let $\underline{\psi}_i \colon \Sigma \to \mathbb{Z}$ be the piecewise-linear function given by $\underline{\psi}_i(-A_{\{j\}}) = \delta_{ij}$. More generally, for $I \subset \{1,\ldots,n\}$, we put $\underline{\psi}_I = \sum_{i\in I}\underline{\psi}_i$. We clearly have

$$\underline{\psi}_I(-A_J) = \#(I\cap J).$$

These functions can be used to describe the terms in the Koszul resolution of the structure sheaf $\mathcal{O}_{\bar{C}_I}$ of the orbit closure C_I:

$$M_I^\bullet = \left(L(\underline{\psi}_I) \to \cdots \to \bigoplus_{\substack{K\subset I\\ \#K=k}} L(\underline{\psi}_K) \to \cdots \to \bigoplus_{\substack{K\subset I\\ \#K=2}} L(\underline{\psi}_K) \to \bigoplus_{i\in I} L(\underline{\psi}_i) \to \mathcal{O}_X\right). \tag{12.4}$$

Using this, we can describe the dualizing complex $\omega_{\bar{C}_I} = Ri^!_{C_I}\omega_X$, as follows. We know that $\omega_{\bar{C}_I}$ is a shift of a line bundle, concentrated in degree $\dim X - \dim C_I = \#I$. Similar remarks apply to $i_{C_I*}\omega_{\bar{C}_I}$, which is represented by the chain complex $R^\bullet = \mathcal{H}\mathrm{om}(M_I^\bullet, \mathcal{O}_X)$. Explicitly, the terms of $R^\bullet$ are given by

$$R^k = \bigoplus_{\substack{K\subset I\\ \#K=k}} L(-\underline{\psi}_K).$$

In particular, $R^k = 0$ for $k > \#I$, so $h^{\#I}(R^\bullet)$ is a quotient of $R^{\#I} \cong L(-\underline{\psi}_I)$. Since $i_{C_I*}\omega_{\bar{C}_I}$ is supported on $\bar{C}_I$, $h^{\#I}(R^\bullet)$ must actually be a quotient of $\mathcal{O}_{\bar{C}_I}\otimes L(-\underline{\psi}_I)$, but no proper quotient of the latter is a line bundle over $\bar{C}_I$. We conclude that

$$i_{C_I*}\omega_{\bar{C}_I} \cong \mathcal{O}_{\bar{C}_I}\otimes L(-\underline{\psi}_I)[-\#I]. \tag{12.5}$$

Let us now compute $Li^*_{C_J}\mathcal{L}_{I,\underline{\psi}}$. This clearly vanishes if $I \not\subset J$, so let us assume henceforth that $I \subset J$. We begin by forming the chain complex

$$\begin{aligned} N^\bullet &= M_I^\bullet \otimes L(\underline{\psi}) \\ &= \left(L(\underline{\psi}_I + \underline{\psi}) \to \cdots \to \bigoplus_{\substack{K\subset I\\ \#K=k}} L(\underline{\psi}_K + \underline{\psi}) \to \cdots\right. \\ &\qquad \left. \to \bigoplus_{\substack{K\subset I\\ \#K=2}} L(\underline{\psi}_K + \underline{\psi}) \to \bigoplus_{i\in I} L(\underline{\psi}_i + \underline{\psi}) \to L(\underline{\psi})\right). \end{aligned} \tag{12.6}$$

This is a free resolution of $\mathcal{L}_{I,\underline{\psi}}$, so $Li^*_{C_J}\mathcal{L}_{I,\underline{\psi}}|_{C_J}$ is represented by the chain complex $Q^\bullet = (\mathcal{O}_{C_J}\otimes N^\bullet)|_{C_J}$. The terms of this chain complex are of the form

$$Q^{-k} = \left(\mathcal{O}_{C_J}\otimes \bigoplus_{\substack{K\subset I\\ \#K=k}} L(\underline{\psi}_K + \underline{\psi})\right)\Bigg|_{C_J}.$$

Since $I \subset J$, we have $-\underline{\psi}_K(-A_J) = -\#K = -k$ for each K occurring in the direct sum. Using the fact that $\underline{\psi}$ is I-steady, we also have $-\underline{\psi}(-A_J) = -\underline{\psi}(-A_I - A_{J\smallsetminus I}) = -\underline{\psi}(-A_I)$. We now see using (12.2) that Q^{-k} is s-pure with

$$\operatorname{step} Q^{-k} = -\underline{\psi}_K(-A_J) - \underline{\psi}(-A_J) = -k - \underline{\psi}(-A_I).$$

It follows that the cohomology sheaf $h^{-k}(Q^\bullet) \cong h^{-k}(Li^*_{C_J}\mathcal{L}_{I,\underline{\psi}}|_{C_J})$ is s-pure of step $-k - \underline{\psi}(-A_I)$ as well. Now, $h^{-k}(Q^\bullet)[-k - \underline{\psi}(-A_I) - \#J]$ is a staggered sheaf on C_J that is skew-pure of skew degree $-2k - 2\underline{\psi}(-A_I) - \#J$, so $h^{-k}(Q^\bullet)[k]$ is a skew-pure object in $\mathcal{D}^{\mathrm{b}}_G(X)$ of skew degree $(-2k - 2\underline{\psi}(-A_I) - \#J) + (2k + \underline{\psi}(-A_I) + \#J)$. We conclude that $Li^*_{C_J}\mathcal{L}_{I,\underline{\psi}}|_{C_J}$ is skew-pure of skew-degree

$$-\underline{\psi}(-A_I).$$

Finally, using (12.5), we see that

$$\mathbb{D}\mathcal{L}_{I,\underline{\psi}} \cong \mathcal{H}\mathrm{om}(\mathcal{O}_{\bar{C}_I} \otimes L(\underline{\psi}), L(-\underline{\psi}_I))[-\#I] \cong \mathcal{L}_{I,-\underline{\psi}_I-\underline{\psi}}[-\#I].$$

