Compactification of Strata of Abelian Differentials I

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Joint work with

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Question: How can one compactify $\Omega\mathcal{M}_g(\mu)$?

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- If μ is a meromorphic signature, the Hodge bundle compactification can be defined via twisting by the negative part of μ.

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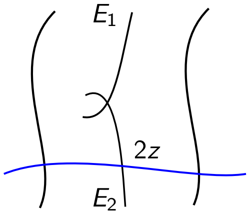

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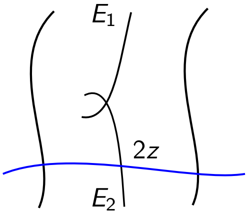

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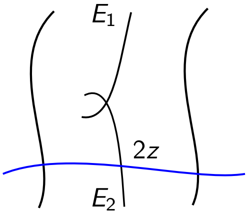

The stable differential is vanishing entirely on $E_2 \Longrightarrow$ No information for the limiting position of z.

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- Taking the closure of $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ in $\overline{\mathcal{M}}_{g,n}$ is called the *Deligne-Mumford compactification*.
- This closure is the *main component* of the moduli space of twisted canonical divisors ([Farkas-Pandharipande, 2015]).

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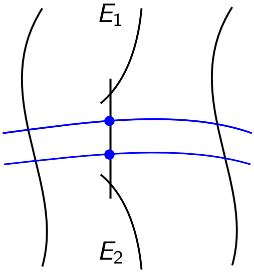

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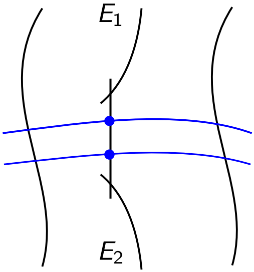

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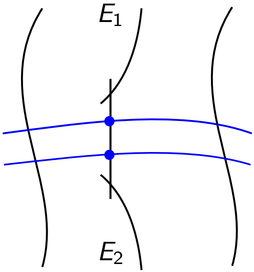

The twisted canonical divisor is trivial on E_1 and $E_2 \Longrightarrow$ As flat tori, E_1 and E_2 can have different sizes.

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- Forgetting the differentials, $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu) \to \overline{\mathcal{M}}_{g,n}$ determines the Deligne-Mumford compactification.
- The incidence variety compactification was first studied in ([Gendron, 2015]).

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- A *twisted differential of type* μ on $(X, z_1, \ldots, z_n)$ is a collection of (possibly meromorphic) differentials η_v on the irreducible components X_v of X such that the following conditions hold:

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(2) (Matching residues) If q is a simple polar node joining X_1 and X_2, then $\operatorname{Res}_q \mu_1 + \operatorname{Res}_q \mu_2 = 0$.

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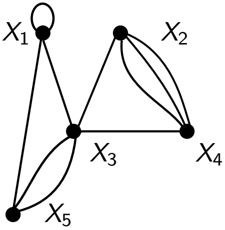

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Remark

The partial order condition only applies to two components that intersect.

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- Recall that the incidence variety compactification $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{\mathrm{inc}}(\mu) \subset \mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$ parameterizes pointed stable differentials $(X, \omega, z_1, \ldots, z_n)$.

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- Suppose η is a twisted differential compatible with a level graph $\overline{\Gamma}$ on $(X, z_1, \ldots, z_n) \in \overline{\mathcal{M}}_{g,n}$.

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- Define a stable differential ω on X

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- *A stable differential only keeps the information on the top level components.*
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- *A stable differential (X, ω) lies in the Hodge bundle strata compactification iff there exists a pointed stable differential in $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu)$ that maps to (X, ω) by forgetting the markings.*

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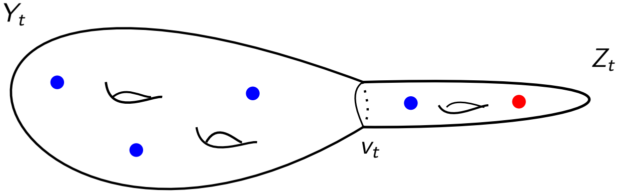

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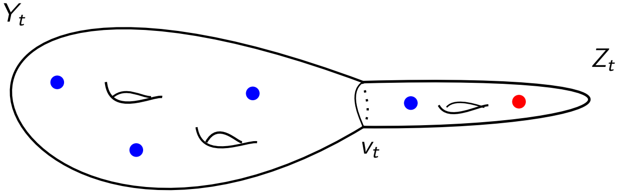

