MORI'S PROGRAM FOR THE KONTSEVICH MODULI SPACE $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$

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Abstract. We run Mori's program for the Kontsevich space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ and give modular interpretations of all the intermediate spaces appearing in the process. In particular, we show that the closure of the twisted cubic curves in the Hilbert scheme $\mathcal{H}^{3m+1}(\mathbb{P}^3)$ is the flip of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ over the Chow variety.

Contents

1. Introduction

In the field of higher dimensional geometry, due to the recent success of Mori's minimal model program, cf. [BCHM], people have become interested in carrying out the minimal model program explicitly for various moduli spaces. In [Has], Brendan Hassett initiated a framework to understand the canonical model

$$\mathrm{Proj}\Big(\bigoplus_{m\geq 0} H^0(\overline{\mathcal{M}}_g, mK_{\overline{\mathcal{M}}_g})\Big)$$

of the moduli space $\overline{\mathcal{M}}_g$ of stable genus g curves, by using the log canonical models

$$\mathrm{Proj}\Big(\bigoplus_{m\geq 0} H^0(\overline{\mathcal{M}}_g, m(K_{\overline{\mathcal{M}}_g} + \alpha\delta))\Big),$$

where δ is the total boundary divisor corresponding to the locus of singular curves. The eventual goal is to decrease α from 1 to 0 and describe the log canonical models appearing in this process. Surprisingly, it seems that all these models may be geometrically meaningful spaces, by which we mean for instance, moduli spaces parameterizing curves with suitable singularities, or GIT quotients of some Hilbert schemes and Chow varieties of curves. This phenomenon has been already verified for special cases, cf. [Has], [HH], [HL] and [S]. But the whole picture is still far from complete.

Similarly, if we consider an arbitrary effective divisor other than the canonical divisor, a much broader question can be raised as follows.

Question 1.1. *Given a moduli space $\overline{\mathcal{M}}$ and an effective divisor D, study*

$$Proj\Big(\bigoplus_{m\geq 0} H^0(\overline{\mathcal{M}}, mD)\Big)$$

and enable its modular interpretation.

We call this Mori's program for moduli spaces. For instance, we can consider running this program on the Kontsevich space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is a useful compactification for the scheme parameterizing smooth degree d rational curves in $\mathbb{P}^r$. It has finite quotient singularities. Weil divisors determine elements of $\mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$. In particular, when $r = d$ the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is completely known [CHS2, Thm 1.5]. When d is 2, it is well-known that $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the blow-up of the Hilbert scheme of plane conics along the locus of double lines, i.e., $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2) \cong \mathrm{Bl}_{V_2}\mathbb{P}^5$, where V_2 is the image of the degree 2 Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. So the first unknown case is $d = 3$.

In this article, we run Mori's program for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ and give geometric interpretations of all the intermediate models. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is a compactification of the space of twisted cubics. It has dimension 12. Before we state the main result, let us recall some basic divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$. One can refer to [P1], [CHS1], and [CHS2] for a general description of the divisor theory on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

On $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$, H is the class of the divisor of maps whose images intersect a fixed line in $\mathbb{P}^3$; Δ is the class of the boundary divisor consisting of maps with reducible domains; T is the class of the tangency divisor consisting of maps $[C, \mu]$ such that $\mu^{-1}(\Pi)$ is not 3 distinct points, where Π is a fixed plane in $\mathbb{P}^3$; D_{deg} is the class of the divisor parameterizing stable maps whose set-theoretic images do not span $\mathbb{P}^3$. Call D_{deg} the divisor of degenerate maps. We also let K denote the canonical divisor of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$.

Here we need one more divisor.

Definition 1.2. Fix a point p and a plane Λ containing p in $\mathbb{P}^3$. Define a divisor F as the closure of $\{[C, \mu] \in \mathcal{M}_{0,0}(\mathbb{P}^3, 3) \mid \exists\ p_1, p_2 \in \mu(C) \cap \Lambda$ such that p, p_1 and p_2 are collinear$\}$.

We call $(p \in \Lambda)$ the defining point-plane flag for F. Note that F does not contain the tangency divisor T. Since if p_1 and p_2 tend to the same point q on C, the tangent line at q should pass through p.

Proposition 1.3.
(i). Pic $(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)) \otimes \mathbb{Q}$ is generated by H and Δ;
(ii). The effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is generated by Δ and D_{deg};
(iii). The nef cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is generated by H and T;
(iv). $T = \frac{2}{3}(H + \Delta)$;
(v). $D_{deg} = \frac{2}{3}(H - \frac{1}{2}\Delta)$;
(vi). $K = -\frac{8}{3}(H + \frac{1}{4}\Delta)$;
(vii). $F = \frac{5}{3}(H - \frac{1}{5}\Delta)$.

Proof. (i), (iv) and (vi) are special cases in [P1] and [P2]. (ii), (iii) and (v) have been proved in [CHS1] and [CHS2] also in a more general setting. For (vii), write F as the linear combination $aH + b\Delta$. Take two test curves in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$:
B_1: a plane conic attached to a pencil of lines at the base point.
B_2: a line attached to a pencil of plane conics at one of the four base points.

