# MT822: INTRODUCTION TO ALGEBRAIC GEOMETRY

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## 1. Algebraic varieties

The main objects in (classical) algebraic geometry are called *algebraic varieties* or simply *varieties*. As a set, a variety consists of the common zero locus of a group of polynomial equations in an affine or projective space.

### 1.1. **Affine varieties.**

Let $K$ be a field (e.g. $\mathbb{C}$). Consider the $n$-dimensional vector space $\mathbb{A}^n$ over $K$. Every point of $\mathbb{A}^n$ has affine coordinate $(z_1, \ldots, z_n)$, where $z_i \in K$. For $K = \mathbb{C}$, locally we can use standard complex coordinate charts and identify the affine space $\mathbb{A}^n$ as a smooth $n$-dimensional complex manifold. For any $m < n$, an $m$-dimensional *linear subspace* is a subvariety of $\mathbb{A}^n$ (with induced algebraic structure), cut out by $n - m$ linearly independent forms, say, $z_i = 0$ for $i = 1, \ldots, n - m$. In general, an *affine subvariety* in $\mathbb{A}^n$ is the common zero locus of a collection of polynomials $f_1, \ldots, f_l \in K[z_1, \ldots, z_n]$.

For beginners, it does not hurt to think of (smooth) varieties as (smooth, complex) manifolds. Then a (compact) projective variety can be glued using affine coordinate charts, where the transition functions are holomorphic.

### 1.2. **Projective varieties.**

The $n$-dimensional projective space

$$\mathbb{P}^n \cong (\mathbb{A}^{n+1} \backslash \{0\})/K^*,$$

i.e. $(Z_0, \ldots, Z_n) \sim (Z'_0, \ldots, Z'_n)$ if and only if there exists $\lambda \neq 0$ such that $\lambda Z_i = Z'_i$ for all $i$. Use the *homogeneous coordinates* $[Z_0, \ldots, Z_n]$ to denote the corresponding equivalence class. A *homogeneous polynomial* of degree $m$ on $\mathbb{P}^n$ is

$$F(Z_0, \ldots, Z_n) = \sum_{i_1 + \cdots + i_n = m} a_{i_1 \cdots i_n} Z_0^{i_0} \cdots Z_n^{i_n}.$$

Note that

$$F(\lambda Z_0, \ldots, \lambda Z_n) = \lambda^m F(Z_0, \ldots, Z_n),$$

hence $F$ is *not* a well-defined function on $\mathbb{P}^n$. Nevertheless, its zero locus is well-defined and called a *hypersurface* of degree $m$. In general, a *projective subvariety* in $\mathbb{P}^n$ is the common zero locus of a collection of homogenous polynomials, or equivalently, the intersection of a collection of hypersurfaces. The degree of the defining homogeneous polynomial is called the *degree* of the hypersurface.

**Example 1.1** ($\mathbb{P}^1$ and the Riemann Sphere)**.** For $n = 1$, we have $\mathbb{P}^1 = [Z_0, Z_1]$. Let $U_i$ be the open subset defined by $Z_i \neq 0$ for $i = 0, 1$. Let $x = Z_1/Z_0$ and $y = Z_0/Z_1$. Then $x$ and $y$ are global (affine) coordinates for $U_0$ and $U_1$, respectively, and $U_i \cong \mathbb{A}^1$. For $x \neq 0$ and $y \neq 0$, one can glue the one-punctured $U_i$'s together via the transition function $y = 1/x$ to realize $\mathbb{P}^1$ as the Riemann sphere.

**Exercise 1.2.** For $\mathbb{P}^n = [Z_0, \ldots, Z_n]$, let $U_i$ be the open subset defined by $Z_i \neq 0$ for $i = 0, \ldots, n$. Show that each $U_i$ is isomorphic to $\mathbb{A}^n$. How do you glue $U_i$'s together to form $\mathbb{P}^n$ as a $n$-dimensional manifold?

**Example 1.3** (Conics in $\mathbb{P}^2$)**.** The zero locus of $Z_0^2 - Z_1 Z_2$ defines a degree 2 hypersurface $C$, called a (smooth) plane conic, in the projective plane $\mathbb{P}^2 = [Z_0, Z_1, Z_2]$.

**Exercise 1.4.** Over the complex number field, can you prove that $C$ is isomorphic to the Riemann sphere (as a complex manifold)?

**Exercise 1.5.** What is the (projective) dimension of space of degree 2 plane curves? In general, what is the dimension of space of degree $k$ hypersurfaces in $\mathbb{P}^n$?

**Exercise 1.6.** Prove that a general chosen line in $\mathbb{P}^n_{\mathbb{C}}$ intersects a degree $k$ hypersurface at $k$ points (counted with multiplicity). This provides another definition for the degree of a hypersurface.

1.3. **Zariski topology.** Let $X \subset \mathbb{A}^n$ be an affine variety. Define a basis of open sets by $U_f = \{p \in X : f(p) \neq 0\}$ for polynomials $f$ on $\mathbb{A}^n$. Namely, closed sets are defined by subvarieties of $X$. If $X \subset \mathbb{P}^n$ is a projective variety, define $U_F = \{p \in X : F(p) \neq 0\}$ for homogeneous polynomials $F$ on $\mathbb{P}^n$. Note that the Zariski topology is not Hausdorff.

**Exercise 1.7.** Show that any two open sets of $\mathbb{P}^n$ under the Zariski topology intersect.

It is convenient to formulate many results in algebraic geometry by the language of Zariski topology. But when we visualize an algebraic variety, perhaps we still see its geometry under the classical analytic topology.

1.4. **Algebraic geometry and analytic geometry.** In $\mathbb{P}^n_{\mathbb{C}}$, one can identify algebraic subvarieties with complex analytic subvarieties (cut out by globally holomorphic functions). Note that polynomials are holomorphic, hence an algebraic subvariety in $\mathbb{P}^n$ is obviously an analytic subvariety. But some holomorphic (transcendental) functions are not polynomials.

**Theorem 1.8** (Chow's Theorem)**.** *If $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a (closed) complex analytic subvariety, then $X$ is an algebraic subvariety.*

1.5. **Singular varieties.** In general, algebraic geometers like to study (closed) varieties, and sometimes a variety could be *singular*, *reducible* or even *non-reduced* (as a scheme, explained later). We take these "pathologies" into account, because they naturally arise even if we only care about smooth things. Moreover, this setting provides powerful tools, both conceptually and practically, which are not seen in the category of open, smooth manifolds.

For example, two non-parallel affine lines $L_1$ and $L_2$ in $\mathbb{A}^2$ intersects at a unique point. Turn $L_1$ gradually to make it parallel to $L_2$. Then they are disjoint. It implies that the *intersection number* does not behave well (under deformation, say, moving the lines) for an open space. Nevertheless, for any two projective lines in $\mathbb{P}^2$, the intersection number is always one. This can be made sense even for two identical lines.

Now let us consider a family of plane conics $C_t$ given by $Z_0^2 - tZ_1Z_2 = 0$, where $t$ is a base parameter. For $t \neq 0$, $C_t$ is smooth (prove it) and meets a general line $L$ at two points. For $t = 0$, $C_0$ becomes a non-reduced *double line* supported on $Z_0 = 0$. The intersection number remains to be 2 only if we keep the double structure of $C_0$ instead of the reduced line $Z_0 = 0$.

**Remark 1.9.** The above example implies that it is often better to characterize a variety by its *defining equations* than its zero locus.

**Exercise 1.10** (Classification of plane conics)**.** Consider the space of degree 2 plane curves $C$, i.e. those cut out by degree 2 homogeneous polynomials in $\mathbb{P}^2 = [Z_0, Z_1, Z_2]$. Show that up to projective equivalence, i.e. automorphisms of $\mathbb{P}^2$ by $\mathrm{PGL}(3)$, $C$ is either a smooth conic, or a pair of lines meeting at one point, or a double line.

1.6. **Ideals.** Let $X \subset \mathbb{A}^n$ be a variety. Define the *ideal* of $X$ by

$$I(X) = \{f \in K[z_1, \ldots, z_n] : f \equiv 0 \text{ on } X\},$$

i.e. functions that are vanishing on $X$. Conversely, for an ideal $I$ in $K[z_1, \ldots, z_n]$, define an affine variety $V(I)$ by the common zero locus of polynomials in $I$.

**Example 1.11** (Hypersurfaces)**.** By definition, a hypersurface $X$ defined by a polynomial $f$ has ideal $I(X) = (f)$.

Affine varieties are bijective to *radical ideals*. Recall that $I$ is called radical, if for a power $f^k \in I$ we have $f \in I$.

**Theorem 1.12** (Hilbert Nullstellensatz)**.** *For any ideal $I \subset K[z_1, \ldots, z_n]$, we have*

$$I(V(I)) = \mathfrak{r}(I).$$

*Proof.* Denote by $X$ the variety $V(I)$. Take a element $f \in \mathfrak{r}(I)$. Then some power $f^k \in I$, hence $f^k$ is vanishing on $X$, which implies that $f$ is vanishing on $X$. Therefore we conclude that $f \in V(X)$, hence the inclusion $\mathfrak{r}(I) \subset I(X)$. *Complete or read the proof for the other direction.* □

If $X \subset \mathbb{P}^n$ is a projective variety, define $I(X)$ by homogeneous polynomials that are vanishing on $X$. Conversely let $I$ be a ideal (generated by homogeneous polynomials) in $K[Z_0, \ldots, Z_n]$. Define its *saturation* $\bar{I}$ by

$$\bar{I} = \{F \in K[Z_0, \ldots, Z_n] : (Z_0, \ldots, Z_n)^k \cdot F \subset I \text{ for some } k\}.$$

Saturated ideals play the role of radical ideals that uniquely characterizes subvarieties in $\mathbb{P}^n$.

**Example 1.13.** Consider the ideal $I = (X^2, XY, XZ) \subset K[X, Y, Z]$. In $\mathbb{P}^2$ its zero locus is the line $L$ defined by $X = 0$, which is the same as the zero locus of its saturated ideal $\bar{I} = (X)$.

**Exercise 1.14** (Twisted cubics)**.** Consider the map $f : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ by

$$[X, Y] \mapsto [X^3, X^2Y, XY^2, Y^3].$$

The image curve $R$ is called a *twisted cubic*.

(i) Show that $f$ is an embedding.

(ii) Show that for a generally chosen plane $H$ in $\mathbb{P}^3$, the intersection locus $H \cap R$ consists of three points. (The intersection number defines $R$ as a degree three curve, so it is called a twisted cubic).

(iii) Describe the homogeneous ideal $I(R)$.

1.7. **Regular functions and maps.** Let $X \subset \mathbb{A}^n$ be a variety. Define the *coordinate ring* of $X$ to be

$$A(X) := K[z_1, \dots, z_n]/I(X).$$

Essentially we want to define a regular function as the restriction to $X$ of a polynomial in $K[z_1, \dots, z_n]$ modulo those vanishing on $X$, i.e. an element in $A(X)$. Nevertheless, it is more convenient to first give a local definition.

We say that a function is *regular* at $p \in X$ if in some neighborhood of $p$ it can be expressed as a quotient $f/g$, where $f, g \in K[z_1, \dots, z_n]$ and $g(p) \neq 0$. We say that the funtion is *regular on* $X$ if it is regular at every point of $X$. If $X$ is a projective variety, we can adapt the same definition with the additional assumption that in the quotient $F/G$, $F$ and $G$ are homogeneous polynomials of the same degree.

A *regular map* (also called a *morphism*) $X \to \mathbb{A}^n$ is given by an $n$-tuple of regular functions:

$$x \mapsto (f_1(x), \dots, f_n(x)).$$

A regular map $X \to Y \subset \mathbb{A}^n$ is a map to $\mathbb{A}^n$ with image contained in $Y$. A map $\phi : X \to \mathbb{P}^n$ is regular if it is regular locally, i.e. for the standard charts $U_i \cong \mathbb{A}^n$, $\phi^{-1}(U_i) \to U_i$ is regular, where $U_i = \{[Z_0, \dots, Z_n] : Z_i \neq 0\}$ for $0 \leq i \leq n$.

**Example 1.15.** Consider the smooth conic curve $C \subset \mathbb{P}^2$ given by $X^2 + Y^2 - Z^2$. Define $\phi : C \to \mathbb{P}^1$ by

$$[X, Y, Z] \mapsto [X, Z - Y].$$

I claim that $\phi$ is a regular map.

In the affine chart $U_X = \{X \neq 0\}$, $C$ is defined by the affine equation $z^2 - y^2 = 1$, where $y = Y/X$ and $z = Z/X$. The map $\phi$ is given by

$$(y, z) \mapsto \frac{Z - Y}{X} = z - y,$$

which is regular. In the affine chart $Z - Y \neq 0$ in $\mathbb{P}^1$, make a linear transformation by $Y' = Y + Z$ and $Z' = Z - Y$. Then $C$ is given by $X^2 + Y'Z' = 0$ and $\phi : [X, Y', Z'] \mapsto [X, Z']$. In the affine chart $\{Z' \neq 0\}$, $C$ has equation $x'^2 + y' = 0$ where $x' = X/Z'$ and $y' = Y'/Z'$. The map $\phi$ given by $(x', z') \mapsto X/Z' = x'$ is also regular.

In the affine plane $Z \neq 0$ (over $\mathbb{R}$), the map is given by $(x, y) \mapsto (1 - y)/x$, which corresponds to projecting points on the unit disk from the north pole $(0, 1)$ to the line $y = -1$. The north pole just maps to the "infinity" when we compactify the affine line to $\mathbb{P}^1$.

**Remark 1.16.** Note that at the point $[0, 1, 1]$ the map looks like not well-defined, because $X = Z - Y = 0$. This suggests that writing down an $(n + 1)$-tuple of homogeneous polynomials of the same degree, we may not immediately see whether or not it defines a regular map.

What is the relationship between the locally defined regular functions and elements in the coordinate ring?

**Theorem 1.17.** *The ring of regular functions on $X$ is the coordinate ring $A(X)$. More generally, let $U_f = \{p \in X : f(p) \neq 0\}$. Then the ring of regular functions on $U_f$ is the localization $A(X)[1/f]$.*

*Proof.* Note that $K[z_1, \ldots, z_n]$ is a Noetherian ring, i.e. any ideal is finitely generated. In other words, it satisfies the Descending Chain Condition, i.e. for a chain of closed subsets $Y_1 \supset Y_2 \supset \cdots$ under Zariski topology, for some $m$ we have $Y_m = Y_{m+1} = \cdots$. For any regular function $g$ on $U_f$, we can find a *finite* open cover $\{U_\alpha\}$ of $U_f$ such that in each $U_\alpha$ we can write $g = h_\alpha / k_\alpha$, where $k_\alpha(p) \neq 0$ for any $p \in U_\alpha$. Since $k_\alpha$ is nowhere zero in $U_\alpha$ and $\{U_\alpha\}$ forms a cover, the common zero locus of the $k_\alpha$ must be contained in the zero locus of $f$. By Hilbert Nullstellensatz, we have $f^m \in (\ldots, k_\alpha, \ldots)$ for some $m$, i.e.

$$f^m = \sum_\alpha l_\alpha k_\alpha.$$

Consequently we have

$$f^m \cdot g = \sum_\alpha l_\alpha h_\alpha,$$

i.e. $g = (\sum_\alpha l_\alpha h_\alpha)/f^m \in A(X)[1/f]$.

Now take $f$ to be a constant function. We have $U_f = X$ and $A(X)[1/f] = A(X)$. □

**Corollary 1.18.** *Any regular function on $\mathbb{A}^n$ must be a polynomial. Any regular function on $\mathbb{P}^n$ must be a constant.*

We say two varieties $X$ and $Y$ are *isomorphic* if there exist regular maps $f : X \to Y$ and $g : Y \to X$ inverse to each other. For any two projective subvarieties $X, Y \subset \mathbb{P}^n$, we say that they are *projectively equivalent* if there exists an automorphism in $\mathrm{PGL}_{n+1}(K)$ of $\mathbb{P}^n$ carrying $X$ to $Y$.

**Example 1.19** (Rational normal curves)**.** We have seen conics and twisted cubics, as embeddings of $\mathbb{P}^1$ into $\mathbb{P}^2$ and $\mathbb{P}^3$. This construction can be carried out in general. Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^n$ given by

$$[X, Y] \mapsto [X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n].$$

In affine coordinates, $f$ is given by

$$t \mapsto (t, t^2, \ldots, t^n).$$

It is easy to see that $f$ is an isomorphism onto its image, which is called a rational normal curve in $\mathbb{P}^n$.

**Exercise 1.20.** Let $p_1, \ldots, p_k$ be $k$ points on a rational normal curve in $\mathbb{P}^n$. Show that for $k \leq n + 1$, the linear span of $p_1, \ldots, p_k$ (i.e. the minimal linear subspace containing the $p_i$) is $(k-1)$-dimensional.

## 2. Sheaves and cohomology

Roughly speaking, a sheaf corresponds to a collection of local data on a topological space and cohomology groups of a sheaf contain information about passing from local to global. First, let us consider a motivating example.

2.1. **The Mittag-Leffler problem.** Let $X$ be a Riemann surface and $p \in X$ a point with local coordinate $z$. A *principal part* at $p$ is the polar part of a Laurent series $\sum_{i=1}^{n} a_i z^{-i}$ for some $n > 0$ in a neighborhood of $p$. Let $\{p_i\}$ be a collection of finitely many distinct points on $X$ and $f_i$ a fixed principal part for each $p_i$.

**Question 2.1.** *Can we find a global meromorphic function $f$ on $X$ such that it is holomorphic on $X\backslash\{p_i\}$ and locally around the $p_i$ we have $f = f_i$?*

A possible solution goes as follows. Let us first introduce some notation. For an open set $U$, let $\mathscr{O}(U)$ denotes the ring of holomorphic functions on $U$ and $\mathscr{M}(U)$ the ring of meromorphic functions.

Consider an open covering $\underline{U} = \{U_\alpha\}$ of $X$ such that each $U_\alpha$ contains at most one $p_i$. Choose $f_\alpha \in \mathscr{O}(U_\alpha)$ solving the problem locally. On $U_{\alpha\beta} = U_\alpha \cap U_\beta$, let $f_{\alpha\beta} = f_\alpha - f_\beta$. Note that

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0$$

on $U_\alpha \cap U_\beta \cap U_\gamma$.

Suppose there exists a global solution $f$. Then we have $g_\alpha = f|_{U_\alpha} - f_\alpha \in \mathscr{O}(U_\alpha)$ and $f_{\alpha\beta} = g_\beta - g_\alpha \in \mathscr{O}(U_{\alpha\beta})$. Conversely, if we have a collection of holomorphic functions $g_\alpha \in \mathscr{O}(U_\alpha)$ satisfying $g_\beta - g_\alpha = f_{\alpha\beta}$, then we can define $f = g_\alpha + f_\alpha$ on $U_\alpha$. Since $g_\alpha + f_\alpha = g_\beta + f_\beta$ on $U_{\alpha\beta}$, $f$ is globally defined, hence it is a desired solution.

Now define two groups

$$Z^1(\underline{U}, \mathscr{O}) = \big\{\{f_{\alpha\beta} \in \mathscr{O}(U_{\alpha\beta})\} : f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0\big\},$$

$$C^0(\underline{U}, \mathscr{O}) = \big\{\{g_\alpha \in \mathscr{O}(U_\alpha)\}\big\},$$

and a morphism $\delta : C^0(\underline{U}, \mathscr{O}) \to Z^1(\underline{U}, \mathscr{O})$ by

$$\{g_\alpha\} \mapsto \{f_{\alpha\beta} = g_\beta - g_\alpha\}.$$

By the above analysis, finding a global solution is equivalent to finding $\{g_\alpha \in U_\alpha\}$ such that $g_\beta - g_\alpha = f_{\alpha\beta}$. In other words, the quotient group

$$H^1(\underline{U}, \mathscr{O}) := \frac{Z^1(\underline{U}, \mathscr{O})}{\delta C^0(\underline{U}, \mathscr{O})}$$

is the *obstruction* to solving this problem in general.

2.2. **Sheaves.** Let $X$ be a topological space. A *sheaf* $\mathscr{F}$ on $X$ associates to each open set $U$ an abelian group $\mathscr{F}(U)$, called the *sections* of $\mathscr{F}$ over $U$, along with a *restriction map* $r_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ for any open sets $U \subset V$ (for a section $\sigma \in \mathscr{F}(V)$, we often write $\sigma|_U$ to denote $r_{V,U}(\sigma)$), satisfying the following conditions:

(1) For any open sets $U \subset V \subset W$, $r_{V,U} \circ r_{W,V} = r_{W,U}$;

(2) For any open sets $U, V$ and sections $\alpha \in \mathscr{F}(U), \beta \in \mathscr{F}(V)$, if $\alpha|_{U\cap V} = \beta|_{U\cap V}$, then there exists $\gamma \in \mathscr{F}(U \cup V)$ such that $\gamma|_U = \alpha$ and $\gamma|_V = \beta$;

(3) For any section $\gamma \in \mathscr{F}(U \cup V)$, if $\gamma|_U = 0$ and $\gamma|_V = 0$, then $\gamma = 0$.

**Remark 2.2.** Note that (2) says that two sections can be glued together if they coincide on an open subset and (3) implies that such gluing is unique. For many sheaves we will consider, the restriction maps are the obvious ones by restricting functions from bigger subsets to smaller ones.

**Example 2.3.** We have the *constant sheaf* $\mathbb{Z}$ on a topological space $X$, where $\mathbb{Z}(U)$ is the group of constant functions on an open set $U \subset X$. Similarly one can define constant sheaves $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{K}$ on an algebraic variety defined over a field $K$.

Let $X$ be a complex manifold and $U \subset X$ an open set.

(1) *Sheaf $\mathscr{O}$ of holomorphic functions*:

$$\mathscr{O}(U) = \{\text{holomorphic functions on } U\}.$$

(2) *Sheaf $\mathscr{O}^*$ of nonzero holomorphic functions*:

$$\mathscr{O}^*(U) = \{\text{holomorphic functions } f \text{ on } U : f(p) \neq 0 \text{ for any } p \in U\}.$$

The group structure is given by multiplication.

(3) *Sheaf $\mathscr{M}$ of meromorphic functions*: strictly speaking, a meromorphic function is not a function, even we take $\infty$ into account. If $X$ is compact, we cannot take a meromorphic function as a global quotient of two holomorphic functions, since any holomorphic function is a constant on $X$. Instead, we define $f \in \mathscr{M}(U)$ as local quotient of holomorphic functions. Namely, there exists an open covering $\{U_i\}$ of $U$ such that on each $U_i$, $f$ is given by $g_i/h_i$ for some $g_i, h_i \in \mathscr{O}(U_i)$ satisfying $g_i/h_i = g_j/h_j$, i.e. $g_i h_j = g_j h_i \in \mathscr{O}(U_i \cap U_j)$, hence these local quotients can be glued together over $U$.

(4) *Sheaf $\mathscr{M}^*$ of meromorphic functions not identically zero*: this is defined similarly and the group structure is given by multiplication.

Let $X$ be a variety. Replacing holomorphic functions by regular functions, we also use $\mathscr{O}$ to denote the *sheaf of regular functions* on $X$ and $\mathscr{M}$ the *sheaf of rational functions*. We also call $\mathscr{O}$ the *structure sheaf* of a variety.