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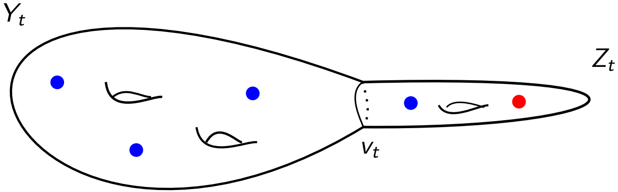

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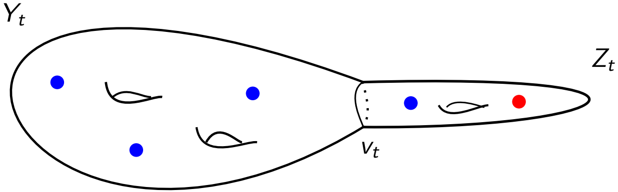

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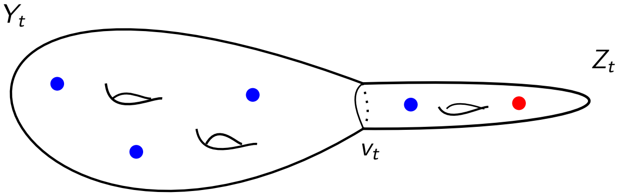

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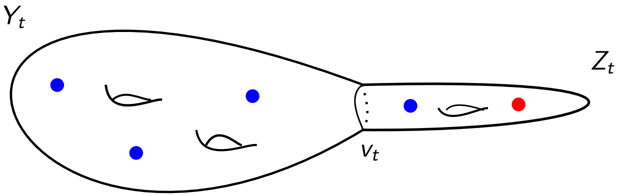

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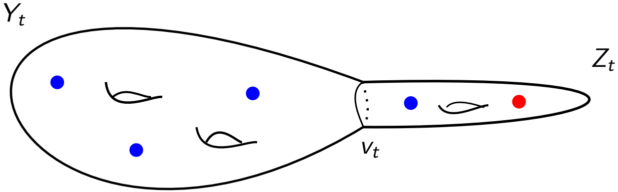

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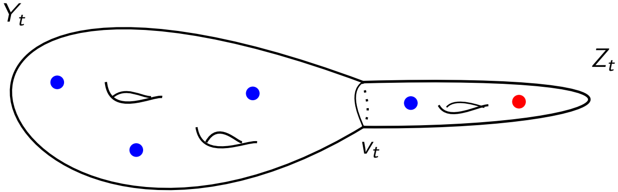

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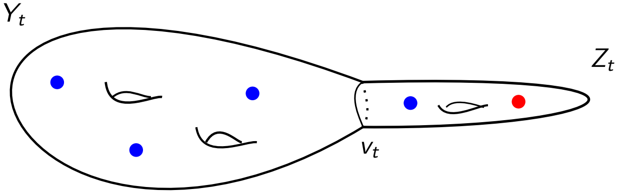

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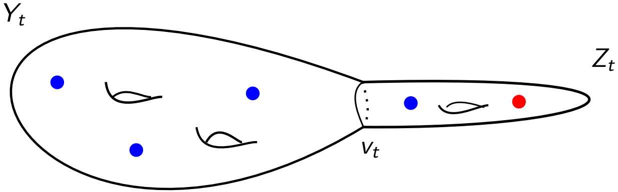

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Corollary

Under the above setting, a pointed stable differential $(X, \omega, z_1, \ldots, z_n)$ is contained in $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{\mathrm{inc}}(\mu)$ iff it is associated to a twisted differential η of type μ such that η has no residue at every node of X.