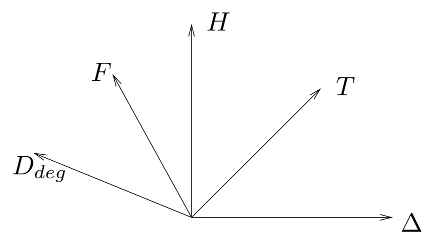
The following intersection numbers are easy to verify:

$$H.\,B_1 = 1,\ \Delta.\,B_1 = -1,\ F.\,B_1 = 2,$$

$$H.\,B_2 = 1,\ \Delta.\,B_2 = 2,\ F.\,B_2 = 1.$$

Hence, we get $a = \frac{5}{3}$ and $b = -\frac{1}{3}$. □

The picture below shows the decomposition of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$ by these divisors, as explained to the author by Izzet Coskun.



The next result provides a theoretical foundation for us to run Mori's program for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$.

Proposition 1.4. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$ *is Fano. In particular, it is a Mori dream space. Its Cox ring is finitely generated and Mori's program can be carried out for any divisor on* $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$*, i.e. the necessary contractions and flips exist and any sequence terminates.*

Proof. From Proposition 1.3, we know that $-K$ is ample, so $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$ is Fano. By [BCHM, 1.3.1], a projective normal Fano variety is always a Mori dream space. All the other consequences in this proposition are equivalent definitions or properties for Mori dream spaces, cf. [HK, 1.10, 1.11, 2.9]. □

To explicitly carry out Mori's program for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$, for a divisor D, define

$$\overline{M}(D) = \mathrm{Proj}\Big(\bigoplus_{m\geq 0} H^0(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3), mD)\Big).$$

In order to give a geometric interpretation of $\overline{M}(D)$, we also need a few other compactifications besides $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$ for the space of twisted cubics. Actually all the compactifications we need have already been investigated in various contexts by other people. We will simply exhibit related results as follows.

The Hilbert scheme $\mathcal{H}^{3m+1}(\mathbb{P}^3)$ of curves of degree 3 and arithmetic genus 0 is well understood [PS] and can be described as follows.

Proposition 1.5. $\mathcal{H}^{3m+1}(\mathbb{P}^3)$ *consists of two smooth irreducible components* $\mathcal{H}$ *and* $\mathcal{H}'$*, of dimension 12 and 15 respectively. The component* $\mathcal{H}$ *is the closure of the locus of twisted cubics. The other component* $\mathcal{H}'$ *is the closure of the locus of plane cubic curves union a point.*

The component $\mathcal{H}$ is a model we would like to compare with $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$. Moreover, if we regard a twisted cubic as a cycle class, we can also take the closure of the space of twisted cubics in the Chow variety. Use $\mathcal{C}$ to denote this closure. There are some fundamental maps among $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$, $\mathcal{H}$ and $\mathcal{C}$.

Proposition 1.6. *$\mathcal{H}$ admits a natural birational morphism g_0 to $\mathcal{C}$ forgetting the scheme structure of a curve but remembering its cycle class. Similarly, there is also a natural birational morphism f_0 from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ to $\mathcal{C}$ forgetting the maps but remembering the image cycles.*

Proof. The maps among $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3), \mathcal{H}$ and $\mathcal{C}$ were already mentioned in [Pi]. The cycle map g_0 from Hilbert to Chow is a standard fact, cf. [GIT, 5.10]. Moreover, in [K, I. 3-4], the Chow functor is defined on the category of reduced and seminormal schemes. Since in our situation $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is a smooth stack and its coarse moduli space has finite quotient singularities, the cycle map f_0 from Kontsevich to Chow also follows immediately. On the open locus parameterizing twisted cubics, the three spaces coincide. □

Corollary 1.7. *The Chow compactification $\mathcal{C}$ of the space of twisted cubics is not normal.*

Proof. Take the union of a conic with a line on the same plane as a Chow point $[C]$ in $\mathcal{C}$. The line and the conic meet at two points p_1 and p_2. $g_0^{-1}([C])$ consists of two different points in $\mathcal{H}$, because we need to impose a spatial embedded point either at p_1 or p_2 to make the arithmetic genus of C equal 0. Similarly, $f_0^{-1}([C])$ consists of two different maps in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$, since the node of the domain curve can either map to p_1 or p_2. Therefore, the fibers of g_0 and f_0 over $[C]$ are both disconnected, so $\mathcal{C}$ is not normal. □

We naturally pass to $\mathcal{C}^\nu$, the normalization of $\mathcal{C}$. The maps f_0 and g_0 factor through $\mathcal{C}^\nu$. Hence, denote the induced morphisms from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ and $\mathcal{H}$ to $\mathcal{C}^\nu$ by f and g respectively. We would like to understand the exceptional loci of these maps.

Definition 1.8. The multi-scheme locus $N \subset \mathcal{H}$ is the locus of curves possessing a nonreduced 1-dimensional component. The multi-image locus $M \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is $\{[C, \mu] \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \mid \exists$ a component C_1 of Im μ such that the map degree of μ over C_1 is greater than $1\}$.

N is irreducible. A general point in N corresponds to a double line of arithmetic genus -1 with a line meeting it and lying in its projective tangent space at the point of intersection, cf. [H, p. 39] and [Lee, 4.4]. It is clear that dim $N = 9$.

M breaks into two components $M_{1,2}$ and M_3. A general point in $M_{1,2}$ corresponds to a map whose domain consists of a nodal union of two $\mathbb{P}^1$'s. One component maps with degree 1 and the other one maps with degree 2. A general point in M_3 is a map whose domain is a single $\mathbb{P}^1$ which maps with degree 3. Apparently M is contained in the degenerate divisor D_{deg}. We also have dim $M_{1,2} = 9$ and dim $M_3 = 8$.