2.3. **Maps of sheaves.** Let $\mathscr{E}, \mathscr{F}$ be sheaves on $X$. A *map* $f : \mathscr{E} \to \mathscr{F}$ is a collection of group homomorphisms

$$\{f_U : \mathscr{E}(U) \to \mathscr{F}(U)\}$$

such that they commute with the restriction maps, i.e. for open sets $U \subset V$ and $\sigma \in \mathscr{E}(V)$ we have

$$f_V(\sigma)|_U = f_U(\sigma|_U).$$

Define the *sheaf of kernel* $\ker(f)$ as

$$\ker(f)(U) = \{\ker(f_U : \mathscr{E}(U) \to \mathscr{F}(U))\}.$$

**Exercise 2.4.** Prove that in the above definition $\ker(f)$ is a sheaf.

*Proof.* Let $U, V$ be two open subsets of $X$ and $\alpha \in \ker(f)(U), \beta \in \ker(f)(V)$. If $\alpha|_{U\cap V} = \beta|_{U\cap V}$, since $\alpha, \beta$ are also sections of $\mathscr{E}$, there exists $\gamma \in \mathscr{E}(U \cup V)$ such that $\gamma|_U = \alpha$ and $\gamma|_V = \beta$ (here we use the gluing property of a sheaf). Moreover, $f(\gamma)|_U = f(\alpha) = 0$ and $f(\gamma)|_V = f(\beta) = 0$ (here we use that $f$ commutes with the restriction maps), hence $f(\gamma) = 0 \in \mathscr{F}(U \cup V)$ (here we use the unique gluing property of sheaves). It implies that $\gamma \in \ker(f)(U \cup V)$.

Now suppose for $\eta \in \ker(f)(U \cup V)$ we have $\eta|_U = 0$ and $\eta|_V = 0$. Since $\eta$ is also a section of $\mathscr{E}(U \cup V)$, as a consequence $\eta = 0$. $\square$

**Example 2.5.** Let $X$ be a complex manifold. Define the *exponential map*

$$\exp : \mathscr{O} \to \mathscr{O}^*$$

by $\exp(h) = e^{2\pi\sqrt{-1}h}$ for any open set $U \subset X$ and section $h \in \mathscr{O}(U)$. It is easy to see that $\ker(\exp)$ is the constant sheaf $\mathbb{Z}$.

The *sheaf of cokernel* is harder to define. Naively, one would like to define $\operatorname{coker}(f)(U) = \operatorname{coker}(f_U : \mathscr{E}(U) \to \mathscr{F}(U))$, but this is problematic. For instance, consider the exponential map $\exp : \mathscr{O} \to \mathscr{O}^*$ on the punctured plane $\mathbb{C}\backslash\{0\}$. The section $z \in \mathscr{O}^*(\mathbb{C}\backslash\{0\})$ is not in the image of $f$, hence it would define a section in the cokernel. Nevertheless, restricted to any contractible open set $U \subset \mathbb{C}\backslash\{0\}$, $z$ lies in the image of $f$. Now cover $\mathbb{C}\backslash\{0\}$ by contractible open sets. By the gluing property of sheaves, $z$ would be zero everywhere, leading to a contradiction.

Instead, we want a section of $\operatorname{coker}(f)(U)$ to be a collection of sections $\sigma_\alpha \in \mathscr{F}(U_\alpha)$ for an open covering $\{U_\alpha\}$ of $U$ such that for all $\alpha, \beta$ we have

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in f_{U_\alpha \cap U_\beta}(\mathscr{E}(U_\alpha \cap U_\beta)).$$

Here the definition still depends on the choice of an open covering. To get rid of this, we pass to the direct limit. More precisely, we further identify two collections $\{(U_\alpha, \sigma_\alpha)\}$ and $\{(V_\beta, \sigma_\beta\}$ if for all $p \in U_\alpha \cap V_\beta$, there exists an open set $W$ satisfying $p \in W \subset U_\alpha \cap V_\beta$ such that

$$\sigma_\alpha|_W - \sigma_\beta|_W \in f_W(\mathscr{E}(W)).$$

This identification yields an equivalence relation and correspondingly we define $\operatorname{coker}(f)(U)$ as the group of equivalence classes of the above sections.

**Exercise 2.6.** Prove that in the above definition $\operatorname{coker}(f)$ is a sheaf.

Consider a sequence of maps

$$0 \longrightarrow \mathscr{E} \xrightarrow{\ \alpha\ } \mathscr{F} \xrightarrow{\ \beta\ } \mathscr{G} \longrightarrow 0.$$

We say that it is a *short exact sequence* if $\mathscr{E} = \ker(\beta)$ and $\mathscr{G} = \operatorname{coker}(\alpha)$. In this case we also say that $\mathscr{E}$ is a *subsheaf* of $\mathscr{F}$ and $\mathscr{G}$ is the *quotient sheaf* $\mathscr{F}/\mathscr{E}$.

If $Y \subset X$ is a subspace and $\mathscr{F}$ is a sheaf on $Y$, we can *extend* $\mathscr{F}$ *by zero* to obtain a sheaf $\mathscr{F}'$ on $X$ by

$$\mathscr{F}'(U) = \mathscr{F}(U \cap Y), \text{ if } U \cap Y \neq \emptyset,$$
$$\mathscr{F}'(U) = \{0\}, \text{ if } U \cap Y = \emptyset.$$

The restriction maps are the obvious ones. In general, we will abuse notation and still use $\mathscr{F}$ to denote the extension of $\mathscr{F}$ by zero.

**Exercise 2.7.** Prove that in the above definition $\mathscr{F}'$ is a sheaf on $X$.

**Example 2.8.** Let $X$ be a complex manifold. We have the exact *exponential sequence*:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \mathscr{O} \xrightarrow{\ \exp\ } \mathscr{O}^* \longrightarrow 0,$$

where $i$ is the natural inclusion and $\exp(f) = e^{2\pi\sqrt{-1}f}$ for $f \in \mathscr{O}(U)$.

**Exercise 2.9.** Prove that the exponential sequence is exact.

**Example 2.10.** Let $X$ be a complex manifold and $Y \subset X$ a submanifold. Define the *ideal sheaf* $\mathscr{I}_{Y/X}$ of $Y$ in $X$ (or simply $\mathscr{I}_Y$ if there is no confusion) by

$$\mathscr{I}_Y(U) = \{\text{holomorphic functions on } U \text{ vanishing on } Y \cap U\}.$$

We have the exact sequence:

$$0 \longrightarrow \mathscr{I}_Y \xrightarrow{\ i\ } \mathscr{O}_X \xrightarrow{\ r\ } \mathscr{O}_Y \longrightarrow 0,$$

where $i$ is the natural inclusion and $r$ is defined by the natural restriction map. Note that here we treat $\mathscr{O}_Y$ as a sheaf on $X$ by extending $\mathscr{O}_Y$ by zero.

The same exact sequence holds in the context of varieties, with holomorphic functions replaced by regular functions.

**Exercise 2.11.** Prove that the above sequence is exact.

2.4. **Stalks and germs.** Let $\mathscr{F}$ be a sheaf on a topological space $X$ and $p \in X$ a point. Suppose $U, V$ are two open subsets, both containing $p$, with two sections $\alpha \in \mathscr{F}(U)$ and $\beta \in \mathscr{F}(V)$. Define an equivalence relation $\alpha \sim \beta$, if there exists an open subset $W$ satisfying $p \in W \subset U \cap V$ such that $\alpha|_W = \beta|_W$. Define the *stalk* $\mathscr{F}_p$ as the union of all sections in open neighborhoods of $p$ modulo this equivalence relation. Namely, $\mathscr{F}_p$ is the direct limit

$$\mathscr{F}_p := \varinjlim_{U \ni p} \mathscr{F}(U) = \Big( \bigsqcup_{U \ni p} \mathscr{F}(U) \Big) / \sim .$$

Note that $\mathscr{F}_p$ is also a group, by adding representatives of two equivalence classes. There is a group homomorphism $r_U : \mathscr{F}(U) \to \mathscr{F}_p$ mapping a section $\alpha \in \mathscr{F}(U)$ to its equivalence class. The image is called the *germ* of $\alpha$.

**Example 2.12.** Let $X$ be a Riemann surface and $p \in X$ a point with local coordinate $z$. Then the stalk of the sheaf of holomorphic functions $\mathscr{O}_X$ at $p$ is the set of Taylor series $\sum_{i=0}^{\infty} a_i z^i$ converging in a neighborhood of $p$. Two holomorphic functions have the same germ if and only if their Taylor expansions are the same at $p$.

**Example 2.13** (Skyscraper sheaf)**.** Let $p \in X$ be a point on a topological space $X$. Define the *skyscraper sheaf* $\mathscr{F}$ at $p$ by $\mathscr{F}(U) = \{0\}$ for $p \notin U$ and $\mathscr{F}(U) = A$ for $p \in U$, where $A$ is a group (or a ring, a vector space, etc). The restriction maps are either the identity map $A \to A$ or the zero map. For $q \neq p$, the stalk $\mathscr{F}_q = \{0\}$. At $p$, we have $\mathscr{F}_p = A$. Note that $\mathscr{F}$ can also be obtained by extending the constant sheaf $A$ at $p$ by zero to $X$.

**Exercise 2.14.** Let $X$ be a Riemann surface and $p \in X$ a point. Let $\mathscr{I}_p$ be the ideal sheaf of $p$ in $X$ parameterizing holomorphic functions vanishing at $p$. We have the exact sequence

$$0 \longrightarrow \mathscr{I}_p \xrightarrow{\ i\ } \mathscr{O}_X \xrightarrow{\ r\ } \mathscr{O}_p \longrightarrow 0.$$

Show that the quotient sheaf $\mathscr{O}_p$ is isomorphic to the skyscraper sheaf with stalk $\mathbb{C}$ at $p$.

It is more convenient to verify injections and surjections for maps of sheaves by the language of stalks.

**Proposition 2.15.** *Let $\phi : \mathscr{E} \to \mathscr{F}$ be a map for sheaves $\mathscr{E}$ and $\mathscr{F}$ on a topological space $X$.*

*(1) $\phi$ is injective if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is injective for the stalks at every point $p$.*

*(2) $\phi$ is surjective if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is surjective for the stalks at every point $p$.*

*(3) $\phi$ is an isomorphism if and only if the induced map $\phi_p : \mathscr{E}_p \to \mathscr{F}_p$ is an isomorphism for the stalks at every point $p$.*

*Proof.* The claim (3) follows from (1) and (2). Let us prove (1) only, and one can easily find the proof of (2) in many books, e.g. Hartshorne.

Suppose $\phi$ is injective. Take a section $\sigma \in \mathscr{E}(U)$ on an open subset $U$. If $\phi([\sigma]) = 0 \in \mathscr{F}_p$, there exists a smaller open subset $V \subset U$ such that $\phi_V(\sigma) = 0 \in \mathscr{F}(V)$, hence $\sigma|_V = 0 \in \mathscr{E}(V)$. Consequently the equivalence class $[\sigma] = 0 \in \mathscr{E}_p$ and we conclude that $\phi_p$ is injective.

Conversely, suppose $\phi_p$ is injective for every point $p$. Take a section $\sigma \in \mathscr{E}(U)$. If $\phi(\sigma) = 0 \in \mathscr{F}(U)$, then for every point $p \in U$, $[\phi(\sigma)] = 0 \in \mathscr{F}_p$. Since $\phi_p$ is injective, it implies that $[\sigma] = 0 \in \mathscr{E}_p$ i.e. there exists an open subset $U_p \ni p$ such that $\sigma|_{U_p} = 0 \in \mathscr{E}(U_p)$. Applying the gluing property to the open covering $\{U_p\}$ of $U$, we conclude that $\sigma = 0 \in \mathscr{E}(U)$. $\square$

**Remark 2.16.** The image of $\phi$ does not automatically form a sheaf. In general, it is only a *presheaf*, i.e. without the gluing and unique gluing properties in the definition of sheaves. If the *sheafification* of $\mathrm{Im}(\phi)$ equals $\mathscr{F}$, we say that $\phi$ is surjective. In particular, it does not mean $\mathscr{E}(U) \to \mathscr{F}(U)$ is surjective for every open set $U$. Sometimes one has to pass to a smaller open set in order to obtain a surjection between sections.

**Example 2.17.** Consider the exponential map $\exp : \mathscr{O} \to \mathscr{O}^*$ on the punctured plane $\mathbb{C}\backslash\{0\}$. This map is surjective, but the section $z$ over the total space does not have an inverse image. It does have an inverse over any contractible open subset.

2.5. **Cohomology of sheaves.** Let $\mathscr{F}$ be a sheaf on a topological space $X$. Take a locally closed open covering $\underline{U} = \{U_\alpha\}$ of $X$. Define the $k$-th *cochain group*

$$C^k(\underline{U}, \mathscr{F}) := \prod_{\alpha_0,\ldots,\alpha_k} \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}).$$

An element $\sigma$ of $C^k(\underline{U}, \mathscr{F})$ consists of a section $\sigma_{\alpha_0,\ldots,\alpha_k} \in \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$ for every $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$, satisfying the cochain condition, i.e. skew-symmetric: $\sigma_{\alpha_0,\ldots,\alpha_k} = -\sigma_{\alpha_0,\ldots,\alpha_{i-1},\alpha_{i+1},\alpha_i,\alpha_{i+2},\ldots,\alpha_k}$.

Define a *coboundary map* $\delta : C^k(\underline{U}, \mathscr{F}) \to C^{k+1}(\underline{U}, \mathscr{F})$ by

$$(\delta\sigma)_{\alpha_0,\ldots,\alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \sigma_{\alpha_0,\ldots,\hat{\alpha}_j,\ldots,\alpha_{k+1}}|_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_{k+1}}}.$$

**Example 2.18.** Consider $\underline{U} = \{U_1, U_2, U_3\}$ as an open covering of $X$. Take a cochain element $\sigma \in C^0(\underline{U}, \mathscr{F})$, i.e. $\sigma$ is a collection of a section $\sigma_i \in \mathscr{F}(U_i)$ for every $i$. Then we have

$$(\delta\sigma)_{ij} = (\sigma_j - \sigma_i)|_{U_i \cap U_j} \in \mathscr{F}(U_i \cap U_j).$$

Now take $\tau \in C^1(\underline{U}, \mathscr{F})$, i.e. $\tau$ is a collection of a section $\tau_{ij} \in \mathscr{F}(U_i \cap U_j)$ for every pair $i, j$. Then we have

$$(\delta\tau)_{123} = (\tau_{23} - \tau_{13} + \tau_{12})|_{U_1 \cap U_2 \cap U_3} \in \mathscr{F}(U_1 \cap U_2 \cap U_3).$$

A cochain $\sigma \in C^k(\underline{U}, \mathscr{F})$ is called a *cocyle* if $\delta\sigma = 0$. We say that $\sigma$ is a *coboundary* if there exists $\tau \in C^{k-1}(\underline{U}, \mathscr{F})$ such that $\delta\tau = \sigma$.

**Lemma 2.19.** *A coboundary is a cocycle, i.e.* $\delta \circ \delta = 0$.

*Proof.* Let us prove it for the above example. The same idea applies in general with messier notation. Under the above setting, we have

$$\begin{aligned}((\delta\circ\delta)\sigma)_{123} &= (\delta\sigma)_{23}-(\delta\sigma)_{13}+(\delta\sigma)_{12}\\ &= (\sigma_3-\sigma_2)-(\sigma_3-\sigma_1)+(\sigma_2-\sigma_1)\\ &= 0\in\mathscr{F}(U_1\cap U_2\cap U_3).\end{aligned}$$

Here we omit the restriction notation, since it is obvious. □

**Exercise 2.20.** Prove in full generality that $\delta\circ\delta=0$.

For the coboundary map $\delta_k : C^k(\underline{U},\mathscr{F})\to C^{k+1}(\underline{U},\mathscr{F})$, define the $k$-th cohomology group (respect to $\underline{U}$) by

$$H^k(\underline{U},\mathscr{F}) := \frac{\ker(\delta_k)}{\mathrm{Im}(\delta_{k-1})}.$$

This is well-defined due to the above lemma.

**Example 2.21.** For $k=0$, we have $H^0(\underline{U},\mathscr{F})=\ker(\delta_0)$. Take an element $\{\sigma_i\in\mathscr{F}(U_i)\}$ in this group. Because it is a cocycle, it satisfies

$$\sigma_i|_{U_i\cap U_j}=\sigma_j|_{U_i\cap U_j}\in\mathscr{F}(U_i\cap U_j).$$

By the gluing property of sheaves, there exists a global section $\sigma\in\mathscr{F}(X)$ such that $\sigma|_{U_i}=\sigma_i$. Conversely, if $\sigma$ is a global section, then define $\sigma_i=\sigma|_{U_i}\in\mathscr{F}(U_i)$. In this way we obtain a cocycle in $C^1(\underline{U},\mathscr{F})$. From the discussion we see that $H^0(\underline{U},\mathscr{F})=\mathscr{F}(X)$, which is independent of the choice of an open covering. Hence $H^0(\underline{U},\mathscr{F})$ is called the *group of global sections* of $\mathscr{F}$ and we often denote it by $H^0(X,\mathscr{F})$ or simply $H^0(\mathscr{F})$.

In general, we would like to define cohomology independent of open coverings. Take two open coverings $\underline{U}=\{U_\alpha\}_{\alpha\in I}$ and $\underline{V}=\{V_\beta\}_{\beta\in J}$. We say that $\underline{U}$ is a refinement of $\underline{V}$ if for every $U_\alpha$ there exists a $V_\beta$ such that $U_\alpha\subset V_\beta$ and we write it as $\underline{U}<\underline{V}$. Then we also have an index map $\phi: I\to J$ sending $\alpha$ to $\beta$. It induces a map

$$\rho_\phi : C^k(\underline{V},\mathscr{F})\to C^k(\underline{U},\mathscr{F})$$

given by

$$\rho_\phi(\sigma)_{\alpha_0,\ldots,\alpha_k}=\sigma_{\phi(\alpha_0),\ldots,\phi(\alpha_p)}|_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_k}}.$$

One checks that it commutes with the coboundary map $\delta$, i.e. $\delta\circ\rho_\phi=\rho_\phi\circ\delta$. Moreover, it induces a map

$$\rho : H^k(\underline{V},\mathscr{F})\to H^k(\underline{U},\mathscr{F}),$$

which is independent of the choice of $\phi$. Finally, we define the $k$-th (Čech) cohomology group by passing to the direct limit:

$$H^k(X,\mathscr{F}) := \varinjlim H^k(\underline{U},\mathscr{F}).$$

The definition involves direct limit, which is inconvenient to use in practice. Nevertheless, we can simplify the situation once the open covering $\underline{U}$ is fine enough. We say that $\underline{U}=\{U_i\}_{i\in I}$ is *acyclic* respect to $\mathscr{F}$, if for any $k>0$ and $i_1,\ldots,i_l\in I$ we have

$$H^k(U_{i_1}\cap\cdots\cap U_{i_l},\mathscr{F})=0.$$

**Theorem 2.22** (Leray's Theorem)**.** *If the open covering $\underline{U}$ is acyclic respect to $\mathscr{F}$, then $H^*(\underline{U},\mathscr{F})\cong H^*(X,\mathscr{F})$.*

**Example 2.23.** Let us compute the cohomology of the structure sheaf $\mathscr{O}$ on $\mathbb{P}^1_{\mathbb{C}}$. It is clear that $H^0(\mathbb{P}^1,\mathscr{O})=\mathbb{C}$, since any global holomorphic function on a compact complex manifold is constant. For higher cohomology, use $[X,Y]$ to denote the coordinates of $\mathbb{P}^1$. Take the standard open covering $U=\{[X,Y]:X\neq 0\}$ and $V=\{[X,Y]:Y\neq 0\}$. It is acyclic respect to the structure sheaf $\mathscr{O}$ (morally because $U\cap V=\mathbb{C}^*$ is contractible). Let $s=Y/X$ and $t=X/Y$ as affine coordinates of $U$ and $V$, respectively. Suppose $h$ is an element in $C^1(\{U,V\},\mathscr{O})$, i.e. $h\in\mathscr{O}(U\cap V)$. We can write

$$h=\sum_{i=-\infty}^{\infty}a_is^i.$$

Now take

$$f=-\sum_{i=0}^{\infty}a_is^i\in\mathscr{O}(U),$$

$$g=\sum_{i=-\infty}^{-1}a_is^i=\sum_{i=-\infty}^{-1}a_it^{-i}\in\mathscr{O}(V).$$

Then we have $(f,g)\in C^1(\{U,V\},\mathscr{O})$ and $\delta((f,g))=g-f=h$. It implies that $H^1(\mathbb{P}^1,\mathscr{O})=0$. All the other $H^k(\mathbb{P}^1,\mathscr{O})=0$ for $k>1$, since there are only two open subsets in the covering.

**Remark 2.24.** In the context of complex manifolds, if $U_i$'s are contractible, then $\underline{U}$ is acyclic respect to the sheaves we will consider. While for varieties, if $U_i$'s are affine, then $\underline{U}$ is acyclic.

**Example 2.25.** Let $\Omega$ denote the *sheaf of holomorphic one-forms* on a Riemann surface, i.e. locally a section of $\Omega$ can be expressed as $f(z)dz$, where $z$ is local coordinate and $f(z)$ a holomorphic function. Let us compute the cohomology of $\Omega$ on $\mathbb{P}^1$. Take the above open covering. Suppose $\omega$ is a global holomorphic one-form. Then on the open chart $U$, it can be written as

$$\Big(\sum_{i=0}^{\infty}a_is^i\Big)ds.$$

Using the relation $s=1/t$ and $ds=-dt/t^2$, on $V$ it can be expressed as

$$-\Big(\sum_{i=0}^{\infty}a_it^{-i-2}\Big)dt,$$

which is holomorphic if and only if $a_i=0$ for all $i$. Hence $w$ is the zero one-form and $H^0(\mathbb{P}^1,\Omega)=0$. Now take $\omega\in C^1(\{U,V\},\Omega)$, i.e. $\omega\in\Omega(U\cap V)=\Omega(\mathbb{C}^*)$, we express it as

$$\omega=\Big(\sum_{i=-\infty}^{\infty}a_it^i\Big)dt.$$

Note that any $\alpha\in\Omega(U)$ and $\beta\in\Omega(V)$ can be written as

$$\alpha=\Big(\sum_{i=0}^{\infty}b_is^i\Big)ds,$$

$$\beta=\Big(\sum_{i=0}^{\infty}c_it^i\Big)dt.$$

Hence on $U \cap V$ we have

$$\delta((\alpha,\beta)) = \beta - \alpha = -\Big(\sum_{i=0}^{\infty} b_i t^{-i-2}\Big)dt + \Big(\sum_{i=0}^{\infty} c_i t^i\Big)dt.$$

Note that only the term $t^{-1}$ is missing from the expression. We conclude that $H^1(\mathbb{P}^1, \Omega) = \{a_{-1}t^{-1}dt\} \cong \mathbb{C}$.

