

THIRD HOMOLOGY OF SOME SPORADIC FINITE GROUPS

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ABSTRACT. We compute the integral third homology of most of the sporadic finite simple groups and of their central extensions.

1. INTRODUCTION

In this paper we compute the third homology of some of the sporadic simple groups, and of their central extensions. For many of these groups we are able to name elements (characteristic classes) that generate $H^4(G; \mathbf{Z})$, the Pontryagin dual of $H_3(G)$. In the following table we write $n.G$ for the Schur covering of the sporadic group G — for a sporadic simple group, the covering is always by a cyclic group $n = H_2(G)$ — and have left empty spaces where $G = n.G$.

	M ₁₁	M ₁₂	M ₂₂	M ₂₃	M ₂₄			
$n = H_2(G)$	1	2	12	1	1			
$H_3(G)$	8	2×24	1	1	12			
$H_3(n.G)$		8×24	24					
	HS	J ₂	Co ₁	Co ₂	Co ₃	McL	Suz	
$H_2(G)$	2	2	2	1	1	3	6	
$H_3(G)$	2×2	30	12	4	6	1	4	
$H_3(n.G)$	2×8	120	24			1	24	
	J ₁	O’N	J ₃	Ru	J ₄	Ly		
$H_2(G)$	1	3	3	2	1	1		
$H_3(G)$	30	8	15	?	1	1		
$H_3(n.G)$		8	3×15	?				
	He	HN	Th	Fi ₂₂	Fi ₂₃	Fi’ ₂₄	B	M
$H_2(G)$	1	1	1	6	1	3	2	1
$H_3(G)$	12	?	?	1	?	?	?	$24 \times [\leq 4]$
$H_3(n.G)$				$3 \times [\leq 4]$?	?	

An expression like “ $a \times b$ ” is short for $\mathbf{Z}/a \oplus \mathbf{Z}/b$. Question marks in the table denote groups for which we do not know the answer, and “[≤ 4]” denotes an unknown, possibly trivial, group of order dividing 4. Further partial results for the groups HN, Th, Fi₂₃, and Fi’₂₄ are listed in §8.

Only some entries in the table are original. The Schur multiplier row (the first row in the table) was computed over many years, partly in service of the classification of finite simple groups, and is available in the ATLAS [CCN+85]. With \mathbf{F}_2 -coefficients, the entire cohomology rings of many of the smaller sporadic groups are listed in [AM04]. The Mathieu entries are reviewed in [DSE09]. Significantly, $H_3(M_{24})$ was first computed in that paper using Graham Ellis’s software package “HAP,” which we have found can also determine $H_3(G)$ for $G \in \{\text{HS}, 2\text{HS}, J_2, 2J_2, J_1, J_3, \text{McL}\}$ using the permutation models given in the ATLAS. For the larger groups G , although HAP cannot

calculate $H_3(G)$ on its own, it played an essential role in our calculations, as did the ‘‘Cohomolo’’ package by Derek Holt.

1.1. Motivation. If G is a compact simple Lie group, or a finite cover of a compact simple Lie group, the cohomology of its classifying space can be complicated at small primes but one always has $H^4(BG; \mathbf{Z}) \cong \mathbf{Z}$. If G is a finite group of Lie type, split and simply connected and defined over the field \mathbf{F}_q , then Jesper Grodal has shown that with finitely many exceptions, $H^4(G; \mathbf{Z}) \cong \mathbf{Z}/(q^2 - 1)$. Part of our motivation has been to see whether we could discern any patterns in $H^4(G; \mathbf{Z})$ when G is sporadic.

We have also been inspired by the idea that 3-cocycles $G \times G \times G \rightarrow U(1)$ (when G is finite, these represent classes in $H^4(G; \mathbf{Z})$) can explain and predict some features of moonshine [Gan09, Gan16, CdLW16, GPRV13]. Such a cocycle can arise as the gauge anomaly of a group action on a conformal field theory. Even in the newer examples of moonshine where no conformal-field-theoretic explanation is known, there are some numerical hints about this cocycle.

To some extent these hints can be pursued in an elementary way in pure group theory. If s and t are a pair of commuting elements in a finite group G , we may define the following infinite group:

$$\Gamma(s, t) := \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, g \right) \in \mathrm{SL}_2(\mathbf{Z}) \times G \mid gsg^{-1} = s^a t^b \text{ and } gtg^{-1} = s^c t^d \right\}.$$

It is the fundamental group of one of the components of the moduli stack of pairs (E, T) , where E is an elliptic curve and T is a G -torsor over E . If there is a natural family of McKay–Thompson series attached to G , one expects that their modularity properties (and more ambiguously, their mock modularity properties) can be expressed in terms of a holomorphic line bundle on this space, or equivalently in terms of a $\Gamma(s, t)$ -equivariant line bundle on the upper-half plane. The topological types of such line bundles are parametrized by the finite group $H^2(\Gamma(s, t); \mathbf{Z})$, which is the target of a transgression map $H^4(G; \mathbf{Z}) \rightarrow H^2(\Gamma(s, t); \mathbf{Z})$. Duncan–Mertens–Ono have used our calculations to explore a version of this idea in their ‘‘O’Nan moonshine’’ [DMO17, §3].

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2. METHODS

In this section we review some standard constructions in group cohomology, which we return to repeatedly in the following sections. The first of these is computing the p -primary part of $H^4(G; \mathbf{Z})$, which we denote by $H^4(G; \mathbf{Z})_{(p)}$, one prime at a time. An upper bound for the p -primary part is provided by the following lemma:

Lemma 2.1. *Let G be a finite group and let $S \subset G$ be a subgroup that contains a Sylow p -subgroup for some prime p . The restriction map $\alpha \mapsto \alpha|_S : H^k(G; \mathbf{Z})_{(p)} \rightarrow H^k(S; \mathbf{Z})_{(p)}$ is an injection onto a direct summand. \square*

Lemma 2.1 together with some basic properties (which we review in some detail in §3.1) of $H^4(\mathbf{Z}/p; \mathbf{Z})$ and $H^4(\mathbf{Z}/p \times \mathbf{Z}/p; \mathbf{Z})$, allow us to dispose of many of the larger primes, at least for sporadic groups:

Lemma 2.2. *Let $p \geq 5$ be a prime and let G be a sporadic simple group whose Sylow p -subgroup has order p or p^2 . Then either $(p, G) = (5, J_2)$ or else the p -part of $H^4(G; \mathbf{Z})$ vanishes.*

Proof. Let $H \subset G$ be a Sylow p -subgroup. Suppose first that H has order p . Then G has strictly fewer than $(p - 1)/2$ conjugacy classes of order p . (We checked this by inspecting the ATLAS's tables: it is not true for every non-sporadic group, e.g. $\mathrm{SL}_2(\mathbf{F}_{32})$ has 15 conjugacy classes of order 31.) It follows that for a generator $h \in H$ (or indeed for any element h of order p in G), there is an $x \in G$ with $xhx^{-1} = h^a$, where a is neither 1 nor $-1 \pmod p$. Conjugation by such an x acts trivially on $\mathrm{H}^\bullet(G; \mathbf{Z})$ but nontrivially on $\mathrm{H}^4(H; \mathbf{Z})$, so the image of the restriction map $\mathrm{H}^4(G; \mathbf{Z}) \rightarrow \mathrm{H}^4(H; \mathbf{Z})$ must vanish. After Lemma 2.1, $\mathrm{H}^4(G; \mathbf{Z})$ must vanish as well.

For all sporadic groups, if the p -Sylow has order p^2 then it is isomorphic to $\mathbf{Z}/p \times \mathbf{Z}/p$. (Note this is not true for a general simple group, e.g. the 3-Sylow in $\mathrm{SL}_2(\mathbf{F}_{16})$ is cyclic of order 9). Furthermore by inspecting the tables one sees that such a sporadic group has strictly less than $(p - 1)/2$ conjugacy classes of order p , unless $p = 5$ and $G = \mathrm{J}_2$. With this single exception, we conclude that $\mathrm{H}^4(G; \mathbf{Z}) \rightarrow \mathrm{H}^4(\langle g \rangle; \mathbf{Z})$ is zero whenever g has order p — in particular $\mathrm{H}^4(G; \mathbf{Z}) \rightarrow \mathrm{H}^4(\langle g \rangle; \mathbf{Z})$ is zero whenever g is an element of $H \cong \mathbf{Z}/p \times \mathbf{Z}/p$. But when p is odd, $\mathrm{H}^4(\mathbf{Z}/p \times \mathbf{Z}/p; \mathbf{Z})$ is detected on its cyclic subgroups. \square

In many cases not covered by Lemma 2.2, there is a maximal subgroup $S \subset G$ that contains a p -Sylow, and that has shape $S = E.J$ where E is either an elementary abelian or an extraspecial p -group. (See [Wil17] for a survey of maximal subgroups of finite groups.) Sometimes we know $\mathrm{H}^4(J; \mathbf{Z})$, either by induction or by computer. The Lyndon-Hochschild-Serre (LHS) spectral sequence

$$E_2^{ij} = \mathrm{H}^i(J; \mathrm{H}^j(E; \mathbf{Z})) \implies \mathrm{H}^{i+j}(S; \mathbf{Z})$$

gives an upper bound for $\mathrm{H}^4(S; \mathbf{Z})$ (and therefore for $\mathrm{H}^4(G; \mathbf{Z})$), in terms of the cohomology groups (with twisted coefficients)

$$(2.1) \quad \mathrm{H}^0(J; \mathrm{H}^4(E; \mathbf{Z})) \quad \mathrm{H}^1(J; \mathrm{H}^3(E; \mathbf{Z})) \quad \mathrm{H}^2(J; \mathrm{H}^2(E; \mathbf{Z}))$$

We describe the groups $\mathrm{H}^j(E; \mathbf{Z})$ for $j \leq 4$ as $\mathrm{Aut}(E)$ -modules in §3. We used extensively Derek Holt's software package "Cohomolo" to determine the groups $\mathrm{H}^1(J; -)$ and $\mathrm{H}^2(J; -)$, but sometimes the following vanishing criterion can be employed instead:

Lemma 2.3. *Suppose that the center $Z(J)$ acts on $\mathrm{H}^j(E; \mathbf{Z})$ through a nontrivial character $Z(J) \rightarrow \mathbf{F}_p^\times$. Then $\mathrm{H}^i(J; \mathrm{H}^j(E; \mathbf{Z})) = 0$ for all i .*

Proof. In that case the trivial J -module and $\mathrm{H}^j(E; \mathbf{Z})$ are in different blocks. \square

The LHS spectral sequence allows us a comparison between the cohomology of a group and of its Schur cover. Recall that a finite group is called "perfect" if its abelianization is trivial.

Lemma 2.4. *Let G be a finite perfect group and choose a cyclic subgroup $n \subset \mathrm{H}_2(G)$. Then the pullback map $\mathrm{H}^4(G; \mathbf{Z}) \rightarrow \mathrm{H}^4(n.G; \mathbf{Z})$ is an injection. The cokernel has order dividing n if n is odd, or dividing $2n$ if n is even.*

Proof. We treat the case $n = \mathrm{H}_2(G)$ for clarity; the general case is no harder. Consider the LHS spectral sequence for the extension $n.G$. Its E_2 page, in total degree ≤ 5 , reads:

$$\begin{array}{ccccccc}
 & 0 & & & & & \\
 & (\mathbf{Z}/n)y^2 & 0 & & & & \\
 & 0 & 0 & 0 & & & \\
 & (\mathbf{Z}/n)y & 0 & (\mathbf{Z}/n) & [(\mathbf{Z}/n)xy] \cdot \mathrm{hom}(\mathrm{H}_3(G), \mathbf{Z}/n) & & \\
 & 0 & 0 & 0 & 0 & 0 & \\
 \mathbf{Z} & 0 & 0 & & (\mathbf{Z}/n)x & \mathrm{H}^4(G; \mathbf{Z}) & \mathrm{H}^5(G; \mathbf{Z})
 \end{array}$$

The rows are filled in using the universal coefficient theorem. We have indicated some of the multiplicative structure: we labeled a generator of $\mathrm{H}^3(G; \mathbf{Z}) \cong \mathbf{Z}/n$ by the name "x" and we

labeled a generator of $H^2(n; \mathbf{Z}) \cong \mathbf{Z}/n$ by the name “ y .” In particular, y^2 and xy generate (\mathbf{Z}/n) s in degrees $(i, j) = (0, 4)$ and $(3, 2)$ respectively.

The even differentials in the spectral sequence vanish automatically, so $E_3^{ij} = E_2^{ij}$. The d_3 differential takes $y \mapsto x$, and so $d_3(y^2) = 2xy$. It follows that if n is odd, then $d_3 : E_3^{04} \rightarrow E_3^{32}$ is injective, and if n is even, $d_3 : E_3^{04} \rightarrow E_3^{32}$ has kernel of order 2. Thus the E_4 page reads:

$$\begin{array}{cccccc} 0 & & & & & \mathbf{Z}/2 \\ 0 & 0 & 0 & & & 0 & 0 & 0 \\ 0 & 0 & X & & \text{or} & 0 & 0 & X \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ \mathbf{Z} & 0 & 0 & 0 & H^4(G) & \mathbf{Z} & 0 & 0 & 0 & H^4(G) \end{array}$$

depending on the parity of n , where $X \subset (\mathbf{Z}/n)$ is the kernel of $d_3 : E_3^{22} \rightarrow E_3^{50}$. \square

Corollary 2.5. *Let p be an odd prime such that $H^1(G; \mathbf{Z}/p) = 0$ and $H^2(G; \mathbf{Z}/p) = p$. Let pG denote a nontrivial central extensions of G by the group \mathbf{Z}/p . Suppose that $S \subset G$ also has $H^1(S; \mathbf{Z}/p) = 0$, and that the central extension pG , when restricted to S , is nonsplit. Then the pullback map $H^4(pG; \mathbf{Z}) \rightarrow H^4(pS; \mathbf{Z})$ induces an injection*

$$\text{coker}(H^4(G; \mathbf{Z}) \rightarrow H^4(pG; \mathbf{Z})) \hookrightarrow \text{coker}(H^4(S; \mathbf{Z}) \rightarrow H^4(pS; \mathbf{Z})).$$

This injection is an isomorphism if S contains the p -Sylow of G .

Proof. By Lemma 2.4, $\text{coker}(H^4(G) \rightarrow H^4(pG))$ and $\text{coker}(H^4(S) \rightarrow H^4(pS))$ are each of order either 0 or p . We need only to show that if $\text{coker}(H^4(G) \rightarrow H^4(pG)) = p$, then $\text{coker}(H^4(S) \rightarrow H^4(pS)) = p$. By the proof of Lemma 2.4, $\text{coker}(H^4(G) \rightarrow H^4(pG)) = p$ if and only if the $d_3 : E_2^{22} \rightarrow E_2^{50}$ vanishes. Let $\alpha \in H^2(G; p)$ denote the generator classifying the extension pG . Then $d_3 : \alpha \mapsto \text{Bock}(\alpha^2)$, where $\text{Bock} : H^4(G; p) \rightarrow H^5(G)$ denotes the integral Bockstein. But then $\text{Bock}((\alpha|_S)^2) = \text{Bock}(\alpha^2)|_S$ also vanishes, and so $\text{coker}(H^4(S) \rightarrow H^4(pS)) = p$ by the spectral sequence for the extension pS .

Conversely, assuming S contains the p -Sylow in G , if $\text{Bock}(\alpha^2)|_S = 0$, then $\text{Bock}(\alpha^2) = 0$ by Lemma 2.1. \square

As we have mentioned, each page of the LHS spectral sequence provides an upper bound for $H^4(G)_{(p)}$. We can improve this upper bound whenever we can show that the images of the two maps

$$H^4(J; \mathbf{Z}) \rightarrow H^4(S; \mathbf{Z}) \leftarrow H^4(G; \mathbf{Z})$$

have trivial intersection — in that case only the groups (2.1) can contribute to $H^4(G; \mathbf{Z})$.

With the improved upper bound in hand, the last step is to give a lower bound for $H^4(G; \mathbf{Z})$. In almost all cases these come from the characteristic class of a representation $G \rightarrow K$, where K is a Lie group. For $K = \text{U}(n)$, resp. $\text{Spin}(n)$, this characteristic class is c_2 , resp. $\frac{p_1}{2}$. In two cases we appeal to $K = \text{E}_6$ and E_8 . For some of the Monster sections, it is not possible for such representations to give a strong enough lower bound, and we instead appeal to the construction of [JF17].