Based on the above results, it is not hard to check the following proposition.

Proposition 1.9. *For the maps g and f, we have Exc$(g) = N$ and Exc$(f) = M$. Moreover, there is a birational map $\phi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \dashrightarrow \mathcal{H}$ which is a canonical isomorphism away from M and N. In particular, ϕ is an isomorphism in codimension two and Pic$(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)) \otimes \mathbb{Q}$ is isomorphic to Pic$(\mathcal{H}) \otimes \mathbb{Q}$.*

Therefore, we will keep the same notation for divisors on both spaces $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ and $\mathcal{H}$. However, the reader should be aware that they may differ in loci of higher codimension. For instance, D_{deg} on $\mathcal{H}$ only consists of curves whose 1-dimensional components as schemes are planar. A triple line whose ideal is given by the square of the ideal of a reduced line is not contained in D_{deg}. But a triple covering map of a reduced line is contained in D_{deg} on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$.

Later on we will see that the Hilbert component $\mathcal{H}$ is the flip of the Kontsevich space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ over Chow. The flipping loci are N and M respectively. An interesting fact is that the flipping locus $M = M_{1,2} \cup M_3$ is reducible but N is irreducible.

Let us continue to introduce more compactifications for the space of twisted cubics. A twisted cubic curve C can be parameterized by a unique net of quadrics $H^0(\mathcal{I}_C(2))$ inside the space of quadrics $H^0(\mathcal{O}_{\mathbb{P}^3}(2))$. So the space of twisted cubics can be realized as a subscheme in the Grassmannian $\mathbb{G}(2, 9)$. Denote its closure by $\mathcal{H}(2)$. Moreover, one can check that $H^0(\mathcal{I}_C(2))$ is always 3-dimensional for any $[C] \in \mathcal{H}$, by using [PS, Lemma 2]. Hence, we get a morphism $h : \mathcal{H} \to \mathcal{H}(2)$. Two curves have the same image under h if and only if they share a common point-plane flag $(p \in \Lambda)$, namely, they have a spatial embedded point at p pointing out of Λ and their 1-dimensional components are contained in Λ. All nets of quadrics vanishing on both Λ and the embedded point form a locus called the point-plane incidence correspondence $\{(p \in \Lambda)\}$ in $\mathcal{H}(2)$. The space $\mathcal{H}(2)$ and the morphism h have been investigated intensively in [EPS]. In particular, they showed the following result, among others.

Proposition 1.10. *$\mathcal{H}(2)$ is smooth and h is the blow-up of $\mathcal{H}(2)$ along the locus of the point-plane incidence correspondence $\{(p \in \Lambda)\}$. The Picard numbers of $\mathcal{H}$ and $\mathcal{H}(2)$ are 2 and 1 respectively.*

The last space we need is less well-known. It is the space of 2-stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$, which is originally introduced in [MM] and also studied in [Pa]. For the reader's convenience, we include the definition for the space of k-stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k)$ as follows. One can refer to [MM, Def 1.2] and [Pa, Cor 4.6] for more details.

Definition 1.11. Let k be a natural number, $0 \le k \le d$. Fix a rational number ϵ such that $0 < \epsilon < 1$. The moduli space of k-stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k)$ parameterizes the following data $(\pi : C \to S,\ \mu : C \to \mathbb{P}^r,\ \mathfrak{L},\ e)$ where:
(1) $\pi : C \to S$ is a flat family of rational nodal curves over the scheme S.
(2) $\mathfrak{L}$ is a line bundle on C of degree d on each fiber C_s which, together with the morphism $e : \mathcal{O}_C^{r+1} \to \mathfrak{L}$ determines the rational map $\mu : C \dashrightarrow \mathbb{P}^r$.
(3) $\omega_{C/S}^{d-k+\epsilon} \otimes \mathfrak{L}$ is relatively ample over S.
(4) $\mathfrak{G} :=$ coker e, restricted to each fiber C_s, is a skyscraper sheaf, and dim $\mathfrak{G}_p \le d - k$ for any $p \in C_s$, where $\mathfrak{G}_p$ is the stalk of $\mathfrak{G}$ at p. If $0 <$ dim $\mathfrak{G}_p$, then $p \in C_s$ is a smooth point of C_s.

Condition (3) rules out those tails which map with degree less than or equal to $d - k$. Instead, we allow the appearance of base points by (4). Here, a tail means a component of a rational curve whose removal does not disconnect the curve. For instance, the middle $\mathbb{P}^1$ of a length-3 chain of $\mathbb{P}^1$'s is not a tail.

When $k = d$, we just get the ordinary Kontsevich space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. Furthermore, there are natural morphisms among these spaces, cf. [MM, Prop 1.3]. We cite the result in the following.

Proposition 1.12. *There is a natural morphism $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k+1) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, k)$ that contracts maps with tails of degree $d - k$.*

As for our application, the difference between $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$ is that tails of degree 1 are replaced by simple base points at the attaching places of the tails. We have the following corollary.

Corollary 1.13. *There is a morphism $\theta : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$ that contracts the boundary divisor Δ.*

Given all the above spaces and maps among them, now we can state the main result about the models $\overline{M}(D)$. Since the result depends on which chamber the effective divisor D lies in, we will use the notation $(D_1, D_2), [D_1, D_2], [D_1, D_2)$ and $(D_1, D_2]$ to denote the open, closed, left-closed right-open, and left-open right-closed chambers bounded by two divisors D_1, D_2 respectively. Namely, $D \in (D_1, D_2)$ means that D can be expressed in the form $aD_1 + bD_2$ for $a, b > 0$, and $D \in (D_1, D_2]$ means $a > 0, b \geq 0$, etc.