**Remark 2.26.** If $\mathbb{P}^1$ is defined over an algebraically closed field $K$, replacing holomorphic functions by regular functions, we have $H^0(\mathbb{P}^1, \mathscr{O}) \cong K$ and $H^1(\mathbb{P}^1, \mathscr{O}) = 0$. Indeed, we have seen that the coordinate ring $A(U \cap V)$ is given by $K[s, 1/s]$, hence the above argument goes word by word. Similarly replacing holomorphic one-forms by *regular differentials*, i.e. in the expression $f(z)dz$, $f(z)$ is regular and $dz$ satisfies the usual differentiation rules, we have $H^0(\mathbb{P}^1, \Omega) = 0$ and $H^1(\mathbb{P}^1, \Omega) \cong K$. In general, the rank of $H^1(X, \mathscr{O}) \cong H^0(X, \Omega)$ (by Serre Duality) is called the *genus* of a Riemann surface (or an algebraic curve) $X$.

**Exercise 2.27.** Let $D = p_1 + \cdots + p_n$ be a collection of $n$ points in $\mathbb{P}^1$. We say that $D$ is an *effective divisor of degree* $n$. Define the sheaf $\mathscr{O}(D)$ on $\mathbb{P}^1$ by

$$\mathscr{O}(D)(U) = \{f \in \mathscr{M}(U) : f \in \mathscr{O}(U \backslash \{p_1, \ldots, p_n\}) \text{ and has at most simple pole at } p_i\}.$$

Assume that the standard covering of $\mathbb{P}^1$ is acyclic respect to $\mathscr{O}(D)$. Use it to calculate the cohomology groups $H^*(\mathbb{P}^1, \mathscr{O}(D))$.

As many other homology/cohomology theories, one can associate a *long exact sequence of cohomology* to a short exact sequence. Suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathscr{E} \xrightarrow{\ \alpha\ } \mathscr{F} \xrightarrow{\ \beta\ } \mathscr{G} \longrightarrow 0.$$

Then $\alpha$ and $\beta$ induce maps

$$\alpha : C^k(\underline{U}, \mathscr{E}) \to C^k(\underline{U}, \mathscr{F}), \quad \beta : C^k(\underline{U}, \mathscr{F}) \to C^k(\underline{U}, \mathscr{G}).$$

Since the coboundary map $\delta$ is given by alternating sums of restrictions, $\alpha$ and $\beta$ commute with $\delta$, hence they send a cocycle to cocycle and a coboundary to coboundary. Consequently they induce maps for colomology

$$\alpha_* : H^k(X, \mathscr{E}) \to H^k(X, \mathscr{F}), \quad \beta_* : H^k(X, \mathscr{F}) \to H^k(X, \mathscr{G}).$$

Next we define the coboundary map

$$\delta_* : H^k(X, \mathscr{G}) \to H^{k+1}(X, \mathscr{E}).$$

For $\sigma \in C^k(\underline{U}, \mathscr{G})$ satisfying $\delta\sigma = 0$, after refining $\underline{U}$ (still denoted by $\underline{U}$) such that there exists $\tau \in C^k(\underline{U}, \mathscr{F})$ satisfying $\beta(\tau) = \sigma$, because $\beta$ is surjective. Then $\beta(\delta\tau) = \delta(\beta(\tau)) = \delta\sigma = 0$, hence after refining further there exists $\mu \in C^{k+1}(\underline{U}, \mathscr{E})$ satisfying $\alpha(\mu) = \delta\tau$. Note that $\mu$ is a cocycle. It is because $\alpha(\delta\mu) = \delta(\alpha(\mu)) = \delta\delta(\tau) = 0$ and $\alpha$ is injective, hence $\delta\mu = 0$ and $\mu \in \ker(\delta)$. We thus take $\delta_*\sigma := [\mu] \in H^{k+1}(X, \mathscr{E})$. One checks that this is independent of the choice of $\tau$ and $\mu$.

We say that a sequence of maps

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\ \alpha_{n-1}\ } A_n \xrightarrow{\ \alpha_n\ } A_{n+1} \longrightarrow \cdots$$

is *exact* if $\operatorname{Im}(\alpha_{n-1}) = \ker(\alpha_n)$.

**Proposition 2.28.** *The long sequence of cohomology associated to a short exact sequence of sheaves is exact.*

*Proof.* We prove it under an extra assumption that there exists an acyclic open covering $\underline{U}$ such that for any $U = U_{i_1} \cap \cdots \cap U_{i_k}$ we have the short exact sequence:

$$0 \to \mathscr{E}(U) \to \mathscr{F}(U) \to \mathscr{G}(U) \to 0.$$

At least for sheaves considered in this course, this assumption always holds. It further implies the following sequence is exact:

$$0 \to C^k(\underline{U}, \mathscr{E}) \to C^k(\underline{U}, \mathscr{F}) \to C^k(\underline{U}, \mathscr{G}) \to 0.$$

Let us prove that

$$H^k(\underline{U}, \mathscr{F}) \xrightarrow{\beta_*} H^k(\underline{U}, \mathscr{G}) \xrightarrow{\delta_*} H^{k+1}(\underline{U}, \mathscr{E})$$

is exact. The other cases are easier.

Consider $\tau \in Z^k(\underline{U}, \mathscr{F})$. In the definition of $\delta_*$, take $\sigma = \beta(\tau)$. Then there exists $\mu \in C^k(\underline{U}, \mathscr{E})$ such that $\alpha(\mu) = \delta\tau = 0$. Then we have $\mu = 0$ since $\alpha$ is injective. Consequently $\delta_*\beta_*(\tau) = \delta_*(\sigma) = \mu = 0$, hence $\delta_*\beta_* = 0$ and $\mathrm{Im}(\beta_*) \subset \ker(\delta_*)$.

Conversely, suppose $\delta_*\sigma = 0$ for $\sigma \in Z^k(\underline{U}, \mathscr{G})$. In the definition of $\delta_*$, it implies that $\mu = 0 \in H^{k+1}(\underline{U}, \mathscr{E})$, hence there exists $\gamma \in C^k(\underline{U}, \mathscr{E})$ such that $\delta\gamma = \mu$. Since $\alpha(\mu) = \delta\tau$, we have $\delta\tau = \delta\alpha(\gamma)$ and $\tau - \alpha(\gamma) \in Z^k(\underline{U}, \mathscr{F})$ is a cocycle. Moreover, $\beta(\tau - \alpha(\gamma)) = \beta(\tau) = \sigma$, hence $\beta_*(\tau - \alpha(\gamma)) = \sigma$. We conclude that $\ker(\delta_*) \subset \mathrm{Im}(\beta_*)$. $\square$

**Exercise 2.29.** Prove in general the cohomology sequence is exact.

**Example 2.30.** Consider the short exact sequence

$$0 \longrightarrow \mathscr{I}_p \xrightarrow{\ i\ } \mathscr{O}_{\mathbb{P}^1} \xrightarrow{\ r\ } \mathscr{O}_p \longrightarrow 0.$$

Its long exact sequence of cohomology is as follows:

$$0 \to H^0(\mathscr{I}_p) \to H^0(\mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathscr{O}_p) \to H^1(\mathscr{I}_p) \to H^1(\mathscr{O}_{\mathbb{P}^1}) \to 0.$$

The last term is zero because $p$ is a point so it does not have higher cohomology. We have $H^0(\mathscr{O}_{\mathbb{P}^1}) = K$ because any global regular function on $\mathbb{P}^1$ is constant. Note that $H^0(\mathscr{I}_p) = 0$, because vanishing at $p$ forces such a constant function to be zero. Moreover we have seen that $H^1(\mathscr{O}_{\mathbb{P}^1}) = 0$. Altogether it implies $H^1(\mathscr{I}_p) = 0$, because $H^0(\mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathscr{O}_p)$ is an isomorphism by evaluating at $p$.

**Exercise 2.31.** Let $D$ be an effective divisor of degree $n$ on $\mathbb{P}^1$. We have the short exact sequence

$$0 \longrightarrow \mathscr{I}_D \xrightarrow{\ i\ } \mathscr{O}_{\mathbb{P}^1} \xrightarrow{\ r\ } \mathscr{O}_D \longrightarrow 0.$$

Use the associated long exact sequence to calculate the cohomology $H^*(\mathbb{P}^1, \mathscr{I}(D))$.

The cohomology of constant sheaves contains topological information for the underlying space. Let $K$ be a simplicial complex with the underlying topological space $X$. Let $H^*(K, \mathbb{Z})$ denote the *singular cohomology* with coefficient $\mathbb{Z}$.

**Proposition 2.32.** *The singular cohomology of $K$ and the Čech cohomology of the constant sheaf on $X$ are isomorphic:*

$$H^*(K, \mathbb{Z}) \cong H^*(X, \mathbb{Z}).$$

*Proof.* For each vertex $v_\alpha$ of $K$, associate to it an open set $U_\alpha$ by the interior of the union of all simplices with $v_\alpha$ as a vertex. Then $\underline{U} = \{U_\alpha\}$ is an open covering of $X$. Note that $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ is nonempty and connected if $v_{\alpha_0}, \ldots, v_{\alpha_k}$ are the vertices of a $k$-simplex. Otherwise it is empty. Then an element $\sigma \in \mathbb{Z}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$ is either an integer if $v_{\alpha_0}, \ldots, v_{\alpha_k}$ span a $k$-simplex or it is zero otherwise.

Given a cochain $\sigma \in C^k(\underline{U}, \mathbb{Z})$, define a simplicial cochain $\sigma' \in C^k(K, \mathbb{Z})$ by

$$\sigma'(\langle v_{\alpha_0}, \ldots, v_{\alpha_k} \rangle) = \sigma_{\alpha_0, \ldots, \alpha_k}.$$

The association $\sigma \mapsto \sigma'$ induces an isomorphism $C^k(\underline{U}, \mathbb{Z}) \xrightarrow{\sim} C^k(K, \mathbb{Z})$. Moreover, the coboundary maps satisfy

$$\begin{aligned}
\delta\sigma'(\langle v_{\alpha_0}, \ldots, v_{\alpha_{k+1}} \rangle) &= \sum_{i=0}^{k+1} (-1)^i \sigma'(\langle v_{\alpha_0}, \ldots, \hat{v}_{\alpha_i}, \ldots, v_{\alpha_{k+1}} \rangle) \\
&= \sum_{i=0}^{k+1} (-1)^i \sigma_{\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1}} \\
&= (\delta\sigma)_{\alpha_0, \ldots, \alpha_{k+1}} \\
&= (\delta\sigma)'(\langle v_{\alpha_0}, \ldots, v_{\alpha_{k+1}} \rangle),
\end{aligned}$$

so we obtain an isomorphism for the chain complexes $C^*(\underline{U}, \mathbb{Z}) \xrightarrow{\sim} C^*(K, \mathbb{Z})$ and consequently an isomorphism $H^*(\underline{U}, \mathbb{Z}) \xrightarrow{\sim} H^*(K, \mathbb{Z})$. Finally we subdivide $K$ such that $\underline{U}$ can be arbitrarily fine. Then the claim follows, since $H^*(K, \mathbb{Z})$ does not change. $\square$

## 3. Vector bundles, line bundles and divisors

3.1. **Holomorphic vector bundles.** Let $k$ be a positive integer. Consider $\pi : E \to X$ a holomorphic map between complex manifolds, such that for every $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is isomorphic to $\mathbb{C}^k$ and there exists an open neighborhood $U$ of $x$ along with an isomorphism

$$\phi_U : E_U = \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^k$$

mapping $E_x$ to $\{x\} \times \mathbb{C}^k$ which is a linear isomorphism between vector spaces. Then $E$ is called a *holomorphic vector bundle of rank $k$* on $X$ and has a *trivialization* $\{(U, \phi_U)\}$. If $E$ is of rank 1, we say that $E$ is a *line bundle*.

We give an equivalent characterization of vector bundles based on transition functions. Note that for two trivializations $\phi_U$ and $\phi_V$, they induce an isomorphism $g_{UV} = \phi_V \circ \phi_U^{-1} : (U \cap V) \times \mathbb{C}^k \xrightarrow{\sim} (U \cap V) \times \mathbb{C}^k$ preserving the vector space structure fiber by fiber. In other words,

$$g_{UV} \in \operatorname{Hom}(U \cap V, \operatorname{GL}(\mathbb{C}^k))$$

specifies (holomorphically) a linear isomorphism $E_x \xrightarrow{\sim} E_x$ over each base point $x \in U \cap V$. We call such $g_{UV}$ *transition functions* with respect to the trivialization $\{(U, \phi_U)\}$. The transition functions satisfy the following identities:

$$g_{UV}(x) \cdot g_{VU}(x) = I, \quad \text{for all } x \in U \cap V,$$

$$g_{UV}(x) \cdot g_{VW}(x) \cdot g_{WU}(x) = I, \quad \text{for all } x \in U \cap V \cap W.$$

Conversely suppose $\underline{U} = \{U_\alpha\}$ is an open covering of $X$. Given holomorphic functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \operatorname{GL}(\mathbb{C}^k)$$

satisfying the above equalities, we can construct a vector bundle $E$ by gluing $U_\alpha \times \mathbb{C}^k$ together. More precisely,

$$E = \bigcup (U_\alpha \times \mathbb{C}^k)$$

as a complex manifold is defined by identifying $(x, v)$ with $(x, g_{\alpha\beta}(v))$ for $x \in U_\alpha \cap U_\beta$ and $v \in \mathbb{C}^k$ and $E \to X$ is given by projection to the bases $U_\alpha$.

Define the *dual bundle* $E^*$ of $E$ by taking $\phi_U^* : E_U^* \to U \times \mathbb{C}^{k*} \cong U \times \mathbb{C}^k$ for open subsets $U \subset X$. Each fiber of $E^*$ is given by $\mathrm{Hom}(E_x, \mathbb{C}) \cong \mathbb{C}^{k*}$. One checks that the transition functions of $E^*$ satisfy $g_{\alpha\beta}^*(x) = ((g_{\alpha\beta}(x))^{-1})^t$.

Suppose $F$ is another vector bundle of rank $l$ with transition functions $h_{\alpha\beta}$. Define the *direct sum* $E \oplus F$ as a vector bundle of rank $k + l$ with transition functions $\mathrm{diag}(g_{\alpha\beta}(x), h_{\alpha\beta}(x)) \in \mathrm{GL}(\mathbb{C}^{k+l})$.

Define the *tensor product* $E \otimes F$ as a vector bundle of rank $kl$ with transition functions $g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x) \in \mathrm{GL}(\mathbb{C}^k \otimes \mathbb{C}^l)$. Note that if $E$ is a line bundle, then $E^* \otimes E$ is the trivial line bundle $X \times \mathbb{C}$ (explained in detail later), since the transition functions are identity everywhere.

Define the *wedge product* $\wedge^r E$ as a vector bundle of rank $\binom{k}{r}$ with transition functions $\wedge^r g_{\alpha\beta}(x) \in \mathrm{GL}(\wedge^r \mathbb{C}^k)$ for $1 \le r \le k$. In particular, $\wedge^k E$ is a line bundle with transition functions $\det(g_{\alpha\beta}(x)) \in \mathrm{GL}(\mathbb{C})$. Therefore, $\wedge^k E$ is called the *determinant line bundle* of $E$ and often denoted by $\det(E)$.

Let $f : X \to Y$ be a holomorphic map and $E$ a vector bundle on $Y$. We define the *pullback bundle* $f^*E$ by setting $(f^*E)_x = E_{f(x)} \cong \mathbb{C}^k$. If $\phi_U : E_U \to U \times \mathbb{C}^k$ is a trivialization in a neighborhood of $f(x)$, then $f^*\phi_U : (f^*E)_{f^{-1}(U)} \to f^{-1}(U) \times \mathbb{C}^k$ defines $f^*E$ as a complex manifold and moreover as a vector bundle on $X$ with transition functions given by $f^*\phi_U$.

Let $F \subset E$ be a collection of $l$ dimensional subspaces $\{F_x \subset E_x\}$ for every $x \in X$ such that there exists a neighborhood $U$ of $x$ and a trivialization $\phi_U : E_U \to U \times \mathbb{C}^k$ satisfying

$$\phi_U|_{F_U} : F_U \to U \times \mathbb{C}^l \subset U \times \mathbb{C}^k.$$

Then we say that $F$ is a rank $l$ *subbundle* of $E$. Up to changing coordinates, the transition functions of $E$ respect to these trivializations can be written as an upper triangular matrix

$$g_{UV}(x) = \begin{pmatrix} f_{UV}(x) & j_{UV}(x) \\ 0 & h_{UV}(x) \end{pmatrix}$$

with $f_{UV}(x)$ being the transition functions of $F$. Similarly we define the *quotient bundle* $E/F$ with $(E/F)_x = E_x/F_x$ as well as transition functions given by $h_{UV}(x)$.

**Exercise 3.1.** Verify that the above definitions give rise to vector bundles with the claimed transition functions.

A *map* between vector bundles $E, F$ on $X$ is given by a holomorphic map $f : E \to F$ such that $f(E_x) \subset F_x$ and $f_x = f|_{E_x} : E_x \to F_x$ is linear. Note that if $f(E_x)$ has the same rank for every $x$, then $\ker(f)$ and $\mathrm{Im}(f)$ are naturally subbundles of $E$ and $F$, respectively. We say that $E$ and $F$ are *isomorphic* if $f_x$ is an isomorphism for every $x$. A vector bundle is called *trivial* if it is isomorphic to the direct product $X \times \mathbb{C}^k$.

**Exercise 3.2.** Give an example of a map between vector bundles $f : E \to F$ on $X$ such that the rank of $\mathrm{Im}(f|_{E_x})$ is not a constant for all $x \in X$.

**Exercise 3.3.** Let $L$ be a line bundle on $X$. Prove that $L \otimes L^*$ is a trivial line bundle.

Define a *section* $\sigma$ as a holomorphic map $\sigma : X \to E$ such that $\sigma(x) \in E_x$ for every $x \in X$, i.e. $\pi \circ \sigma$ is identity. If $\sigma(x) = 0 \in E_x$, we say that $\sigma$ is vanishing on $x$.

**Exercise 3.4.** Let $L$ be a line bundle on $X$. Prove that $L$ is trivial if and only if it possesses a nowhere vanishing section.

**Example 3.5** (Holomorphic tangent bundles)**.** Let $X$ be a $n$-dimensional complex manifold. Suppose $\phi_U : U \to \mathbb{C}^n$ are coordinate charts of $X$. Define the (holomorphic) *tangent bundle* $T_X$ by setting $T_X = \bigcup T_x$ with

$$T_x = \mathbb{C}\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\} \cong \mathbb{C}^n$$

as well as transition functions $g_{UV} = J(\phi_V \phi_U^{-1})$, where $J$ denotes the Jacobian matrix $(\frac{\partial w_i}{\partial z_j})$ for $1 \le i, j \le n$. The dual bundle $T_X^*$ is called the *cotangent bundle* of $X$. The determinant $\det(T_X^*)$ is called the *canonical line bundle* of $X$.

**Remark 3.6.** Alternatively, one can define vector bundles on a topological space, a differential manifold and an algebraic variety. The above definitions and properties go through literally after replacing "holomorphic map" by "homomorphism", "smooth map" or "regular map".

3.2. **Vector bundles on a variety and locally free sheaves.** In the context of varieties (more rigorously, schemes), there is a *one-to-one correspondence* between isomorphism classes of vector bundles of rank $n$ and isomorphism classes of locally free sheaves of rank $n$ on a variety $X$. Here we briefly explain the idea. The reader can refer to Hartshorne II 5, especially Ex. 5.18 for more details.

Let $\mathscr{O}_X$ be the structure sheaf of a variety $X$. Note that $\mathscr{O}_X(U)$ has a *ring* structure (not only a group) for any open set $U$. A *sheaf of $\mathscr{O}_X$-modules* is a sheaf $\mathscr{F}$ on $X$ such that for each open set $U$, the group $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module. An $\mathscr{O}_X$-module $\mathscr{F}$ is called *free* if it is isomorphic to a direct sum of copies of $\mathscr{O}_X$. It is called *locally free* if there is an open covering $\underline{U} = \{U_\alpha\}$ such that for each open subset $U_\alpha$, $\mathscr{F}|_{U_\alpha}$ is a free $\mathscr{O}_X|_U$-module. The *rank* of $\mathscr{F}$ on $U$ is the number of copies of $\mathscr{O}$ in the summation. In this course we only consider the rank being finite (such sheaves are objects in the category of *coherent sheaves*). If $X$ is connected, the rank of $\mathscr{F}$ does not vary with the open subsets. In particular, a locally free sheaf of rank 1 is also called an *invertible sheaf.*

Roughly speaking, if $\mathscr{F}$ is locally free of rank $n$, we can choose a set of $n$ generators $x_1, \dots, x_n$ for the $\mathscr{O}_X(U)$-module $\mathscr{F}(U)$. They span an $n$-dimensional affine space $A[x_1, \dots, x_n]$ over $U$, where $A$ is the coordinate ring of $U$. By changing to a different set of generators over another open subset, one can write down the transition functions, hence it associates to $\mathscr{F}$ a vector bundle structure. Conversely if $F$ is a vector bundle on $X$, locally we have $F|_U \cong U \times \mathbb{A}^n$ with $x_1, \dots, x_n$ a basis (i.e. $n$ linearly independent sections) of $\mathbb{A}^n$ over $U$. Then we can associate to $F|_U$ an $\mathscr{O}_X(U)$-module of rank $n$ using $x_1, \dots, x_n$ as generators.

**Example 3.7.** Let $X \subset \mathbb{P}^n$ be a (smooth) variety and $Y \subset X$ a hypersurface, i.e. $Y$ is cut out (transversely) by a hypersurface $F$ in $\mathbb{P}^n$ with $X$. We have the short exact sequence

$$0 \to \mathscr{I}_{Y/X} \to \mathscr{O}_X \to \mathscr{O}_Y \to 0.$$

The ideal sheaf $\mathscr{I}_{Y/X}$ is an invertible sheaf. Indeed, for an open subset $U \subset X$, $\mathscr{I}_{Y/X}(U)$ can be expressed as $(F|_U) \cdot \mathscr{O}_X(U)$, hence is locally free of rank 1. The sheaf $\mathscr{O}_Y$ (extended to $X$ by zero) is *not* locally free. For $U \cap Y = \emptyset$, $\mathscr{O}_Y(U) = 0$ and for $U \cap Y \neq \emptyset$, $\mathscr{O}_Y(U)$ is non-zero. Later we will see how to construct a line bundle corresponding to $\mathscr{I}_{Y/X}$.

3.3. **Divisors.** Let $X$ be a variety (complex or algebraic). Suppose $Y \subset X$ is an irreducible subvariety of codimension one. We say that $Y$ is an *irreducible divisor* of $X$. More precisely, for every $p \in Y$ there exists an open neighborhood $U \subset X$ of $p$ such that $U \cap Y$ is cut out by a (holomorphic or regular) function $f$. We call $f$ a local defining equation for $Y$ near $p$. A *divisor* $D$ on $X$ is a formal linear combination of irreducible divisors:

$$D = \sum a_i Y_i,$$

where $a_i \in \mathbb{Z}$ (or $\mathbb{Q}$, $\mathbb{R}$ depending on the context). If $a_i \geq 0$ for all $i$, we say that $D$ is *effective* and denote it by $D \geq 0$. The divisors on $X$ form an additive group $\mathrm{Div}(X)$.