3. ELEMENTARY ABELIAN AND EXTRASPECIAL p -GROUPS

3.1. Elementary abelian groups.

Lemma 3.1. *Let $E = p^n$ be an elementary abelian p -group and let $E^* := \text{Hom}(E, \mu_p)$, where μ_p denotes the group of p th roots of unity in \mathbf{C}^* .*

(1) *If $p = 2$, we have isomorphisms of $\text{GL}(E)$ -modules*

$$H^2(E; \mathbf{Z}) = E^* \quad H^3(E; \mathbf{Z}) = \text{Alt}^2(E^*) \quad H^4(E; \mathbf{Z}) = E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*)$$

where the last group on the right denotes a filtered $\mathrm{GL}(E)$ -module whose subquotients of E^* , $\mathrm{Alt}^2(E^*)$, and $\mathrm{Alt}^3(E^*)$. The submodule $E^* \cdot \mathrm{Alt}^2(E^*)$ is $\mathrm{GL}(E)$ -isomorphic to $\mathrm{Sym}^2(E^*)$.

(2) If p is odd, we have isomorphisms of $\mathrm{GL}(E)$ -modules

$$\mathrm{H}^2(E; \mathbf{Z}) = E^* \quad \mathrm{H}^3(E; \mathbf{Z}) = \mathrm{Alt}^2(E^*) \quad \mathrm{H}^4(E; \mathbf{Z}) = \mathrm{Sym}^2(E^*) \oplus \mathrm{Alt}^3(E^*)$$

Proof. See [Mas07, Proposition 2.2] or [JFT18, Lemma 4.4]. \square

If V is an elementary abelian p -group, we regard it as an \mathbf{F}_p -vector space in the obvious way. We may identify E^* with the usual dual \mathbf{F}_p -vector space to E by fixing at the outset an isomorphism $\mu_p \cong \mathbf{Z}/p$. We use $\mathrm{Sym}^n(V)$ and $\mathrm{Alt}^n(V)$ for the symmetric and exterior powers of V ; recall in positive characteristic these are defined as quotients of $V^{\otimes n}$ in the following way:

- $\mathrm{Sym}^n(V) := \mathrm{H}_0(S_n; V^{\otimes n})$ are the coinvariants of $V^{\otimes n}$ by the symmetric group action
- $\mathrm{Alt}^n(V)$ is the quotient of $V^{\otimes n}$ by the subspace spanned by tensors with a repeated tensorand (tensors $v_1 \otimes \cdots \otimes v_n$ with $v_i = v_j$ for some $i \neq j$).

Though $\mathrm{Sym}^n(E^*)$ and $\mathrm{Sym}^n(E)^*$ are not isomorphic as $\mathrm{GL}(E)$ -modules if $p \leq n$ (instead the dual of $\mathrm{Sym}^n(E^*)$ is the space of divided powers of E), let us record:

Lemma 3.2. *If p is a prime and E is an \mathbf{F}_p -vector space, there is an isomorphism*

$$\mathrm{Alt}^n(E^*) \cong \mathrm{Alt}^n(E)^*$$

of $\mathrm{GL}(E)$ -modules.

Proof. The pairing $V^{\otimes n} \otimes (V^*)^{\otimes n} \rightarrow \mathbf{Z}/p$ given by

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n \rangle = \sum_{\sigma \in S_n} (-1)^\sigma \langle v_1, w_{\sigma(1)} \rangle \cdots \langle v_n, w_{\sigma(n)} \rangle,$$

where $(-1)^\sigma$ denotes the sign of the permutation σ , is $\mathrm{GL}(V)$ -equivariant and descends to a perfect pairing between $\mathrm{Alt}^n(V)$ and $\mathrm{Alt}^n(V^*)$. \square

3.2. Extraspecial p -groups for p odd. If p is prime, $E = p^n$ is an elementary abelian p -group and ω is a function $E \times E \rightarrow \mathbf{Z}/p$, we define a multiplication on the set of formal monomials of the form $z^i t^v$ (where $i \in \mathbf{Z}/p$ and $v \in E$) by the formula

$$(z^i t^v)(z^j t^w) := z^{i+j+\omega(v,w)} t^{v+w}$$

If ω is bilinear, this multiplication is associative, $z^0 t^0$ is a two-sided unit, and $z^{-i+\omega(v,v)} t^{-v}$ is the two-sided inverse to $z^i t^v$: we defined a group that we denote by $(p.E)_\omega$. The groups associated to $(p.E)_\omega$ and $(p.E)_{\omega'}$ are isomorphic if $\omega - \omega'$ can be written as $j(v+w) - j(v) - j(w)$ for some function $j : E \rightarrow \mathbf{Z}/p$ — in particular if p is odd then $\omega(v,w)$ and

$$\frac{1}{2}(\omega(v,w) - \omega(w,v)) = \omega(v,w) - \frac{1}{2}(\omega(v+w, v+w) - \omega(v,v) - \omega(w,w))$$

determine isomorphic groups, so when p is odd we may as well assume that $\omega \in \mathrm{Alt}^2(E^*)$ is skew-symmetric. The center contains z , and if p is odd it is generated by z if and only if ω is nondegenerate; in that case $n = 2m$ and $(p.E)_\omega = p^{1+2m}$ is a copy of the extraspecial p -group of exponent p . (The extraspecial group of exponent p^2 comes from a non-bilinear cocycle $\omega : E \times E \rightarrow \mathbf{F}_p$. The extraspecial groups of order 2^{1+2m} will be treated in §3.3; the group $(p.E)_\omega$ that we have defined is always elementary abelian when $p = 2$).

The automorphism group of p^{1+2m} is $E : \mathrm{GSp}(E, \omega)$, where E acts by inner automorphisms $z^i t^v \mapsto z^{i+2\omega(v,v)} t^v$ and

$$\mathrm{GSp}_{2m}(E, \omega) = \{(g, a) \mid g : E \rightarrow E, a \in \mathrm{GL}_1(\mathbf{F}_p), \omega(gv, gw) = a\omega(v, w)\}$$

acts by $(g, a)(z^i t^v) = z^{ai} t^{gv}$. The scalar $a = a(g)$ is determined by g .

Let $L_\omega \subset \text{Alt}^2(E^*)$ denote the line spanned by ω . It is a one-dimensional GSp -submodule by construction, and if ω is nondegenerate then $E^* \cong E \otimes L_\omega$. Provided p is odd, we have a splitting

$$\text{Alt}^2(E^*) = L_\omega \oplus \text{Alt}^2(E^*)_\omega$$

where $\text{Alt}^2(E^*)_\omega$ is the kernel of the projection $\text{Alt}^2(E^*) \cong \text{Alt}^2(E \otimes L_\omega) \cong \text{Alt}^2(E^*)^* \otimes L_\omega^2 \rightarrow L_\omega^{-1} \otimes L_\omega^2$ dual to the inclusion $L_\omega \rightarrow \text{Alt}^2(E^*)$.

If $m \geq 2$ we also have an inclusion $E^* \otimes L_\omega \rightarrow \text{Alt}^3(E^*)$ sending $f \in E^*$ to $f \wedge \omega$.

Lemma 3.3. *Let p be an odd prime, let $E = p^{2m}$ be an elementary p -group and let $\omega \in \text{Alt}^2(E^*)$ be a nondegenerate symplectic form. Then if $m \geq 2$,*

$$\mathbf{H}^2(p^{1+2m}; \mathbf{Z}) \cong E^*, \quad \mathbf{H}^3(p^{1+2m}; \mathbf{Z}) \cong \text{Alt}^2(E^*)_\omega,$$

as GSp_{2m} -modules. If $m \geq 3$,

$$\mathbf{H}^4(p^{1+2m}; \mathbf{Z}) \cong \text{Sym}^2(E^*) \oplus \text{Alt}^3(E^*) / (E^* \otimes L_\omega),$$

while if $m = 2$,

$$\mathbf{H}^4(p^{1+4}; \mathbf{Z}) \cong \text{Sym}^2(E^*) \cdot (\text{Alt}^2(E^*)_\omega \otimes L_\omega),$$

a possibly nontrivial extension of $\text{Alt}^2(E^*)_\omega$ by $\text{Sym}^2(E^*)$.

Proof. We consider the action of GSp on the LHS spectral sequence

$$\mathbf{H}^s(E; \mathbf{H}^t(p)) \Rightarrow \mathbf{H}^{s+t}(p.E).$$

We have $\mathbf{H}^2(p) = L_\omega$ and $\mathbf{H}^4(p) = L_\omega^2$ in the left $s = 0$ column. The bottom $t = 0$ row is computed in Lemma 3.1. To compute the $t = 2$ row, recall that, provided p is odd, $\mathbf{H}^\bullet(E; \mathbf{F}_p)$ is the graded-commutative \mathbf{F}_p -algebra generated by a copy of E^* in degree 1 and a second copy of E^* in degree 2; in particular:

$$\begin{aligned} \mathbf{H}^1(E; \mathbf{F}_p) &\cong E^*, & \mathbf{H}^2(E; \mathbf{F}_p) &\cong \text{Alt}^2(E^*) \oplus E^*, \\ \mathbf{H}^3(E; \mathbf{F}_p) &\cong \text{Alt}^3(E^*) \oplus E^* \otimes E^* \cong \text{Alt}^3(E^*) \oplus \text{Alt}^2(E^*) \oplus \text{Sym}^2(E^*). \end{aligned}$$

All together, we have on the E_2 -page:

$$\begin{array}{cccccc} L_\omega^2 & & & & & \\ 0 & 0 & & 0 & & \\ L_\omega & E^* \otimes L_\omega & (\text{Alt}^2(E^*) \oplus E^*) \otimes L_\omega & \text{Alt}^2(E^*) \otimes L_\omega \oplus \dots & & \\ 0 & 0 & 0 & 0 & & 0 \\ \mathbf{Z} & 0 & E^* & \text{Alt}^2(E^*) & \text{Sym}^2(E^*) \oplus \text{Alt}^3(E^*) & \end{array}$$

The d_2 differential vanishes and the d_3 differentials $L_\omega \rightarrow \text{Alt}^2(E^*)$, $E^* \otimes L_\omega \rightarrow \text{Alt}^3(E^*)$, and $L_\omega^2 \rightarrow \text{Alt}^2(E^*) \otimes L_\omega$ are the injections discussed above. It remains to understand $d_3 : (\text{Alt}^2(E^*) \oplus E^*) \otimes L_\omega \rightarrow \mathbf{H}^5(E)$. We claim that this map is an injection when $m \geq 3$, and that when $m = 2$ its kernel is $\text{Alt}^2(E^*)_\omega \otimes L_\omega \subset \text{Alt}^2(E^*)$. Note also that when $m = 2$, the map $E^* \otimes L_\omega \rightarrow \text{Alt}^3(E^*)$ is an isomorphism. In this range of degrees, the sequence stabilizes after page 4, and so on the E_∞ page we see

$$\begin{array}{cccccc} 0 & & & & & \\ 0 & 0 & & 0 & & \\ 0 & 0 & \text{Alt}^2(E^*)_\omega \otimes L_\omega & & & \\ 0 & 0 & 0 & 0 & & 0 \\ \mathbf{Z} & 0 & E^* & \text{Alt}^2(E^*)_\omega & \text{Sym}^2(E^*) & \end{array}$$

if $m = 2$ and

$$\begin{array}{ccccccc} 0 & & & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & 0 & \\ \mathbf{Z} & 0 & E^* & \text{Alt}^2(E^*)_\omega & \text{Sym}^2(E^*) \oplus \text{Alt}^3(E^*) & / (E^* \otimes L_\omega) & \end{array}$$

if $m \geq 3$. □

3.3. Extraspecial 2-groups. The case of the extraspecial 2-groups may be treated as follows. If E is elementary abelian then any central extension $2.E$ is determined up to isomorphism by the function

$$Q : E \rightarrow \mathbf{F}_2, \quad Q(v) = \begin{cases} 1 & \text{if the lifts of } v \text{ in } 2.E \text{ have order 4,} \\ 0 & \text{otherwise,} \end{cases}$$

which is a quadratic form. It is not usually possible to write the multiplication explicitly in terms of Q — indeed if Q is nondegenerate and E has rank 6 or more the orthogonal group of Q (which we denote by $O(Q)$) does not act on $2.E$ [Gri73]. But $O(Q)$ still acts on the cohomology of $2.E$.

The LHS spectral sequence begins:

$$\begin{array}{ccccccc} 2 & & & & & & \\ 0 & 0 & 0 & & & & \\ 2 & E^* & \text{Sym}^2(E^*) & & & & \\ 0 & 0 & 0 & 0 & & 0 & \\ \mathbf{Z} & 0 & E^* & \text{Alt}^2(E^*) & E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*) & & \end{array}$$

We first wish to describe the d_3 differential. To do so, recall first that $\mathbf{H}^\bullet(E)$ injects into $\mathbf{H}^\bullet(E; \mathbf{F}_2) \cong \text{Sym}^\bullet(E^*)$ as the subalgebra in the kernel of the derivation $\text{Sq}^1 : \text{Sym}^\bullet(E^*) \rightarrow \text{Sym}^{\bullet+1}(E^*)$. Identifying $\mathbf{H}^\bullet(E)$ with its image in $\mathbf{H}^\bullet(E; \mathbf{F}_2)$, the d_3 differential sends $f \in E_2^{i2} \cong \text{Sym}^i(E^*)$ to $\text{Sq}^1(fQ) \in \text{Sym}^{i+3}(E^*)$. In particular, it sends the generator of the 2 in degree $(0, 2)$ to $\text{Sq}^1(Q) \in \text{Sym}^3(E^*)$. The image of $\text{Sq}^1 : \text{Sym}^2(E^*)$ to $\text{Sym}^3(E^*)$ is isomorphic to $\text{Alt}^2(E^*)$, and under this isomorphism Sq^1 takes Q to its underlying alternating form $B_Q(x, y) = Q(x + y) - Q(x) - Q(y)$.

Let us suppose that Q is nondegenerate and $E = 2^{2m}$. Then in particular $B_Q \neq 0$, so that $d_3 : 2 \rightarrow \text{Alt}^2(E^*)$ is an injection. Let $f \in E^*$ in degree $(1, 2)$ and consider $d_3(f) = \text{Sq}^1(fQ) = f^2Q + f \text{Sq}^1(Q)$. Since $\text{Sym}^\bullet(E^*)$ has no zero-divisors, if $f \neq 0$ but $d_3(f) = 0$, then we must have $\text{Sq}^1(Q) = fQ$. This cannot happen when $m \geq 2$, and so $d_3 : E^* \rightarrow E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*)$ is an injection in this case. (When $m = 1$, it is an injection when Q has Arf invariant -1 and is not an injection when Q has Art invariant $+1$.) Thus, provided $m \geq 2$, we find:

$$\mathbf{H}^1(2.E) \cong E^*, \quad \mathbf{H}^2(2.E) = \text{Alt}^2(E^*)/B_Q.$$

The d_3 differential emitted by the $\text{Sym}^2(E^*)$ in degree $(2, 2)$ always has kernel — Q itself — and nothing more provided $m \geq 2$. Finally, if $m \geq 3$, then $d_5 : E_5^{04} \rightarrow E_5^{50}$ is nonzero, and the E_∞ page looks like

$$\begin{array}{ccccccc} 0 & & & & & & \\ 0 & 0 & & & & & \\ 0 & 0 & Q & & & & \\ 0 & 0 & 0 & 0 & & & \\ \mathbf{Z} & 0 & E^* & \text{Alt}^2(E^*)/B_Q & X & & \end{array}$$

with

$$X \cong (E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*)) / E^*.$$

This can be simplified slightly. The inclusion $E^* \rightarrow E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*)$, sending $f \mapsto \text{Sq}^1(fQ)$, does not land within the $E^* \cdot \text{Alt}^2(E^*) \cong \text{Sym}^2(E^*)$ submodule, and so the composition $E^* \rightarrow$

$E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*) \rightarrow \text{Alt}^3(E^*)$ is nonzero. But E^* is simple as an $O(Q)$ -module, and so this map $E^* \rightarrow \text{Alt}^3(E^*)$ is an injection. (It sends $f \mapsto f \wedge B_Q$.) Thus we can write:

$$X \cong E^* \cdot \text{Alt}^2(E^*) \cdot (\text{Alt}^3(E^*)/E^*).$$

All together, provided $m \geq 3$,

$$\mathbb{H}^4(2.E) \cong (E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*)/E^*) \cdot 2.$$

The group X is elementary abelian, although the extensions written above do not split $O(Q)$ -equivariantly. The group $\mathbb{H}^4(2.E)$ is not elementary abelian; it is isomorphic to $(\mathbf{Z}/2)^n \times (\mathbf{Z}/4)$ for $n = \dim(X) - 1 = \binom{m}{2} + \binom{m}{3} - 1$ when $m \geq 3$.