Theorem 1.14. *Let D be an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$, and let*

$$\overline{M}(D) = Proj\Big(\bigoplus_{m\geq 0} H^0(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3), mD)\Big).$$

Then we have the following models for $\overline{M}(D)$.
(i). $\overline{M}(D) \cong \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ for $D \in (H, T)$;
(ii). $\overline{M}(H) \cong \mathcal{C}^{\nu}$ and $f : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \to \mathcal{C}^{\nu}$ is a small contraction contracting the multi-image locus M;
(iii). $g : \mathcal{H} \to \mathcal{C}^{\nu}$ is the flip of f associated to a divisor D and $\overline{M}(D) \cong \mathcal{H}$ for $D \in (F, H)$. The flipping locus on $\mathcal{H}$ is the multi-scheme locus N;
(iv). $h : \mathcal{H} \to \mathcal{H}(2)$ is a divisorial contraction contracting the locus of degenerate curves D_{deg} and $\overline{M}(D) \cong \mathcal{H}(2)$ for $D \in (D_{deg}, F]$;
(v). $\theta : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$ is a divisorial contraction contracting the boundary Δ and $\overline{M}(D) \cong \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$ for $D \in [T, \Delta)$;
(vi). $\overline{M}(D_{deg})$ and $\overline{M}(\Delta)$ are both the one point space.

The above result can be best seen from the following diagram:

$$\begin{tikzcd}
 & \mathcal{H} \arrow[dl, "h"'] \arrow[dr, "g"] & \text{flip} & \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3) \arrow[ll, dashed, "\phi"'] \arrow[dl, "f"'] \arrow[dr, "\theta"] & \\
\mathcal{H}(2) & & \mathcal{C}^{\nu} & & \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)
\end{tikzcd}$$

This paper is organized as follows. In section 2 we study the stable base locus of an effective divisor. The main theorem is proved in section 3. A few remarks and further questions are discussed in the last section. Throughout the paper, we work over an algebraically closed field of characteristic 0. All the divisors are $\mathbb{Q}$-Cartier divisors and the equivalence relation means numerical equivalence.

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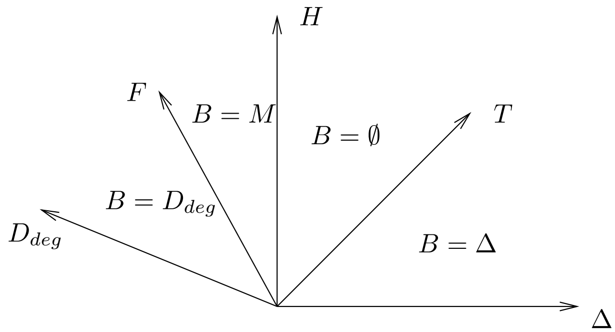
2. Stable Base Loci of Effective Divisors

In order to study the model $\overline{M}(D)$ associated to an effective divisor D on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$, it would be useful to figure out the stable base locus $\mathrm{B}(D)$, in the sense of [L, Def 2.1.20]. In the introduction section, we already saw a chamber decomposition for the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$. Actually $\mathrm{B}(D)$ only depends on which chamber D lies in.

Theorem 2.1. *Let D be an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$.*
(i). D is base point free for $D \in [H, T]$;
(ii). B(D) is the boundary Δ for $D \in (T, \Delta]$;
(iii). B(D) is the multi-image locus M for $D \in [F, H)$;
(iv). B(D) is the locus of the degenerate maps D_{deg} for $D \in [D_{deg}, F)$.

The picture below provides a quick view of the above result.



Proof. (i) is trivial since the nef cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is generated by H and T. H and T are base point free by their definitions. For a map $[C, \mu]$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$, we can take a defining line of H away from Im μ, and take a defining plane of T away from the branch locus of μ, so $[C, \mu]$ is not contained in the corresponding loci of H and T.

For (ii), take the test curve B_1 we used in the proof of Proposition 1.3. $T. B_1 = \frac{2}{3}(H+\Delta). B_1 = 0$ and $\Delta. B_1 = -1$, so $D. B_1 < 0$ for any $D \in (T, \Delta]$. Such B_1's cover an open set of Δ. Hence, $\Delta \subset \mathrm{B}(D)$. After removing the base locus Δ from D, only the part T is left and T is base point free.

For (iii), if $[C, \mu]$ is not a multi-image map, we can always exhibit a line defining H and a flag defining F such that $[C, \mu]$ is neither contained in H nor in F. Hence, $\mathrm{B}(D) \subset M$ for $D \in [F, H)$.