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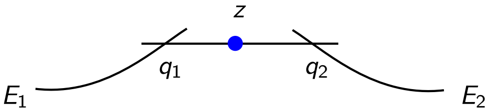

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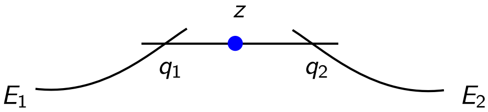

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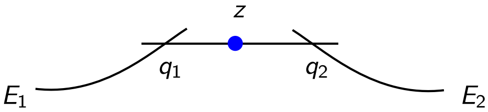

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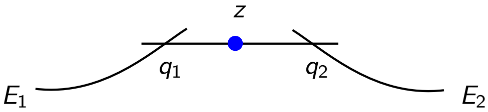

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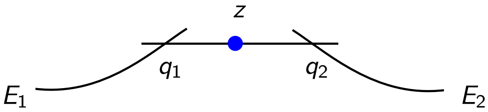

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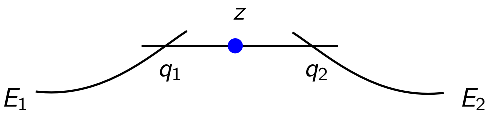

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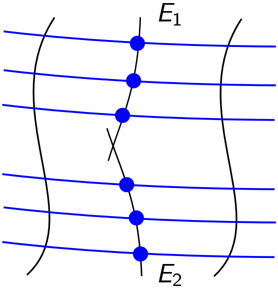

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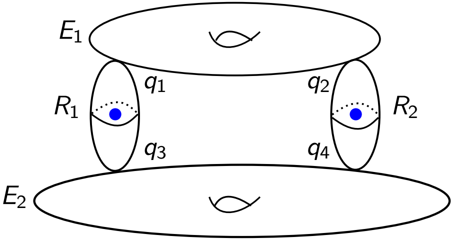

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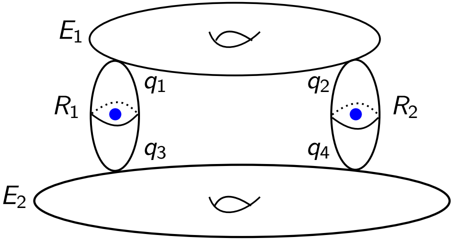

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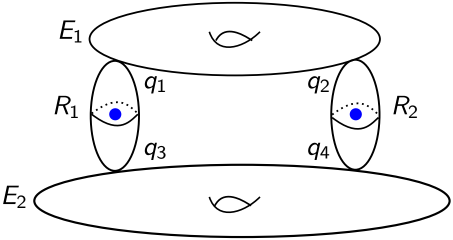

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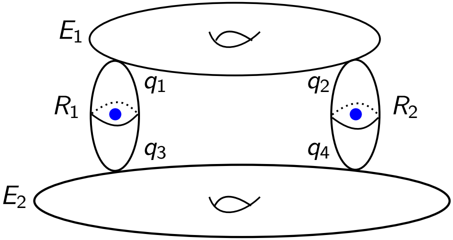

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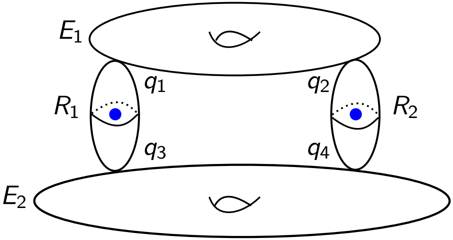

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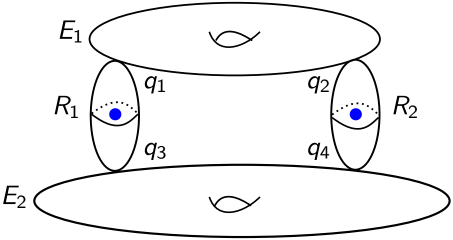

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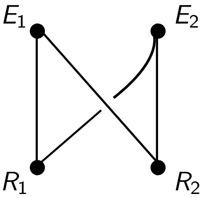

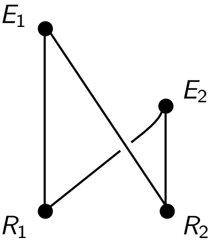

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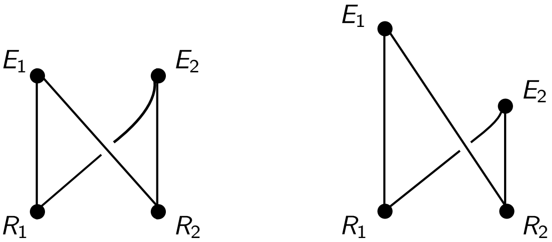

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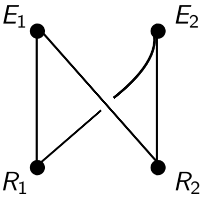