Finally, when $m = 2$, whether $d_5 : E_5^{04} \rightarrow E_5^{50}$ vanishes or not depends on the Arf invariant of Q . Indeed,

$$\mathbb{H}^4(2_+^{1+4}) = X \cdot 4 \cong 2^9 \times 8, \quad \mathbb{H}^4(2_-^{1+4}) = X \cdot 2 \cong 2^9 \times 4.$$

(Both cases are extensions of $X = E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*)/E^* \cong 2^{10}$.)

4. DEMPWOLFF GROUPS, CHEVALLEY GROUPS AND THEIR EXOTIC SCHUR COVERS

4.1. Dempwolff and Alperin groups. In [Dem73], Dempwolff determined that there were no nontrivial extensions of $\text{GL}_n(\mathbf{F}_2)$ by its defining representation on 2^n , unless $n \leq 5$. Conversely, nontrivial extensions exist for $n = 3, 4, 5$; up to isomorphism there is a unique group which can serve as the extension, which we will call

$$2^3 \cdot \text{GL}_3(\mathbf{F}_2), \quad 2^4 \cdot \text{GL}_4(\mathbf{F}_2), \quad 2^5 \cdot \text{GL}_5(\mathbf{F}_2).$$

The largest of these is studied in [Dem72]. A similar group is the nonsplit Alperin-type group

$$4^3 \cdot \text{GL}_3(\mathbf{F}_2).$$

Lemma 4.1. *If $n = 3, 4, 5$, then $\mathbb{H}_3(\text{GL}_n(\mathbf{F}_2)) = \mathbf{Z}/12$. Furthermore,*

- (1) $\mathbb{H}_3(2^3 \cdot \text{GL}_3(\mathbf{F}_2)) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/3$.
- (2) $\mathbb{H}_3(2^4 \cdot \text{GL}_4(\mathbf{F}_2)) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/3$.
- (3) $\mathbb{H}_3(2^5 \cdot \text{GL}_5(\mathbf{F}_2)) \cong \mathbf{Z}/8 \oplus \mathbf{Z}/3$.
- (4) $\mathbb{H}_3(4^3 \cdot \text{GL}_3(\mathbf{F}_2)) \cong (\mathbf{Z}/2)^2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/3$.

Proof. HAP can handle all of these groups except the largest $2^5 \cdot \text{GL}_5(\mathbf{F}_2)$. (In Derek Holt's library of perfect groups, available in GAP, $2^3 \cdot \text{GL}_3(\mathbf{F}_2)$ is `PerfectGroup(1344, 2)`, $2^4 \cdot \text{GL}_4(\mathbf{F}_2)$ is `PerfectGroup(322560, 5)`, and $4^3 \cdot \text{GL}_3(\mathbf{F}_2)$ is `PerfectGroup(10752, 4)`.)

We will obtain $\mathbb{H}_3(2^5 \cdot \text{GL}_5(\mathbf{F}_2)) \cong \mathbb{H}^4(2^5 \cdot \text{GL}_5(\mathbf{F}_2))$ from the LHS spectral sequence. Given Lemma 3.1, the E_2 page of this spectral sequence is

$$\begin{array}{cccccc} 0 & & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & \mathbf{Z}/2 & & & \\ 0 & 0 & 0 & 0 & 0 & \\ \mathbf{Z} & 0 & 0 & 0 & \mathbf{Z}/12 & \end{array}$$

The (2, 2) entry $\mathbb{H}^2(\text{GL}_5(\mathbf{F}_2); (2^5)^*) = \mathbf{Z}/2$ is Dempwolff's computation, and is easily confirmed in Cohomolo. The remaining entries were computed with Cohomolo.

To complete the proof of (3), it suffices to give an element of $\mathbb{H}^4(2^5 \cdot \text{GL}_5(2))$ whose order is divisible by 8. There is a famous embedding of $2^5 \cdot \text{GL}_5(2)$ into the compact Lie group E_8 . Let us write e for the generator of $\mathbb{H}^4(BE_8)$. We will prove that the restriction $e|_{2^5 \cdot \text{GL}_5(2)}$ is such an element.

For the remainder of the proof, let V denote the 248-dimensional adjoint representation of E_8 . The dual Coxeter number of E_8 is $h^\vee = 30$. For any simple simply connected Lie group G , the dual

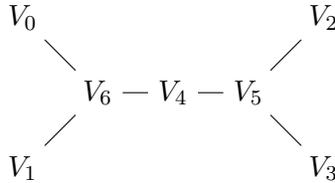
Coxeter number measures the ratio of the fractional Pontryagin class of the adjoint representation of G with the generator of $H^4(BG)$:

$$\frac{21}{2}(\text{adj}) = h^\vee \in \mathbf{Z} \cong H^4(BG).$$

In particular, $c_2(V) = -60e$. Since 60 is divisible by 4, to show that the order of $e|_{2^5 \cdot \text{GL}_5(2)}$ is divisible by 8, it suffices to show that the order $c_2(V)|_{2^5 \cdot \text{GL}_5(2)}$ is divisible by 2.

We will do so by finding a binary dihedral group $2D_8 \subset 2^5 \cdot \text{GL}_5(2)$ such that $c_2(V)|_{2D_8}$ is nonzero. To find such a group, we look inside the normalizer of an order-8 element. There are three conjugacy classes of elements of order 8 in $2^5 \cdot \text{GL}_5(2)$. The normalizer of class 8c is `SmallGroup(64, 151)` in the GAP library. It can be built directly in GAP: the `ATLASRep` package includes a copy of $2^5 \cdot \text{GL}_5(2)$ as a permutation group on 7440 points; GAP can compute orders of centralizers and normalizers, and so in particular can identify class 8c; then GAP can build the normalizer of an element of conjugacy class 8c as a subgroup of $2^5 \cdot \text{GL}_5(2)$. There are four conjugacy classes of order-8 elements in `SmallGroup(64, 151)`, and GAP quickly checks that all four merge in $2^5 \cdot \text{GL}_5(2)$ to conjugacy class 8c.

Finally, `SmallGroup(64, 151)` contains a copy of the binary dihedral group $2D_8$ of order 16. Since $2D_8$ is a finite subgroup of $\text{SU}(2)$, its cohomology is easy to compute: in particular, $H^4(2D_8)$ is cyclic of order $|2D_8| = 16$ and is generated by c_2 of the “defining” two-dimensional representation. As in [JFT18, Section 6], let us index the irreps by:



In particular, V_0 is the trivial representation, V_6 is the “defining” two-dimensional irrep, V_5 is the other faithful irrep, V_4 is the two-dimensional real irrep of D_8 , and $V_1, V_2,$ and V_3 are the nontrivial one-dimensional irreps.

Character table constraints provide a unique fusion map $2D_8 \rightarrow 2^5 \cdot \text{GL}_5(2)$ sending the elements of order 8 to conjugacy class 8c. Along this map, the 248-dimensional irrep V of $2^5 \cdot \text{GL}_5(2)$ decomposes as:

$$V|_{2D_8} = 15V_0 \oplus 15V_1 \oplus 15V_2 \oplus 15V_3 \oplus 30V_4 \oplus 32V_5 \oplus 32V_6.$$

Lemma 6.1 of [JFT18] gives a formula for the second Chern class of any representation of $2D_8$ in which the representations V_1 and V_2 appear with the same coefficient. That formula is:

$$c_2\left(\bigoplus n_i V_i\right) = 4n_4 + 9n_5 + n_6 \pmod{16}, \quad \text{if } n_1 = n_2,$$

where we have identified $H^4(2D_8) = \mathbf{Z}/16$ by identifying $1 \in \mathbf{Z}/16$ with $c_2(V_6)$. Applying this formula to the 248-dimensional representation V gives:

$$c_2(V)|_{2D_8} = 8 \pmod{16}.$$

In particular, $c_2(V)$ is nonzero in $H^4(2D_8)$. As explained above, this implies that $H^4(2^5 \cdot \text{GL}_5(2))$ contains an element of order divisible by 8 (namely, the restriction of the generator of $H^4(\text{BE}_8)$), and so must be isomorphic to $\mathbf{Z}/24$. \square

4.2. A few exotic Chevalley groups. For the most part, any central extension of a finite Chevalley group $G(\mathbf{F}_q)$ is the group of \mathbf{F}_q -points of a central extension of the algebraic group G . In particular if G is of simply connected type then the multiplier $H_2(G(\mathbf{F}_q))$ is usually zero. The finitely many exceptions were classified by Steinberg and Griess. Many of these exotic central extensions occur as centralizers in the sporadic groups.

Lemma 4.2. $H_3(\mathrm{Sp}_6(\mathbf{F}_2)) = \mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/3$ and $H_3(2 \cdot \mathrm{Sp}_6(\mathbf{F}_2)) = \mathbf{Z}/2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/3$.

Proof. We computed these using HAP. The computation of $H_3(\mathrm{Sp}_6(\mathbf{F}_2))$ is fast, but computing $H_3(2 \cdot \mathrm{Sp}_6(\mathbf{F}_2))$ took many hours. Two of the faithful permutation representations of $2 \cdot \mathrm{Sp}_6(\mathbf{F}_2)$ have degrees 240 and 276 (the latter coming from the embedding $2 \cdot \mathrm{Sp}_6(\mathbf{F}_2) \subset \mathrm{Co}_3$). Our machine ran out of memory running HAP on the degree 240 model, and gave the above output after six hours for the degree 276 model. \square

Lemma 4.3. *We have*

$$H_3(\mathrm{G}_2(2)) = \mathbf{Z}/2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/3, \quad H_3(\mathrm{G}_2(3)) = \mathbf{Z}/8 \oplus \mathbf{Z}/3, \quad H_3(\mathrm{G}_2(5)) = \mathbf{Z}/8 \oplus \mathbf{Z}/3.$$

Jesper Grodal has shown that $H^4(\mathrm{G}_2(\mathbf{F}_q))$ is cyclic of order $q^2 - 1$ if $q = p^r$ with either p or r sufficiently large. The computations in the theorem show that this holds also for $q = 5$, but not $q = 3$ or $q = 2$.

Proof. We computed $\mathrm{G}_2(2)$ and $\mathrm{G}_2(3)$ with HAP.

The order of $\mathrm{G}_2(5)$ is $2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$. The proof of Lemma 2.2 applies to this group — for $p = 7$ and 31, there are strictly fewer than $(p - 1)/2$ conjugacy classes of order p — and so we must compute $H^4(\mathrm{G}_2(5))_{(p)}$ for $p = 2, 3$, and 5.

The 2-Sylow in $\mathrm{G}_2(5)$ is contained in the nonsplit extension $2^3 \cdot \mathrm{GL}_3(2)$ whose cohomology, per Lemma 4.1 part (1), is $H^4(2^3 \cdot \mathrm{GL}_3(2))_{(2)} = 2 \times 8$. According to [Mil98], for $q = 1 \pmod{4}$,

$$H^1(\mathrm{G}_2(q); \mathbf{F}_2) \cong H^2(\mathrm{G}_2(q); \mathbf{F}_2) \cong 0, \quad H^3(\mathrm{G}_2(q); \mathbf{F}_2) \cong H^4(\mathrm{G}_2(q); \mathbf{F}_2) \cong \mathbf{F}_2, \\ \mathrm{Sq}^1 = 0 : H^3(\mathrm{G}_2(q); \mathbf{F}_2) \rightarrow H^4(\mathrm{G}_2(q); \mathbf{F}_2).$$

It follows that $H^4(\mathrm{G}_2(q))_{(2)}$ is cyclic of order at least 4, and so $H^4(\mathrm{G}_2(5))_{(2)} = 8$.

The 3-Sylow in $\mathrm{G}_2(5)$ is contained in a maximal subgroup of shape $\mathrm{U}_3(3) : 2$. HAP quickly computes $H_3(\mathrm{U}_3(3) : 2) = 2 \times 8 \times 3$. Conjugacy class $3b \in \mathrm{G}_2(5)$ acts on the 124-dimensional irrep with trace 1, and so $c_2(124\text{-dim rep})|_{(3b)} \neq 0$. It follows that $H^4(\mathrm{G}_2(5))_{(3)} = 3$.

The 5-Sylow in $\mathrm{G}_2(5)$ is contained in a maximal subgroup of shape $5^{1+4} : \mathrm{GL}_2(5)$. The central 4 $\subset \mathrm{GL}_2(5)$ acts on all of 5^4 with the same faithful central character. It therefore acts with nontrivial central characters on $H^j(5^{1+4})$ for $j \in \{1, 2, 3, 4\}$, and so $H^i(\mathrm{GL}_2(5), H^j(5^{1+4})) = 0$ for these j . Since $H^4(\mathrm{GL}_2(5)) = 4 \times 8 \times 3$ has no five part, we find that $H^4(5^{1+4} : \mathrm{GL}_2(5))_{(5)}$, and hence also $H^4(\mathrm{G}_2(5))_{(5)}$, vanishes. \square

The group called “ $\mathrm{O}_n(q)$ ” in the ATLAS is not the $n \times n$ orthogonal group over \mathbf{F}_q . Rather, it is the simple subquotient of the orthogonal group. To avoid confusion, we will follow Dieudonné and write “ $\Omega_n(q)$ ” for this simple group. When $n \geq 5$ and q are odd, $\Omega_n(q)$ is the commutator subgroup of $\mathrm{SO}_n(\mathbf{F}_q) = \Omega_n(q) : 2$, and is the image of $\mathrm{Spin}_n(\mathbf{F}_q)$ in $\mathrm{SO}_n(\mathbf{F}_q)$, and is the kernel of the “spinor norm” $\mathrm{SO}_n(\mathbf{F}_q) \rightarrow \mathbf{F}_q^\times / \{\text{squares}\} \cong \mathbf{Z}/2$.

Lemma 4.4. $H_3(\Omega_7(3)) \cong \mathbf{Z}/4$ and $H_3(2.\Omega_7(3)) \cong \mathbf{Z}/8$.

Proof. The criterion in Lemma 2.2 applies for the primes $p \geq 5$. The 2-Sylow is contained in $\mathrm{Sp}_6(\mathbf{F}_2)$, giving an upper bound of $H^4(\mathrm{Sp}_6(\mathbf{F}_2))_{(2)} = 2 \times 4$ for $H_3(\Omega_7(3))$, and an upper bound of $H^4(2\mathrm{Sp}_6(\mathbf{F}_2))_{(2)} = 2 \times 8$ for $H_3(\mathrm{Spin}_7(3))$, both from Lemma 4.2. It is straightforward to check that second Chern class of the 78-dimensional real representation of $\Omega_7(3)$ restricts with order 4 to conjugacy classes 4a, and so the fractional Pontryagin class of this representation has order 8.

To show that $H_3(\mathrm{Spin}_7(3))_{(2)}$ is exactly $\mathbf{Z}/8$ (which implies in turn that $H_3(\Omega_7(3))_{(2)}$ is exactly $\mathbf{Z}/4$) it suffices to give a class in $H^4(2.\mathrm{Sp}_6(\mathbf{F}_2))_{(2)}$ not in the image of restriction $H_3(\mathrm{Spin}_7(3))_{(2)} \rightarrow H^4(2.\mathrm{Sp}_6(\mathbf{F}_2))_{(2)}$. We claim that the fractional Pontryagin class of the 15-dimensional irrep of $\mathrm{Sp}_6(\mathbf{F}_2)$ is such a class. (This representation is not Spin over $\mathrm{Sp}_6(\mathbf{F}_2)$, but is Spin over $2\mathrm{Sp}_6(\mathbf{F}_2)$. We will henceforth call its fractional Pontryagin class $\frac{p_1}{2}(15) \in H^4(2\mathrm{Sp}_6(\mathbf{F}_2))$.) To prove this, we

consider the conjugacy classes 2b and 2d in $\mathrm{Sp}_6(\mathbf{F}_2)$. They act on the 15-dimensional irrep with traces 7 and -1 respectively; equivalently, 2b acts with spectrum $1^{11}(-1)^4$ whereas 2d acts with spectrum $1^7(-1)^8$. These two classes lift with order 2 to $2\mathrm{Sp}_6(2)$. The fractional Pontryagin classes are therefore $\frac{p_1}{2}(15)|_{2b} = 1 \in \mathbf{H}^4(\langle 2b \rangle) \cong \mathbf{Z}/2$ and $\frac{p_1}{2}(15)|_{2d} = 0$. But 2b and 2d both fuse to class $2c \in \Omega_7(3)$. It follows that $\frac{p_1}{2}(15) \in \mathbf{H}^4(2.\mathrm{Sp}_6(\mathbf{F}_2))$ is not the restriction of any class in $\mathbf{H}^4(\mathrm{Spin}_7(3))$.