On the other hand, take two general quadrics Q_1, Q_2 on a plane Λ and a point $p \in \Lambda$. For a line L_λ in the pencil of lines passing through p on Λ, define a degree 2 map from L_λ to a line $L = [X, Y, 0, 0] \subset \mathbb{P}^3$ by $L_\lambda \to [Q_1|_{L_\lambda}, Q_2|_{L_\lambda}, 0, 0]$. If L_λ is spanned by p and a base point $b \in Q_1 \cap Q_2$, we need to blow up b and the domain curve breaks to a nodal union of two lines each of which maps isomorphically to L. We attach another line to this pencil at p and map it isomorphically to a line L' in $\mathbb{P}^3$ which passes through $[Q_1(p), Q_2(p), 0, 0]$. Then we obtain a 1-dimensional family $E_{1,2}$ in $M_{1,2}$. $E_{1,2}. H = 0$ and $E_{1,2}. \Delta = 3$, so $E_{1,2}. (aF + bH) < 0$ for any $a > 0$. Moreover, such $E_{1,2}$'s sweep out an open set of $M_{1,2}$. Therefore $M_{1,2}$ must be contained in $\mathrm{B}(D)$ for any $D \in [F, H)$.

Furthermore, take two general cubics C_1, C_2 on Λ. For a line L_λ in the pencil of lines passing through p on Λ, define a degree 3 map from L_λ to $L = [X, Y, 0, 0] \subset \mathbb{P}^3$ by $L_\lambda \to [C_1|_{L_\lambda}, C_2|_{L_\lambda}, 0, 0]$. If L_λ is spanned by p and a base point $b \in C_1 \cap C_2$, we blow up b and the domain curve breaks to a nodal union of two lines. One of them admits a double cover to L and the other one maps isomorphically to L. Now we get a 1-dimensional family $E_3 \subset M_3$. $E_3. H = 0$ and $E_3. \Delta = 8$, so $E_3. (aF + bH) < 0$ for any $a > 0$. Moreover, such E_3's cover an open set of M_3. Therefore M_3 must be contained in $\mathrm{B}(D)$ for $D \in [F, H)$.

For (iv), take a pencil R of plane rational nodal cubics passing through a fixed node p. Obviously we have $R.\,H = 1$. Besides p, there are five other base points $b_1, \ldots, b_5$ of this pencil. If C is a degree 3 reducible plane curve singular at p which also passes through $b_1, \ldots, b_5$, then C must be the union of a conic and a line. The conic goes through p and four other base points while the line goes through p and the remaining base point. Hence, there are five reducible curves in this pencil, which implies that $R.\,\Delta = 5$. Another way to calculate the intersection is by blowing up p and $b_1, \ldots, b_5$ from the plane to get a surface S as a rational fibration over $\mathbb{P}^1$. The Euler characteristic $\chi(S) = \chi(\mathbb{P}^2) + 6 = 9$, but $\chi(S)$ also equals $\chi(\mathbb{P}^1) \cdot \chi(\mathbb{P}^1)$ plus the number of reducible fibers. Hence, we get five reducible curves in R again. Therefore, $R.\,(aD_{deg} + bF) = \frac{1}{3}(2a + 5b)(R.\,H) - \frac{1}{3}(a + b)(R.\,\Delta) = -a < 0$ for any $a > 0$, which implies $D_{deg} \subset \mathrm{B}(D)$ for $D \in [D_{deg}, F)$. Now remove D_{deg} from D and only the component F is left. We already verified in (iii) that $\mathrm{B}(F) = M$ and $M \subset D_{deg}$. Hence, $\mathrm{B}(D) = D_{deg}$ for $D \in [D_{deg}, F)$. $\square$

The chamber decomposition for the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ also holds for the effective cone of $\mathcal{H}$. For an effective divisor D on $\mathcal{H}$, we have a similar result about the stable base locus of D.

Theorem 2.2. *Let D be an effective divisor on $\mathcal{H}$.*
(i). The nef cone of $\mathcal{H}$ is $[F, H]$, and D is base point free for $D \in [F, H]$;
(ii). B(D) is the boundary Δ for $D \in (T, \Delta]$;
(iii). B(D) is the multi-scheme locus N for $D \in (H, T]$;
(iv). B(D) is the locus of the degenerate curves D_{deg} for $D \in [D_{deg}, F)$.

The above result can be verified easily by using the same method in the proof of Theorem 2.1. We leave it to the reader.

3. Intermediate Models for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$

This section is devoted entirely to prove Theorem 1.14. Recall that we want to describe the model

$$\overline{M}(D) = \mathrm{Proj}\Big(\bigoplus_{m\geq 0} H^0(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3), mD)\Big)$$

associated to an effective divisor D on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$. $\overline{M}(D)$ relies on which chamber D lies in. We will frequently use the following fact.

Lemma 3.1. *Let $f : X \to Y$ be a birational morphism between two normal projective varieties. Let D be an ample divisor on Y. Then Proj $\Big(\bigoplus_{m\geq 0} H^0(X, mf^*D)\Big) = Y$.*

Proof. Quite generally, $H^0(X, mf^*D) = H^0(Y, f_*(mf^*D))$. We also have $f_*\mathcal{O}_X = \mathcal{O}_Y$, e.g., cf. [Ha, III, Cor 11.4]. So by the projection formula,

$$H^0(Y, f_*(mf^*D)) = H^0(Y, f_*\mathcal{O}_X \otimes (mD)) = H^0(Y, mD).$$

Hence, it follows that

$$\mathrm{Proj}\Big(\bigoplus_{m\geq 0} H^0(X, mf^*D)\Big) = \mathrm{Proj}\Big(\bigoplus_{m\geq 0} H^0(Y, mD)\Big) = Y.$$

$\square$

Now we will prove Theorem 1.14 step by step.

Proof. (i) is trivial since the ample cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$ is the interior of the nef cone generated by H and T.