Suppose $f$ is a local defining equation of an irreducible divisor $Y \subset X$ on an open subset $U \subset X$. For another function $g$ on $X$, locally we can write

$$g = f^a \cdot h$$

such that the regular function $h$ is coprime with $f$ in $\mathscr{O}_X(U)$. We say that $a$ is the *vanishing order* of $g$ along $Y \cap U$. Note that the vanishing order is locally a constant, hence is independent of $U$. We use

$$\mathrm{ord}_Y(g) = a$$

to denote the vanishing order of $g$ along $Y$.

For two regular functions $g, h$ on $X$, we have

$$\mathrm{ord}_Y(gh) = \mathrm{ord}_Y(g) + \mathrm{ord}_Y(h).$$

For a meromorphic function $f = g/h$, we define

$$\mathrm{ord}_Y(f) = \mathrm{ord}_Y(g) - \mathrm{ord}_Y(h).$$

If $\mathrm{ord}_Y(f) > 0$, we say that $f$ has a *zero* along $Y$. If $\mathrm{ord}_Y(f) < 0$, we say that $f$ has a *pole* along $Y$. We also define the *divisor associated to* $f$ by

$$(f) = \sum_Y \mathrm{ord}_Y(f),$$

as well as the *divisor of zeros*

$$(f)_0 = \sum_Y \mathrm{ord}_Y(g)$$

and the *divisor of poles*

$$(f)_\infty = \sum_Y \mathrm{ord}_Y(h).$$

They satisfy

$$(f) = (f)_0 - (f)_\infty.$$

If $D = (f)$ is the associated divisor of a global meromorphic function $f$, $D$ is called a *principal divisor*.

Let $\mathscr{M}^*$ be the multiplicative sheaf of (not identically zero) meromorphic functions and $\mathscr{O}^*$ the multiplicative sheaf of nowhere vanishing regular functions, which is a subsheaf of $\mathscr{M}^*$.

**Proposition 3.8.** *We have a correspondence* $\mathrm{Div}(X) \cong H^0(X, \mathscr{M}^*/\mathscr{O}^*)$.

*Proof.* Suppose $\{f_\alpha\}$ represents a global section of $\mathscr{M}^*/\mathscr{O}^*$ with respect to an open covering $\underline{U} = \{U_\alpha\}$. Associate to it a divisor $D_\alpha = (f_\alpha)$ in $U_\alpha$. We claim that $D_\alpha = D_\beta$ in $U_\alpha \cap U_\beta$. This is due to

$$\frac{f_\alpha}{f_\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta),$$

hence $f_\alpha$ and $f_\beta$ define the same divisor. Consequently $\{D_\alpha\}$ defines a global divisor. Moreover, if $\{f_\alpha\}$ and $\{g_\alpha\}$ define the same divisor, then $f_\alpha/g_\alpha \in \mathscr{O}^*(U_\alpha)$, hence $\{f_\alpha\}$ and $\{g_\alpha\}$ represent the same section of $\mathscr{M}^*/\mathscr{O}^*$. This shows an injection

$$H^0(X, \mathscr{M}^*/\mathscr{O}^*) \hookrightarrow \mathrm{Div}(X).$$

Conversely, suppose $D = \sum a_i Y_i$ is a divisor on $X$ with $a_i \in \mathbb{Z}$ and $Y_i$ effective. We can choose an open covering $\underline{U} = \{U_\alpha\}$ such that $Y_i$ is locally defined by $g_{i\alpha} \in \mathscr{O}(U_\alpha)$. Consider

$$f_\alpha = \prod_i (g_{i\alpha})^{a_i} \in \mathscr{M}^*(U_\alpha).$$

Then we have

$$\frac{f_\alpha}{f_\beta} = \prod_i \Big(\frac{g_{i\alpha}}{g_{i\beta}}\Big)^{a_i}.$$

Both $g_{i\alpha}$ and $g_{i\beta}$ cut out the same divisor $Y_i|_{U_\alpha \cap U_\beta}$ in $U_\alpha \cap U_\beta$, hence we conclude that

$$\frac{g_{i\alpha}}{g_{i\beta}} \in \mathscr{O}^*(U_\alpha \cap U_\beta), \quad \frac{f_\alpha}{f_\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta).$$

Then $\{f_\alpha\}$ defines a global section of $\mathscr{M}^*/\mathscr{O}^*$. Finally if $D$ determines the zero section of $\mathscr{M}^*/\mathscr{O}^*$ (which is 1 since the group structure is multiplicative), it means locally $f_\alpha \in \mathscr{O}^*(U_\alpha)$ (after refining the open covering). Then it does not have zeros or poles, hence $D|_{U_\alpha} = 0$ for each $U_\alpha$ and $D$ is globally zero. This shows the other injection

$$\mathrm{Div}(X) \hookrightarrow H^0(X, \mathscr{M}^*/\mathscr{O}^*).$$

□

3.4. **Line bundles.** Recall that a line bundle $L$ on $X$ is a vector bundle of rank 1. Equivalently, it is a locally free sheaf of rank 1. Define the *Picard group* $\mathrm{Pic}(X)$ parameterizing isomorphism classes of line bundles on $X$. The group law is given by tensor product. We can interpret $\mathrm{Pic}(X)$ as a cohomology group.

**Proposition 3.9.** *There is a one-to-one correspondence between the isomorphism classes of line bundles on $X$ and $H^1(X, \mathscr{O}^*)$, i.e.*

$$\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}^*).$$

*Proof.* Take an open covering $\underline{U} = \{U_\alpha\}$ of $X$ with respect to the trivialization of a line bundle $L$. The transition function

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$$

can be regarded as a section of $\mathscr{O}^*(U_\alpha \cap U_\beta)$, satisfying

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1,$$

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1.$$

Therefore, $\{g_{\alpha\beta}\}$ is a cocycle in $C^1(\underline{U}, \mathscr{O}^*)$, hence represents a cohomology class in $H^1(X, \mathscr{O}^*)$.

Suppose $M$ is another line bundle with transition functions $\{h_{\alpha\beta}\}$. If $M$ and $L$ are isomorphic, then $L \otimes M^*$ is trivial, i.e. $\{g_{\alpha\beta}/h_{\alpha\beta}\}$ are transition functions of $L \otimes M^*$, which has a nowhere vanishing section $\sigma$. Suppose on $U_\alpha$ we have $\sigma_\alpha : U_\alpha \to \mathbb{C}^*$ as the restriction of $\sigma$. Then on $U_\alpha \cap U_\beta$ we have

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} \cdot \sigma_\alpha = \sigma_\beta.$$

Therefore we conclude that

$$\frac{g_{\alpha\beta}}{h_{\alpha\beta}} = \frac{\sigma_\beta}{\sigma_\alpha} \in \delta C^0(\underline{U}, \mathscr{O}^*).$$

□

Now we describe another important correspondence between line bundles and divisors. Suppose $D$ is a divisor on $X$ with local defining equations $\{f_\alpha\}$ such that $f_\alpha \in \mathscr{M}^*(U_\alpha)$. Define

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Then we have $g_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta)$. Moreover, $\{g_{\alpha\beta}\}$ satisfy the assumptions imposed to transition functions, hence they define a line bundle, denoted by $L = [D]$ or $L = \mathscr{O}_X(D)$. We have a group homomorphism

$$\mathrm{Div}(X) \to \mathrm{Pic}(X)$$

induced by

$$D + D' \mapsto [D] \otimes [D'].$$

We say that $D$ and $D'$ are *linearly equivalent*, if $[D]$ and $[D']$ are isomorphic line bundles. We denote linear equivalence by

$$D \sim D'.$$

**Proposition 3.10.** *The associated line bundle $[D]$ is trivial if and only if $D$ is a principal divisor, i.e. $D = (f)$ for some $f \in \mathscr{M}^*(X)$. Two divisors $D \sim D'$ if and only if $D - D'$ is a principal divisor.*

*Proof.* Suppose $D = (f)$ is the associated divisor of a meromorphic function $f$ on $X$. Then $D$ has local defining equations $\{f_\alpha = f|_{U_\alpha}\}$. The transition functions associated to $[D]$ are all equal to 1, hence $[D]$ is a trivial line bundle. Conversely, suppose $[D]$ is trivial. Then it has a nowhere vanishing section $\sigma$ whose restriction to $U_\alpha$ is denoted by $\sigma_\alpha$. The transition functions $g_{\alpha\beta} = f_\beta/f_\alpha$ defined above satisfy

$$g_{\alpha\beta} \cdot \sigma_\alpha = \sigma_\beta,$$

hence we have

$$\frac{f_\alpha}{\sigma_\alpha} = \frac{f_\beta}{\sigma_\beta} \in \mathscr{M}^*(U_\alpha \cap U_\beta).$$

We can glue $\{f_\alpha/\sigma_\alpha\}$ to form a global function $f \in \mathscr{M}^*(X)$. Since $\sigma$ is nowhere vanishing, we obtain that $(f) = D$. □

Let us summarize, using the short exact sequence

$$0 \to \mathscr{O}^* \to \mathscr{M}^* \to \mathscr{M}^*/\mathscr{O}^* \to 0.$$

Recall that

$$\mathrm{Div}(X) \cong H^0(X, \mathscr{M}^*/\mathscr{O}^*), \quad \mathrm{Pic}(X) \cong H^1(X, \mathscr{O}^*).$$

Then we have the long exact sequence

$$0 \to H^0(X, \mathscr{O}^*) \to H^0(X, \mathscr{M}^*) \xrightarrow{(\cdot)} \mathrm{Div}(X) \xrightarrow{[\cdot]} \mathrm{Pic}(X) \to \cdots$$

which encodes all information in the above discussions.

3.5. **Sections of a line bundle.** Let $L$ be a line bundle on $X$ with transition functions $\{g_{\alpha\beta}\}$. A *holomorphic section* $s$ of $L$ has restriction $s_\alpha \in \mathscr{O}(U_\alpha)$, satisfying

$$g_{\alpha\beta} s_\alpha = s_\beta.$$

Conversely, a collection $\{s_\alpha \in \mathscr{O}^*(U_\alpha)\}$ such that $s_\beta/s_\alpha = g_{\alpha\beta}$ determines a section of $L$.

Similarly, we define a *meromorphic section* $s$ to be a collection

$$\{s_\alpha \in \mathscr{M}(U_\alpha)\}$$

such that $g_{\alpha\beta} s_\alpha = s_\beta$. Suppose $t \not\equiv 0$ is another meromorphic section with collection $\{t_\alpha\}$. We have

$$\frac{s_\beta}{t_\beta} = \frac{s_\alpha}{t_\alpha},$$

hence the quotient $s/t$ is a global meromorphic function. Conversely, if $f$ is a global meromorphic function, then $\{f \cdot s_\alpha\}$ defines another meromorphic section of $L$.

For a meromorphic section $s \not\equiv 0$, consider the divisor

$$(s_\alpha)$$

associated to the local section $s_\alpha$ in $U_\alpha$. Since

$$\frac{s_\beta}{s_\alpha} = g_{\alpha\beta}$$

is nowhere vanishing, $\{(s_\alpha)\}$ form a global divisor $(s)$ on $X$. Conversely, suppose $D$ is a divisor. Consider the construction of the associated line bundle $[D]$. Suppose the local defining equations of $D$ are given by $\{s_\alpha\}$. Then the transition functions of $[D]$ are $\{g_{\alpha\beta} = s_\beta/s_\alpha\}$ and consequently the collection $\{s_\alpha\}$ gives rise to a meromorphic section of $[D]$. Note that for a section $s$ of $L$, the divisor $(s)$ is effective if and only if $s$ is a holomorphic section. We thus obtain the following result.

**Proposition 3.11.** *For any section $s$ of $L$, we have $L \cong [(s)]$. A line bundle $L$ is associated to a divisor $D$ if and only if it has a meromorphic section $s$ such that $(s) = D$. In particular, $L$ has a holomorphic section if and only if it is associated to an effective divisor.*

Now we treat a line bundle as a locally free sheaf of rank 1 and reinterpret the above correspondence. Let $D$ be a divisor on $X$. Define a sheaf $\mathscr{O}_X(D)$ or simply $\mathscr{O}(D)$ by

$$\mathscr{O}(D)(U) = \{f \in \mathscr{M}(U) : (f) + D|_U \geq 0\}.$$

It has a vector space structure since $(f) + D|_U \geq 0$ and $(g) + D|_U \geq 0$ implies that $(af + bg) + D|_U \geq 0$ for any $a, b$ in the base field.

**Proposition 3.12.** *The space of holomorphic sections of* $[D]$ *can be identified with* $H^0(X, \mathscr{O}(D))$.

*Proof.* A global section $s \in H^0(X, \mathscr{O}(D))$ is a meromorphic function satisfying

$$(s) + D \geq 0.$$

Suppose $D$ is locally defined by $\{f_\alpha\}$. The associated line bundle $[D]$ has transition functions

$$\{g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}\}.$$

Then the collection $\{s \cdot f_\alpha\}$ defines a section $\sigma$ of $[D]$. Since

$$(s) + (f_\alpha) \geq 0$$

in every $U_\alpha$, $\sigma$ is a holomorphic section of $[D]$. Moreover, the associated divisor $D' = (s) + D$ of the section is linearly equivalent to $D$, since $D' - D = (s)$ is principal.

Conversely, given a holomorphic section $\sigma$ of $[D]$, i.e. a collection $\{h_\alpha \in \mathscr{O}(U_\alpha)\}$ such that

$$\frac{h_\beta}{h_\alpha} = g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Then $\{h_\alpha/f_\alpha\}$ defines a global meromorphic function $h$. Since $(h_\alpha) \geq 0$ in every $U_\alpha$, we have

$$(h|_{U_\alpha}) + (f_\alpha) = (h_\alpha) \geq 0,$$

hence $(h) + D \geq 0$ globally on $X$ and $h \in H^0(X, \mathscr{O}(D))$. □

**Remark 3.13.** Replacing $X$ by any open subset $U$, the proposition implies that the sheaf $\mathscr{O}(D)$ can be regarded as gathering local holomorphic sections of the line bundle $[D]$. If $D \sim D'$, i.e. $D' - D = (f)$ for a global meromorphic function $f$, then for any $g \in \mathscr{O}(D')(U)$, we have

$$0 \leq (g) + D' = (g) + (f) + (D) = (fg) + (D)$$

restricted to $U$. So we obtain an isomorphism

$$\mathscr{O}(D')(U) \xrightarrow{\cdot f} \mathscr{O}(D)(U)$$

for any open subset $U$, compatible with the sheaf restriction maps. In this sense, the sheaf $\mathscr{O}(D)$ and the line bundle $[D]$ have a one-to-one correspondence up to isomorphism and linear equivalence (assuming that every line bundle can be associated to a divisor).

Let $|D|$ be the set of effective divisors that are linearly equivalent to $D$. We call $|D|$ the *linear system* associated to $D$.

**Proposition 3.14.** *Let* $X$ *be compact and* $D$ *a divisor on* $X$. *Then we have*

$$\mathbb{P}H^0(X, \mathscr{O}(D)) = |D|,$$

*i.e. an effective divisor in* $|D|$ *and a holomorphic section of* $[D]$ *(up to scalar) determine each other.*

*Proof.* For any $D' \in |D|$, by definition $D'-D = (f)$ is principal for some $f \in \mathscr{M}(X)$, hence $(f)+D = D' \geq 0$ and $f \in H^0(X, \mathscr{O}(D))$. Since $X$ is compact, if $g$ is another function such that $D' - D = (g)$, then $(f/g) = 0$, i.e. $f/g$ is holomorphic, hence it is a constant.

Conversely, any $f \in H^0(X, \mathscr{O}(D))$ defines an effective divisor $D' = (f) + D$. If $(f) + D = (g) + D$, then $(f/g) = 0$ and $f/g$ is a constant, since $X$ is compact. □

**Exercise 3.15.** Let $D = \sum a_i p_i$ be a divisor on $\mathbb{P}^1$ with $a_i \in \mathbb{Z}$ and $p_i \in \mathbb{P}^1$. Define the *degree* of $D$ by $\deg(D) = \sum a_i$.

(1) Prove that $D \sim D'$ if and only if $\deg(D) = \deg(D')$.

(2) Calculate the cohomology $H^*(\mathbb{P}^1, \mathscr{O}(D))$ in terms of $\deg(D)$.

## 4. Algebraic curves

In this section, we apply the techniques developed before to study algebraic curves.

### 4.1. **The Riemann-Roch formula.** 

Let $X$ be a compact Riemann surface or a closed algebraic curve. Define its *arithmetic genus* by

$$g := h^1(\mathscr{O}_X) = \dim_{\mathbb{C}} H^1(\mathscr{O}_X).$$

**Theorem 4.1** (Riemann-Roch Formula)**.** *Let $D$ be a divisor on $X$ and $\mathscr{O}(D)$ the associated line bundle or locally free sheaf of rank 1. Then we have*

$$h^0(\mathscr{O}(D)) - h^1(\mathscr{O}(D)) = 1 - g + \deg(D).$$

**Remark 4.2.** Define the (holomorphic) *Euler characteristic* of a sheaf $\mathscr{F}$ by

$$\chi(\mathscr{F}) := \sum_{i \geq 0} (-1)^i h^i(\mathscr{F}).$$

The Riemann-Roch formula can be written as

$$\chi(\mathscr{O}(D)) - \chi(\mathscr{O}_X) = \deg(D).$$

*Proof.* Let us first prove it for effective divisors of degree $\geq 0$. Do induction on $n$. The formula obviously holds for $\mathscr{O}_X$. Suppose it is true for $\deg(D) < n$. Consider $D = p + D'$ with $D'$ an effective divisor of degree $n-1$. We have the short exact sequence

$$0 \to \mathscr{O}(D') \to \mathscr{O}(D) \to \mathbb{C}_p \to 0,$$

where $\mathbb{C}_p$ is the skyscraper sheaf with a single stalk $\mathbb{C}$ supported at $p$. The exactness can be easily checked. The map $\mathscr{O}(D') \to \mathscr{O}(D)$ is an inclusion, since

$$(f) + D' \geq 0$$

implies that

$$(f) + D = (f) + D' + p \geq 0.$$

The quotient corresponds to germs of functions $f$ at $p$ such that

$$(f)|_U + D'|_U = -p$$

in arbitrarily small neighborhoods $U$ of $p$. In other words, if $\mathrm{ord}_p(D') = m \geq 0$, we can write

$$f = z^{-m-1} h(z),$$

where $h \in \mathscr{O}^*(U)$. So the quotient sheaf is given by $\mathbb{C} \cdot \{z^{-m-1}\} \cong \mathbb{C}$ supported at $p$. Since the associated cohomology sequence is long exact, we have

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D')) + 1 = 1 - g + (n-1) + 1 = 1 - g + n.$$

In general, write a divisor $D = D_1 - D_2$, where $D_1$ and $D_2$ are both effective divisors of degree $d_1$ and $d_2$, respectively, and $d_1 - d_2 = \deg(D)$. By the same token, we have the short exact sequence

$$0 \to \mathscr{O}(D) \to \mathscr{O}(D_1) \to \mathbb{C}^{d_2} \to 0.$$

Then we obtain that

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D_1)) - d_2 = 1 - g + d_1 - d_2 = 1 - g + \deg(D).$$

□

**Remark 4.3.** Assuming the Serre duality

$$H^1(\mathscr{O}(D)) \cong H^0(K \otimes \mathscr{O}(-D)),$$

where $K$ is the canonical line bundle or the cotangent line bundle or the dualizing sheaf of $X$, then we can rewrite the Riemann-Roch formula as

$$h^0(L) - h^0(K \otimes L^*) = 1 - g + \deg(L),$$

where $L$ is a line bundle on $X$. Note that $K$ is a degree $2g-2$ line bundle (to be discussed later). We conclude that

$$h^0(K) = g, \quad h^1(K) = h^0(\mathscr{O}) = 1.$$

It implies that the space of holomorphic one-forms on a genus $g$ Riemann surface is $g$-dimensional.

4.2. **The Riemann-Hurwitz formula.** A *branched cover* $\pi : X \to Y$ between two (compact, connected) Riemann surfaces is a (surjective) regular morphism. For a general point $q \in Y$, $\pi^{-1}(q)$ consists of $d$ distinct points. Call $d$ the *degree* of $\pi$. Locally around $p \mapsto q$, if the map is given by

$$x \mapsto y = x^m,$$

where $x, y$ are local coordinates of $p, q$, respectively, call $m$ the *vanishing order* of $\pi$ at $p$ and denote it by

$$\mathrm{ord}_p(\pi) = m.$$

If $\mathrm{ord}_p(\pi) > 1$, we say that $p$ is a *ramification point*. If $\pi^{-1}(q)$ contains a ramification point, then $q$ is called a *branch point*. Define the *pullback*

$$\pi^*(q) = \sum_{p \in \pi^{-1}(q)} \mathrm{ord}_p(\pi) \cdot p.$$

Note that $\pi^*(q)$ is a degree $d$ effective divisor on $X$.

**Theorem 4.4** (Riemann-Hurwitz Formula)**.** *Let $\pi : X \to Y$ be a branched cover between two Riemann surfaces. Then we have*

$$K_X \sim \pi^* K_Y + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1) \cdot p,$$

*where $K_X$ and $K_Y$ are canonical divisors on $X$ and $Y$, respectively.*

*Proof.* Take a one-form $\omega$ on $Y$ locally expressed as $f(w)dw$ around a point $q = \pi(p)$. Suppose the covering at $p$ is given by

$$z \mapsto w = z^m,$$

then we have

$$\pi^*(f(w)dw) = mf(z^m)z^{m-1}dz.$$

Namely, the associated divisors satisfy the relation

$$(\pi^*\omega)|_U = (\pi^*(\omega))|_U + (\mathrm{ord}_p(\pi) - 1) \cdot p$$

in a local neighborhood $U$ of $p$. So globally it implies that

$$(\pi^*\omega) = \pi^*(\omega) + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1) \cdot p.$$

Since $\pi^*\omega$ is a one-form on $X$, $(\pi^*\omega)$ is a canonical divisor of $X$ and the claimed formula follows. □

We can interpret the (numerical) Riemann-Hurwitz formula from a topological viewpoint. Let $\chi(X)$ denote the *topological Euler characteristic* of $X$. If $X$ is a Riemann surface of genus $g$, take a triangulation of $X$ and suppose the number of $k$-dimensional edges is $c_k$ for $k = 0, 1, 2$. Then we have

$$\chi(X) = c_0 - c_1 + c_2 = 2 - 2g.$$

**Proposition 4.5.** *Let $\pi : X \to Y$ be a degree $d$ branched cover between two Riemann surfaces. Then we have*

$$\chi(X) = d \cdot \chi(Y) - \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1).$$

*Proof.* Take a triangulation of $Y$ such that every branch point is a vertex. Pull it back as a triangulation of $X$. Note that it pulls back a face to $d$ faces, an edge to $d$ edges and a vertex $v$ to $|\pi^{-1}(v)|$ vertices. Note that if

$$\pi^{-1}(v) = \sum_{i=1}^{k} m_i p_i$$

for distinct points $p_i$, then $|\pi^{-1}(v)| = m$. In other words, we have

$$|\pi^{-1}(v)| = d - \sum_{p \in \pi^{-1}(v)} (\mathrm{ord}_p(\pi) - 1).$$

Then the claimed formula follows right away. □

**Corollary 4.6** (Numerical Riemann-Hurwitz)**.** *Let $\pi : X \to Y$ be a degree $d$ branched cover between two Riemann surfaces of genus $g$ and $h$, respectively. Then we have*

$$2g - 2 = d(2h - 2) + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1).$$

*In particular, if $g < h$, such branched covers do not exist.*

**Corollary 4.7.** *The canonical line bundle of a genus $g$ Riemann surface $X$ has degree equal to $2g - 2$.*

*Proof.* Every Riemann surface $X$ possesses a nontrivial meromorphic function, say by the Riemann-Roch formula. It induces a branched cover $\pi : X \to \mathbb{P}^1$ of degree $d$. By the Riemann-Hurwitz Formula we know

$$\deg(K_X) = d(-2) + \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1),$$

since we have seen that $\deg(K_{\mathbb{P}^1}) = -2$. By the Numerical Riemann-Hurwitz we have

$$2 - 2g = 2d - \sum_{p \in X} (\mathrm{ord}_p(\pi) - 1).$$

Then the claim follows immediately. □

**Exercise 4.8.** Let $X$ be a Riemann surface or algebraic curve of genus $g$. If $X$ admits a branched cover of degree 2 to $\mathbb{P}^1$, we say that $X$ is a *hyperelliptic curve.* Prove that every $g \leq 2$ curve is hyperelliptic. For $g \geq 2$, can you calculate the dimension of the parameter space of genus $g$ hyperelliptic curves?