It remains to handle the prime $p = 3$. In general, the p -Sylow in a characteristic- p group of Lie type is the nilpotent subgroup, and so is contained in any parabolic. We will use two maximal parabolics of SO_7 , corresponding to the Dynkin subdiagrams $B_2 \subset B_3$ and $A_1 \times A_1 \subset B_3$. These lead to two maximal subgroups of $\Omega_7(3)$ that contain the 3-Sylow:

$$3^5 : \mathrm{SO}_5(\mathbf{F}_3), \quad 3_+^{1+6} : (2A_4 \times A_4).2$$

There is one more maximal subgroup of $\Omega_7(3)$ containing the 3-Sylow, corresponding to the Dynkin diagram inclusion $A_2 \subset B_3$, which we will not use in the present proof, but will use in the proof of Corollary 4.5.

The spectral sequence for $3^5 : \mathrm{SO}_5(\mathbf{F}_3)$ has E_2 page:

$$\begin{array}{cccccc} 3 & & & & & \\ 0 & 0 & 3 & & & \\ 0 & 3 & 0 & & & \\ 0 & 0 & 0 & 0 & & 0 \\ \mathbf{Z} & 0 & 0 & 2 & 2^2 \times 4 \times 3 & \end{array}$$

The bottom line was computed in HAP, and the middle entries in Cohomolo. The entry $E_2^{04} = 3$ corresponds to the symmetric pairing on 3^5 .

We claim that the maps $\mathbf{H}^4(\Omega_7(3))_{(3)} \rightarrow \mathbf{H}^4(3^5 : \mathrm{SO}_5(\mathbf{F}_3))_{(3)}$ and $\mathbf{H}^4(\mathrm{SO}_5(\mathbf{F}_3))_{(3)} \rightarrow \mathbf{H}^4(3^5 : \mathrm{SO}_5(\mathbf{F}_3))$ have trivial intersection. To see this, first note that $\mathrm{SO}_5(\mathbf{F}_3) \cong \mathrm{Weyl}(E_6)$ has a 6-dimensional irrep, on which the conjugacy class 3b acts with trace 3. It follows that

$$c_2(6\text{-dim irrep})|_{(3b)} \neq 0.$$

But $3b \in \mathrm{SO}_5(\mathbf{F}_3)$ fuses along the inclusion $\mathrm{SO}_5(\mathbf{F}_3) \rightarrow 3^5 : \mathrm{SO}_5(\mathbf{F}_3) \rightarrow \Omega_7(3)$ to class 3b therein, which meets $3^5 \subset 3^5 : \mathrm{SO}_5(\mathbf{F}_3)$. It follows that $c_2(6\text{-dim irrep}) \in \mathbf{H}^4(\mathrm{SO}_5(\mathbf{F}_3))$, when pulled back along $3^5 : \mathrm{SO}_5(\mathbf{F}_3) \rightarrow \mathrm{SO}_5(\mathbf{F}_3)$, distinguished conjugate-in- $\Omega_7(3)$ elements, and so is not the restriction of a class in $\mathbf{H}^4(\Omega_7(3))$.

It follows that the restriction map $\mathbf{H}^4(\Omega_7(3))_{(3)} \rightarrow \mathbf{H}^0(\mathrm{SO}_5(\mathbf{F}_3); \mathbf{H}^4(3^5))$ is an injection. The order-3 conjugacy classes in $\Omega_7(3)$ that meet 3^5 are classes 3a, 3b, and 3c, corresponding to vectors in 3^5 of norm 0 and ± 1 . The nonzero class in $\mathbf{H}^0(\mathrm{SO}_5(\mathbf{F}_3); \mathbf{H}^4(3^5)) \cong \mathbf{Z}/3$ corresponds to the symmetric pairing, and so restricts trivially to $\langle 3a \rangle$ but nontrivially to $\langle 3b \rangle$ and $\langle 3c \rangle$. In particular the restriction map $\mathbf{H}^4(\Omega_7(3))_{(3)} \rightarrow \mathbf{H}^4(\langle 3b \rangle)$ is an injection.

The other maximal subgroup we consider is the one of shape $3_+^{1+6} : (2A_4 \times A_4).2$. It is the normalizer of conjugacy class 3a. GAP can work with $\Omega_7(3)$ by using its degree-351 permutation representation, and quickly find this subgroup. In particular, GAP finds that the action of $(2A_4 \times A_4).2$ on 3^6 is generated by the following three matrices:

$$\begin{pmatrix} \cdot & 1 & \cdot & \cdot & 2 & 2 \\ 2 & \cdot & \cdot & 2 & \cdot & 1 \\ 1 & 1 & \cdot & 1 & 2 & 2 \\ \cdot & 2 & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & 2 \\ 2 & \cdot & 2 & 1 & 2 & 2 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 2 & 2 & \cdot & 1 \\ \cdot & 1 & 2 & \cdot & 1 & 1 \\ \cdot & 2 & \cdot & \cdot & 2 & \cdot \\ 1 & \cdot & 1 & 2 & \cdot & 1 \\ \cdot & 2 & 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & 1 & \cdot \end{pmatrix} \quad \begin{pmatrix} 2 & \cdot & 2 & 1 & \cdot & 2 \\ \cdot & \cdot & 2 & \cdot & 2 & 2 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 2 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & 2 & \cdot & 1 & 1 \\ \cdot & 2 & \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

Recall from Lemma 3.3 that $H^4(3^{1+6}) \cong \text{Sym}^2(3^6) \oplus (\text{Alt}^3(3^6)/3^6)$. The group $(2A_4 \times A_4).2$ contains the matrix -1 , and which acts by -1 on $(\text{Alt}^3(3^6)/3^6)$. Furthermore, the representation 3^6 is not symmetrically self-dual. (In fact it is not self-dual: its antisymmetric pairing changes by a sign under the odd elements of $(2A_4 \times A_4).2$.) It follows that $H^0((2A_4 \times A_4).2; H^4(3^{1+6})) = 0$.

But this means in particular that the restriction map $H^4(\Omega_7(3)) \rightarrow H^4(3^{1+6})$ vanishes. Conjugacy class $3b \in \Omega_7(3)$ meets 3^{1+6} . It follows that restriction $H^4(\Omega_7(3)) \rightarrow H^4(\langle 3b \rangle)$ is the zero map. But we showed above that $H^4(\Omega_7(3))_{(3)} \rightarrow H^4(\langle 3b \rangle)$ is an injection. So $H^4(\Omega_7(3))_{(3)} = 0$. \square

The Chevalley group $\Omega_7(3)$ has an exceptional cover: its multiplier is 6, whereas the multiplier of $\Omega_7(q)$ is generically $2 = \pi_1(\text{SO}(7, \mathbf{C}))$.

Corollary 4.5. $H_3(3.\Omega_7(3)) \cong \mathbf{Z}/12$ and $H_3(6.\Omega_7(3)) \cong \mathbf{Z}/24$.

Proof. It must calculate $H^4(3.\Omega_7(3))_{(3)}$. To do so we use the third maximal parabolic of $\Omega_7(3)$, of shape $3^{3+3} : \text{SL}(3, 3)$. For the remainder of the proof we will call this subgroup S . Since S contains the 3-Sylow, the extension $3.\Omega_7(3)$ restricts to a nontrivial central extension $3.S$. One can show, for instance by running a LHS spectral sequence, that $H^1(S; 3) = 0$ and $H^2(S; 3) = 3$. In particular, there is a unique nonsplit extension $3.S$.

By Lemma 4.4, $H^4(\Omega_7(3))_{(3)} = 0$, and so $H^4(3.\Omega_7(3))_{(3)} = \text{coker}(H^4(\Omega_7(3)) \rightarrow H^4(3.\Omega_7(3)))$. This is in turn isomorphic to $\text{coker}(H^4(S) \rightarrow H^4(3.S))$ by Corollary 2.5.

The smallest complex representations of $\Omega_7(3)$ and $3.\Omega_7(3)$ have dimensions 78 and 27 respectively, equal to the smallest representations of the simple Lie group $E_6^{adj}(\mathbf{C})$ and its simply connected cover $E_6^{sc} = 3.E_6^{adj}$. However, $\Omega_7(3)$ does not preserve the Lie bracket on the 78-dimensional representation. It does preserve a lattice, and preserves the Lie bracket “modulo 2”: in fact, $\Omega_7(3)$ embeds into the twisted Chevalley group ${}^2E_6(2) \subset E_6^{adj}(\mathbf{F}_4)$. The finite subgroups of the Lie group $E_6^{adj}(\mathbf{C})$ were classified in [CW97]. In particular, $E_6^{adj}(\mathbf{C})$ and $\Omega_7(3)$ intersect along the subgroup S , lifting to the nonsplit extension $3.S \subset E_6^{sc}(\mathbf{C})$.

Both $H^4(BE_6^{adj})$ and $H^4(BE_6^{sc})$ are infinite cyclic, but the restriction map $H^4(BE_6^{adj}) \rightarrow H^4(BE_6^{sc})$ is not an isomorphism: its cokernel has order 3. By Corollary 2.5, this forces the inclusion $H^4(S) \rightarrow H^4(3.S)$ to have cokernel of order 3. \square

The Chevalley group $G_2(3)$ has an exceptional multiplier of order 3.

Corollary 4.6. $H_3(3.G_2(3))$ has order 72.

Proof. The 7-dimensional representation of G_2 provides an inclusion $G_2(3) \subset \Omega_7(3)$, and the exceptional triple cover of $\Omega_7(3)$ restricts to the exceptional triple cover of $G_2(3)$. Corollary 2.5 then forces the inclusion $H^4(G_2(3)) \rightarrow H^4(3.G_2(3))$ to have cokernel of order 3. \square

5. MATHIEU GROUPS

The low-degree homology groups of all Mathieu groups can be computed in HAP, and are listed in [DSE09]; that paper was the first to compute $H_3(M_{24})$. In [GPRV13] it is shown that the restriction map $H^4(M_{24}) \rightarrow H^4(\langle 12b \rangle)$ is an isomorphism, and that $H^4(M_{24})$ is generated by the “gauge anomaly” of “ M_{24} moonshine.” In [JFT18, Theorems 5.1 and 5.2] we gave direct proofs of the results $H^4(M_{23}) = 0$ and $H^4(M_{24}) = 12$ following the method outlined in §2, and we also recognized that the generator of $H^4(M_{24})$ from [GPRV13] is more simply described as the fractional Pontryagin class of the defining degree-24 permutation representation. We remark that the same holds for M_{11} :

Proposition 5.1. $H^4(M_{11}) \cong \mathbf{Z}/8$ is generated by the fractional Pontryagin class of the defining degree-11 permutation representation.

Proof. Let Perm denote the permutation representation of M_{11} . The two conjugacy classes of order 8 in M_{11} have the same spectrum on Perm: they act by $\text{diag}(1, 1, 1, \zeta, i, \zeta^3, -1, -1, \zeta^{-3}, -i, \zeta^{-1})$, where $\zeta = \exp(2\pi i/8)$. Let $t \in H^2(\mathbf{Z}/8)$ denote a generator of $H^\bullet(\mathbf{Z}/8) = \mathbf{Z}[t]/(8t)$. The total Chern character of Perm, restricted to a cyclic group of order 8, is therefore

$$c(\text{Perm})|_{\langle 8a \rangle} = 1^3(1-t)(1-2t)(1-3t)(1-4t)^2(1+3t)(1+2t)(1+t) = 1 + 2t^2 + \dots$$

In particular, $c_2(\text{Perm})$ has order divisible by 4. But Perm is a real and therefore Spin representation, and so $\frac{p_1}{2}(\text{Perm})$ has order divisible by 8. \square

The Schur cover of M_{12} is studied in [CdLW16]; they compute $H^4(2M_{12}) = 8^2 \times 3$ with HAP, and show that the map $H^4(2M_{12}) \rightarrow \prod_{g \in 2M_{12}} H^4(\langle g \rangle)$ has kernel of order 2. To fully describe $H^4(2M_{12})$ requires moving slightly beyond cyclic groups, and also requires some notation. Let Perm denote (a choice of either) degree-12 permutation representation of M_{12} , and write V_{12} for the unique 12-dimensional faithful irrep of $2M_{12}$. Then V_{12} is real, and hence Spin, as a $2M_{12}$ -module (since $2M_{12}$ has no central extensions). Perm $\otimes \mathbf{R}$ is not Spin as an M_{12} -module, but is automatically Spin as a $2M_{12}$ -module. Write $\frac{p_1}{2}(\text{Perm})$ and $\frac{p_1}{2}(V_{12})$ for their fractional Pontryagin classes. The group $2M_{12}$ has two conjugacy classes of elements of order 3: class 3b acts on Perm with cycle structure 3^4 . There are also four conjugacy classes of elements of order 8. Classes 8a and 8b differ by the central element and act on Perm with cycle structure $1^2 2^1 8^1$; classes 8c and 8d differ by the central element and act with cycle structure $4^1 8^1$. Finally, there is a unique conjugacy class of quaternion subgroups $Q_8 \subset 2M_{12}$ in which the center of Q_8 maps to the center of $2M_{12}$.

Proposition 5.2. $H^4(2M_{12})$ is spanned by the classes $\frac{p_1}{2}(\text{Perm})$ and $\frac{p_1}{2}(V_{12})$. The restriction map

$$H^4(2M_{12}) \rightarrow H^4(Q_8) \times H^4(\langle 8a \rangle) \times H^4(\langle 8c \rangle) \times H^4(\langle 3b \rangle) \cong 8^3 \times 3$$

is an injection.

We remark that the outer automorphism of $2M_{12}$ switches the two degree-12 permutation representations and also switches 8ab with 8cd.

Proof. We choose the following generators of $H^4(Q_8)$ and $H^4(\langle 8a \rangle) \cong H^4(\langle 8c \rangle) \cong H^4(C_8)$ and $H^4(\langle 3b \rangle) \cong H^4(C_3)$: the generator of $H^4(Q_8)$ is the fractional Pontryagin class of the 4-dimensional real representation (equal to the negative second Chern class of the 2-dimensional complex irrep); if n divides 24, we take the unique generator of $H^4(C_n)$ which is a cup square (it is unique by what Conway and Norton call “the defining property of 24”).

It is straightforward to compute the images of $\frac{p_1}{2}(\text{Perm})$ and $\frac{p_1}{2}(V_{12})$ to $H^4(Q_8) \times H^4(\langle 8a \rangle) \times H^4(\langle 8c \rangle) \times H^4(\langle 3b \rangle)$. They are:

$$\begin{aligned} \frac{p_1}{2}(\text{Perm}) &\mapsto (3, 1, 1, -1), \\ \frac{p_1}{2}(V_{12}) &\mapsto (4, -1, 1, -1). \end{aligned}$$

But $(3, 1, 1, -1)$ and $(4, -1, 1, -1)$ together generate a subgroup isomorphic to $8^2 \times 3 \cong H^4(2M_{12})$ inside $8^3 \times 3$. \square

The covers of M_{22} are not directly computable by HAP, since they do not have sufficiently small permutation representations.

Proposition 5.3. The covers of M_{22} have the following third homologies:

$$H_3(2M_{22}) = 4, \quad H_3(3M_{22}) = 3, \quad H_3(4M_{22}) = 8, \quad H_3(6M_{22}) = 12, \quad H_3(12M_{22}) = 24.$$

Proof. Given Lemma 2.4 together with the computer computation $H_3(M_{22}) = 0$, it suffices to give lower bounds $H^4(2M_{22}) \geq 4$, $H^4(8M_{22}) \geq 8$, and $H^4(3M_{22}) \geq 3$. The first two can be handled

simultaneously as follows. $2M_{22}$ has a unique faithful 210-dimensional irrep V . Conjugacy class $4c \in 2M_{22}$ acts on V with spectrum $1^{50}(-1)^{50}i^{55}(-i)^{55}$. Let $t \in H^2(\langle 4c \rangle)$ denote a generator. Then

$$c_2(V)|_{\langle 4c \rangle} = t^2 \in H^4(\langle 4c \rangle)$$

has order 4. This gives the lower bound for $H^4(2M_{22})$. The representation V is real, but it is not Spin as a $2M_{22}$ -module (since, indeed, $c_2(V)$ is not divisible by 2). It is, however, Spin as a $4M_{22}$ -module. Since $H^4(2M_{22}) \rightarrow H^4(4M_{22})$ is an injection, $c_2(V)|_{4M_{22}}$ has order divisible by 4, so $\frac{p_1}{2}(V) \in H^4(4M_{22})$ has order divisible by 8. This provides the lower bound for $H_3(4M_{22})$. Finally, for $3M_{22}$, we may use either 21-dimensional faithful representation W . Element $3c \in 3M_{22}$ acts on W with trace 0, and so $c_2(W)|_{\langle 3c \rangle} = -\frac{21}{3}t^2 \neq 0 \in H^4(\langle 3c \rangle)$. Thus $c_2(W)$ has order divisible by 3 in $H^4(3M_{22})$. \square

6. LEECH LATTICE GROUPS

6.1. Higman–Sims group and Janko group 2. The smallest faithful permutation representations of the Higman–Sims group HS and its double cover 2HS have degrees 100 and 704 respectively. Using these representations, we find that HAP can compute $H_3(\text{HS})$ and $H_3(2\text{HS})$ without further human assistance:

$$H_3(\text{HS}) \cong (\mathbf{Z}/2)^2, \quad H_3(2\text{HS}) \cong \mathbf{Z}/2 \times \mathbf{Z}/8.$$

The proof of Lemma 2.4 implies that the only way for $H^4(G) \subset H^4(2G)$ to have index 4 is if the latter contains elements with nontrivial restriction to the central $2 \subset 2G$. It is not hard to check that all complex representations V of 2HS have $c_2(V)|_2 = 0$, and all real representations have $\frac{p_1}{2}(V)|_2 = 0$. In particular, we do not know generators for $H^4(2\text{HS}) \cong \mathbf{Z}/2 \times \mathbf{Z}/8$.