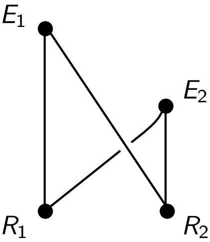

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- *Differentials in* $\Omega\mathcal{M}_3(2,2)$ *degenerate to* R_1 *and* R_2 *in the same speed.*
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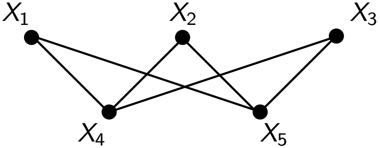

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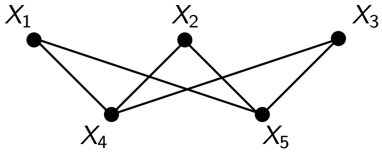

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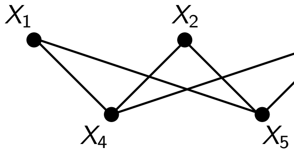

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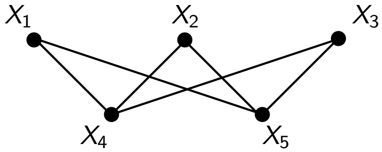

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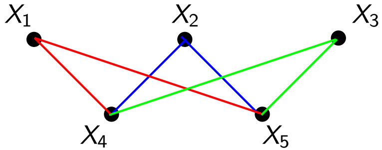

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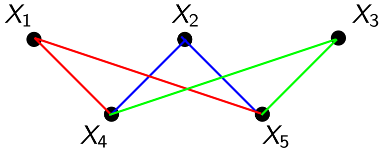

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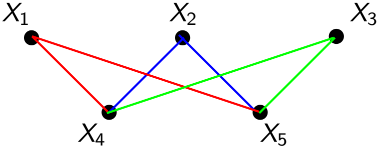

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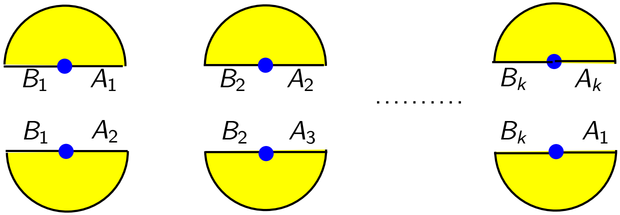

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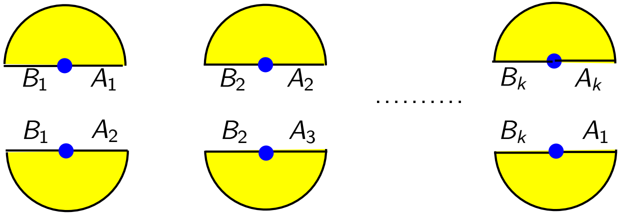

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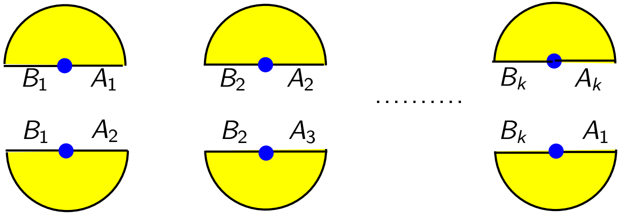

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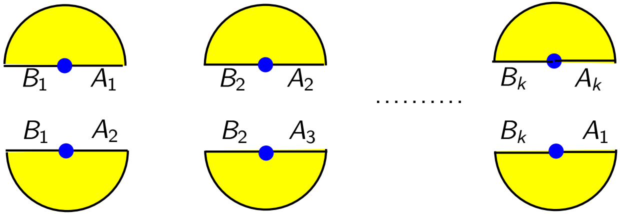

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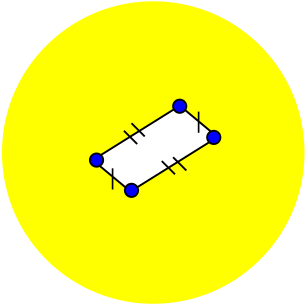

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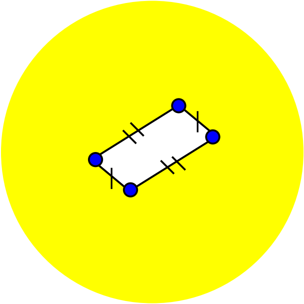

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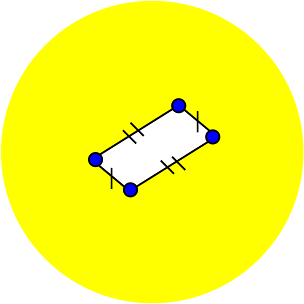

- [Boissy, 2012] described meromorphic differentials using broken half-planes and classified connected components of strata of meromorphic differentials.