**Remark 4.9.** The moduli space of genus $g$ curves has dimension $3g - 3$, which is bigger than $2g - 1$ for $g > 2$. Hence a general $g > 2$ curve is *not* hyperelliptic.

4.3. **Genus formula of plane curves.** Suppose $F(Z_0, Z_1, Z_2)$ is a general degree $d$ homogeneous polynomial whose vanishing locus is a plane curve $C \subset \mathbb{P}^2$. Since $F$ is general, $C$ is smooth. In other words, the singularities of $C$ locate at the common zeros of $F = 0$ and $\partial F / \partial Z_i = 0$ for all $i$, which are empty for a general $F$.

**Theorem 4.10.** *In the above setting, the genus $g$ of $C$ is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

*Proof.* We give two proofs. The first one is more algebraic. Note that all degree $d$ curves are linearly equivalent in $\mathbb{P}^2$. Hence it makes sense to use $\mathscr{O}_{\mathbb{P}^2}(d)$ to denote the associated line bundle. In particular, $\mathscr{O}(1)$ is the associated line bundle of a line $L$. Then we have the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_L \to 0.$$

Tensor it with $\mathscr{O}_{\mathbb{P}^2}(1 - m)$. We obtain that

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-m) \to \mathscr{O}_{\mathbb{P}^2}(-(m-1)) \to \mathscr{O}_{\mathbb{P}^2}(1-m)|_L \to 0.$$

Since $\mathscr{O}_{\mathbb{P}^2}(1-m)|_L$ is the line bundle associated to a degree $1 - m$ divisor on $L$ and $L \cong \mathbb{P}^1$, we conclude that

$$\chi(\mathscr{O}_{\mathbb{P}^2}(-(m-1))) - \chi(\mathscr{O}_{\mathbb{P}^2}(m)) = \chi(\mathscr{O}_{\mathbb{P}^1}(1-m)) = 2 - m,$$

where we apply the Riemann-Roch formula to $\mathbb{P}^1$ in the last equality. Then we obtain that

$$\chi(\mathscr{O}_{\mathbb{P}^2}) - \chi(\mathscr{O}_{\mathbb{P}^2}(-d)) = \sum_{m=1}^{d} (2 - m) = -\frac{d(d-3)}{2}.$$

Now by the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-d) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_C \to 0,$$

we have

$$1 - g = \chi(\mathscr{O}_C) = -\frac{d(d-3)}{2},$$

hence the genus formula follows.

The other proof is an application of the Riemann-Hurwitz formula. Without loss of generality, suppose $o = [0,0,1] \notin C$. Let $L$ be the line $Z_2 = 0$ and project $C$ to $L$ from $o$, i.e.

$$[Z_0, Z_1, Z_2] \mapsto [Z_0, Z_1].$$

In affine coordinates $x = Z_1/Z_0$ and $y = Z_2/Z_0$, this map is given by

$$(x, y) \mapsto x,$$

i.e. we project $C$ vertically to the $x$-axis. This yields a degree $d$ branched cover

$$\pi : C \to L \cong \mathbb{P}^1.$$

A point $p$ is a ramification point of $\pi$ if and only if there exists a vertical line tangent to $C$ at $p$, i.e. $p$ is a common zero of $F$ and $\partial F/\partial Z_2$. Since $F$ and $\partial F/\partial Z_2$ have degree $d$ and $d-1$, respectively, they intersect at $d(d-1)$ points. By Riemann-Hurwitz, we have

$$2g - 2 = d(-2) + d(d-1),$$

hence the genus formula follows. In order to make sure all the ramifications are simple, we may choose a general projection direction such that it is different from those of the (finitely many) lines with higher tangency to $C$. $\square$

**Remark 4.11.** In the first proof, indeed we did not use the smoothness of $C$. So the (arithmetic) genus formula holds for an arbitrary plane curve, even if it is singular, reducible or non-reduced. In the second proof, even if the projection has higher ramification points, a detailed local study plus Riemann-Hurwitz still gives rise to the desired formula.

4.4. **Base point free and very ample line bundles.** Let $L$ be a line bundle or a locally free sheaf of rank 1 on $X$. We say that $L$ has a *base point* at $p \in X$ if $p$ belongs to the vanishing locus of every regular section of $L$. If the base locus of $L$ is empty, then $L$ is called *base point free*.

For a base point free line bundle $L$, let $\sigma_0, \ldots, \sigma_n$ be a basis of the space $H^0(L)$ of regular sections. Locally around a point $p \in X$, treat $\sigma_i$ as a regular function and associate to $p$ the point

$$[\sigma_0(p), \ldots, \sigma_n(p)] \in \mathbb{P}^n.$$

This is well-defined, since if we take a different chart, then we get $\sigma_i'(p) = g_{\alpha\beta}\sigma_i(p)$ where $\{g_{\alpha\beta}\}$ are transition functions of $L$. Therefore, we obtain a regular map

$$\phi_L : X \to \mathbb{P}^n.$$

We can give a more conceptual and coordinate free description of $\phi_L$. Since $L$ is base point free, the space of regular sections $\sigma$ vanishing at $p$ forms a hyperplane $H_p \subset H^0(L) \cong \mathbb{C}^{n+1}$. Then one can define $\phi_L(p) = [H_p] \in (\mathbb{P}^n)^*$ in the dual projective space parameterizing hyperplanes.

**Proposition 4.12.** *In the above setting, there is a one-to-one correspondence between hyperplane sections of $X$ and effective divisors in the linear system $|L|$.*

*Proof.* This is just a reformulation of the one-to-one correspondence

$$|L| = \mathbb{P}H^0(L),$$

which we proved before. In other words, an effective divisor in $|L|$ uniquely determines a regular section $\sigma = \sum a_i \sigma_i$ mod scalar. $\square$

**Example 4.13.** If $L = \mathscr{O}$, then we have $\phi_{\mathscr{O}}$ maps $X$ to a single point.

**Example 4.14.** Let $X = \mathbb{P}^1$ and $L = \mathscr{O}(2p)$ where $p = [0, 1]$. Then $H^0(L)$ is 3-dimensional and we can choose a basis by

$$1, \quad \frac{Y}{X}, \quad \frac{Y^2}{X^2}.$$

Recall that around $p$ the sections of $L$ are given by $f \cdot (X/Y)^2$. Hence we obtain that

$$\phi_L([X, Y]) = [X^2, XY, Y^2],$$

which is a smooth conic in $\mathbb{P}^2$. The genus formula for a plane curve of degree 2 also implies that the image has $g = 0$.

**Exercise 4.15.** A variety $X \subset \mathbb{P}^n$ is called *non-degenerate* if it is not contained in any hyperplane.

(1) Show that any non-degenerate smooth rational curve in $\mathbb{P}^n$ has degree $\geq n$.

(2) For $d \geq n \geq 3$, show that there exist non-degenerate smooth degree $d$ rational curves in $\mathbb{P}^n$.

**Example 4.16.** Let $E$ be an elliptic curve and $L = \mathscr{O}(2p)$. By Riemann-Roch, $h^0(L) = 2$. Moreover, $L$ is base point free. Otherwise if $q$ is a base point, then $q$ has to be $p$ and there exists another effective divisor $p + r \in |2p|$ such that $p + r \sim 2p$. But this implies $r - p$ is principal and $E \cong \mathbb{P}^1$, leading to a contradiction. Now $\phi_L : E \to \mathbb{P}^1$ is a branched cover of degree 2. Two points $s$ and $t$ lie in the same fiber if and only if $s + t \sim 2p$.

The above example indicates that $\phi_L$ is *not* always an embedding. We say that $L$ is *very ample* if $\phi_L$ is an embedding and that $L$ is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

**Example 4.17.** The line bundle $\mathscr{O}(d)$ is very ample on $\mathbb{P}^1$ if and only if $d > 0$. The induced map $\phi$ embeds $\mathbb{P}^1$ into $\mathbb{P}^d$ as a degree $d$ smooth rational curve, i.e. a rational normal curve.

Let us give a criterion when $L$ is base point free or very ample on an algebraic curve.

**Proposition 4.18.** *Let $L$ be a line bundle on a curve $X$.*

*(1) $L$ is base point free if and only if*

$$h^0(L \otimes \mathscr{O}(-p)) = h^0(L) - 1$$

*for any $p \in X$.*

*(2) $L$ is very ample if and only $L$ is base point free and for any $p, q \in X$ (not necessarily distinct)*

$$h^0(L \otimes \mathscr{O}(-p - q)) = h^0(L \otimes \mathscr{O}(-p)) - 1 = h^0(L \otimes \mathscr{O}(-q)) - 1.$$

*Proof.* Treat $L$ as a locally free sheaf of rank 1. By the short exact sequence

$$0 \to L \otimes \mathscr{O}(-p) \to L \to \mathbb{C}_p \to 0,$$

we have

$$h^0(L) - 1 \leq h^0(L \otimes \mathscr{O}(-p)) \leq h^0(L).$$

Then $L$ has a base point at $p$ if and only if all sections of $L$ vanish at $p$, i.e. $H^0(L \otimes \mathscr{O}(-p)) = H^0(L)$. This proves (1).

For (2), a very ample line bundle is necessarily base point free by definition. If $p \neq q \in X$ have the same image under $\phi_L$, it is equivalent to saying that the subspace of sections vanishing at $p$ is the same as the subspace of sections vanishing at $q$, which is further equivalent to

$$h^0(L \otimes \mathscr{O}(-p)) = h^0(L \otimes \mathscr{O}(-p-q)) = h^0(L \otimes \mathscr{O}(-q)).$$

Moreover, $\phi_L$ induces an injection restricted to the tangent space $T_p(X)$ if and only if there exists a hyperplane such that it cuts out $X$ locally a simple point at $p$, namely, if and only if there is a section vanishing at $p$ with multiplicity 1, i.e.

$$h^0(L \otimes \mathscr{O}(-2p)) < h^0(L \otimes \mathscr{O}(-p)).$$

But we have

$$h^0(L \otimes \mathscr{O}(-2p)) \geq h^0(L \otimes \mathscr{O}(-p)) - 1.$$

Hence (2) follows by combining the two cases. $\square$

**Remark 4.19.** In (2), for $p \neq q$ the condition geometrically means the sections of $L$ separate any two points. When $p = q$, it says that the sections of $L$ separate tangent vectors at $p$.

**Example 4.20.** Let $E$ be an elliptic curve, i.e. a torus as a Riemann surface of genus 1. Fix a point $p \in E$. The morphism

$$\tau : E \to J(E) \cong \mathrm{Pic}^0(E)$$

by $\tau(q) = [q - p]$ is an isomorphism. This defines a group law on $E$ with respect to $p$, i.e. $q + r = s$, where $s \in E$ is the unique point satisfying

$$(q - p) + (r - p) \sim s - p.$$

Now consider the linear system $|3p|$ on $E$. Since

$$h^0(\mathscr{O}(3p)) = 3, \quad h^0(\mathscr{O}(2p)) = 2, \quad h^0(\mathscr{O}(p)) = 1,$$

$\mathscr{O}(3p)$ is very ample. It induces an embedding of $E$ into $\mathbb{P}^2$ as a plane cubic. A line cuts out a divisor of degree 3 in $E$, say, $q + r + s$ (not necessarily distinct) if and only if

$$q + r + s \sim 3p.$$

Note that the tangent line $L$ of $E$ at $p$ is a *flex line*, i.e. the tangency multiplicity $(L \cdot E)_p = 3$. Such $p$ is called a *flex point*.

**Exercise 4.21.** Show that there are in total 9 flex points on a smooth plane cubic.

Let $V \subset |L|$ be a linear subspace. We say that $V$ is a *linear series* of $L$. The linear system $|L|$ is also called a *complete linear series*. The above definitions and properties go through similarly for the induced map $\phi_V$.

**Exercise 4.22.** Write down a linear series of $|\mathscr{O}(3)|$ on $\mathbb{P}^1$ such that it maps $\mathbb{P}^1$ into $\mathbb{P}^2$ as a *singular* plane cubic. How many different types of such singular plane cubics can you describe?

4.5. **Canonical maps.** Let $K$ be the canonical line bundle a curve $X$. If $X$ is $\mathbb{P}^1$, $\deg(K) = -2$ and $K$ is not effective. If $X$ is an elliptic curve, then $K \cong \mathscr{O}$ and the induced map $\phi_K$ is onto a point. From now on, assume that the genus $g$ of $X$ satisfies $g \geq 2$. Recall that $X$ is called *hyperelliptic* if it admits a degree 2 branched cover of $\mathbb{P}^1$. Two points $p, q \in X$ are called *conjugate* if they have the same image in $\mathbb{P}^1$. A ramification point of the double cover is called a *Weierstrass point* of $X$, i.e. it is self conjugate. By Riemann-Hurwitz, a genus $g \geq 2$ hyperelliptic curve possesses $2g+2$ Weierstrass points.

**Lemma 4.23.** *If $X$ is a hyperelliptic curve of genus $\geq 2$, then $X$ admits a unique double cover of $\mathbb{P}^1$.*

*Proof.* Otherwise suppose $h^0(\mathscr{O}(p+q)) = 2$ and $h^0(\mathscr{O}(p+r)) = 2$ for $q \neq r$. Note that $h^0(\mathscr{O}(p+q+r)) < 3$, since $X$ cannot be a plane cubic by the genus formula. Then we conclude that

$$H^0(\mathscr{O}(p+q)) = H^0(\mathscr{O}(p+r)) = H^0(\mathscr{O}(p+q+r)),$$

which implies that both $q, r$ are base points of $|p+q+r|$ and $h^0(\mathscr{O}(p)) = 2$, $X \cong \mathbb{P}^1$, leading to a contradiction. $\square$

**Proposition 4.24.** *Let $X$ be a curve of genus $g \geq 2$. Then the canonical line bundle $K$ is base point free. The induced map*

$$\phi_K : X \to \mathbb{P}^{g-1}$$

*is an embedding if and only if $X$ is not hyperelliptic. If $X$ is hyperelliptic, $\phi_K$ is a double cover of a rational normal curve in $\mathbb{P}^{g-1}$.*

*Proof.* First, let us show that $K$ is base point free. For any point $p \in X$, by Riemann-Roch we have

$$h^0(K \otimes \mathscr{O}(-p)) - h^0(\mathscr{O}(p)) = 1 - g + (2g-3),$$
$$h^0(K \otimes \mathscr{O}(-p)) = g - 1 = h^0(K) - 1.$$

Hence $K$ satisfies the criterion of base point freeness.

Next, $K$ fails to separate $p, q$ (not necessarily distinct) if and only if

$$h^0(K \otimes \mathscr{O}(-p-q)) = h^0(K \otimes \mathscr{O}(-p)) = g-1,$$

which is equivalent to, by Riemann-Roch again, that

$$h^0(\mathscr{O}(p+q)) = 2.$$

In other words, the linear system $|p+q|$ induces a double cover $X \to \mathbb{P}^1$.

Finally, if $X$ is hyperelliptic of genus $\geq 2$, it admits a unique double cover of $\mathbb{P}^1$. By the above analysis, two points $p, q$ have the same image under the canonical map if and only if $h^0(p+q) = 2$, i.e. $p, q$ are conjugate. Then the canonical map is a double cover of a rational curve of degree $\deg(K)/2 = g-1$ in $\mathbb{P}^{g-1}$, i.e. a rational normal curve. A hyperplane section of $\phi_K(X)$ pulls back to $X$ a divisor

$$\sum_{i=1}^{g-1}(p_i + q_i),$$

where $p_i, q_i$ are conjugate or $p_i = q_i$ a Weierstrass point. $\square$

**Remark 4.25.** For a non-hyperelliptic curve $X$, $\phi_K$ is called the *canonical embedding* of $X$ and its image is called a *canonical curve*.

**Example 4.26.** Let $X$ be a genus 2 curve. Then $h^0(K) = 2$, hence $X$ is hyperelliptic and the double cover of $\mathbb{P}^1$ is induced by the canonical line bundle, as we have seen.

**Example 4.27.** A genus 3 non-hyperelliptic curve admits a canonical embedding to $\mathbb{P}^2$ as a plane quartic. An effective canonical divisor corresponds to a line section of the quartic. By the genus formula, any smooth plane quartic also has genus equal to 3. Moreover, a smooth plane quartic $X$ gives rise to a line bundle $L$ of degree 4 on $X$ by restricting $\mathscr{O}_{\mathbb{P}^2}(1)$. By Riemann-Roch, $h^0(K \otimes L^*) \geq 1$, but $\deg(K \otimes L^*) = 0$, hence $L = K$. So any plane quartic is a canonical embedding of a genus 3 non-hyperelliptic curve.

**Example 4.28.** Let $X$ be a genus 4 non-hyperellipic curve. Then its canonical embedding is a degree 6 curve in $\mathbb{P}^3$. Let $\mathscr{O}_{\mathbb{P}^3}(1)$ denote the line bundle on $\mathbb{P}^3$ associated to a hyperplane class. Its restriction to $X$ is the canonical line bundle $K_X$. We have the exact sequence

$$0 \to \mathscr{I}_C \otimes \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_C(2) \to 0.$$

Since $h^0(\mathscr{O}_{\mathbb{P}^3}(2)) = 10 > h^0(\mathscr{O}_C(2)) = 9$, we conclude that $C$ is contained in a quadric surface $Q$ in $\mathbb{P}^3$. Indeed $Q$ is unique, because otherwise $\deg(C) \leq 4$.

If $Q$ is smooth, it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Up to projective equivalence, it has coordinates

$$[XZ, XW, YZ, YW],$$

i.e. its equation is

$$Z_0Z_3 - Z_1Z_2 = 0.$$

Note that

$$Z_0 - aZ_1 = Z_2 - aZ_3 = 0$$

defines a family $A$ of lines in $Q$ parameterized by the value of $a$. Any two lines in $A$ are disjoint. Similarly,

$$Z_0 - bZ_2 = Z_1 - bZ_3 = 0$$

defines another family $B$ of lines in $Q$ parameterized by $b$, and any two lines in $B$ are disjoint. Moreover, a line $L_1$ in $A$ and a line $L_2$ in $B$ intersect at a unique point $[ab, b, a, 1]$. Hence they span a plane $H$ in $\mathbb{P}^3$. Suppose $H \cdot X = D_1 + D_2$, where $D_i$ is a divisor in $L_i$. Since $L_i$ varies in $Q$ in a $\mathbb{P}^1$ family, we have $h^0(\mathscr{O}_X(D_i)) \geq 2$ for $i = 1, 2$. Moreover, because $D_1 + D_2 \sim K_X$, we have $\deg(D_1) + \deg(D_2) = 6$. Since $X$ is not hyperelliptic, we conclude that $\deg(D_i) = 3$ for $i = 1, 2$. This can also be seen since $X$ is a complete intersection of $Q$ with a cubic surface, hence its class in $Q$ is linearly equivalent to $3L_1 + 3L_2$. Therefore, we obtain two distinct line bundles of degree 3 given by $\mathscr{O}_X(D_1)$ and $\mathscr{O}_X(D_2)$ such that $h^0(\mathscr{O}_X(D_i)) = 2$. In other words, $X$ admits two triple covers of $\mathbb{P}^1$. This can be seen by projecting $X$ along the direction of $L_i$ for $i = 1, 2$.

If $Q$ is singular, since $X$ is non-degenerate, $Q$ is a quadric cone whose equation is given by

$$Z_0^2 - Z_1Z_2 = 0.$$

It has a unique singular point $v = [0, 0, 0, 1]$ as the vertex and $Q$ can be viewed as the cone over a plane conic $C$ defined by the same equation. Every line $L$ contained in $Q$ passes through $v$, hence $C$ parameterizes a $\mathbb{P}^1$-family of lines. Similarly, one checks that $L \cdot X = 3$ and $X$ admits a unique triple cover of $\mathbb{P}^1$ by projecting from $v$ to $C$.

**Exercise 4.29.** Suppose $g \geq 3$. Let $X \subset \mathbb{P}^{g-1}$ be a non-degenerate smooth genus $g$ curve of degree $2g-2$. Show that $X$ is a canonical curve.

**Exercise 4.30.** Let $X$ be a genus 4 non-hyperelliptic curve. If $D = p + q + r$ is a degree 3 effective divisor on $X$ such that $h^0(\mathscr{O}(D)) = 2$, show that $p$, $q$ and $r$ are collinear in the canonical embedding of $X$ in $\mathbb{P}^3$.

**Exercise 4.31.** A hyperelliptic curve $X$ of genus $g$ can be explicitly written as the locus of $(z, w)$ satisfying

$$w^2 = (z - a_1) \cdots (z - a_{2g+2}).$$

One can treat $(w, z) \mapsto z$ as the double cover of $\mathbb{P}^1$ which is branched at $a_1, \ldots, a_{2g+2}$. Prove that a basis for the space $H^0(K)$ of holomorphic one-forms on $X$ is

$$\frac{dz}{w}, z\frac{dz}{w}, \ldots, z^{g-1}\frac{dz}{w}.$$

4.6. **An informal moduli counting.** In this section we present some (heuristic) properties about moduli of curves. We want to find a "moduli space" $\mathcal{M}_g$ parameterizing isomorphism classes of genus $g$ curves and study its geometry. If $g = 0$, we just get a pointed space representing the unique isomorphism class of $\mathbb{P}^1$. For $g = 1$, the $j$-invariant characterizes elliptic curves, so their parameter space should be of 1-dimensional. Alternatively, we can take the space of smooth plane cubics modulo PGL(3), which gives rise to a 1-dimensional parameter space of elliptic curves. For higher genus, let us first consider the moduli space of hyperelliptic curves.