The smallest permutation representations of Janko’s second group J_2 (also called the Hall–Janko group HJ) and of its double cover $2J_2$ have degrees 100 and 200 respectively, and HAP computes:

$$H_3(J_2) \cong \mathbf{Z}/30, \quad H_3(2J_2) \cong \mathbf{Z}/120.$$

Proposition 6.1. $H^4(2J_2) \cong \mathbf{Z}/120$ is generated by $c_2(V)$, where V is either 6-dimensional irrep of $2J_2$.

Proof. Given the traces

$$\text{trace}(2a; V) = -6, \quad \text{trace}(3b; V) = 0, \quad \text{trace}(5a; V) = -2\zeta - 2\zeta^{-1},$$

where ζ is a primitive 5th root of unity, it is easy to compute that $c_2(V)$ has nontrivial restriction to the cyclic subgroups $\langle 2a \rangle$, $\langle 3b \rangle$, and $\langle 5a \rangle$. The latter two restrictions immediately give that $c_2(V)$ has order divisible by 15. Furthermore, by inspecting the proof of Lemma 2.4, we see that any class in $H^4(2J_2)$ with nontrivial restriction to the center must have order divisible by 8. \square

We remark that $2J_2$ contains two conjugacy classes of copies of $\text{SL}(2, 5)$, but that $c_2(V)|_{\text{SL}(2,5)}$ has order merely 40. Thus the restriction map $H^4(2J_2) \cong \mathbf{Z}/120 \rightarrow H^4(\text{SL}(2, 5)) \cong \mathbf{Z}/120$ is not an isomorphism.

6.2. Conway groups. In [JFT18, Theorems 0.1 and 5.3] we showed that

$$H^4(\text{Co}_1) \cong \mathbf{Z}/12, \quad H^4(2.\text{Co}_1) \cong \mathbf{Z}/24,$$

and that these groups are generated by the fractional Pontryagin classes of the 276- and 24-dimensional representations, respectively. Let us denote the 24-dimensional real representation of $2.\text{Co}_1$ by the name Leech. The second and third Conway groups Co_2 and Co_3 are subgroups of $2.\text{Co}_1$, and so Leech restricts to representations of each (where it splits as a trivial representation plus a 23-dimensional irrep).

Theorem 6.2. $H^4(\text{Co}_2) \cong \mathbf{Z}/4$ is generated by the restriction of $\frac{p_1}{2}(\text{Leech})$.

Proof. In [JFT18, Theorem 7.1] we gave a formula for $\frac{p_1}{2}(\text{Leech})|_{\langle g \rangle}$ for all elements $g \in 2.\text{Co}_1$ in terms of the Frame shape of g in the Leech representation. The conjugacy class $4g \in \text{Co}_2$ has Frame shape 4^6 , and so $\frac{p_1}{2}(\text{Leech})$ restricts with order 4 to this conjugacy class. This gives the lower bound $H^4(\text{Co}_2) \geq 4$.

For the upper bound, Lemma 2.2 handles the primes ≥ 7 . For the primes 3 and 5, we note that Co_2 contains a subgroup isomorphic to McL , which in turn contains the 3- and 5-Sylows. Since $H^4(\text{McL}) = 0$ (see §6.3), we learn that $H^4(\text{Co}_2)_{(p)} = 0$ for p odd.

It remains to give an upper bound for the 2-part of $H^4(\text{Co}_2)$. The 2-Sylow in Co_2 is contained in a subgroup isomorphic to $2^{10} : (\text{M}_{22} : 2)$, where $E = 2^{10}$ is an irreducible $(\text{M}_{22} : 2)$ -module over \mathbf{F}_2 . The subgroup E contains elements with Frame shape 2^{12} . By [JFT18, Theorem 7.1], $\frac{p_1}{2}(\text{Leech})$ restricts nontrivially to such an element, and so $\frac{p_1}{2}(\text{Leech})|_E \in H^0(\text{M}_{22} : 2; H^4(E))$ is nonzero. There are two irreducible 10-dimensional $(\text{M}_{22} : 2)$ -modules over \mathbf{F}_2 , which we will call V_a and V_b , where the letters “a” and “b” match the notation in the ATLAS. They enjoy:

$$\begin{aligned} H^0(\text{M}_{22} : 2; V_a) &= H^0(\text{M}_{22} : 2; \text{Alt}^2(V_a)) = H^0(\text{M}_{22} : 2; \text{Alt}^3(V_a)) = 0, \\ H^0(\text{M}_{22} : 2; V_b) &= H^0(\text{M}_{22} : 2; \text{Alt}^2(V_b)) = 0, \quad H^0(\text{M}_{22} : 2; \text{Alt}^3(V_b)) \cong \mathbf{Z}/2 \end{aligned}$$

Since $H^4(E) \cong E^* \cdot \text{Alt}^2(E^*) \cdot \text{Alt}^3(E^*)$ by Lemma 3.1, the only way for $H^4(E)$ to have a nontrivial $(\text{M}_{22} : 2)$ -fixed point is if $E^* \cong V_b$.

With this isomorphism in hand, we can compute the E_2 page of the LHS spectral sequence for $E : (\text{M}_{22} : 2)$:

$$\begin{array}{ccccccc} & & & & & & 2 \\ & & & & & 0 & 2 & 2^2 \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & \mathbf{Z} & 0 & 2 & 2 & 2^2 \end{array}$$

The bottom row is computed by HAP, and the middle rows by Cohomolo. The dashed line reminds that the extension $E : (\text{M}_{22} : 2)$ splits, and so $H^4(\text{M}_{22} : 2)$ is a direct summand of $H^4(E : (\text{M}_{22} : 2))$.

To complete the proof, it suffices to show that $H^4(\text{Co}_2) \rightarrow H^4(E : \text{M}_{22} : 2)$ and $H^4(\text{M}_{22} : 2) \rightarrow H^4(E : \text{M}_{22} : 2)$ have trivial intersection. There are three conjugacy classes of order 2 in $\text{M}_{22} : 2$, with cycle structures $1^6 2^8$, $1^8 2^7$, and 2^{11} in the degree-22 permutation representation. Together, these three classes detect $H^4(\text{M}_{22} : 2)$: if $\alpha \in H^4(\text{M}_{22} : 2)$, then there is an element $g \in \text{M}_{22} : 2$ of order 2 such that $\alpha|_{\langle g \rangle} \neq 0$. (Indeed, the images of $H^4(2) \rightarrow H^4(\text{M}_{22} : 2)$ and $H^4(\text{M}_{24}) \rightarrow H^4(\text{M}_{22} : 2)$ are transverse, and one can quickly compute the restrictions of their images to the three elements of order 2.)

Given $\alpha \neq 0 \in H^4(\text{M}_{22} : 2)$, choose $g \in \text{M}_{22} : 2$ of order 2 such that $\alpha|_{\langle g \rangle} \neq 0$. Choose also an order-2 lift \tilde{g} of g in $E : \text{M}_{22} : 2$, and let $\tilde{\alpha} \in H^4(E : \text{M}_{22} : 2)$ denote the pullback of α . Then $\tilde{\alpha}|_{\langle \tilde{g} \rangle} = \alpha|_{\langle g \rangle} \neq 0$. But Co_2 has only three conjugacy classes of order 2, distinguished by their traces on Leech, and all three classes meet E . Since $\tilde{\alpha}|_E = 0$, we find that $\tilde{\alpha}$ takes different values on conjugate-in- Co_2 elements, and so cannot be the restriction of a class in $H^4(\text{Co}_2)$. This completes the proof that $H^4(\text{Co}_2) \cong \mathbf{Z}/4$. \square

Theorem 6.3. $H^4(\text{Co}_3) \cong \mathbf{Z}/6$ is generated by the restriction of $\frac{p_1}{2}(\text{Leech})$.

Proof. The conjugacy class $6e \in \text{Co}_3$ has Frame shape 6^4 in the Leech representation. It follows from [JFT18, Theorem 7.1] that $\frac{p_1}{2}(\text{Leech})|_{\langle 6e \rangle}$ has order 6, giving the claimed lower bound $H^4(\text{Co}_3) \geq 6$. Lemma 2.2 handles the primes ≥ 7 , and Co_3 contains a copy of McL , which contains the 5-Sylow.

The 3-Sylow in Co_3 is contained in a subgroup of shape $3^{11} : (2 \times \text{M}_{11})$. There are two irreducible 11-dimensional representations of M_{11} over \mathbf{F}_3 , dual to each other. They lead to LHS spectral

sequences with E_2 pages

$$\begin{array}{cccccc}
 0 & & & & & 0 \\
 0 & 0 & 0 & & & 0 & 3 & 3 \\
 0 & 0 & 0 & & & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \mathbf{Z} & 0 & 2 & 0 & 2 \times 8 & \mathbf{Z} & 0 & 2 & 0 & 2 \times 8
 \end{array}
 \quad \text{and}$$

Only the latter of these is consistent with the lower bound $H^4(\text{Co}_3) \geq 3$, and provides the desired upper bound.

To complete the proof we must verify that $H^4(\text{Co}_3)_{(2)} \leq 2$. The 2-Sylow in Co_3 is contained in three maximal subgroups: one the form $2^4 \cdot \text{GL}_4(\mathbf{F}_2)$, one of the form $2 \cdot \text{Sp}_6(\mathbf{F}_2)$, and one of order $2^{10} \cdot 3^3$. Lemma 4.1 part (2) and Lemma 4.2 give:

$$H^4(2^4 \cdot \text{GL}_4(\mathbf{F}_2); \mathbf{Z}) = \mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/3$$

$$H^4(2 \cdot \text{Sp}_6(\mathbf{F}_2); \mathbf{Z}) = \mathbf{Z}/2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/3$$

By Lemma 2.1, $H^4(\text{Co}_3)_{(2)}$ is a direct summand of both $\mathbf{Z}/2 \oplus \mathbf{Z}/4$ and of $\mathbf{Z}/2 \oplus \mathbf{Z}/8$, which forces $H^4(\text{Co}_3)_{(2)} \subset \mathbf{Z}/2$. \square

6.3. McLaughlin group. HAP is able to directly compute

$$H_3(\text{McL}) = 0$$

by using the permutation representation of degree 275. HAP is unable to directly compute $H_3(3\text{McL})$ because the smallest faithful permutation representation of 3McL has degree 66825. Lemma 2.4 only provides the upper bound $H_3(3\text{McL}) \leq 3$. Nevertheless, with some human involvement, we do have:

Theorem 6.4. $H_3(3\text{McL}) = 0$.

Proof. The computer calculation of $H_3(\text{McL})$ leaves only the 3-part of $H_3(3\text{McL})$ to be computed. But we can also dispense with the other parts directly. The 2-Sylow in McL is contained in a maximal subgroup of shape M_{22} , and $H_3(M_{22})_{(2)} = 0$ by computer calculation (see Proposition 5.3). The 5-Sylow is contained in a group of shape $5^{1+2} : 3 : 8$, and HAP quickly computes

$$H^4(5^{1+2} : 3 : 8)_{(5)} = 0.$$

(The ATLAS contains generators for most maximal subgroups of sporadic groups.) Lemma 2.2 handles the primes $p \geq 7$.

Only the prime 3 is left. The 3-Sylow in 3McL is contained in two maximal subgroups, one of shape $3^5.M_{10}$ and the other of shape $3^{2+4} : 2S_5$. The latter is more useful, and for the remainder of the proof we will call it S . The quotient $2S_5$ is the one listed in the ATLAS under the names “ $2S_5$ ” and “Isoclinic($2.A_5.2$)”; it is the group of shape $2S_5$ that contains elements of order 12. This $2S_5$ has a faithful 4-dimensional representation over \mathbf{F}_3 . The quotient of S in McL has shape $3^{1+4} : 2S_5$, and the “central” 3 is not central, but rather transforms by the sign representation of $2S_5$. In terms of the 4-dimensional module, it corresponds to a symplectic form on 3^4 which is $2A_5$ -but not $2S_5$ -fixed. There is also a symplectic form which is $2S_5$ -fixed, and the 3^{2+4} subgroup of S extends 3^4 by both symplectic forms simultaneously.

After a multi-hour computation, HAP reports

$$H_1(S; \mathbf{F}_3) = H_2(S; \mathbf{F}_3) = 0, \quad H_3(S; \mathbf{F}_3) = 3,$$

from which we learn that

$$H_1(S)_{(3)} = H_2(S)_{(3)} = 0, \quad H_3(S)_{(3)} \text{ is cyclic.}$$

On the other hand,

$$H_3(2S_5)_{(3)} = 3,$$

and since the extension $S = 3^{2+4} : 2S_5$ splits, $H_3(S)_{(3)}$ contains $H_3(2S_5)_{(3)}$ as a direct summand. Passing to cohomology, we learn that the pullback map

$$H^4(2S_5) \rightarrow H^4(S)$$

is an isomorphism.

There is a unique conjugacy class of order 3 in $2S_5$, and the restriction map $H^4(2S_5)_{(3)} \rightarrow H^4(\langle 3a \rangle)$ is an isomorphism. Take any element $g \in 2S_5$ of order 3 and lift it to an order-3 element $\tilde{g} \in S$. Then the composition $H^4(2S_5)_{(3)} \xrightarrow{\sim} H^4(S)_{(3)} \rightarrow H^4(\langle \tilde{g} \rangle)$ is an isomorphism. On the other hand, all conjugacy classes of order-3 elements in 3McL meet the normal subgroup $3^{2+4} \subset S$, and the composition $H^4(2S_5)_{(3)} \xrightarrow{\sim} H^4(S)_{(3)} \rightarrow H^4(3^{2+4})$ is zero.

Thus the image of $H^4(2S_5)_{(3)} \xrightarrow{\sim} H^4(S)_{(3)}$ has trivial intersection with the restriction map $H^4(3\text{McL})_{(3)} \hookrightarrow H^4(S)_{(3)}$, and so $H^4(3\text{McL})_{(3)} = 0$. \square

6.4. Suzuki group. The Schur cover of the Suzuki group is the beginning of a famous sequence of subgroups of 2Co_1 centralizing actions of binary alternating groups on the Leech lattice; 6Suz centralizes an action of $2A_3 \cong \mathbf{Z}/6$, which corresponds to a “complex structure” on the Leech lattice. In particular, 6Suz has a conjugate pair of 12-dimensional irreducible complex representations, (either one of) which we will call V throughout this section. The underlying real representation of V is $(\text{Leech} \otimes \mathbf{R})|_{6\text{Suz}}$.

Unlike the case of the Conway groups, the 2-Sylow subgroup of Suz is not contained in an extension of shape (elementary abelian).(smaller). It is, however, contained in an extension of shape (extraspecial).(smaller); specifically, of shape $2^{1+6} \cdot \text{SWeyl}(E_6)$, where by “SWeyl” we mean the index-2 subgroup of the Weyl group consisting of even reflections. The extension does not split. Two other groups with this feature are: Conway’s group Co_1 , whose 2-Sylow lives in a nonsplit extension of shape $2^{1+8} \cdot \text{SWeyl}(E_8)$; and the Monster group M , whose 2-Sylow lives in $2^{1+24} \cdot \text{Co}_1$. From the point of view of this paper, the difference between Co_1 and the other two is that the former also contains a 2-Sylow-containing subgroup of shape (elementary abelian) : (simple), making it comparatively computable.

Our goal in this section is to prove:

Theorem 6.5. *The Suzuki group and its Schur covers have the following fourth cohomologies:*

$$H^4(\text{Suz}) = \mathbf{Z}/4, \quad H^4(2\text{Suz}) = \mathbf{Z}/8, \quad H^4(3\text{Suz}) = \mathbf{Z}/12, \quad H^4(6\text{Suz}) = \mathbf{Z}/24.$$

$H^4(6\text{Suz})$ is generated by $c_2(V)$, where V denotes either 12-dimensional complex irrep.

We will split the proof into a series of lemmas.