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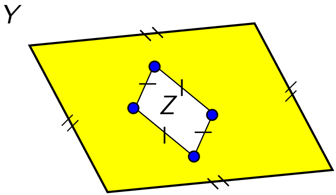

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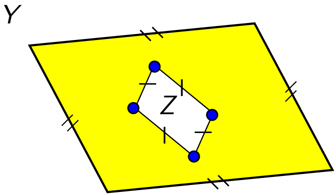

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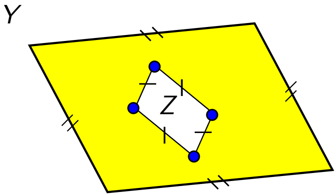

- Shrinking Z to a point gives a flat surface $(Y, \eta_Y) \in \Omega\mathcal{M}_1(0)$.
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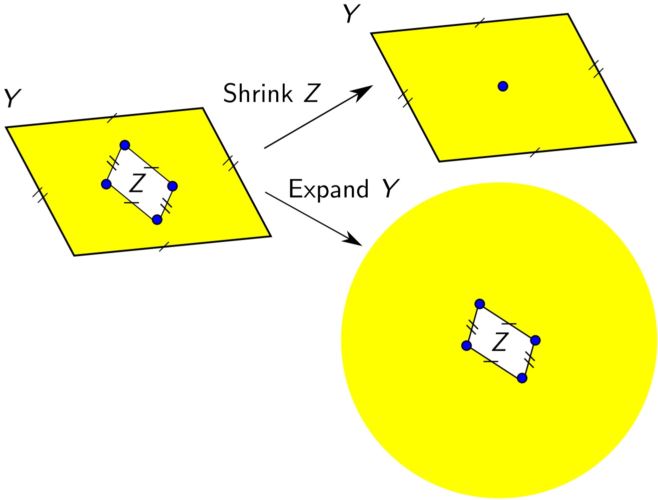

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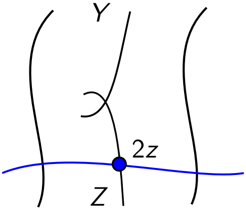

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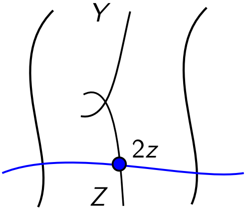

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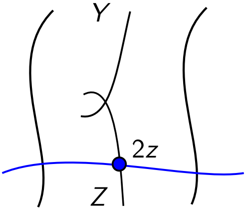

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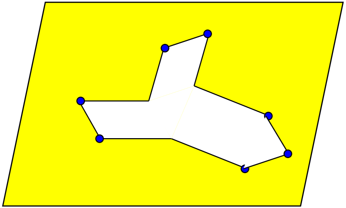

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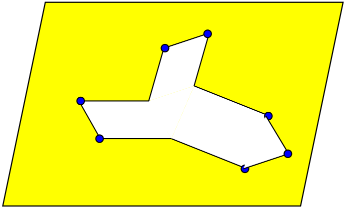

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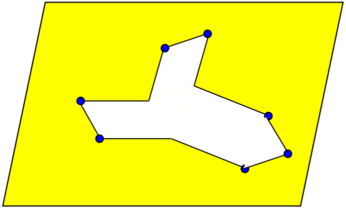

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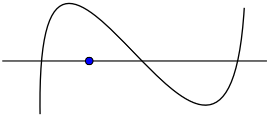

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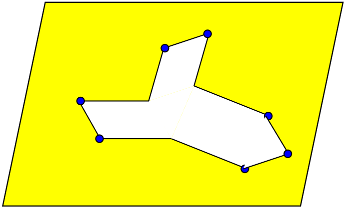

Thank you!