**Proposition 4.32.** *The space of genus $g \geq 2$ hyperelliptic curves has dimension equal to $2g - 1$.*

*Proof.* If $X$ is hyperelliptic of genus $g \geq 2$, it admits a unique double cover of $\mathbb{P}^1$ with $2g + 2$ branch points. Conversely, fix $2g + 2$ points on $\mathbb{P}^1$. There exists a unique genus $g$, connected double cover of $\mathbb{P}^1$ branched at the fixed points, by the Hurwitz monodromy description, i.e. the monodromy around each branch point is the unique simple transposition $(12) \in S_2$. Therefore, up to $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}(2)$ we have a one-to-one correspondence between hyperelliptic curves and $(2g+2)$-pointed $\mathbb{P}^1$. Hence the dimension of the space of hyperelliptic curve is $(2g + 2) - 3 = 2g - 1$ for $g \geq 2$. Along the way we also see how to construct the moduli space, by

$$(\mathrm{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)/\,\mathrm{PGL}(2),$$

where $\Delta$ is the big diagonal of the symmetric product. □

**Example 4.33.** Every genus 2 curve is hyperelliptic. In particular, the moduli space $\mathcal{M}_2$ is 3-dimensional.

**Example 4.34.** Every smooth plane quartic corresponds to a canonical curve of genus 3. Consequently the moduli space of (non-hyperelliptic) genus 3 curves can be identified with (an open subset of) $\mathbb{P}^{14}/\,\mathrm{PGL}(3)$. In particular, it is 6-dimensional. On the other hand, the space of hyperelliptic curves of genus 3 is 5-dimensional, hence a general genus 3 curve is *not* hyperelliptic.

In general, one approach to count the dimension of moduli of genus $g$ curves is by using branched covers of $\mathbb{P}^1$ with sufficiently high degree.

**Proposition 4.35.** *The moduli space $\mathcal{M}_g$ of genus $g \geq 2$ curves has dimension equal to $3g - 3$. In particular, a general genus $g \geq 3$ curve is not hyperelliptic.*

*Proof.* Fix $d > 2g-2$ and $2g-2+2d$ distinct points on $\mathbb{P}^1$ and consider degree $d$, connected, genus $g$ simply branched covers of $\mathbb{P}^1$ branching at the fixed points. Due to the monodromy description, there exist finitely many (not necessarily one) such covers. Conversely, to construct such a cover on a genus $g$ curve $X$, we can first take a degree $d$ effective divisor $D$ which maps to $\infty$. Then the map is induced by a meromorphic function $f$ on $X$ such that

$$(f) + D \geq 0,$$

i.e. $f$ is a section of $\mathscr{O}(D)$. Since $D$ is non-special, we have

$$h^0(\mathscr{O}(D)) = 1 - g + d.$$

Putting everything together, the space $\mathcal{M}_g$ depends on

$$2g - 2 + 2d - [d + (1 - g + d)] = 3g - 3$$

parameters. Here a scalar acting on $f$ matters, since if $f = g/h$ and $[g, h]$ is the induced map, it is different from the map $[\lambda g, h]$, where $\lambda$ is a constant.

For $g \geq 3$, we have

$$3g - 3 > 2g - 1,$$

hence a general genus $g \geq 3$ curve is non-hyperelliptic. □

**Remark 4.36.** Note that the naive dimension count fails for $g = 0, 1$. For $g = 0$, $\mathrm{Aut}(\mathbb{P}^1)$ is 3-dimensional, hence the choice of the pole divisor $D$ depends on $d - 3$ parameters only. While for $g = 1$, due to the translation of an elliptic curve, it depends on $d - 1$ parameters. Taking this correction into account, we recover the right moduli for both cases. Of course here we implicitly used the assumption that the automorphism group of a genus $g \geq 2$ curve is finite (discussed later).

4.7. **Special linear systems.** Let $D = p_1 + \cdots + p_d$ be an effective divisor of degree $d$ on a genus $g$ curve $X$. Recall that the linear system $|D|$ can be identified with $\mathbb{P}H^0(\mathscr{O}(D))$ parameterizing effective divisors linearly equivalent to $D$. Suppose as a projective space

$$r = \dim |D| = h^0(\mathscr{O}(D)) - 1.$$

By Riemann-Roch, we have

$$\dim |K \otimes \mathscr{O}(-D)| = r + g - d - 1.$$

Note that $|K \otimes \mathscr{O}(-D)|$ is the linear system of canonical divisors that contain $D$. By the canonical map

$$\phi_K : X \to \mathbb{P}^{g-1},$$

it says that the space of hyperplanes of $\mathbb{P}^{g-1}$ that contain $\phi_K(p_1), \ldots, \phi_K(p_d)$ is $(r + g - d - 1)$-dimensional. In other words, the linear span of $\phi_K(p_1), \ldots, \phi_K(p_d)$ is a

$$(g - 2) - (r + g - d - 1) = (d - 1) - r$$

dimensional subspace in $\mathbb{P}^{g-1}$. Since we expect $d$ points to span a $(d-1)$-dimensional linear subspace, geometrically it says that $\phi_K(D)$ fails to impose

$$r = \dim |D|$$

independent conditions. We summarize the discussion as a geometric version of the Riemann-Roch formula.

**Theorem 4.37** (Geometric Riemann-Roch)**.** *In the above setting, let $\overline{\phi_K(D)}$ be the linear span of the image of $D$ under the canonical map. Then we have*

$$\dim|D| = \deg(D) - 1 - \dim\overline{\phi_K(D)}.$$

**Remark 4.38.** Even if $D$ contains multiple point, the formulation still holds, by taking the multiplicity into account. Say, if $D$ contains $2p$, then $2p$ span the tangent line at $p$. If $D$ contains $3p$, then $3p$ spans an osculating 2-plane at $p$.

**Example 4.39.** Let us revisit the canonical embedding of a genus 4 non-hyperelliptic curve $X$ in $\mathbb{P}^3$. Recall that $X$ is contained in a unique quadric surface $Q$ and we showed that $X$ admits a triple cover of $\mathbb{P}^1$ corresponding to a family of lines in $Q$. Conversely, if $D = p + q + r$ induces a triple cover of $\mathbb{P}^1$, i.e. if $\dim|D| = 1$, by Geometric Riemann-Roch, we have

$$\dim\overline{\phi_K(D)} = 3 - 1 - 1 = 1,$$

i.e. $p$, $q$ and $r$ are collinear in a line $L$ in $\mathbb{P}^3$. Because $L \cdot Q = 2$ unless $L$ is contained in $Q$, any triple cover of $\mathbb{P}^1$ on $X$ corresponds to a family of lines in $Q$. We have seen that if $Q$ is smooth, there are two such families of lines, i.e. $X$ admits two distinct triple covers of $\mathbb{P}^1$, while if $Q$ is singular, such a triple cover is unique.

Let us study in detail the dimension of a linear system.

**Lemma 4.40.** *Let $D$ be a divisor on a curve $X$. Then $\dim|D| \geq k$ if and only if for every $k$ points $p_1, \ldots, p_k \in X$ there exists an effective divisor in $|D|$ containing all of them.*

*Proof.* Since $\sum_{i=1}^k p_i$ varies in a $k$-dimensional family, then $\dim|D| \geq k$ is obvious. Alternatively, we may also prove it by induction. Suppose it holds for $\leq k$. Assume for every $p_1, \ldots, p_{k+1}$, there exists $D' \in |D|$ containing all of them. Then we conclude that $\dim|D - p| \geq k$ for any $p \in X$. Since $|D|$ cannot contain entirely $X$ in its base locus, choose a point $p$ not in the base locus of $|D|$. Consequently we have

$$\dim|D| = \dim|D - p| + 1 \geq k + 1.$$

Now suppose $\dim|D| \geq k$. Then we have

$$h^0(\mathscr{O}(D - \sum_{i=1}^k p_i)) \geq h^0(\mathscr{O}(D)) - k \geq 1.$$

It implies that there exists a non-zero meromorphic function $f$ such that

$$(f) + D - \sum_{i=1}^k p_i \geq 0,$$

hence $(f) + D = D'$ is an effective divisor in $|D|$ containing $p_1, \ldots, p_k$. □

**Corollary 4.41.** *For any two effective divisors $D_1$ and $D_2$ on $X$, we have*

$$\dim|D_1| + \dim|D_2| \leq \dim|D_1 + D_2|.$$

*Proof.* Suppose $\dim|D_i| = k_i$ for $i = 1, 2$. Take any $k_1 + k_2$ points

$$p_1, \ldots, p_{k_1}, q_1, \ldots, q_{k_2}$$

in $X$. By the above lemma, there exist $D_1' \in |D_1|$ and $D_2' \in |D_2|$ such that $D_1'$ contains all the $p_i$ and $D_2'$ contains all the $q_j$. Then $D_1' + D_2' \in |D_1 + D_2|$ contains all the $p_i$, $q_j$, hence we obtain

$$\dim |D_1 + D_2| \geq k_1 + k_2$$

by the lemma again. □

Note that if $h^0(K \otimes (-D)) = 0$, then Riemann-Roch determines that

$$h^0(\mathscr{O}(D)) = 1 - g + \deg(D).$$

Some subtlety may occur if

$$h^0(K \otimes (-D)) \neq 0$$

and we call such a divisor $D$ a *special divisor* and the associated linear system $|D|$ a *special linear system*. By Riemann-Roch, any $D$ with $\deg(D) > 2g - 2$ is non-special. By Geometric Riemann-Roch, $D$ is non-special if and only if the linear span of $\phi_K(D)$ is the entire space $\mathbb{P}^{g-1}$.

**Theorem 4.42** (Clifford's Theorem). *Let $D$ be a special effective divisor on $X$. Then we have*

$$\dim |D| \leq \frac{1}{2} \cdot \deg(D).$$

*Proof.* Since $D$ is special, there exists an effective divisor $D'$ such that $D + D' \sim K$, hence we have

$$\dim |D| + \dim |D'| \leq \dim |K| = g - 1.$$

By Riemann-Roch, we have

$$\dim |D| - \dim |D'| = 1 - g + \deg(D).$$

The desired inequality follows by combining the two relations. □

**Remark 4.43.** Indeed, the equality holds only if $D = 0$, $D = K$ or $X$ is hyperelliptic. If $D = 0$ or $D = K$, one easily checks that the equality holds. If $X$ is hyperelliptic, we may choose $D = p + q$ where $p, q$ are conjugate and $\dim |p + q| = 1$. To prove that these are the only possibilities, we need the *uniform position theorem* regarding a general hyperplane section of a non-degenerate space curve, cf. [Griffiths-Harris, p. 249].

**Exercise 4.44.** Let $X$ be a genus $\geq 2$ hyperelliptic curve. For $0 < 2k \leq g$, find an effective divisor $D$ of degree $2k$ on $X$ such that $\dim |D| = k$. Classify all such divisors up to linear equivalence.

4.8. **Weierstrass points.** We want to generalize the concept of Weierstrass points on a hyperelliptic curve to an arbitrary curve. Let $X$ be a curve of genus $g \geq 2$ and $p \in X$ a point. Set $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$ and define $H_p \subset \mathbb{N}$ by

$$H_p = \{n : \exists f \in \mathscr{M}(X) \text{ such that } (f)_\infty = np\}.$$

Note that if $(f)_\infty = np$ and $(h)_\infty = mp$, then $(fh)_\infty = (m + n)p$. We say that $H_p$ is the *Weierstrass semigroup* of $p$. Define $G_p \subset \mathbb{N}$ as the complement of $H_p$, i.e.

$$G_p = \{n : \nexists f \in \mathscr{M}(X) \text{ such that } (f)_\infty = np\}.$$

We say that $G_p$ is the *Weierstrass gap sequence* of $p$.

**Lemma 4.45.** *We have $n \in H_p$ if and only if*

$$h^0(\mathscr{O}(np)) = h^0(\mathscr{O}((n-1)p)) + 1$$

*and $n \in G_p$ if and only if*

$$h^0(\mathscr{O}(np)) = h^0(\mathscr{O}((n-1)p)).$$

*In other words, $n \in G_p$ (resp. $H_p$) if and only if $p$ is (resp. not) a base point of the linear system $|np|$. Moreover, $G_p$ is a subset of $\{1, \dots, 2g-1\}$ with cardinality $g$.*

*Proof.* By the exact sequence

$$0 \to \mathscr{O}((n-1)p) \to \mathscr{O}(np) \to \mathbb{C}_p \to 0,$$

we conclude that

$$h^0(\mathscr{O}((n-1)p)) \leq h^0(\mathscr{O}(np)) \leq h^0(\mathscr{O}((n-1)p)) + 1.$$

The right hand side equality holds if and only if $p$ is not a base point of $|np|$, namely, if and only if there exists $f \in \mathscr{M}(X)$ such that $(f) + np \geq 0$ but $(f) + (n-1)p \ngeq 0$, which is equivalent to saying that $(f)_\infty = np$, i.e. if and only if $n \in H_p$.

For any $n \geq 2g$, by Riemann-Roch we have

$$h^0(np) = h^0((n-1)p) + 1.$$

Hence we conclude that $G_p \subset \{1, \dots, 2g-1\}$. Moreover, we have

$$\begin{aligned} g-1 &= h^0(\mathscr{O}((2g-1)p)) - h^0(\mathscr{O}_X) \\ &= \sum_{n=0}^{2g-1} \Big( h^0(\mathscr{O}(np)) - h^0(\mathscr{O}((n-1)p)) \Big) \end{aligned}$$

So there are $g-1$ elements of $\{1, \dots, 2g-1\}$ belonging to $H_p$. In other words, the cardinality of $G_p$ is

$$(2g-1) - (g-1) = g.$$

□

**Lemma 4.46.** *We have $n \in G_p$ if and only if there exists a section of the canonical line bundle, i.e. a holomorphic differential $\omega$, such that $\mathrm{ord}_p(\omega) = n-1$. As a consequence we conclude again that $G_p$ is a subset of $\{1, \dots, 2g-1\}$ of cardinality $g$.*

*Proof.* By Riemann-Roch, $h^0(\mathscr{O}(np)) = h^0(\mathscr{O}((n-1)p))$ if and only if

$$h^0(K \otimes \mathscr{O}(-(n-1)p)) = h^0(K \otimes \mathscr{O}(-np)) + 1.$$

Note that $H^0(K \otimes \mathscr{O}(-mp))$ parameterizes holomorphic one-forms $\omega$ such that $(\omega) \geq mp$. So the above equality holds if and only if there exists $\omega$ such that $(\omega) \geq (n-1)p$ but $(\omega) \ngeq np$, namely, if and only if $\mathrm{ord}_p(\omega) = (n-1)p$.

Since $h^0(K) = g$, one can choose a basis $\omega_1, \dots, \omega_g$ such that $\mathrm{ord}_p(\omega_i) = a_i$ and

$$a_1 < a_2 < \cdots < a_g.$$

Then we obtain that

$$G_p = \{a_1 + 1, a_2 + 1, \dots, a_g + 1\}.$$

Because $\deg(K) = 2g-2$, we have $0 \geq a_i \leq 2g-2$ for all $i$, hence $G_p$ is a subset of $\{1, \dots, 2g-1\}$. □

We say that $p$ is a *Weierstrass point* of $X$ if $G_p \neq \{1, 2, \ldots, g\}$ and $p$ is a *normal Weierstrass point* if $G_p = \{1, 2, \ldots, g-1, g+1\}$. Define the *weight* of $p$ by

$$\begin{aligned} w(p) &= \Big( \sum_{n \in G_p} n \Big) - (1 + 2 + \cdots + g) \\ &= \Big( \sum_{n \in G_p} n \Big) - \frac{g(g+1)}{2}. \end{aligned}$$

Then $p$ is a Weierstrass point if and only if $w(p) > 0$ and $p$ is a normal Weierstrass point if and only if $w(p) = 1$.

Now we introduce some basic properties of a semigroup of $\mathbb{N}$. In general, if $H \subset \mathbb{N}$ is a semigroup whose complement $G = \mathbb{N} - H$ consists of $g$ elements, define the *weight* of $H$ by

$$w(H) = \Big( \sum_{n \in G} n \Big) - \frac{g(g+1)}{2}.$$

**Lemma 4.47.** *In the above setting, we have*

$$w(H) \leq \frac{(g-1)g}{2}.$$

*The equality holds if and only if* $2 \in H$.

*Proof.* Let $H = \{a_1, a_2, \ldots,\}$ such that $a_1 < a_2 < \cdots$. Suppose $b \in G$ is the smallest element such that $b > a_n$. Then for $1 \leq i \leq n$, we have

$$b - a_i \in G, \quad b - a_i \leq a_n.$$

The two disjoint sets

$$\{b - a_n, b - a_{n-1}, \ldots, b - a_1\}, \quad \{a_1, \ldots, a_n\}$$

are both contained in $\{1, 2, \ldots, a_n\}$. Hence we conclude that

$$a_n \geq n + n = 2n.$$

Now suppose $G = \{b_1, \ldots, b_g\}$. Then $H$ can be expressed as

$$\{1, \ldots, b_1 - 1; b_1 + 1, \ldots, b_2 - 1; b_3 + 1, \ldots, b_4 - 1; \ldots\}.$$

Using the inequality, we have

$$b_k - 1 = a_{b_k - k} \geq 2(b_k - k),$$

hence we conclude that $b_k \leq 2k - 1$.

Now by definition, we have

$$\begin{aligned} w(H) &= \sum_{k=1}^{g} (b_k - k) \\ &\leq \sum_{k=1}^{g} (k - 1) \\ &= \frac{(g-1)g}{2}. \end{aligned}$$

Moreover, the equality holds if and only if $b_k = 2k - 1$ for $1 \leq k \leq g$, hence $G = \{1, 3, \ldots, 2g - 1\}$ and $2 \in H$. $\square$

**Proposition 4.48.** *Let $X$ be a curve of genus $g \geq 2$. Then we have*

$$\sum_{p \in X} w(p) = (g-1)g(g+1).$$

*In particular, $X$ has finitely many Weierstrass points and the number of distinct Weierstrass points is at least $2g+2$. This minimum number occurs if and only if $X$ is hyperelliptic.*

*Proof.* Choose a basis $\omega_1, \ldots, \omega_g$ of $H^0(X, K)$ and write $\omega_i = f_i(z)dz$ for $i = 1, \ldots, g$ in terms of a local coordinate $z$ around $p$. Consider the Wronskian

$$W(z) = \det \begin{pmatrix} f_1(z) & \cdots & f_g(z) \\ f_1'(z) & \cdots & f_g'(z) \\ \vdots & & \vdots \\ f_1^{(g-1)}(z) & \cdots & f_g^{(g-1)}(z) \end{pmatrix}$$

It is a basic fact that for linearly independent functions $f_1, \ldots, f_g$, $W(z)$ is non-zero.

Suppose $\tilde{z}$ is another coordinate around $p$ such that $\omega_i = \tilde{f}_i(\tilde{z})d\tilde{z}$. Let

$$\psi = \frac{d\tilde{z}}{dz} \in \mathscr{O}^*(U \cap \tilde{U}).$$

For $f = \psi \tilde{f}$, we have

$$\begin{aligned} \frac{df}{dz} &= \psi \frac{d\tilde{f}}{d\tilde{z}} \frac{d\tilde{z}}{dz} + \frac{d\psi}{dz} \tilde{f} \\ &= \psi^2 \frac{d\tilde{f}}{d\tilde{z}} + \frac{d\psi}{dz} \tilde{f} \end{aligned}$$

and in general

$$\frac{d^n f}{dz^n} = \psi^{n+1} \frac{d^n \tilde{f}}{d\tilde{z}^n} + \cdots$$

for higher derivatives. Denote by

$$N = 1 + 2 + \cdots + g = g(g+1)/2.$$

Then we conclude that

$$W(z) = \left(\frac{d\tilde{z}}{dz}\right)^N W(\tilde{z}).$$

For instance, if $g = 2$, we have

$$\begin{aligned} \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} &= \det \begin{pmatrix} \psi \tilde{f}_1 & \psi \tilde{f}_2 \\ \psi^2 \tilde{f}_1' + \psi' \tilde{f}_1 & \psi^2 \tilde{f}_2' + \psi' \tilde{f}_2 \end{pmatrix} \\ &= \psi^{1+2} \det \begin{pmatrix} \tilde{f}_1 & \tilde{f}_2 \\ \tilde{f}_1' & \tilde{f}_2' \end{pmatrix}. \end{aligned}$$

Consequently it implies that $W(z)(dz)^N$ defines a global section of $H^0(X, K^{\otimes N})$.

Moreover, from the expression of $W(z)$ we have

$$\begin{aligned}\mathrm{ord}_p(W(z)) &= \Big(\sum_{i=1}^{g} a_i\Big) - (0+1+\cdots+(g-1)) \\ &= \Big(\sum_{i=1}^{g}(a_i+1)\Big) - (1+2+\cdots+g) \\ &= \Big(\sum_{n\in G_p} n\Big) - g(g+1)/2 \\ &= w(p).\end{aligned}$$

Since $\deg(K^{\otimes N}) = (g-1)g(g+1)$, the desired formula follows right away.

By definition, a Weierstrass point has strictly positive weight, so $X$ has finitely many Weierstrass points. Since $w(p) \leq (g-1)g/2$, we conclude that $X$ has at least $2g+2$ distinct Weierstrass points. If this minimum number occurs, then each Weierstrass point $p$ satisfies $2 \in H_p$, i.e. $X$ admits a double cover of $\mathbb{P}^1$ with $p$ as a ramification point, hence $X$ is hyperelliptic. $\square$

**Exercise 4.49.** Let $X$ be a hyperelliptic curve of genus $\geq 2$ with a double cover $\pi : X \to \mathbb{P}^1$ and $p \in X$ a point. Prove directly that

(1) $p$ is a ramification point of $\pi$ if and only if $G_p = \{1, 3, \ldots, 2g-1\}$;

(2) $p$ is not a ramification point of $\pi$ if and only if $G_p = \{1, 2, \ldots, g\}$.

*Hint: use either the canonical map of $X$ or an explicit basis of $H^0(K_X)$.*

**Exercise 4.50.** Determine all possible Weierstrass gap sequences that may occur for a point on a genus 3 curve.

As an application, we have the following result.

**Proposition 4.51.** *The automorphism group of a genus $g \geq 2$ curve is finite.*

*Proof.* Let $X$ be a genus $g$ curve. Since $g \geq 2$, $X$ has finitely many Weierstrass points. In particular, an automorphism $\tau$ of $X$ sends a Weierstrass point to a Weierstrass point.

If $X$ is hyperelliptic, there are $2g+2$ Weierstrass points, corresponding to the $2g+2$ ramification points of the associated double cover $\pi : X \to \mathbb{P}^1$. Since the space of canonical divisors containing $p$ is the same of that containing $\tau(p)$, $\tau$ factorizes through the canonical map. Modulo the hyperelliptic involution exchanging the two sheets of $\phi$, $\tau$ is induced by an automorphism of $\mathbb{P}^1$ that sends the $2g+2$ branch points to themselves. In other words, $\mathrm{Aut}(X)$ is a $(\mathbb{Z}/2)$-extension of the automorphism group of a $(2g+2)$-pointed $\mathbb{P}^1$. Since $2g+2 > 3$, such automorphisms of $\mathbb{P}^1$ are of finitely many, because any automorphism of $\mathbb{P}^1$ fixing three points must be the identity.