Lemma 6.6. $c_2(V)$ has order 24 in $H^4(6\text{Suz})$.

Proof. Since the underlying real representation of V is $\text{Leech} \otimes \mathbf{R}$, we have $c_2(V) = -\frac{p_1}{2}(\text{Leech})|_{6\text{Suz}}$. Then [JFT18, Theorem 0.1] gives an upper bound of 24 on the order of $c_2(V)$.

The action of 6Suz on the Leech lattice includes elements with Frame shape 3^8 ; according to [JFT18, Theorem 7.1], $\frac{p_1}{2}(\text{Leech})$ has nontrivial restriction to such elements. This gives a lower bound of 3 on the order of $c_2(V)$.

6Suz contains a maximal subgroup of shape $6A_7$. As observed in [JFT18, Lemma 4.1], there is a unique conjugacy class of subgroups $D_8 \subset A_6$, where D_8 denotes the dihedral group of order 8. Along the standard inclusion $A_6 \subset A_7$, the 6-fold cover pulls back to the cover $3 \times 2D_8$ of D_8 , where $2D_8$ denotes the binary dihedral group of order 16. This group $2D_8$ is the one used in [JFT18, Lemma 4.1], where it is shown that $\frac{p_1}{2}(\text{Leech})|_{2D_8}$ has order 8. This gives a lower bound of 8 on the order of $c_2(V)$. \square

Lemma 6.7. $H^4(\text{Suz})_{(3)} = 0$.

Proof. The 3-Sylow in Suz is contained in a maximal subgroup of shape $3^5 : M_{11}$. There are two nontrivial 5-dimensional M_{11} -modules over \mathbf{F}_3 . They lead to LHS spectral sequences with E_2 pages:

$$\begin{array}{cccccc} 0 & & & & & 0 \\ 0 & 3 & 3 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & \text{or} & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{Z} & 0 & 0 & 0 & 8 & \mathbf{Z} & 0 & 0 & 0 & 8 \end{array}$$

The former is incompatible with $H^3(\text{Suz})_{(3)} = 3$, and the latter immediate gives $H^4(\text{Suz})_{(3)} = 0$. \square

As mentioned above, the 2-Sylow in Suz is contained in an extension of shape $2^{1+6}.\text{SWeyl}(E_6)$, where by $\text{SWeyl}(E_6)$ we mean the even subgroup of the Weyl group. The ATLAS calls this group $\text{SWeyl}(E_6) = U_4(2)$. We will write it as a Weyl group to make the action on 2^6 transparent: it is the mod-2 reduction of the action of $\text{Weyl}(E_6)$ on the E_6 -lattice. In particular, the quadratic form on 2^6 defining the extension 2^{1+6} has Arf invariant -1 .

Lemma 6.8. $H^0(\text{SWeyl}(E_6); H^4(2^{1+6}))$ is congruent to either $\mathbf{Z}/2$ or $\mathbf{Z}/4$.

Proof. Let us write J for $\text{SWeyl}(E_6)$ and E for the 6-dimensional $\text{SWeyl}(E_6)$ -module over \mathbf{F}_2 . In §3.3 we identified $H^4(2.E)$ as

$$(E^*.\text{Alt}^2(E^*).\text{Alt}^3(E^*)/E^*).2.$$

This group has a unique nonzero element which is divisible by 2; it lives in the subgroup $X = E^*.\text{Alt}^2(E^*).\text{Alt}^3(E^*)/E^*$, and so $H^0(J; X) \geq \mathbf{Z}/2$. On the other hand,

$$H^0(J; E^*) = H^0(J; \text{Alt}^3(E^*)/E^*) = 0, \quad H^0(J; \text{Alt}^2(E^*)) = \mathbf{Z}/2.$$

Indeed, E^* and $\text{Alt}^3(E^*)/E^*$ are simple J -modules of dimensions 6 and 14 respectively, and the unique fixed point in J -fixed point in $\text{Alt}^2(E^*)$ is the underlying alternating form of the quadratic form defining the extension $2.E$. From the long exact sequence $H^\bullet(J; A) \rightarrow H^\bullet(J; A.B) \rightarrow H^\bullet(J; B) \rightarrow H^{\bullet+1}(J; A) \rightarrow \dots$, we find

$$H^0(J; X) \leq \mathbf{Z}/2$$

and

$$H^0(J; X.2) \leq (\mathbf{Z}/2).(\mathbf{Z}/2) = \mathbf{Z}/4. \quad \square$$

We were unable to determine the exact value of $H^0(\text{SWeyl}(E_6); H^4(2^{1+6}))$. We remark that the order-2 class therein has many descriptions. It arises as c_2 of the unique 2^3 -dimensional complex irrep of 2^{1+6} . It also arises as follows. The nonsplit extension $2^6.\text{SWeyl}(E_6)$ is naturally a subgroup of the compact Lie group of type E_6 (adjoint form); in this realization, 2^6 is the group of 2-torsion points in the maximal torus. The generator of $H^4(BE_6)$ restricts to 2^6 to the E_6 quadratic form living in $\text{Sym}^2(2^6) \subset H^4(2^6)$, and this form pulls back to 2^{1+6} to the $\text{SWeyl}(E_6)$ -fixed order-2 class.

We may now compute the E_2 page of the LHS spectral sequence for the extension $2^{1+6}.\text{SWeyl}(E_6)$ using HAP and Cohomolo:

$$\begin{array}{cccccc} 2 \text{ or } 4 & & & & & \\ 0 & 2 & 0 & & & \\ 0 & 0 & 2 & & & \\ 0 & 0 & 0 & 0 & 0 & \\ \mathbf{Z} & 0 & 0 & 2 & 4 & \end{array}$$

Let $2\text{SWeyl}(E_6)$ denote the Spin double cover of $\text{SWeyl}(E_6) \subset \text{SO}(6, \mathbf{R})$. From the above spectral sequence, we learn:

Lemma 6.9. *The 2-Sylow in 2Suz is contained in a maximal subgroup of shape $2^{1+6}.2\text{SWeyl}(E_6)$.*

We claim that $H^4(2\text{SWeyl}(E_6)) \rightarrow H^4(Q)$ is an isomorphism. Indeed, consider either 4-dimensional faithful complex representations of $2\text{SWeyl}(E_6)$. Class 2a acts on this representation with trace 0. It follows that this representation decomposes over Q as one copy of the 2-dimensional irrep plus two copies of the same 1-dimensional irrep, and so $c_2(4\text{-dim rep})|_Q$ has order 8. We furthermore learn that $c_2(4\text{-dim rep})$ generates $H^4(2\text{SWeyl}(E_6))$.

We henceforth write $\alpha \in H^4(2\text{SWeyl}(E_6)) \cong \mathbf{Z}/8$ for this generator. Let $\tilde{\alpha}$ denote its pullback to $2^{1+6}.\text{SWeyl}(E_6)$. To prove the Proposition, it suffices to show that $4\tilde{\alpha}$ is not in the image of $H^4(2\text{Suz})$.

The central element of Q' is an order-2 lift of class $2a \in \text{SWeyl}(E_6)$. Any such lift fuses to class $2a \in \text{Suz}$. But $2^{1+6}.\text{SWeyl}(E_6)$ is the centralizer of an element of class $2a \in \text{Suz}$. It follows that Q' is conjugate in 2Suz to some other quaternion group $Q'' \subset 2^{1+6}.\text{SWeyl}(E_6)$ whose central element is central covers the center of $2\text{SWeyl}(E_6)$.

Since Q' is a lift of Q , we find that $\tilde{\alpha}|_{Q'}$ is a generator of $H^4(Q')$, and so $4\tilde{\alpha}|_{Q'} \neq 0$. On the other hand, since the center of Q'' maps to something central in 2SWeyl , the 4-dimensional representation of 2SWeyl breaks up over the image of Q'' as either four 1-dimensional representations or two copies of the 2-dimensional representation, and in either case we find that $\tilde{\alpha}|_{Q''} = c_2(4\text{-dim rep})|_{Q''}$ has order at most 4, so that $4\tilde{\alpha}|_{Q''} = 0$. Since Q' and Q'' are conjugate in 2Suz , the class $4\tilde{\alpha}$ cannot be the restriction of a class in $H^4(2\text{Suz})$. \square

Proof of Theorem 6.5. $H^4(\text{Suz})_{(p)}$ vanishes for $p \geq 5$ by Lemma 2.2. Lemma 6.7 gave $H^4(\text{Suz})_{(3)} = H^4(2\text{Suz})_{(3)} = 0$, and so Lemma 2.4 gives an upper bound $H^4(3\text{Suz})_{(3)} \leq 3$. But Lemma 6.6 provides the lower bound $H^4(3\text{Suz})_{(3)} \geq 3$, and so:

$$H^4(3\text{Suz})_{(3)} \cong H^4(6\text{Suz})_{(3)} \cong \mathbf{Z}/3,$$

generated by the 3-part of $c_2(V)$.

We now argue that $H^4(2\text{Suz}) = 8$. Lemma 6.6 implies that $H^4(2\text{Suz})$ contains an element of order 8, namely the 2-part of $c_2(V)$, where V denotes the 12-dimensional irrep of 6Suz . Lemma 6.11 implies that $H^4(2\text{Suz})$ has order at most 16. Furthermore, since the 2-part of $c_2(V)$ has order 8, its restriction to 2^{1+6} must be nonzero. On the other hand, for every representation W of 2^{1+6} , $c_2(W) \in H^4(2^{1+6})$ has order dividing 2. (Indeed, for the one-dimensional irreps of 2^{1+6} this is automatic, and for the unique irrep of dimension 2^3 it is a straightforward computation.) Thus $c_2(V)$ restricts to the unique class in $H^4(2^{1+6})$ which is divisible by 2. From this we learn that the only way for $H^4(2\text{Suz})$ to have order 16 is if $c_2(V)$ is divisible by 2.

Suppose that there were a class “ $\frac{1}{2}c_2(V)$.” Since $c_2(V)$ restricts to $2D_8$ with order 8, this class $\frac{1}{2}c_2(V)$ would have to have order 16 when restricted to $2D_8$. The order-16 classes in $H^4(2D_8)$ are the ones that have nontrivial restriction to the center of $2D_8$, which by construction is the center of 2Suz . But all classes in $H^4(2^{1+6}.\text{SWeyl}(E_6))$, hence all classes in $H^4(2\text{Suz})$, restrict trivially to the center. This proves:

$$H^4(2\text{Suz}) \cong \mathbf{Z}/8,$$

generated by the 2-part of $c_2(V)$.

Finally, we argue that $H^4(\text{Suz})_{(2)} \cong H^4(3\text{Suz})_{(2)} \cong \mathbf{Z}/4$ by repeating the logic from [JFT18, Theorem 5.3]. Namely, we have a commutative diagram

$$\begin{array}{ccccc} 2D_8 & \hookrightarrow & 6\text{Suz} & \hookrightarrow & 2C_{01} \\ \downarrow & & \downarrow & & \downarrow \\ D_8 & \hookrightarrow & 3\text{Suz} & \hookrightarrow & C_{01} \end{array}$$

which, upon applying $H^4(-)_{(2)}$, gives the diagram:

$$\begin{array}{ccccc} \mathbf{Z}/16 & \longleftarrow & \mathbf{Z}/8 & \longleftarrow & \mathbf{Z}/8 \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{Z}/4 \times (\mathbf{Z}/2)^2 & \longleftarrow & H^4(3\text{Suz})_{(2)} & \longleftarrow & \mathbf{Z}/4. \end{array}$$

The north-then-west compositions are injective, and so both southern arrows must be injective. It follows that

$$H^4(\text{Suz}) \cong H^4(3\text{Suz})_{(2)} \cong \mathbf{Z}/4. \quad \square$$

7. PARIAS

7.1. Janko groups 1 and 3. Using the permutation representations listed in the ATLAS, HAP is able to compute:

$$H_3(J_1) \cong \mathbf{Z}/30, \quad H_3(J_3) \cong \mathbf{Z}/15.$$

HAP is unable to compute $H_3(3J_3)$ directly. Nevertheless, we do have:

Theorem 7.1. $H_3(3J_3) \cong (\mathbf{Z}/3)^2 \times \mathbf{Z}/5$. Both $H^4(J_3)$ and $H^4(3J_3)$ consist entirely of Chern classes, and are detected on cyclic subgroups.

Proof. Let V_{323} denote a choice of complex irrep of J_3 of dimension 323. (The two choices are the characters χ_4 and χ_5 when listed by increasing dimension; both are real, and are exchanged by the outer automorphism of J_3 .) Then $c_2(V_{323})$ restricts nontrivially to all cyclic subgroups of order 5 in J_3 . For the 3-parts of $H^4(J_3)$, we focus on the conjugacy classes 3a and 9a, and any choice of 1920-dimensional irrep V_{1920} (there are three such irreps, all real, with characters χ_{14} , χ_{15} , and χ_{16}). Then $c_2(V_{1920})$ restricts nontrivially to both $\langle 3a \rangle$ and $\langle 9a \rangle$.

Choose any lift of $3a \in J_3$ to $3J_3$, for example $3c \in 3J_3$. (The classes $3a, 3b \in 3J_3$ are central.) The class $9a \in J_3$ lifts to a single conjugacy class in $3J_3$, also called 9a. With these new names, we still have that $c_2(V_{1920})|_{3c}$ and $c_2(V_{1920})|_{9a}$ are nontrivial. Finally, consider the smallest faithful representation V_{18} of $3J_3$, with dimension 18 and character χ_{22} . Then $c_2(V_{18})|_{3c} = 0$, but $c_2(V_{18})$ restricts with order 3 to $\langle 9a \rangle$.

It follows that the image of the map $H^4(3J_3)_{(3)} \rightarrow H^4(\langle 3c \rangle) \times H^4(\langle 9a \rangle)$ is not cyclic. On the other hand, the HAP computation of $H_3(J_3)_{(3)}$ together with Lemma 2.4 imply that the domain has order at most 9. So $H^4(3J_3)_{(3)} \cong (\mathbf{Z}/3)^2$. \square

7.2. O’Nan group.

Theorem 7.2. $H_3(O’N) \cong H_3(3O’N) \cong \mathbf{Z}/8$.

Proof. The p -parts of $H^4(O’N)$ vanish for $p = 5$ and $p \geq 11$ by Lemma 2.2. The 7-Sylow is contained in a subgroup isomorphic to $\text{PSL}_3(7)$, and a HAP computation gives $H^4(\text{PSL}_3(7)) \cong \mathbf{Z}/16$.

The 3-Sylow in $3O’N$ is an extraspecial group of shape 3^{1+4} , and its normalizer N is a maximal subgroup of shape $N = 3^{1+4} : 2^{1+4}.D_{10}$. HAP quickly computes:

$$H_3(N) \cong \mathbf{Z}/4 \times \mathbf{Z}/8 \times \mathbf{Z}/5.$$

It follows that $H^4(3O’N)_{(3)}$, and hence also $H^4(O’N)_{(3)}$, vanishes.

The 2-Sylow in $O’N$ is contained inside the nonsplit extension $4^3 \cdot \text{GL}_3(2)$. According to Lemma 4.1 part (4),

$$H^4(4^3 \cdot \text{GL}_3(2))_{(2)} \cong (\mathbf{Z}/2)^2 \times \mathbf{Z}/8.$$

The \mathbf{F}_2 -cohomology ring of $O’N$, including the action of Steenrod squares, was computed by [AM95]. They find that

$$H^1(O’N; \mathbf{F}_2) = H^2(O’N; \mathbf{F}_2) = 0, \quad H^3(O’N; \mathbf{F}_2) \cong H^4(O’N; \mathbf{F}_2) \cong \mathbf{F}_2,$$

but

$$\mathrm{Sq}^1 = 0 : H^3(\mathrm{O}'\mathrm{N}; \mathbf{F}_2) \rightarrow H^4(\mathrm{O}'\mathrm{N}; \mathbf{F}_2).$$

It follows that $H^4(\mathrm{O}'\mathrm{N})_{(2)}$ is cyclic of order strictly greater than 2. Since $H^4(\mathrm{O}'\mathrm{N})_{(2)}$ is a direct summand of $H^4(4^3 \cdot \mathrm{GL}_3(2))_{(2)}$, the only option is $H^4(\mathrm{O}'\mathrm{N})_{(2)} \cong \mathbf{Z}/8$. \square

7.3. Janko group 4 and Lyons group. The two largest Pariahs are Janko's fourth group J_4 and Lyons' group Ly . Both have trivial Schur multiplier [Gr71], and so their cohomologies vanish in degrees ≤ 3 . We find that in fact their cohomologies vanish in degrees ≤ 4 . Only one other sporadic group has this property: the cohomology of M_{23} vanishes in degrees ≤ 5 [Mil00].

Theorem 7.3. $H^4(J_4)$ is trivial.