From the above calculation, we see that $Li^*_{C_J}\mathbb{D}\mathcal{L}_{I,\underline{\psi}}|_{C_J}$ is skew-pure of skew degree

$$\underline{\psi}_I(-A_I) + \underline{\psi}(-A_I) - \#I = \underline{\psi}(-A_I).$$

Therefore, $Ri^!_{C_J}\mathcal{L}_{I,\underline{\psi}} \cong \mathbb{D}Li^*_{C_J}\mathbb{D}\mathcal{L}_{I,\underline{\psi}}$ is skew-pure of skew degree $-\underline{\psi}(-A_I)$ as well, and $\mathcal{L}_{I,\underline{\psi}}$ is pointwise pure, as desired. □

As an application, we have the following positivity result in the equivariant K-theory of a smooth toric variety.

Theorem 12.5. *Let X be a smooth toric variety for a torus G. Let $K_G(X)$ denote the Grothendieck group of $\mathcal{D}^{\mathrm{b}}_G(X)$, and consider the subset*

$$\mathcal{B} = \{(-1)^{\text{skew degree of } \mathcal{F}}[\mathcal{F}] \mid \mathcal{F} \text{ a simple staggered sheaf}\} \subset K_G(X).$$

Then $K_G(X)$ is a free abelian group with basis $\mathcal{B}$. Moreover, if we endow $K_G(X)$ with the ring structure induced by $\otimes^L$, then for $b_1, b_2 \in \mathcal{B}$, we have

$$b_1 \cdot b_2 = \sum_{b\in\mathcal{B}} a_b b \quad \text{with } a_b \geqslant 0. \tag{12.7}$$

Proof. It is clear that $\mathcal{B}$ is a basis for $K_G(X)$. Now, recall that every pointwise pure object of weight 0 is automatically skew-pure of skew degree 0. Conversely, every skew-pure object is semisimple by Theorem 11.5 and therefore pointwise pure by Proposition 12.4. Thus, on a smooth toric variety, skew-purity and pointwise purity are equivalent. In particular, every indecomposable pointwise pure object is a shift of a simple staggered sheaf. More precisely, we have that $\mathcal{F}$ is a simple staggered sheaf of skew degree w if and only if $\mathcal{F}[-w]$ is an indecomposable pointwise pure object of weight 0. In $K_G(X)$, of course, we have $(-1)^w[\mathcal{F}] = [\mathcal{F}[-w]]$, so $\mathcal{B}$ can be regarded as the set of classes of indecomposable pointwise pure objects of weight 0. Now, suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two indecomposable pointwise pure objects of weight 0. It follows from Propositions 12.2 and 12.3 that $\mathcal{G}_1 \otimes^L \mathcal{G}_2$ is a direct sum of various indecomposable pointwise pure objects of weight 0. The non-negativity of the coefficients in (12.7) follows. □

12.3. An example

To illustrate the above results, we will explicitly calculate a tensor product of pointwise pure objects on the variety $X = \boldsymbol{A}^3(\mathbb{C})$. We will use the notation and conventions introduced in the proof of Proposition 12.4 for the study of affine spaces. A piecewise-linear function $\underline{\psi}\colon \Sigma \to \mathbb{Z}$ is determined by the three integers $\underline{\psi}(-A_{\{1\}}), \underline{\psi}(-A_{\{2\}}), \underline{\psi}(-A_{\{3\}})$, so we will specify such functions by writing down triples of integers.

We also set $R = \mathbb{C}[x, y, z]$, so that $X = \operatorname{Spec} R$. We will pass freely between the language of G-equivariant coherent sheaves on X and that of A-modules with a compatible G-action. In the notation of (12.3), we see that

$$\mathcal{L}_{\{2,3\},(0,0,0)} = R/(y, z)$$

is the structure sheaf of the x-axis in $\mathbb{A}^3$. Similarly, $\mathcal{L}_{\{1,2\},(0,0,0)} = R/(x, y)$ is the structure sheaf of the z-axis. Both are pointwise pure of weight 0. To compute their tensor product, we use the following free resolution of $\mathcal{L}_{\{2,3\},(0,0,0)}$:

$$yzR \to yR \oplus zR \to R \quad \text{or} \quad L(0,1,1) \to L(0,1,0) \oplus L(0,0,1) \to R.$$

Then $\mathcal{L}_{\{2,3\},(0,0,0)} \otimes^L \mathcal{L}_{\{1,2\},(0,0,0)}$ is represented by the chain complex

$$yzR/(xyz, y^2z) \to yR/(xy, y^2) \oplus zR/(xz, yz) \to R/(x, y).$$

Taking cohomology, one finds explicitly that

$$\mathcal{L}_{\{2,3\},(0,0,0)} \otimes^L \mathcal{L}_{\{1,2\},(0,0,0)} \cong \mathcal{L}_{\{1,2,3\},(0,0,0)} \oplus \mathcal{L}_{\{1,2,3\},(0,1,0)}[1].$$

In fact, it can be checked that $\mathcal{L}_{\{1,2,3\},(0,0,0)}[-3]$ is a simple staggered sheaf of skew degree 3, and $\mathcal{L}_{\{1,2,3\},(0,1,0)}[-4]$ is a simple staggered sheaf of skew degree 5.

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