Suppose $X$ is non-hyperelliptic. It is sufficient to show that if an automorphism $\tau$ fixes all Weierstrass points of $X$, then $\tau$ is the identity. Suppose $\tau$ is not identity and fixes all Weierstrass points of $X$. Take general points $p_1, \ldots, p_{g+1}$ on $X$. Then there exists $f \in \mathscr{M}(X)$ such that

$$(f)_\infty = p_1 + \cdots + p_{g+1}.$$

Consider $h = f - \tau^* f \in \mathscr{M}(X)$. Since $p_i$'s are general, $h$ is not identically zero. Since $\deg(h) \leq g+1$, $h$ can have at most $g+1$ zeros and $g+1$ poles. If $p$ is a fixed point of $\tau$, then $p$ is either a zero or a pole of $h$. So the number of fixed points of $\tau$

is bounded above by $2g+2$. But $X$ has more than $2g+2$ Weierstrass points, since it is not hyperelliptic, leading to a contradiction. □

**Remark 4.52.** We used the fact that for general $p_1, \dots, p_{g+1}$ on $X$, the linear system $|p_1 + \cdots + p_{g+1}|$ has dimension $\geq 1$ and is base point free. The dimension bound follows directly from Riemann-Roch. The base point freeness follows if we can show that $\dim |p_1 + \cdots + p_g| \geq 1$ only happens for special collections of $p_i$'s. By Riemann-Roch again, this is equivalent to saying that there is a hyperplane in $\mathbb{P}^{g-1}$ that contains $p_1, \dots, p_g$ in the canonical image of $X$. We just need to take general $p_i$'s such that any $g$ of them span the total space $\mathbb{P}^{g-1}$ other than a hyperplane.

4.9. **Plane curves and their duals.** Let $C \subset \mathbb{P}^2$ be a smooth degree $d$ plane curve defined by a homogeneous polynomial $F$. For every point $p \in C$, its tangent line $T_p(C)$ is a point in the dual plane $\mathbb{P}^{2*}$. Varying $p$ in $C$, the union of all tangent lines of $C$ forms a curve $C^*$ in $\mathbb{P}^{2*}$. We say that $C^*$ is the *dual curve* of $C$. At a point $[Z_0, Z_1, Z_2] \in C$, the tangent line equation is given by

$$\sum_{i=0}^{2} \frac{\partial F}{\partial Z_i} Z_i = 0,$$

i.e. $C^*$ has coordinates $[\partial F/\partial Z_0, \partial F/\partial Z_1, \partial F/\partial Z_2]$.

**Example 4.53.** Let $Q$ be a smooth plane conic defined by

$$XZ - Y^2 = 0.$$

At a point $[X, Y, Z]$, suppose the tangent line of $Q$ is given by $AX + BY + CZ$. One checks that the coefficients $A$, $B$ and $C$ satisfy

$$4AC - B^2 = 0.$$

In other words, the dual curve $Q^*$ is a smooth plane conic in the dual plane.

**Proposition 4.54.** *Let $C$ be a smooth degree $d$ plane curve. Then its dual curve $C^*$ in $\mathbb{P}^{2*}$ has degree equal to $d(d-1)$.*

*Proof.* Let $p \in \mathbb{P}^2$ be a point. All lines passing through $p$ form a line in the dual plane $\mathbb{P}^{2*}$. Conversely, every line in $\mathbb{P}^{2*}$ arises in this way. In other words, we have the incidence correspondence variety

$$I = \{([X, Y, Z], [A, B, C]) \mid AX + BY + CZ = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$$

parameterizing the pairs $(p, q)$, where the line representing by $q$ (resp. $p$) passes through $p$ (resp. $q$).

Let $L_p$ be a line in the dual plane $\mathbb{P}^{2*}$ parameterizing all lines in $\mathbb{P}^2$ passing through $p$. Note that $q$ lies in the intersection of $L_p$ and $C^*$ if and only if there exists a tangent line $L_q$ of $C$ such that $L_q$ contains $p$. Namely, the degree of $C^*$ is equal to the number of tangent lines of $X$ through $p$.

Fix a line $L_0$ in $\mathbb{P}^2$ and project $C$ from $p$ to $L_0$. We obtain a degree $d$ branched cover $\pi : C \to L_0 \cong \mathbb{P}^1$. If $p$ is a general point, we can make sure that the tangent lines of $X$ through $p$ are simple, i.e. $\pi$ has only simple ramification points. By Riemann-Hurwitz, the number of ramification points of $\pi$ is

$$2g - 2 + 2d = d(d-1).$$

Since each simple ramification point gives rise to a tangent line through $p$ and vice versa, we conclude that

$$\deg(C^*) = d(d-1).$$

□

**Remark 4.55.** If $C$ has isolated singularities, the dual curve can still be defined by the closure of the union of its tangent lines at smooth points.

**Proposition 4.56.** *The dual of the dual is the original curve, i.e.* $C^{**} = C$.

*Proof.* Let $p_0 \in C$ be a regular point. The tangent line $p_0^* \in \mathbb{P}^{2*}$ to $C$ at $p_0$ is the limit of the secant lines $\overline{p_0 p}$ as $p$ approaches $p_0$. Similarly, $p_0^{**}$ is the limit of the secant lines $\overline{p^* p_0^*}$ as $p^*$ approaches $p_0^*$. Note that the dual of the secant line $\overline{p^* p_0^*}$ is the intersection point of the two lines $p^*$ and $p_0^*$ in $\mathbb{P}^2$. In other words, $p_0^{**} \in \mathbb{P}^{2**} = \mathbb{P}^2$ is the limit of the intersection point of the tangent lines to $C$ at $p$ and $p_0$ as $p$ approaches $p_0$. It is clear that this limit is just $p_0$. □

**Remark 4.57.** The above two propositions seem to contradict each other. Indeed, nothing is wrong, because the dual curve $C^*$ is not always smooth, even if $C$ is smooth. For instance, if $C$ has a bitangent line $L$, i.e. $L$ is tangent to $C$ at two distinct points $p$ and $q$, then $C^*$ will self-intersect at a node $[L] \in \mathbb{P}^{2*}$. If $C$ has a flex line $L$, i.e. $L$ is tangent to $C$ at $p$ with tangency multiplicity $\geq 3$, then $C^*$ will have a cusp at $[L] \in \mathbb{P}^{2*}$. Hence it makes sense to discuss these basic plane curve singularities.

Let $C$ be a plane curve and $C^*$ its dual. Suppose the defining equations of $C$ and $C^*$ are $f$ and $f^*$, respectively. We say that $C$ has *traditional singularities* if every point $p \in C$ is one of the following:

1. A *regular point*. Namely, $p$ is a smooth point of $C$ and the tangent line to $C$ at $p$ is a smooth point of $C^*$. In local coordinates, $f$ can be expressed as

$$[1, z + \cdots, z^2 + \cdots],$$

i.e. $f = y - x^2$ modulo the ideal $(y^2, xy, x^3)$.

2. An *ordinary flex*. Namely, $p$ is a smooth point of $C$ and the tangent line to $C$ at $p$ has contact of order three. Locally $f$ can be expressed as

$$[1, z + \cdots, z^3 + \cdots],$$

i.e. $f = y - x^3$ modulo $(y^2, xy, x^4)$. The tangent lines around $p$ have equation

$$\begin{aligned} \frac{y - (z^3 + \cdots)}{x - (z + \cdots)} &= \frac{\partial y}{\partial x} \\ &= \frac{3z^2 + \cdots}{1 + \cdots}. \end{aligned}$$

Writing it as $AX + BY + CZ$, the dual curve $C^*$ has coordinates

$$[-3z^2 + \cdots, 1 + \cdots, 2z^3 + \cdots],$$

i.e. it has a singularity of type $y^2 - x^3 = 0$, which is a cusp.

3. A *cusp*. Locally $f$ is expressed as

$$[1, z^2 + \cdots, z^3 + \cdots],$$

i.e. $f = y^2 - x^3$ modulo $(y^3, xy, x^4)$. The tangent lines around $p$ have equation

$$\begin{aligned} \frac{y-(z^3+\cdots)}{x-(z^2+\cdots)} &= \frac{\partial y}{\partial x} \\ &= \frac{3z^2+\cdots}{2z+\cdots}. \end{aligned}$$

Writing it as $AX + BY + CZ$, the dual curve $C^*$ has coordinates

$$[-3z^2+\cdots, 2z+\cdots, z^4+\cdots] = [-3z+\cdots, 2+\cdots, z^3+\cdots],$$

i.e. $C^*$ has a flex line.

**Remark 4.58.** We see that cusps and flexes correspond to each other between a curve and its dual.

4. A *bitangent*. Namely, the tangent line at $p$ is also tangent to $X$ at another point $q$ with both contact orders equal to two.
5. A *node* (or *ordinary double point*). Namely, $X$ has two branches intersecting transversely at $p$. Locally $f$ has expression $f = xy$ modulo $(x^2, y^2)$.

**Remark 4.59.** Geometrically it is clear that nodes and bitangents correspond to each other between a curve and its dual.

Let $C$ be a plane curve and $C^*$ its dual with traditional singularities. Let $g$ be the *geometric genus* of $C$, i.e. the genus of its normalization $\tilde{C}$. Let $d$ be the degree of $C$, $b$ the number of bitangents, $f$ the number of flexes, $\kappa$ the number of cusps and $\delta$ the number of nodes. We also write $g^*$, $d^*$ etc to denote the corresponding quantities for $C^*$. Based on the above discussion, we have

$$\begin{aligned} b = \delta^*, &\quad \delta = b^*, \\ \kappa = f^*, &\quad f = \kappa^*. \end{aligned}$$

**Proposition 4.60.** *We have the following classical Plücker formulas:*

$$\begin{aligned} g &= \frac{(d-1)(d-2)}{2} - \delta - \kappa, \\ d^* &= d(d-1) - 2\delta - 3\kappa, \\ g^* &= \frac{(d^*-1)(d^*-2)}{2} - b - f, \\ d &= d^*(d^*-1) - 2b - 3f. \end{aligned}$$

*Moreover, we have* $g = g^*$.

*Proof.* By assumption $C$ may have nodes and cusps. Each singularity makes the geometric genus drop by one compared to the arithmetic genus. Hence the geometric genus of $C$ equals

$$g = \frac{(d-1)(d-2)}{2} - \delta - \kappa.$$

Recall how we determine the degree of $C^*$ for a smooth plane curve $C$. We project $C$ from a general point $p$ to a line $L$. Then the degree of $C^*$ is equal to the number of ramification points of the map. In case when $C$ has nodes and cusps, the projection $\pi : C \to L$ induces a branched cover $\tilde{\pi} : \tilde{C} \to L$. If the projection goes through a cusp, the inverse image of the cusp is a simple ramification point of $\tilde{\pi}$.

If the projection passes a node, we see two distinct points in $\tilde{C}$, both unramified. By Riemann-Hurwitz we conclude that

$$2g - 2 + 2d = d^* + \kappa.$$

Hence we obtain that

$$d^* = d(d-1) - 2\delta - 3\kappa.$$

The other two identities follow from the duality relation. Since $C$ and $C^*$ are isomorphic away from finitely many points, they have the same normalization by the Riemann extension theorem, hence $g = g^*$. □

**Remark 4.61.** Let $\pi : \tilde{C} \to C$ be the normalization map. Then we have an exact sequence

$$0 \to \mathscr{O}_C \to \pi_* \mathscr{O}_{\tilde{C}} \to \mathscr{F} \to 0,$$

where $\mathscr{F}$ has 0-dimensional support at the singularities of $C$. For example, if $C$ has a cusp, locally

$$\mathscr{O}_C = k[x,y]/(y^2 - x^3), \quad \mathscr{O}_{\tilde{C}} = k[t].$$

The map on the left side is given by

$$x \mapsto t^2, \quad y \mapsto t^3,$$

hence the cokernel is one-dimensional spanned by $t$.

The cohomology of $\pi_* \mathscr{O}_{\tilde{C}}$ is the same as that of $\mathscr{O}_{\tilde{C}}$. Hence we have

$$\begin{aligned} g(\tilde{C}) &= 1 - \chi(\mathscr{O}_{\tilde{C}}) \\ &= 1 - \chi(\mathscr{O}_C) - h^0(\mathscr{F}) \\ &= p_a(C) - h^0(\mathscr{F}). \end{aligned}$$

Note the the structure of $\mathscr{F}$ is determined by the local type of the singularities. Since a plane rational cubic with a node or a cusp has arithmetic genus equal to one but geometric genus equal to zero, we conclude that a node or a cusp makes the genus of a plane curve drop by one.

**Example 4.62.** Let $C$ be a smooth plane cubic. Then we have

$$d = 3, \quad g = 1, \quad \delta = 0, \quad \kappa = 0.$$

Moreover, by the degree reason $C$ does not possess a bitangent, hence $b = 0$. By the Plücker formulas, we conclude that $d^* = 6$, hence

$$3f = d^*(d^* - 1) - d = 27$$

and $f = 9$. We have seen that the nine flexes correspond to the nine torsion points in the Jacobian of an elliptic curve.

Now suppose $C$ is a plane rational nodal cubic. Then we have

$$d = 3, \quad g = 0, \quad \delta = 1, \quad \kappa = 0, \quad b = 0.$$

We conclude that

$$d^* = 4, \quad f = 3,$$

i.e. the number of flexes of $C$ drops to three.

If $C$ is a plane rational cuspidal cubic, we have

$$d = 3, \quad g = 1, \quad \delta = 0, \quad \kappa = 1, \quad b = 0.$$

We conclude that

$$d^* = 3, \quad f = 1,$$

i.e. $C$ has a unique flex at the cusp.

**Example 4.63.** Let $C$ be a smooth plane quartic such that its dual has traditional singularities. Then we have

$$d = 4, \quad g = 3, \quad \delta = 0, \quad \kappa = 0.$$

By the Plücker formulas, we have

$$d^* = 12,$$

$$b + f = 52,$$

$$2b + 3f = 128,$$

hence we obtain that

$$b = 28, \quad f = 24.$$

In other words, $C$ possesses 28 bitangents and 24 flexes. Note that the 24 flexes correspond to the 24 simple Weierstrass points on a general curve of genus three.

**Exercise 4.64.** Let $C$ be a smooth degree $d$ plane curve whose dual curve has traditional singularities. Calculate the number of bitangents and flexes of $C$.

## 5. Chern classes

Chern classes are objects associated to vector bundles. They can be defined in various ways. Here our approach is to treat them as elements in the Chow ring of the base variety.

### 5.1. **Chow ring and rational equivalence.**

Let $X$ be an $n$-dimensional variety, smooth or with mild singularities. Let $Z^d(X)$ be the free abelian group generated by irreducible subvarieties of codimension $d$. Elements of $Z^d(X)$ are called *$d$-cocycles* or *$(n-d)$-cycles.*

Let $Y \subset X$ be a $(k+1)$-dimensional subvariety. For a $k$-cycle $Z$, if there exists $f \in \mathscr{M}(Y)$ such that $(f) = Z$, we say that $Z$ is a *principal $k$-cycle*. Define *rational equivalence* $\sim$ between two $k$-cycles if their difference is a principal $k$-cycle. Define

$$A^d(X) = Z^d(X)/\sim$$

as the *dth Chow group* of $X$ and use $[Z]$ to denote the equivalence class of $Z$.

**Remark 5.1.** Geometrically, if two $k$-cycles $Z$ and $W$ are rationally equivalent, then $f$ induces a map $Y \to \mathbb{P}^1$. We can construct an incidence correspondence

$$\tilde{Y} = \{(y, t) \mid f(y) = t\} \subset X \times \mathbb{P}^1.$$

In other words, $\tilde{Y}$ is a $(k+1)$-cycle of $X \times \mathbb{P}^1$ such that the fibers of $\tilde{Y} \to \mathbb{P}^1$ over two points are $Z$ and $W$, respectively. Hence rational equivalence is analogous to cobordism in topology.

**Proposition 5.2.** *Let $X$ be an $n$-dimensional irreducible variety.*

*(1) $A^0(X) \cong \mathbb{Z}$ is generated by $[X]$.*
*(2) $A^d(X) = 0$ for $d > n$.*
*(3) If $X$ is smooth, then $A^1(X) \cong \mathrm{Pic}(X)$.*
*(4) There exists a degree map $A^n(X) \to \mathbb{Z}$.*

*Proof.* The first three trivially hold by definition. For (4), note that $A^n(X)$ is generated by rational equivalence classes of points. Since the degree of a principal divisor on a curve is zero, the degree map $Z^n(X) \to \mathbb{Z}$ factorizes through $A^n(X)$. $\square$

**Remark 5.3.** Rational equivalence could be subtle even for 0-dimensional cycles. Two points are rationally equivalent if and only if there exists a chain of rational curves (not necessarily smooth) connecting them. For instance, any two points on $\mathbb{P}^1$ are rationally equivalent. But any two points on a positive genus curve are not rationally equivalent.

**Example 5.4.** One can show that any two $k$-dimensional linear subspaces of $\mathbb{P}^n$ are rationally equivalent. Moreover, any two hypersurfaces of the same degree are rationally equivalent in $\mathbb{P}^n$. In general, let $Y$ be a subvariety of $\mathbb{P}^n$ of degree $k$. Take a general point $p$ and project $Y$ from $p$ to a hyperplane $\mathbb{P}^{n-1}$. We can choose $p$ such that the projection $\pi : Y \to \mathbb{P}^{n-1}$ is generically finite, i.e. for a general point $q$ in the image of $\pi$, the inverse of $q$ consists of finitely many points. Suppose $p$ has coordinate $[1, 0, \ldots, 0]$ and the hyperplane is given by $X_0 = 0$. Then $\pi$ is given by

$$[X_0, \ldots, X_n] \mapsto [0, X_1, \ldots, X_n].$$

It implies that $[X_0, \ldots, X_n] \in Y$ varies in a $\mathbb{P}^1$ family $[tX_0, X_1, \ldots, X_n] \in Y_t$ to its projection image as $t$ varies from 1 to 0. Then we conclude that $Y$ is rationally equivalent to $\deg(\pi) \cdot \pi(Y)$.

Let $[H]$ be the hyperplane class of $\mathbb{P}^n$. By induction we thus know that $A^d(\mathbb{P}^n) = \mathbb{Z}$ generated by $[H^d]$, where $[H^d]$ is the class of a linear subspace of codimension $d$. This notation makes sense because we can take $d$ hyperplanes $H_1, \ldots, H_d$ such that they cut out a linear subspace of codimension $d$. Moreover, $[H_i] = [H]$ for all $i$. This example suggests that the union of $A^d(X)$ for all $d$ possesses a graded ring structure induced by intersection between cycles.

**Exercise 5.5.** Prove directly that two linear subspaces of $\mathbb{P}^n$ of the same dimension are rationally equivalent.

Let $V$ and $W$ be two irreducible subvarieties. If

$$\operatorname{codim}(V \cap W) = \operatorname{codim}(V) + \operatorname{codim}(W),$$

we say that $V$ and $W$ intersect *properly*. This induces an intersection product

$$A^d(X) \times A^e(X) \to A^{d+e}(X)$$

by $[V] \cdot [W] = [V \cap W]$ with appropriate multiplicities on each component of the intersection. Define the *Chow ring* of $X$ as

$$A^*(X) = \bigoplus_{d=0}^{n} A^d(X).$$

This is a graded ring with unit $1 = [X]$.

**Remark 5.6.** The fact that the intersection product is well-defined is based on *Chow's moving lemma*. It says that if $Y, Z$ are two cycles, then there exists $Y'$ rationally equivalent to $Y$ such that $Y'$ and $Z$ intersect properly. Moreover, if $Y''$ is another such cycle, then $Y' \cap Z$ and $Y'' \cap Z$ are rationally equivalent.

**Example 5.7.** Let $H$ be the hyperplane class of $\mathbb{P}^n$. Then we have

$$A^*(\mathbb{P}^n) \cong \mathbb{Z}[H]/H^{n+1}.$$

Note that the Chow ring and the cohomology ring of $\mathbb{P}^n$ coincide over $\mathbb{C}$. But in general this is not the case.

5.2. **Formal calculation of Chern classes.** Let $E$ be a vector bundle of rank $m$ on $X$. Assign to $E$ an element $c_k(E) \in A^k(X)$ for $1 \leq k \leq m$. Call $c_k(E)$ the *kth Chern class* of $E$ and define the *total Chern class* by

$$c(E) = 1 + c_1(E) + \cdots + c_m(E).$$

We want these assignments to satisfy the following properties:

(1) If $L$ is a line bundle associated to a divisor $D$, then $c_1(L) = [D]$.

(2) If there exists an exact sequence $0 \to E \to F \to G \to 0$, then

$$c(F) = c(E)c(G).$$

(3) If $f : Y \to X$ is a map and $E$ a vector bundle on $X$, then $c(f^*E) = f^*c(E)$.

The actual definition of Chern classes will be introduced later by degeneracy loci. Here we do some calculation first.

**Example 5.8.** Let $E_i = \mathscr{O}(d_i)$ be a line bundle of degree $d_i$ on $\mathbb{P}^n$ for $i = 1, 2$. Then we have

$$\begin{aligned} c(E_1 \oplus E_2) &= c(E_1)c(E_2) \\ &= (1 + d_1H)(1 + d_2H) \\ &= 1 + (d_1 + d_2)H + d_1d_2H^2. \end{aligned}$$

Hence we conclude that

$$c_1(E_1 \oplus E_2) = (d_1 + d_2)H,$$

$$c_2(E_1 \oplus E_2) = d_1d_2H^2.$$

**Example 5.9.** Let us compute the Chern class of the tangent bundle on $\mathbb{P}^n$. Let $S$ be the *tautological line bundle* such that the fiber $S|_{[L]}$ represents the 1-dimensional subspace $L \subset \mathbb{C}^{n+1}$. Then $S^*$ is the *universal line bundle* such that the fiber $S^*|_{[L]}$ is the space of linear functionals $\mathrm{Hom}(L, \mathbb{C})$. Take a non-zero linear map $\phi : \mathbb{C}^{n+1} \to \mathbb{C}$ and define $\Phi([L], v) = \phi(v)$, where $v$ is a vector in $L$. Then $\Phi$ can be regarded as a section of $S^*$. Moreover, $\Phi$ is the zero map at $[L]$ if and only if $L \subset \ker(\phi)$, i.e. the zero locus of $\Phi$ in $\mathbb{P}^n$ is a hyperplane given by the projectivization of $\ker(\phi)$. It implies

$$S^* = \mathscr{O}_{\mathbb{P}^n}(1), \quad S = \mathscr{O}_{\mathbb{P}^n}(-1).$$

We have an exact sequence

$$0 \to S \to V \to Q \to 0,$$

where $V = \mathbb{C}^{n+1} \times X$ is the trivial bundle of rank $n + 1$, $S \to V$ is the natural inclusion $L \subset \mathbb{C}^{n+1}$ fiber by fiber and $Q$ is the quotient bundle with fiber $Q|_{[L]} = \mathbb{C}^{n+1}/L$. Tensor it by $S^*$ and apply the identification $S = \mathscr{O}_{\mathbb{P}^n}(-1)$. We thus obtain

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^{n+1} \mathscr{O}_{\mathbb{P}^n}(1) \to S^* \otimes Q \to 0.$$

Note that the fiber of the tangent bundle $T_{\mathbb{P}^n}|_{[L]}$ is canonically isomorphic to $\mathrm{Hom}(L, \mathbb{C}^{n+1}/L)$. More precisely, given $\phi \in \mathrm{Hom}(L, \mathbb{C}^{n+1})$ and $v \in L$, one can define an arc $v(t) = v + t\phi(v) \in \mathbb{C}^{n+1}$. The part $\phi(v)$ modulo $L$ is a tangent vector at $[L]$. Conversely, one can lift a tangent vector as an equivalence class of arcs through $v \in L$ and deduce $\phi$ accordingly. Globally it implies that

$$T_{\mathbb{P}^n} \cong S^* \otimes Q.$$

Hence we obtain the following *Euler sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^{n+1} \mathscr{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n} \to 0.$$

The total Chern class of $T_{\mathbb{P}^n}$ is

$$\begin{aligned} c(T_{\mathbb{P}^n}) &= c(\mathscr{O}_{\mathbb{P}^n}(1))^{n+1} \\ &= (1+H)^{n+1} \\ &= 1 + (n+1)H + \binom{n+1}{2} H^2 + \cdots + \binom{n+1}{n} H^n. \end{aligned}$$

Note that for $n = 2$, $c(T_{\mathbb{P}^2}) = 1 + 3H + 3H^2$ is not a product of two linear forms of $H$. Hence $T_{\mathbb{P}^2}$ cannot be an *extension* of two line bundles on $\mathbb{P}^2$, i.e. there do not exist two line bundles $L$ and $M$ such that they fit in the exact sequence

$$0 \to L \to T_{\mathbb{P}^2} \to M \to 0.$$

5.3. **The splitting principle.** In an ideal world, a vector bundle would split as a direct sum of line bundles. Then we could calculate its Chern class based on that of line bundles. Well, in real world there exist tons of non-splitting vector bundles. But for computing the Chern classes only, one can still pretend that a vector bundle splits and carry out the calculation formally.