Proof. The only primes not covered by Lemma 2.2 are 2, 3, and 11. The 11-Sylow in J_4 is contained in a maximal subgroup of shape $11^{1+2} : (5 \times 2S_4)$. It is easy to check that $H^4(11^{1+2} : (5 \times 2S_4))_{(11)} = 0$; for example by observing that the central $10 \subset 5 \times 2S_4$ acts on 11^2 through the isomorphism $10 \cong (\mathbf{Z}/11)^\times$ and applying Lemma 2.3.

The 3-Sylow is contained in a maximal subgroup of shape $2^{11} : M_{24}$. There are two conjugacy classes of elements of order 3 in M_{24} ; the restriction map $H^4(M_{24})_{(3)} \cong \mathbf{Z}/3 \rightarrow H^4(\langle 3a \rangle)$ is zero, whereas $H^4(M_{24})_{(3)} \cong \mathbf{Z}/3 \rightarrow H^4(\langle 3b \rangle)$ is an isomorphism [GPRV13]. But J_4 has only one conjugacy class of order 3. It follows that $H_4(J_4)_{(3)} = 0$.

The 2-Sylow is contained in $2^{11} : M_{24}$, and also in a maximal subgroup of shape $2^{1+22}.3M_{22}.2$ centralizing conjugacy class $2a \in J_4$. This latter subgroup turns out to be more useful. Using Cohomolo and Proposition 5.3, we find that the E_2 page for the LHS spectral sequence reads:

$$\begin{array}{cccccc} \leq 4 & & & & & \\ 0 & 0 & 2 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & \\ \mathbf{Z} & 0 & 2 & 4 & 2^2 \times 3 & \end{array}$$

The entry " ≤ 4 " in bidegree $(0, 4)$ comes from the LES for the extension

$$H^4(2^{1+12}) = 2^{12} \cdot \mathrm{Alt}^2(2^{12}) \cdot (\mathrm{Alt}^3(2^{12})/2^{12}) \cdot 2$$

from §3.3 and the calculations

$$H^0(3M_{22}.2; 2^{12}) = H^1(3M_{22}.2; 2^{12}) = H^0(3M_{22}.2; \mathrm{Alt}^2(2^{12})) = 0, \quad H^0(3M_{22}.2; \mathrm{Alt}^2(2^{12})) = 2.$$

We showed during the proof of Theorem 6.2 that $H^4(M_{22} : 2)_{(2)} = 2^2$ is detected by restricting to the three conjugacy classes of order 2 in $M_{22} : 2$. All three of these classes have order-2 preimages in $2^{1+22}.3M_{22}.2$. But both conjugacy classes of order 2 in J_4 meet 2^{1+22} . It follows that the images of $H^4(M_{22} : 2)_{(2)} \rightarrow H^4(2^{1+22}.3M_{22}.2)_{(2)}$ and $H^4(J_4)_{(2)} \rightarrow H^4(2^{1+22}.3M_{22}.2)_{(2)}$ have trivial intersection.

In particular, if $H^4(J_4) \neq 0$, then it contains an order-2 class α whose restriction to 2^{1+12} is the unique element $\tilde{q} \in H^4(2^{1+12})$ which is twice some other element. That unique element is pulled back from $H^4(2^{12})$, where it corresponds to the quadratic form $q \in \mathrm{Sym}^2(2^{12})$ defining the extension 2^{1+12} . Choose any pair of vectors $v_1, v_2 \in 2^{12}$ such that $q(v_1) = q(v_2) \neq 0$. Their lifts generate a quaternion group $Q_8 = 2^{1+2} \subset 2^{1+12}$, and $\tilde{q} \in H^4(2^{1+12})$ restricts nontrivially to that quaternion group. (These lifts of v_1, v_2 have order 4 in 2^{1+12} , and so are in conjugacy class $4a$ in $2^{1+12}.3M_{22}.2$.)

Choose $\tilde{g} \in 2^{1+12}.3M_{22}.2$ in conjugacy class $4b$. The character table libraries confirm that this \tilde{g} has the following properties: \tilde{g}^2 is the nontrivial central element in $2^{1+12} \subset 2^{1+12}.3M_{12}.2$; the image g in $3M_{22}.2$ of \tilde{g} is in conjugacy class $2a$. In particular, g acts on 2^{12} fixing 8 dimensions. Regardless of the Arf invariant of q , one can find a basis such that q vanishes on at most one basis vector; thus we can find a vector $v_1 \in 2^{12}$ with $q(v) \neq 0$ and $vg \neq v$. (Following GAP's conventions,

we write the action of $3M_{22}.2$ on 2^{12} from the right.) Set $v_2 = vg$; then $q(v_2) \neq 0$, and so the lifts of v_1, v_2 generate a Q_8 as in the previous paragraph.

Furthermore, we have arranged for the lifts of v_1, v_2 together with \tilde{g} to generate a binary dihedral group $2D_8 \subset 2^{1+12}.3M_{22}.2$ extending this Q_8 . Suppose that $H^4(J_4) \neq 0$, and let α denote its order-2 class. Then $\alpha|_{2D_8}$ has order 2, and so is divisible by 8. But then $\alpha|_{Q_8} = 0$. This contradicts the fact that Q_8 detected \tilde{q} , and so $H^4(J_4)$ must vanish. \square

Theorem 7.4. $H^4(\text{Ly})$ is trivial.

Proof. For $p > 5$, the Sylow p -subgroup of Ly is cyclic, so Lemma 2.2 applies.

Wilson has shown that $G_2(5)$ and $3\text{McL} : 2$ are subgroups (in fact, maximal subgroups) of the Lyons group. The 5-Sylow is contained in $G_2(5)$, so the $H^4(\text{Ly})_{(5)}$ vanishes by Lemma 4.3.

The 2- and 3-Sylows are contained in $3\text{McL} : 2$. By Theorem 6.4, the only cohomology of the latter is pulled back from the quotient $\mathbf{Z}/2$, and so is detected on a conjugacy class of order 2 (specifically, class $2b \in 3\text{McL} : 2$). But Ly has only one conjugacy class of order 2, and it meets $3\text{McL} \subset 3\text{McL} : 2$ (since 3McL has a class of order 2). It follows that the pullback $H^4(2) \xrightarrow{\sim} H^4(3\text{McL} : 2)$ and the restriction $H^4(\text{Ly})_{(2)} \rightarrow H^4(3\text{McL} : 2)$ have nonintersecting images. \square

8. MONSTER SECTIONS

8.1. Held group. The Held group is small enough to be accessible by the methods of §2.

Theorem 8.1. $H^4(\text{He}) \cong \mathbf{Z}/12$. It is spanned by fractional Pontryagin classes.

Proof. The primes not covered by Lemma 2.2 are $p = 2, 3$, and 7.

The normalizer of a 7-Sylow in He has shape $7^{1+2} : (3 \times S_3)$. There are no 7s in its low cohomology: $H^4(7^{1+2} : (3 \times S_3)) = H^4(3 \times S_3) = 2 \times 3^2$.

The 3-Sylow in He is inside a maximal subgroup of shape $3S_7$, with $H^4(3S_7) = 2^2 \times 4 \times 3^2$. We claim that the inclusions $H^4(\text{He})_{(3)} \rightarrow H^4(3S_7)_{(3)}$ and $H^4(S_7)_{(3)} = 3 \rightarrow H^4(3S_7)_{(3)}$ have nonintersecting images, giving an upper bound $H^4(\text{He})_{(3)} \leq 3$. To show this, we first observe that if C_3 is a cyclic group of order 3 then the two nonzero classes in $H^4(C_3) \cong \mathbf{Z}/3$ are canonically distinguished: one, which we will call $t^2 \in H^4(C_3)$, is the cup-square of both nonzero classes in $H^2(C_3) \cong \mathbf{Z}/3$, and the other is not a cup-square. Now consider the class $c_2(\text{Perm}) \in H^4(S_7)$, where Perm denotes the defining permutation representation. There are three conjugacy classes of order 3 in $3S_7$: the ‘‘central’’ one (not actually central — it is inverted by the odd elements of S_7), and two that act on Perm with cycle structures $1^4 3^1$ and $1^1 3^2$, respectively. It follows that $c_2(\text{Perm})|_{\langle 1^4 3^1 \rangle} = -t^2$ whereas $c_2(\text{Perm})|_{\langle 1^1 3^2 \rangle} = +t^2$, meaning that $c_2(\text{Perm})$ takes different values on these two classes. However, these two classes merge to the same class in He, and so $c_2(\text{Perm})$ cannot be the restriction of a cohomology class on He.

To establish the lower bound $H^4(\text{He})_{(3)} \geq 3$, we observe that the smallest irrep of He has dimension 51, and conjugacy class $3b \in \text{He}$ acts with trace 0, and so c_2 of this irrep, when restricted to $\langle 3b \rangle$, does not vanish.

For the prime $p = 2$, we use the 2-Sylow-containing maximal subgroup of shape $2^6 : 3S_6$. The E_2 page (localized at $p = 2$) of the LHS spectral sequence for this extension reads

$$\begin{array}{cccccc} E_2^{04} & & & & & \\ 0 & 2 & 2^2 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & \\ \mathbf{Z} & 0 & 2 & 2 & 2^2 \times 4 & \end{array}$$

with

$$E_2^{04} = H^0(3S_6; V. \text{Alt}^2(V). \text{Alt}^3(V)), \quad V = (2^6)^*.$$

Since

$$H^0(3S_6; V) = H^0(3S_6; \text{Alt}^2(V)) = 0, \quad H^0(3S_6; \text{Alt}^3(V)) \cong \mathbf{Z}/2,$$

we find that:

$$E_2^{04} \leq 2.$$

We claim that the inclusions $H^4(S_6)_{(2)} \cong H^4(3S_6)_{(2)} \rightarrow H^4(2^6 : 3S_6)$ and $H^4(\text{He})_{(2)} \rightarrow H^4(2^6 : 3S_6)$ have trivial intersection. To establish this claim, we first study $H^4(S_6)_{(2)}$. There are four 5-dimensional complex irreps of S_6 , corresponding to the characters χ_3, χ_4, χ_5 , and χ_6 . Their second Chern classes, when restricted to the conjugacy classes 2b, 4a, and 4b in S_6 , are:

	2b	4a	4b
$c_2(\chi_3)$	0	$-t^2$	$+t^2$
$c_2(\chi_4)$	0	$-t^2$	$-t^2$
$c_2(\chi_5)$	1	$-t^2$	$-t^2$
$c_2(\chi_6)$	1	$-t^2$	$+t^2$

Our notation is the following. The two classes in $H^4(\langle 2b \rangle) \cong \mathbf{Z}/2$ are “0” and “1.” If C_4 is a cyclic group of order 4, the two generators of $H^2(C_4)$ have the same cup-square in $H^4(C_4)$, which we call “ $+t^2$ ”; the generator of $H^4(C_4)$ which is not a cup-square is called “ $-t^2$.”

The images of $\{c_2(\chi_3), c_2(\chi_4), c_2(\chi_5), c_2(\chi_6)\}$ in $H^4(\langle 2b \rangle) \times H^4(\langle 4a \rangle) \times H^4(\langle 4b \rangle) \cong \mathbf{Z}/2 \times (\mathbf{Z}/4)^2$ together span a group isomorphic to $(\mathbf{Z}/2)^2 \times \mathbf{Z}/4$. It follows that these four classes span $H^4(S_6)_{(2)} \cong (\mathbf{Z}/2)^2 \times \mathbf{Z}/4$ and that the restriction map $H^4(S_6)_{(2)} \rightarrow H^4(\langle 2b \rangle) \times H^4(\langle 4a \rangle) \times H^4(\langle 4b \rangle)$ is an injection.

On the other hand, the subgroup $2^6 \subset 2^6 : 3S_6 \subset \text{He}$ meets both conjugacy classes of order 2, and the order-4 preimages in $2^6 \subset 2^6 : 3S_6$ of the classes 4a, 4b $\in S_6$ are He-conjugate to preimages of order-2 elements in S_6 . It follows that the image of $H^4(S_6)_{(2)} \cong H^4(3S_6)_{(2)} \rightarrow H^4(2^6 : 3S_6)$ does not intersect $H^4(\text{He})$.

We have so far established the following upper bound on $H^4(\text{He})_{(2)}$: it is a group of order at most 4. The last ingredient needed is a cohomology class of order divisible by 4. Let V denote the irreducible He-module with character χ_{19} : it is real and of dimension 7650. Consider the conjugacy class 4a $\in \text{He}$. It squares to 2a, and

$$\chi_{19}(4a) = 6, \quad \chi_{19}(2a) = 90.$$

Therefore 4a acts with spectrum $1^{1938} i^{1890} (-1)^{1932} (-i)^{1890}$, and so the total Chern class of V , when restricted to $\langle 4a \rangle$, is

$$c(V) = 1^{1938} (1-t)^{1890} (1-2t)^{1932} (1+t)^{1890} = 1 - 2t^2 + \dots \pmod{4t}.$$

In particular, $c_2(V)|_{\langle 4a \rangle} \neq 0$. But V is a real representation, and therefore Spin (since He has trivial Schur multiplier). So it has a fractional Pontryagin class, twice of which is $c_2(V)$. It follows that $\frac{p_1}{2}(V)$ has order divisible by 4, and so $H^4(\text{He})_{(2)} \cong \mathbf{Z}/4$. \square

8.2. Harada–Norton and Thompson groups. We were able to obtain partial results about the Harada–Norton and Thompson groups HN and Th.

Theorem 8.2. $H^4(\text{HN}; \mathbf{Z}[\frac{1}{2}]) \cong \mathbf{Z}/3$. At the prime 2, $H^4(\text{HN})_{(2)}$ has order at most 16 and exponent at most 8.

Proof. Lemma 2.2 handles the primes $p \geq 7$, and so we must study just the primes $p = 5$ and $p = 3$. The 5-Sylow in HN is contained in a maximal subgroup of shape $5^{1+4}.2^{1+4}.5.4$; the LHS spectral sequence gives $H^4(5^{1+4}.2^{1+4}.5.4)_{(5)} = 0$.

The 3-Sylow is inside a maximal subgroup of shape $3^{1+4} : 4A_5$, where by “ $4A_5$ ” we mean the “diagonal” central extension $2.(A_5 \times 2)$. HAP can directly compute:

$$H^4(3^{1+4} : 4A_5)_{(3)} = 3^2.$$

We claim that the generator of $H^4(4A_5)_{(3)} = 3$, when pulled back to $3^{1+4} : 4A_5$, is not the restriction of a class on HN. Indeed, that generator has nontrivial restriction to the elements of order 3 in $4A_5$, and so its pullback has nonzero restriction to some elements of order 3. But both conjugacy classes of order 3 in HN meet $3^{1+4} \subset 3^{1+4} : 4A_5$. Finally, we claim that $H^4(\text{HN})_{(3)} \neq 0$. There is a unique conjugacy class of order 9 in HN, and its traces on either 133-dimensional representation, which characters χ_2 and χ_3 , are $\chi_2(9a) = 1$ and $\chi_2(9a^3) = \chi_2(3b) = -2$. From this one can compute that $c_2(\chi_2)|_{\langle 9a \rangle} \neq 0$.

The 2-Sylow in HN is contained in the centralizer of conjugacy class 2b, which has shape $2_+^{1+8} \cdot (A_5 \wr 2)$. The E_2 page of the corresponding LHS spectral sequence provides an upper bound of $|H^4(\text{HN})_{(2)}| \leq 2^6$. Let $E = 2^8 \cong (2^8)^*$ and $J = (A_5 \wr 2)$; then

$$H^0(J; E) = H^1(J; E) = H^0(J; \text{Alt}^3(E)) = 0, \quad H^2(J; E) \cong \mathbf{Z}/2,$$

while

$$H^0(J; \text{Alt}^2(E)) \cong \mathbf{Z}/2, \quad H^0(J; \text{Alt}^2(E)/2) \cong \mathbf{Z}/2; \quad H^1(J; \text{Alt}^2(E)/2) \cong (\mathbf{Z}/2)^2.$$

Therefore the E_2 page of the LHS spectral sequence, after localizing at 2, reads:

$$\begin{array}{cccccc} & & & & & 2 \\ & & & & & 2 & 2^2 \\ & & & & & 0 & 0 & 2 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ \mathbf{Z} & & & & & 0 & 2 & 2 & 2 & 2^2 \end{array}$$

As usual, the image of $H^4(A_5 \wr 2)_{(2)} \rightarrow H^4(2_+^{1+8} \cdot (A_5 \wr 2))_{(2)}$ does not intersect $H^4(\text{HN})$: the former is detected on elements of order 2, but both conjugacy classes of order 2 in HN meet $2_+^{1+8} \subset 2_+^{1+8} \cdot (A_5 \wr 2)$. \square

Theorem 8.3. $H^4(\text{Th}; \mathbf{Z}[\frac{1}{3}]) \cong \mathbf{Z}/8$.