For (ii), the divisor H is nef and base point free. Obviously it contracts the multi-image locus M. Note that the defining ample line bundle on $\mathcal{C}$ exactly pulls back to H on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$ via the normalization $\mathcal{C}^\nu$. By Lemma 3.1, the resulting space associated to H is $\mathcal{C}^\nu$. Moreover, the two components $M_{1,2}$ and M_3 have dimension 9 and 8 respectively. Hence, f is a small contraction.

(iii) is the central part of the whole proof. The main point is to check that D is g-ample and $-D$ is f-ample for $D \in (F,H)$. The fact that the Picard numbers of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$ and $\mathcal{H}$ are both 2 actually provides a formal way to verify the relative ampleness. We know that (F,H) is the ample cone of $\mathcal{H}$, by Theorem 2.2. So D is not only g-ample, but also ample on $\mathcal{H}$. Similarly, since T is ample on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3,3)$ and H contracts $\mathrm{Exc}(f)$, $-D$ is f-ample for $D \in (F,H)$. Nevertheless, in the following we will still provide an explicit analysis for the restriction of D on the exceptional fibers of f and g. The analysis is interesting for its own sake.

Firstly, let us verify that $-D$ is f-ample. Pick a cycle class $[C]$ in the image of $\mathrm{Exc}(f)$ in $\mathcal{C}^\nu$. $[C]$ can be of type either $[L_1]+2[L_2]$ or $3[L_3]$, where L_1, L_2, L_3 are lines in $\mathbb{P}^3$ and L_1, L_2 meet at one point p.

If $[C]=[L_1]+2[L_2]$, then the fiber $f^{-1}([C])$ is contained in $M_{1,2}$ and is isomorphic to the locus of the evaluation divisor $\mathcal{L}=ev^*(p)$ inside $\overline{\mathcal{M}}_{0,1}(L_2,2)$. The marked point maps to p for the purpose of attaching a copy of L_1. For a general point corresponding to a map $[\mathbb{P}^1,\mu]$ in $\mathcal{L}$, μ branches at two distinct points p_1,p_2 on L_2 away from p. $\mu^{-1}(p)$ consists of two points as the candidates for the marked point. However, the involution of μ switches these two points. So in this case the branch points p_1,p_2 determine uniquely an element in $\mathcal{L}$. When p_1 and p_2 meet away from p, the domain of μ breaks to a nodal union of two $\mathbb{P}^1$'s. Both of them map isomorphically to L_2. The marked point can be contained in either one of the two domain components, but the involution of μ switches them. The most special case is when p_1,p_2 and p all coincide. Then the domain of μ splits to a length-3 chain of $\mathbb{P}^1$'s. The middle $\mathbb{P}^1$ contains the marked point and gets contracted to p. The other two rational tails both map isomorphically to L_2. Therefore, $\mathcal{L}$ is isomorphic to $\mathrm{Sym}^2\mathbb{P}^1 \cong \mathbb{P}^2$. Now we take the family $E_{1,2}$ used in the proof of Theorem 2.1 (iii). $E_{1,2}$ is an effective curve class in $f^{-1}([C])$ since the base point of the pencil maps to a fixed point in $\mathbb{P}^3$. There are 4 reducible curves in this pencil, so $E_{1,2}.\Delta>0$. But $E_{1,2}.H=0$, so $E_{1,2}.D<0$ for any $D\in(F,H)$.

Similarly, if $[C]=3[L_3]$, the fiber $f^{-1}([C])$ is isomorphic to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,3)$. Pic $(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,3))\otimes\mathbb{Q}$ is generated by the boundary divisor $\Delta_{1,2}$. So $\overline{\mathrm{NE}}(f^{-1}([C]))$ is 1-dimensional. We pick the effective curve class E_3 in the proof of Theorem 2.1 (iii). $E_3.H=0$ and $E_3.\Delta=8$, so $E_3.D<0$ for any $D\in(F,H)$.

Overall, we proved that $-D$ is f-ample.

Next, we will show directly that D is g-ample.

If $[C]=[L_1]+2[L_2]$ is in the image of $\mathrm{Exc}(g)$ in $\mathcal{C}^\nu$, $g^{-1}([C])$ consists of four different types of curves up to projective equivalence in $\mathcal{H}$, cf. [H, p. 39-41] and [Lee, 4.4]:

(I). A double line B of genus -1 supported on L_2 meeting L_1 at one point p such that $L_1\subset\mathbb{T}_pB$.

(II). A planar double line B of genus 0 supported on L_2 meeting L_1 at one point p and L_1 not lying on the plane containing B.

(III). A planar double line B of genus 0 supported on L_2 meeting L_1 at one point p with an embedded point $q \neq p$ somewhere on L_2 and L_1 lying on the plane containing B.
(IV). A planar double line B of genus 0 supported on L_2 meeting L_1 at one point p with an embedded point at p and L_1 lying on the plane containing B.
Note that (I) can specialize to either (II) or (III), and all of them can specialize to (IV). Now we will do an explicit calculation to show that $g^{-1}([C]) \cong \mathbb{P}^2$ and D_{deg} restricted to $g^{-1}([C])$ is ample.

Suppose the coordinate of $\mathbb{P}^3$ is given by $[X, Y, Z, W]$. L_1 is defined by $Y = Z = 0$, L_2 is defined by $X = Y = 0$, and $p = L_1 \cap L_2 = [0, 0, 0, 1]$. The double line B in (I) is defined by the ideal $\mathcal{I}_B = ((X, Y)^2, XG - YF)$, where $F = aZ + bW$ and $G = cZ + dW$ are two linear forms on L_2, cf. [N, Prop 1.4]. Since the tangent direction $[F, G, 0, 0]$ at p is contained in the plane $Y = 0$ spanned by L_1, L_2, we get $G(p) = 0$, i.e., $d = 0$ and $G = cZ$. When b, c are both nonzero, it corresponds to (I). If $c = 0$ but $b \neq 0$, $\mathcal{I}_{D\cup L_1} = ((X, Y)^2, Y(aZ + bW)) \cap (Y, Z) = (X, Y, aZ + bW)^2 \cap (Y, X^2) \cap (Y, Z)$, so an embedded point appears at the point $[0, 0, -b, a]$ on L_2, and L_1 is on the plane where the planar double line B lies. Hence, we get case (III). If $c \neq 0$ but $b = 0$, $\mathcal{I}_{B\cup L_1} = ((X, Y)^2, Z(aY - cX)) \cap (Y, Z) = (Y, Z) \cap (aY - cX, Y^2)$, that is, case (II). Finally, if $b = c = 0$, then $\mathcal{I}_{B\cup L_1} = ((X, Y)^2, YZ) \cap (Y, Z) = (Y, Z) \cap (Y, X^2) \cap (X, Y, Z)^2$, which is case (IV). It is clear that $g^{-1}([C])$ is isomorphic to $\mathbb{P}^2$ defined by the parameters $[a, b, c]$ up to scalar. The locus (I) is isomorphic to $\mathbb{P}^2 \backslash (\mathrm{II}) \cup (\mathrm{III}) \cup (\mathrm{IV})$. (II) and (III) are both isomorphic to $\mathbb{A}^1$ and (IV) is their intersection point. We can pick a general line class R in $g^{-1}([C])$. $R.\,H = 0$ and $R.\,D_{deg} = 1$ since (I) and (II) are not contained in D_{deg} but (III) is. Hence, $R.\,D > 0$, so D_{deg} restricted to $g^{-1}([C])$ is ample for $D \in [D_{deg}, H)$.

For the other case, if $[C] = 3[L_3]$, $g^{-1}([C])$ in $\mathcal{H}$ is the locus of triple lines supported on L_3. There are three different types for such triple lines:
(V). A triple line J supported on L_3 lying on a quadric cone.
(VI). A planar triple line J supported on L_3 with an embedded point on it.
(VII). A triple line J whose ideal is given by the square of $\mathcal{I}_{L_3}$.

We will show that $g^{-1}([C])$ is isomorphic to a projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-3))$ contracting a section over $\mathbb{P}^1$, and D_{deg} restricted to $g^{-1}([C])$ is ample.

For J in case (V), suppose that L_3 is defined by $X = Y = 0$ and the multiplicity two structure $\mathcal{I}_2$ associated to J is $(\beta X - \alpha Y, (X, Y)^2)$, where $\beta X - \alpha Y$ is the tangent plane of a quadric cone containing J at L_3. If $\beta \neq 0$, let $t = \alpha/\beta$. Then $\mathcal{I}_2$ can be rewritten as $(X - tY, Y^2)$. As in [N, Prop 2.1], I_J has the form $((X - tY)^2, (X - tY)Y, Y^3, (aW + bZ)(X - tY) - cY^2)$ which is parameterized by $(t; [a, b, c])$. Similarly, if $\alpha \neq 0$, let $s = \beta/\alpha$. Then $I_J = ((sX - Y)^2, (sX - Y)X, X^3, (aW + bZ)(sX - Y) - cX^2)$ which is parameterized by $(s; [a, b, c])$. When we change the coordinates, it is easy to check that $(t; [a, b, c]) \to (1/t; [a, b, c/t^3])$. So after gluing together, we simply get a $\mathbb{P}^2$ bundle $\mathbb{P}V^* \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-3))$ over $\mathbb{P}^1$. The locus S where $c = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}V^*$ which corresponds to case (VI), and S is also the restriction of D_{deg}. However, if $a = b = 0$ but $c \neq 0$, then $\mathcal{I}_J$ is always equal to $(X, Y)^2$ independent of $[\alpha, \beta]$. It implies that $g^{-1}([C])$ is isomorphic to $\mathbb{P}V^*$ contracting the section Γ cut out by $a = b = 0$. Γ is an extremal ray of $\overline{\mathrm{NE}}(\mathbb{P}V^*)$. $S.\,\Gamma = 0$ since S is physically away from Γ. So S is nef on $\mathbb{P}V^*$. After contracting Γ, S becomes ample on $g^{-1}([C])$. Again, $R.\,D > 0$ for every effective curve class $R \in \overline{\mathrm{NE}}(g^{-1}([C]))$, since D is a positive linear combination of H and D_{deg}. Hence, D restricted to $g^{-1}([C])$ is ample for $D \in (F, H)$.

For (iv), since we already flipped $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ to $\mathcal{H}$ for $D \in (F, H)$, to resume Mori's program we need to start from $\mathcal{H}$. When $D = F$, in the proof of Theorem 2.1 (iv) we know that $F.\,R = 0$, where R is a general pencil of plane rational nodal curves on Λ possessing a common node p.