Proof. Lemma 2.2 handles the primes $p \geq 7$. The 5-Sylow is inside $5^{1+5} : 4S_4$, and Lemma 2.3 implies $H^4(5^{1+5} : 4S_4)_{(5)} = 0$. The 3-Sylow does not live in any nice maximal subgroups, and so we cannot compute $H^4(\text{Th})_{(3)}$.

The 2-Sylow in Th is contained in the Dempwolf group of shape $2^5 \cdot \text{GL}_5(2)$. According to Lemma 4.1 part (3), $H^4(2^5 \cdot \text{GL}_5(2))_{(2)} = 8$, and the proof of that Lemma established that $c_2(V) \neq 0$, where V denotes the 248-dimensional irrep of $2^5 \cdot \text{GL}_5(2)$. This irrep V extends to the defining 248-dimensional irrep of Th, and so the restriction map $H^4(\text{Th})_{(2)} \rightarrow H^4(2^5 \cdot \text{GL}_5(2))_{(2)}$ is nonzero. Since the image of that restriction map is a direct summand, we must have $H^4(\text{Th})_{(2)} \cong H^4(2^5 \cdot \text{GL}_5(2))_{(2)} \cong \mathbf{Z}/8$. \square

8.3. Fischer groups. The Fischer groups Fi_{22} , Fi_{23} , and Fi_{24} are a ‘‘third generation’’ version of the Mathieu groups M_{22} , M_{23} , and M_{24} . Specifically, the 2-Sylows in Fi_N , for $N \in \{22, 23, 24\}$, lives in a subgroup of shape $2^{\lceil N/2 \rceil - 1} \cdot M_N$, making the calculation of $H^4(\text{Fi}_N)_{(2)}$ systematic. The extension $2^{\lceil N/2 \rceil - 1} \cdot M_N$ splits for $N = 22$ and does not split for $N = 23$ and 24.

The 3-Sylows in all cases are contained in orthogonal groups over \mathbf{F}_3 . We were able to handle the 3-parts of $H^4(G)$ for $G = \text{Fi}_{22}$ and 3Fi_{22} , but not for the larger Fischer groups.

Theorem 8.4. *The Fischer groups have the following cohomologies away from the prime 3:*

- (1) $H^4(\text{Fi}_{22}; \mathbf{Z}[\frac{1}{3}]) = 0$.
- (2) $H^4(2\text{Fi}_{22}; \mathbf{Z}[\frac{1}{3}])$ has order 2 or 4.
- (3) $H^4(\text{Fi}_{23}; \mathbf{Z}[\frac{1}{3}]) \cong \mathbf{Z}/2$.
- (4) $H^4(\text{Fi}'_{24}; \mathbf{Z}[\frac{1}{3}])$ has order 2 or 4.

Proof. Lemma 2.2 handles all primes $p \geq 5$ except for $H^4(\text{Fi}'_{24})_{(7)}$. But the 7-Sylow in Fi'_{24} is inside a copy of Held's group He , and $H^4(\text{He})_{(7)} = 0$ by Theorem 8.1.

We now inspect the LHS spectral sequences for the extensions $2^{\lceil N/2 \rceil - 1} \cdot M_N \subset \text{Fi}_N$. The three cases are:

$$\begin{array}{ccc}
 \begin{array}{cccccc}
 0 & & & & & \\
 0 & 0 & 2 & & & \\
 0 & 2 & 0 & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \mathbf{Z} & 0 & 0 & 12 & 0 &
 \end{array} &
 \begin{array}{cccccc}
 0 & & & & & \\
 0 & 2 & 2 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \mathbf{Z} & 0 & 0 & 0 & 0 &
 \end{array} &
 \begin{array}{cccccc}
 0 & & & & & \\
 0 & 2 & & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \mathbf{Z} & 0 & 0 & 0 & 12 &
 \end{array} \\
 \\
 2^{10} : M_{22} \subset \text{Fi}_{22} &
 2^{11} \cdot M_{23} \subset \text{Fi}_{23} &
 2^{11} \cdot M_{24} \subset \text{Fi}_{24}
 \end{array}$$

The first proves claim (1) and provides the upper bound for claim (2). The second provides the upper bound for claim (3).

Let $\alpha \in H^4(M_{24})_{(2)} \cong \mathbf{Z}/4$ denote a generator, and let $\tilde{\alpha}$ denote its pullback to $2^{11} \cdot M_{24}$. Consider the conjugacy classes 4b and 4c in M_{24} . The formula given in [GPRV13, §3.3] provides $\alpha|_{\langle 4b \rangle} = 0$ whereas $\alpha|_{\langle 4c \rangle}$ has order 4 in $H^4(\langle 4c \rangle)$. These classes admit preimages in $2^{11} \cdot M_{24}$ living in conjugacy classes 8e and 8a respectively, and so $\tilde{\alpha}|_{8e} = 0$ whereas $\tilde{\alpha}|_{8a}$ has order 2. (The pullback map $H^4(C_4) \rightarrow H^4(C_8)$ along a surjection of cyclic groups $C_8 \rightarrow C_4$ has image of order 2.) Both 8e and 8a fuse in Fi'_{24} to class 8a, and so $\tilde{\alpha}$ cannot be the restriction of a cohomology class on Fi'_{24} . Together with the above spectral sequence, we find the upper bound in claim (4).

All that remain are the lower bounds. Let ω^{\natural} denote the ‘‘gauge anomaly’’ of the Monster CFT, studied in [JF17]; c.f. §8.4. The McKay–Thompson series for class 4b in the Monster group M has a nontrivial multiplier (of order 2), and so $\omega^{\natural}|_{\langle 4b \rangle}$ is nonzero. But 4b meets $2\text{Fi}_{22} \subset \text{Fi}_{23} \subset 3\text{Fi}'_{24}$, and so ω^{\natural} restricts with order at least 2 to all of these groups. \square

Theorem 8.5. $H^4(\text{Fi}_{22}) = \{0\}$ and $H^4(3\text{Fi}_{22}) \cong \mathbf{Z}/3$.

Combined with Theorem 8.4 part (2), we find that $H^4(6\text{Fi}_{22})$ has order either 6 or 12.

Proof. Given Theorem 8.4 part (1), we must only compute $H^4(\text{Fi}_{22})_{(3)}$ and $H^4(3\text{Fi}_{22})_{(3)}$. But the 3-Sylow in Fi_{22} is contained in a maximal subgroup isomorphic to $\Omega_7(3)$, and $H^4(\Omega_7(3))_{(3)} = 0$ by Lemma 4.4. By Corollary 2.5, the inclusions $H^4(\Omega_7(3))_{(3)} \rightarrow H^4(3\Omega_7(3))_{(3)}$ and $H^4(\text{Fi}_{22})_{(3)} \rightarrow H^4(3\text{Fi}_{22})_{(3)}$ have the same cokernel; this cokernel is $\mathbf{Z}/3$ by Corollary 4.5. \square

8.4. Monster. The main result of [JF17] is that $H^4(M)$ contains a class ω^{\natural} , arising as the gauge anomaly of the Moonshine CFT, of exact order 24. It is reasonable to conjecture that ω^{\natural} generates $H^4(M)$. The calculations in this paper allow us to come close to proving that conjecture:

Theorem 8.6. $H^4(M) \cong \mathbf{Z}/24 \oplus X$, where the $\mathbf{Z}/24$ summand is generated by ω^{\natural} and where the order of X divides 4.

Proof. The primes $p = 11$ and $p \geq 17$ are covered by Lemma 2.2. To calculate $H^4(M)$, we must study the p -Sylows for $p = 2, 3, 5, 7, 13$. For these p , the p -Sylow is contained in the normalizer $N(pb)$ of an element of conjugacy class pb . These normalizers are all of shape

$$N(pb) = p^{1+m} \cdot J$$

where $m = 24/(p - 1)$ and $J \subset \text{Co}_1$. Specifically:

p	J
2	Co_1
3	$2\text{Suz}.2$
5	$2\text{J}_2.4$
7	$3 \times 2S_7$
13	$3 \times 4S_4$

The extension $p^{1+m}.J$ splits for $p \geq 5$. When $p = 3$, $3^{1+12}.2.\text{Suz}.2$ does not split, but the quotient $3^{12} : 2\text{Suz}.2$ does split. When $p = 2$, the quotient $2^{24} \cdot \text{Co}_1$ does not split.

The center of the group J in all cases has order $p - 1$, and acts by a faithful central character on p^m . Applying Lemma 2.3, we find that $H^i(J; H^j(p^{1+m})) = 0$ for $0 < j < p$. Combined with the HAP calculation of $H^4(2\text{J}_2)$ from §6.1, the known value $H^4(3 \times 2S_7)_{(7)} = 0$ (which follows easily by the methods of Lemma 2.2), and the trivial result $H^4(3 \times 4S_4)_{(13)} = 0$, we find that $H^4(M; \mathbf{Z}[\frac{1}{6}]) = 0$.

At the prime 3, central character considerations imply that the only potentially nonzero entries of total degree 4 on the LHS spectral sequence for $3^{1+12}.2\text{Suz}.2$ are, in the notation of Lemma 3.3:

$$H^0(2\text{Suz}.2; \text{Sym}^2(3^{12})), \quad H^1(2\text{Suz}.2; \text{Alt}^2(3^{12})_\omega), \quad H^4(2\text{Suz}.2; \mathbf{Z})_{(3)}.$$

The first vanishes because 3^{12} is antisymmetrically but not symmetrically self-dual as a $2.\text{Suz}$ module. Actually, as a $2\text{Suz}.2$ module, 3^{12} is not self-dual at all: the symplectic pairing changes by a sign via the surjection $2\text{Suz}.2 \rightarrow 2$. The last vanishes by Theorem 6.5.

Therefore $H^1(2\text{Suz}.2; \text{Alt}^2(3^{12})_\omega)$ gives an upper bound for $H^4(M)_{(3)}$. The group $2\text{Suz}.2$ is too large for Cohomolo to handle directly, but its 3-Sylow-containing maximal subgroup of shape $3^5 : (M_{11} \times 2)$ is not, and Cohomolo computes:

$$H^1(3^5 : (M_{11} \times 2); \text{Alt}^2(3^{12})_\omega) \cong 3^1.$$

This provides an upper bound for $H^1(2\text{Suz}.2; \text{Alt}^2(3^{12})_\omega)$, and so:

$$H^4(M)_{(3)} \leq 3.$$

On the other hand, [JF17, Lemma 3.2.4] shows that $H^4(M)_{(3)} \geq 3$.

The $p = 2$ part of $H^4(M)$ is more subtle.

We first claim that the $\mathbf{Z}/8 \subset H^4(M)_{(2)}$ generated by ω^{\natural} is a direct summand. Equivalently, we claim that ω^{\natural} is not divisible by 2. Consider the subgroup $N(3b) = 3^{1+12}.2.\text{Suz}.2 \subset M$. Since its quotient $3^{12} : 2\text{Suz}.2$ splits, we can find a copy of $6\text{Suz} \subset 6\text{Suz}.2 \subset M$. The central $3 \subset 6\text{Suz}$ is generated by class $3b$ by construction, and the central $2 \subset 6\text{Suz} \subset M$ is of class $2b$. It follows that this 6Suz is (conjugate to) a subgroup of the normalizer $N(2b) = 2^{1+12} \cdot \text{Co}_1$, where it lives over a copy of $3\text{Suz} \subset \text{Co}_1$. As observed in the proof of Theorem 6.5, the corresponding $6\text{Suz} \subset 2\text{Co}_1$ contains the group $2D_8$ used in [JFT18]; thus the $6\text{Suz} \subset M$ contains a $2D_8$ such that the central element is of class $2b \in M$. This is the $2D_8 \subset M$ used in [JF17, §3.3] to show that the order of ω^{\natural} is divisible by 8. It follows in particular that the order of $\omega^{\natural}|_{6\text{Suz}}$ is divisible by 8. But $H^4(6\text{Suz})_{(2)} = 8$ by Theorem 6.5, and so $\omega^{\natural}|_{6\text{Suz}}$ is not divisible by 2, proving the claim.

The last step of the proof is to study the LHS spectral sequence for the extension $2^{1+24} \cdot \text{Co}_1$:

$$\begin{array}{ccccccc} H^0(\text{Co}_1; H^4(2^{1+24})) & & & & & & \\ H^0(\text{Co}_1; H^3(2^{1+24})) & H^1(\text{Co}_1; H^3(2^{1+24})) & & & & & \\ H^0(\text{Co}_1; H^2(2^{1+24})) & H^1(\text{Co}_1; H^2(2^{1+24})) & H^2(\text{Co}_1; H^2(2^{1+24})) & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \mathbf{Z} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{4} & \end{array}$$

The 2 in the bottom row is $H^3(\text{Co}_1)$, and the 4 in the bottom row is $H^4(\text{Co}_1)_{(2)}$; the latter result is due to [JFT18].

From §3.3, we know

$$\mathbf{H}^2(2^{1+24}) \cong 2^{24}, \quad \mathbf{H}^3(2^{1+24}) \cong \text{Alt}^2(2^{24})/2 \cong 2^{275},$$

while

$$\mathbf{H}^4(2^{1+24}) \cong 2^{24} \cdot \text{Alt}^2(2^{24}) \cdot (\text{Alt}^3(2^{24})/2^{24}) \cdot 2 \cong 2^{2300} \cdot 2$$

has exponent 4.

Without much difficulty, GAP can compute:

$$\mathbf{H}^0(\text{Co}_1; 2^{24}) = \mathbf{H}^0(\text{Co}_1; \text{Alt}^2(2^{24})/2) = \mathbf{H}^0(\text{Co}_1; \text{Alt}^3(2^{24})) = 0.$$

and

$$\mathbf{H}^0(\text{Co}_1; \text{Alt}^2(2^{24})) \cong \mathbf{Z}/2.$$

The groups $\mathbf{H}^i(\text{Co}_1; 2^{24})$ for $i = 0, 1$ were calculated by Derek Holt and reported in [Iva09, Lemma 1.8.8]. They are:

$$\mathbf{H}^1(\text{Co}_1; 2^{24}) = 0, \quad \mathbf{H}^2(\text{Co}_1; 2^{24}) \cong \mathbf{Z}/2.$$

A presentation for Co_1 is given in [Soi87]. Using it, [JF17, §3.5] calculates:

$$\mathbf{H}^1(\text{Co}_1; \text{Alt}^2(2^{24})/2) \cong \mathbf{Z}/2.$$

These calculations almost fill in the E_2 page. The remaining missing ingredient is $E_2^{04} = \mathbf{H}^0(\text{Co}_1; \mathbf{H}^4(2^{1+24}))$. Using the above calculations together with the long exact sequence for cohomology with values in an extension, one finds:

$$\mathbf{H}^0(\text{Co}_1; 2^{24} \cdot \text{Alt}^2(2^{24}) \cdot (\text{Alt}^3(2^{24})/2^{24})) = \mathbf{H}^0(\text{Co}_1; 2^{2300}) \cong \mathbf{Z}/2.$$

It follows that $\mathbf{H}^0(\text{Co}_1; \mathbf{H}^4(2^{1+24}))$ is isomorphic to either $\mathbf{Z}/2$ or $\mathbf{Z}/4$. We suspect the former, but were unable to compute it.

Although we do not know whether $E_2^{04} = 2$ or 4 , we claim that $E_\infty^{04} = 2$. To prove this we quote two facts. First, according to [JF17, §3.5], $\omega^\natural|_{2^{1+24}}$ has exact order 2, and so provides an element of order 2 in E_∞^{04} . Second, we showed above that ω^\natural is not divisible by 2 in $\mathbf{H}^4(\mathbf{M})$. Since the map $\mathbf{H}^4(\mathbf{M}) \rightarrow \mathbf{H}^4(2^{1+24} \cdot \text{Co}_1)$ is an inclusion onto a direct summand, it follows that $\omega^\natural|_{2^{1+24} \cdot \text{Co}_1}$ is not divisible by 2. But $\mathbf{H}^4(2^{1+24} \cdot \text{Co}_1)$ surjects onto E_∞^{04} , and sends ω^\natural to a nonzero value. So the image of ω^\natural in E_∞^{04} cannot be divisible by 2, proving that $E_\infty^{04} \neq 4$.

All together, we find an E_∞ page of the following form:

$$\begin{array}{cccccc} & & & & & 2 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & \mathbf{Z} \end{array} \begin{array}{l} \\ \leq 2 \\ 0 \leq 2 \\ 0 \quad 0 \\ 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 2 \quad 4 \end{array}$$

It follows that $\mathbf{H}^4(\mathbf{M})_{(2)}$ has order at most 32, completing the proof. \square

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