The intersection number $F.\,R = 0$ also holds on $\mathcal{H}$, although a spatial embedded point occurs at p for each curve in R. So after contracting the class R, the resulting space should only remember the information of the point-plane flag ($p \in \Lambda$). Actually this is the space of nets of quadrics $\mathcal{H}(2)$ we defined in the introduction section, cf. Proposition 1.10. Recall that there is a natural morphism h from the open set of $\mathcal{H}$ corresponding to twisted cubics to the Grassmannian $\mathbb{G}(2,9)$ by $h([C]) = [H^0(\mathcal{I}_C(2))] \in \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^3}(2))$. $\mathcal{H}(2)$ is just the closure of the image of h in $\mathbb{G}(2,9)$. This map can extend to $\mathcal{H}$, since for any $[C] \in \mathcal{H}$, $h^0(\mathcal{I}_C(2)) = 3$. Moreover, $[C], [C'] \in D_{deg}$ share the same flag ($p \in \Lambda$) if and only if $H^0(\mathcal{I}_C(2)) = H^0(\mathcal{I}_{C'}(2))$, cf. [PS, Lemma 2]. Hence, $h^*(\mathcal{O}_{\mathcal{H}(2)}(1)).\,R = 0$. $h^*(\mathcal{O}_{\mathcal{H}(2)}(1))$ must be proportional to F, since $F.\,R = 0$ and the Picard number of $\mathcal{H}$ is 2. By Lemma 3.1, we get $\overline{M}(F) = \mathcal{H}(2)$. This also implies that F is nef on $\mathcal{H}$.

Finally for $D \in (D_{deg}, F)$, D_{deg} is always the stable base locus of D. After we remove it, D becomes proportional to F.

For (v), when $D = T$, we know that T is nef on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$. If an extremal ray R satisfies the relation $R.\,T = 0$, R must be contained in the boundary Δ. Moreover, if an image point of a map in R is the image of a node, then it cannot move. Otherwise it must meet the plane Π defining T and $R.\,T$ would not be zero. Similarly, the image of a domain component which maps with degree greater than 1 cannot move. Otherwise the locus of branch points would meet Π somewhere. Hence, only the images of components of map degree 1 can vary. On the other hand, we can choose a general plane defining T so that this plane is away from the images of the finitely many attaching points of those components which map with degree 1, and also away from the branch points on the fixed images of components which map with degree greater than 1. Then clearly we have $T.\,R = 0$. Therefore, this divisorial contraction associated to T contracts all degree 1 tails but only remembers their attaching points. As we mentioned in the introduction section, the resulting space has already appeared in [MM] and [Pa], the space of 2-stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$, cf. Definition 1.11. Actually we can define an ample tangency divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$ which pulls back to T on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$. Again by Lemma 3.1, we get $\overline{M}(T) = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$.

Now for $D \in (T, \Delta)$, since Δ is the stable base locus of D, after removing it, D becomes proportional to T.

(vi) is trivial since the stable base loci of Δ and D_{deg} are themselves. $\square$

4. Remarks and Further Questions

In this section, we will post several remarks and open questions for a further study of the Kontsevich moduli space.

Remark 4.1. A more general contraction contracting the total boundary of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ has been considered in [CHS1] without mentioning the modular interpretation of the resulting space, cf. [CHS1, Thm 1.9].

Remark 4.2. Take a plane Λ and three general points p_1, p_2, p_3 on it. Define a divisor G on $\mathcal{H}$ as the closure of twisted cubics whose intersection points with Λ are contained in a conic that passes through p_1, p_2 and p_3. Let us determine the class of G. Using the test families B_1 and B_2 in the proof of Proposition 1.3, we get the following intersection numbers:

$$H.\,B_1 = 1,\ \Delta.\,B_1 = -1,\ G.\,B_1 = 2,$$

$$H.\,B_2 = 1,\ \Delta.\,B_2 = 2,\ G.\,B_2 = 1,$$

which tells that $G = \frac{5}{3}(H - \frac{1}{5}\Delta) = F$. We saw that F is proportional to $h^*(\mathcal{O}_{\mathcal{H}(2)}(1))$ in the proof of Theorem 1.14 (iv). Actually it is easy to check $G = h^*(\mathcal{O}_{\mathcal{H}(2)}(1))$ directly. $\mathcal{O}(1)$ of the Grassmannian $\mathbb{G}(2,9)$ is defined by the Schubert cycle σ_1. When it pulls back to $\mathcal{H}$, the locus corresponds to twisted cubics whose nets of quadrics always meet a fixed $\mathbb{P}^6$ of quadrics. We can take this $\mathbb{P}^6$ naturally by imposing p_1, p_2 and p_3 on quadrics. So after restricted to Λ, those nets of quadrics become nets of conics. At least one conic in each net comes from the fixed $\mathbb{P}^6$ of quadrics containing p_1, p_2, p_3. This is just the divisor G by its definition.

Remark 4.3. Define another divisor Cu on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ as the closure of twisted cubics possessing a tangent line that passes through a fixed point p in $\mathbb{P}^3$. Namely, if we project such a twisted cubic from p to a plane, the image should be in general a cuspidal plane cubic. One can check that Cu has divisor class equivalent to $2H$.

Inspired by Remark 4.2, 4.3 and the fact that the Cox ring of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is finitely generated, cf. Proposition 1.4, we ask the following question.

Question 4.4. *What is the ring structure of the Cox ring for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$?*

The results obtained in this paper also merit further generalization to other Kontsevich moduli spaces $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

Question 4.5. *How do we decompose the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and describe the intermediate models?*

In a joint work [CC] with Izzet Coskun, we will present a more comprehensive study for these questions.

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