To illustrate what this means, consider a vector bundle $E$ of rank $n$. Suppose $E$ splits as a direct sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_n$$

and let $a_i = c_1(L_i)$. Then we have

$$\begin{aligned} c(E) &= \prod_{i=1}^{n} (1 + a_i) \\ &= 1 + \Big(\sum_{i=1}^{n} a_i\Big) + \Big(\sum_{i \neq j} a_i a_j\Big) + \cdots \end{aligned}$$

hence we conclude that

$$c_1(E) = \sum_{i=1}^{n} a_i, \quad c_2(E) = \sum_{i \neq j} a_i a_j, \quad \ldots$$

Suppose $L$ is a line bundle with first Chern class $c_1(L) = b$. Then

$$\begin{aligned} c(E \otimes L) &= c((L_1 \otimes L) \oplus \cdots \oplus (L_n \otimes L)) \\ &= (1 + a_1 + b) \cdots (1 + a_n + b) \\ &= 1 + \Big(\sum_{i=1}^{n} a_i + nb\Big) + \left(\sum_{i \neq j} a_i a_j + \binom{n}{2} b^2 + (n-1)\Big(\sum_{i=1}^{n} a_i\Big) b\right) + \cdots \\ &= 1 + (c_1(E) + nc_1(L)) + \left(c_2(E) + \binom{n}{2} c_1^2(L) + (n-1)c_1(E)c_1(L)\right) + \cdots \end{aligned}$$

Hence we conclude that

$$c_1(E \otimes L) = c_1(E) + nc_1(L),$$
$$c_2(E \otimes L) = c_2(E) + \binom{n}{2} c_1^2(L) + (n-1)c_1(E)c_1(L)$$

and etc.

The splitting principle says that even if $E$ is not decomposable as a direct sum of line bundles, the above calculation still holds. In other words, one can write

$$c(E) = \prod_{i=1}^{n} (1 + a_i),$$

where $a_i$'s are formal roots of the *Chern polynomial*

$$c_t(E) = 1 + c_1(E)t + \cdots c_n(E)t^n.$$

We call $a_1, \ldots, a_n$ the *Chern roots* of $E$. If $E$ splits as a direct sum of line bundles, then $a_i$'s are just the first Chern classes of the summands, hence belong to $A^1(X)$. In general, $a_i$ may not correspond to a divisor class. But the Chern classes of $E$ are represented by symmetric functions of $a_i$'s, hence eventually one can get rid of $a_i$'s and express the final result of this kind of calculation by the Chern classes of the original vector bundles.

Let us use the splitting principle to study the Chern class of the canonical line bundle on a variety.

**Proposition 5.10.** *Let $K$ be the canonical line bundle and $T$ the tangent bundle on a smooth variety $X$. Then we have*

$$K \sim -c_1(T).$$

*Proof.* If $T$ splits as $\bigoplus_{i=1}^{n} L_i$, then

$$\bigwedge^{n} T = \bigotimes_{i=1}^{n} L_i.$$

Hence we conclude that

$$\begin{aligned} c_1\Big(\bigwedge^{n} T\Big) &= \sum_{i=1}^{n} c_1(L_i) \\ &= c_1(T). \end{aligned}$$

Therefore, the associated divisor of $K$ is the dual, i.e. $-c_1(T)$. The splitting principle ensures that it holds regardless of whether or not $T$ is decomposable. $\square$

**Corollary 5.11.** *On $\mathbb{P}^n$ we have $K = \mathscr{O}(-n-1)$.*

*Proof.* We have seen that
$$c(T_{\mathbb{P}^n}) = (1+H)^{n+1}.$$
In particular, we have $c_1(T_{\mathbb{P}^n}) = (n+1)H$. Then the claim follows from the above proposition. $\square$

Finally let us say a word about why the splitting principle works. Roughly speaking, for a vector bundle $E$ on $X$, one can always find $f : \tilde{X} \to X$ such that $f^*E$ is split and $A^*(X) \to A^*(\tilde{X})$ is injective. Then one can carry out the calculation in $A^*(X)$ regarded as a subring of $A^*(\tilde{X})$.

**Exercise 5.12.** Let $E$ and $F$ be vector bundles of rank two on $X$. Use their Chern classes to calculate the Chern classes of $E^*$, $\mathrm{Sym}^2(E)$, $\bigwedge^2 E$, $E \oplus F$ and $E \otimes F$.

**Exercise 5.13.** Let $X$ be an $k$-dimensional smooth variety embedded in $\mathbb{P}^n$. Suppose the tangent bundle of $X$ is trivial. Prove that $n \geq 2k$ and give an example of such $X$ when the equality holds.

*Hint: use the exact sequence* $0 \to T_X \to T_{\mathbb{P}^n} \to N \to 0$, *where* $N$ *is the normal bundle of* $X$, *and consider their Chern classes.*

5.4. **Determinantal varieties.** Let $M = M(m,n)$ be the space of $m \times n$ matrices. As a variety, $M$ is isomorphic to $\mathbb{A}^{mn}$. For $0 \leq k \leq \min(m,n)$, denote by $M_k = M_k(m,n)$ the locus of matrices of rank at most $k$, that is, cut out by all the $(k+1) \times (k+1)$ minors. We say that $M_k$ is the *kth generic determinantal variety.*

**Proposition 5.14.** *$M_k$ is an irreducible subvariety of $M$ of codimension $(m-k)(n-k)$.*

*Proof.* Define the incidence correspondence
$$\tilde{M}_k = \{(A,W) \mid A \cdot W = 0\} \subset M \times G(n-k,n).$$
Then $\tilde{M}_k$ admits two projections $p_1$ and $p_2$ to $M$ and $G(n-k,n)$, respectively. The map $p_1$ is onto $M_k$ and generically one to one. In other words, $p_1$ has positive dimensional fiber over $A$ if and only if the rank of $A$ is at most $k-1$. On the other hand, for a fixed $W \in G(n-k,n)$, the fiber of $p_2$ over $W$ is isomorphic to $\mathbb{A}^{mk}$. Hence we conclude that
$$\begin{aligned} \dim M_k &= \dim \tilde{M}_k \\ &= \dim G(n-k,n) + mk \\ &= (m+n)k - k^2. \end{aligned}$$
Therefore, the codimension of $M_k$ in $M$ is equal to
$$mn - (m+n)k + k^2 = (m-k)(n-k).$$
Moreover, $\tilde{M}_k$ is a vector bundle on $G(n-k,n)$, hence is irreducible. So is $M_k$. $\square$

**Remark 5.15.** As a vector bundle, $\tilde{M}_k$ is smooth, but the contraction $M_k$ could be singular. With a more detailed study of the tangent space, one can show that the singular locus of $M_k$ is exactly $M_{k-1}$.

Determinantal varieties are useful to the study of maps between vector bundles. Let $E$ and $F$ be two vector bundles on $X$ of rank $n$ and $m$, respectively. Suppose $\phi : E \to F$ is a bundle map. Choose a local trivialization of $E$ and $F$. Then $\phi$ locally corresponds to an $m \times n$ matrix, whose entries are holomorphic functions. Denote by $\Phi_k$ the locus in $X$ where the rank of $\phi$ is at most $k$. Then $\Phi_k$ is the inverse image of the generic determinantal variety $M_k$, hence it has expected codimension $(m-k)(n-k)$ in $X$.

5.5. **Degeneracy loci.** Let $E$ be a vector bundle of rank $n$ on a projective variety $X$. Moreover, suppose $E$ is *globally generated*, i.e. for any $p \in X$, the fiber $E|_p$ is fully spanned by its global sections. We will define Chern classes for such $E$. In general, one can twist a vector bundle by a very ample line bundle to obtain a globally generated vector bundle, define its Chern classes and then twist back to the original bundle.

For $1 \leq i \leq \min\{n, \dim X\}$, let $s_1, \ldots, s_i$ be general sections of $E$. At a point $p \in X$, we expect these sections to be linearly independent. Hence we denote by $D_i$ the *degeneracy locus* in $X$ where $s_1, \ldots, s_i$ are linearly dependent. Equivalently, $D_i$ is the locus of $p$ in $X$ such that

$$s_1(p) \wedge \cdots \wedge s_i(p) = 0.$$

Note that $D_i$ possesses a determinantal structure. Take a trivialization of $E$ in a local neighborhood $U$ and let $e_1, \ldots, e_n$ be sections that generate $E|_U$. Then one can write

$$s_j = \sum_{l=1}^{n} a_{jl} e_l$$

for $j = 1, \ldots, i$, where $a_{jl} \in \mathscr{O}(U)$. Then $D_i|_U$ is defined by all the $i \times i$ minors of the $i \times n$ matrix $(a_{jl})$. Adapting the dimension of the corresponding determinantal variety, we conclude that the (expected) codimension of $D_i$ in $X$ is $n - i + 1$. We thus define the Chern class

$$c_{n-i+1}(E) := [D_i] \in A^{n-i+1}(X).$$

**Remark 5.16.** If the sections $s_1, \ldots, s_i$ are general, then $D_i$ has the right codimension as expected. Moreover, different choices of general sections give rise to rationally equivalent degeneracy loci.

The definition by degeneracy loci satisfies the formal properties of Chern classes. For instance, if $E = L_1 \oplus L_2$ such that $L_i$ is a very ample line bundle, take a general section $s_i$ of $L_i$ such that $(s_i) = S_i$ is an effective divisor. Then $D_1$ is the locus where $(s_1, s_2)$ is zero, i.e.

$$\begin{aligned} c_2(E) &= [D_1] \\ &= [S_1] \cdot [S_2] \\ &= c_1(L_1) c_1(L_2). \end{aligned}$$

Moreover, take $(s_1, 0)$ and $(0, s_2)$ as two sections of $E$. Then $D_2$ is the locus where either $s_1$ or $s_2$ is zero, i.e.

$$\begin{aligned} c_1(E) &= [D_2] \\ &= [S_1] + [S_2] \\ &= c_1(L_1) + c_1(L_2). \end{aligned}$$

Hence we recover that $c(E) = c(L_1)c(L_2)$ as required by the formal properties of Chern classes.

Let us consider some applications along this circle of ideas.

**Example 5.17** (Bézout's Theorem)**.** Consider the vector bundle $E = \mathscr{O}(a) \oplus \mathscr{O}(b)$ on $\mathbb{P}^2$ with $a, b > 0$. We know

$$c(E) = 1 + (a+b)L + abL^2,$$

where $L$ is the line class of $\mathbb{P}^2$. Let $F, G$ be two general homogenous polynomials in three variables of degree $a$ and $b$, respectively. Then $(F, G)$ is a general section of $E$. The degeneracy locus $D_1$ associated to $(F, G)$ consists of points that are common zeros of $F$ and $G$. In other words, let $C$ and $D$ be the corresponding plane curves defined by $F$ and $G$, respectively. Then we conclude that

$$\begin{aligned} \#(C \cap D) &= \deg(c_2(E)) \\ &= ab. \end{aligned}$$

5.6. **Grassmanians and Schubert calculus.** As the dual projective space parameterizes hyperplanes, one can consider the Grassmanian $\mathbb{G}(k, n)$ parameterizing $k$-dimensional linear subspaces of $\mathbb{P}^n$, or equivalently $(k+1)$-dimensional linear subspaces of $\mathbb{C}^{n+1}$. Recall that $\dim \mathbb{G}(k, n) = (k+1)(n-k)$. As we showed before for $\mathbb{P}^n$, there is a similar exact sequence on $\mathbb{G}(k, n)$:

$$0 \to S \to V \to Q \to 0,$$

where $S$ is the tautological bundle of rank $k+1$ whose fiber over $[W] \in \mathbb{G}(k, n)$ represents the subspace $W$, $V$ is the trivial bundle of rank $n+1$ and $Q$ is the quotient bundle of rank $n-k$. The tangent bundle $T_{\mathbb{G}}$ is isomorphic to $S^* \otimes Q$, hence it fits in the Euler exact sequence

$$0 \to \bigoplus^{(k+1)^2} \mathscr{O} \to \bigoplus^{n+1} S^* \to T_{\mathbb{G}} \to 0.$$

In order to compute the Chern class of $T_{\mathbb{G}}$, we need to know the Chow ring of $\mathbb{G}(k, n)$. In general, the Chow ring of a Grassmanian is generated by its subvarieties parameterizing $k$-dimensional linear subspaces satisfying certain interpolation conditions. Below we will use an example to illustrate the idea.

Consider $\mathbb{G}(1, 3)$ parameterizing lines in $\mathbb{P}^3$. This is a 4-dimensional variety. Fix a *flag* $(p \in L \subset \Lambda)$, where $p$ is a point, $L$ is a line and $\Lambda$ is a plane in $\mathbb{P}^3$. Define the *Schubert cycles* of $\mathbb{G}(1, 3)$ as follows:

$\sigma_L$: the locus of lines that intersect $L$;

$\sigma_p$: the locus of lines that passes through $p$;

$\sigma_\Lambda$: the locus of lines that are contained in $\Lambda$.

One can perturb the flag configuration and the resulting cycles are rationally equivalent to the original ones. So those $\sigma_\bullet$ are elements of $A^*(\mathbb{G}(1, 3))$.

**Exercise 5.18.** Show that the codimensions of $\sigma_L$, $\sigma_p$ and $\sigma_\Lambda$ in $\mathbb{G}(1, 3)$ are 1, 2 and 2, respectively.

Since $\sigma_L \in A^1(\mathbb{G}(1, 3))$, let us compute $\sigma_L^2$ in $A^2(\mathbb{G}(1, 3))$. Modulo rational equivalence, we can take two lines $L_1, L_2$ and compute $\sigma_{L_1} \cap \sigma_{L_2}$ instead. Indeed, we specialize $L_1, L_2$ such that they intersect at one point $p$. Then they also span a plane $\Lambda$. In this special configuration, $[L_0] \in \sigma_{L_1} \cap \sigma_{L_2}$ means either $L_0$ is contained in $\Lambda$ or $L_0$ passes through $p$. It implies that

$$\sigma_L^2 = \sigma_p + \sigma_\Lambda$$

in $A^*(\mathbb{G}(1, 3))$. Of course one has check that it does not cause higher multiplicities during the specialization. But it can be verified by a further study of the tangent space of Schubert cycles. In general, the study of intersection products between Schubert cycles is called *Schubert calculus*.

**Exercise 5.19.** Carry out the following calculation in $A^*(\mathbb{G}(1,3))$.

(1) Show that $\sigma_p\sigma_\Lambda = 0$, $\sigma_p^2 = 1$ and $\sigma_\Lambda^2 = 1$.

(2) Calculate the degree of $\sigma_L^4$ in $A^4(\mathbb{G}(1,3)) \cong \mathbb{Z}$. In other words, calculate the number of lines that intersect four general lines in $\mathbb{P}^3$.

Back to the exact sequence $0 \to S \to V \to Q \to 0$ applied to $\mathbb{G}(1,3)$. We know that $S^*|_{[W]} = \mathrm{Hom}(W, \mathbb{C})$, where $W \subset \mathbb{C}^4$ is a 2-dimensional linear subspace. Let us use the degeneracy loci to calculate the Chern classes of $S^*$.

Take two general linear maps $\phi_i : \mathbb{C}^4 \to \mathbb{C}$ for $i = 1, 2$. They induce two general sections $\Phi_i$ of $S^*$ by $\Phi_i([W])(w) = \phi_i(w)$ for $w \in W$, $i = 1, 2$. By definition, the vanishing locus of $\Phi_1$ (or $\Phi_2$) is $c_2(S^*)$. Note that $\Phi_1$ is the zero map on $W$ if and only if $W \subset \ker(\phi_1)$, that is the line $[W]$ is contained in the plane $\Lambda$ obtained by projectivizing $\ker(\phi_1)$. We thus obtain that

$$c_2(S^*) = \sigma_\Lambda.$$

Similarly the locus where $\Phi_1$ and $\Phi_2$ fail to be linearly independent is $c_1(S^*)$. This happens if and only if $W \cap (\ker(\phi_1) \cap \ker(\phi_2)) \neq 0$, i.e. $[W]$ intersects the line $L$ cut out by the projectivization of $\ker(\phi_1) \cap \ker(\phi_2)$. We thus obtain that

$$c_1(S^*) = \sigma_L.$$

In total, we conclude that

$$c(S^*) = 1 + \sigma_L + \sigma_\Lambda.$$

Then we have

$$c(S) = 1 - \sigma_L + \sigma_\Lambda,$$

$$c(Q) = \frac{1}{c(S)} = 1 + \sigma_L + \sigma_p,$$

hence one can readily calculate $c(T_{\mathbb{G}}) = c(S^* \otimes Q)$.

**Example 5.20** (Lines on a cubic surface)**.** Let $X$ be a smooth cubic surface in $\mathbb{P}^3$ defined by a polynomial $F$. Note that $\mathrm{Sym}^3 S^*$ is a rank four vector bundle whose fiber is $H^0(L, \mathscr{O}(3))$, the space of cubic forms on a line $L$ in $\mathbb{P}^3$. The upshot is that $F$ can be regarded as a section of $\mathrm{Sym}^3 S^*$, i.e. over $[L]$ it specifies a cubic form $F|_L \in H^0(L, \mathscr{O}(3))$. Moreover, $L$ is contained in $X$ if and only if $F|_L = 0$. In other words, the locus of lines lying on $X$ as a cycle in the Grassmannian has class $c_4(\mathrm{Sym}^3 S^*) \in A^4(\mathbb{G}(1,3))$. Using the Chern classes of $S^*$, we can show that $c_4(\mathrm{Sym}^3 S^*) = 27$. Modulo some multiplicity check, it implies that a smooth cubic surface contains 27 lines.

**Exercise 5.21.** Do the explicit calculation to show that $c_4(\mathrm{Sym}^3 S^*) = 27$.

**Exercise 5.22.** Use Chern classes to explain that a degree $d \geq 4$ surface in $\mathbb{P}^3$ is expected to contain no lines.

**Exercise 5.23.** Show that the expected number of lines on a quintic 3-fold (i.e. a degree 5 hypersurface in $\mathbb{P}^4$) is 2875.

## 6. Algebraic surfaces

Throughout this section $X$ will be a smooth projective surface. Recall that $\mathrm{Div}(X)$ is the group of all divisors on $X$ whose quotient modulo linear equivalence is the Picard group $\mathrm{Pic}(X)$ of isomorphism classes of line bundles. A curve on $X$ is also a divisor.

6.1. **Intersection theory on surfaces.** Let $C, D$ be two curves on $X$ and $p$ a point in their intersection. We say that $C$ and $D$ meet *transversally* at $p$ if the defining equations $f, g$ of $C, D$ generate the maximal ideal $\mathfrak{m}_p = (x, y)$ in the local ring, where $x, y$ are the local coordinates at $p$. We also use the notation $\mathscr{O}_C(D)$ to denote the line bundle $\mathscr{O}_X(D) \otimes \mathscr{O}_C$ on $C$.

**Exercise 6.1.** Show that if $C$ and $D$ meet transversally at $p$, then $C, D$ are both nonsingular at $p$.

If $C$ and $D$ meet transversally at $r$ distinct points in total, it is natural to define the *intersection number* $C.\,D = r$. We want to generalize this intersection pairing to any two divisors in $\mathrm{Div}(X)$.

**Theorem 6.2.** *There is a unique intersection pairing* $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$*, denoted by* $C.\,D$ *for two divisors* $C, D$ *such that*
*(1) if* $C$ *and* $D$ *meet transversally everywhere, then* $C.\,D = \#(C \cap D)$*,*
*(2) it is symmetric:* $C.\,D = D.\,C$*,*
*(3) it is additive:* $(C_1 + C_2).\,D = C_1.\,D + C_2.\,D$*, and*
*(4) if* $C_1 \sim C_2$*, then* $C_1.\,D = C_2.\,D$*.*

The proof of the theorem is decomposed into several steps.

**Lemma 6.3.** *Suppose two curves* $C$ *and* $D$ *meet transversally. Then*

$$\#(C \cap D) = \deg(\mathscr{O}_C(D)).$$

*Proof.* Note that $\mathscr{O}_X(D) \otimes \mathscr{O}_C$ is a line bundle on $C$ whose associated divisor is $C \cap D$. Since the intersection is transverse everywhere, this divisor thus consists of $\#(C \cap D)$ points, hence its degree equals the degree of the line bundle. □

*Proof of Theorem 6.2.* First let us show the uniqueness. Since $X$ is projective, let $H$ be a fixed ample divisor. For any two divisors $C, D$, we can find $n \gg 0$ such that $C + nH$, $D + nH$ and $nH$ are all very ample. Using certain results of Bertini type, one can choose $C' \in |C + nH|$, $D' \in |D + nH|$, $E' \in |nH|$ and $F' \in |nH|$ such that they are all smooth, $D'$ transversal to $C'$, $E'$ transversal to $D'$ and $F'$ transversal to $C', E'$. Since $C \sim C' - E'$ and $D \sim D' - F'$, by the properties we have

$$C.\,D = \#(C' \cap D') - \#(C' \cap F') - \#(E' \cap D') + \#(E' \cap F').$$

It implies that $C.\,D$ is determined by the natural intersection numbers.

For the existence, first we define for two very ample divisors $C, D$ by setting $C.\,D = \#(C' \cap D')$ where $C' \in |C|$ and $D' \in |D|$ such that $C', D'$ meet transversally. Since $D$ and $D'$ (resp. $C$ and $C'$) represent the same line bundle, by Lemma 6.3 the above setting does not depend on the choices of $C'$ and $D'$. Now we have a well-defined pairing between very ample divisor classes. To define it for any two divisors $C, D$, we can write $C \sim C' - E'$ and $D \sim D' - F'$ such that $C', D', E', F'$ are all very ample. Then $C.\,D$ follows from the intersection pairing between those very ample divisors. One can check that if we choose different expressions for $C$ and

$D$, then the resulting pairings are the same. Moreover, it satisfies all the desired properties. □

It would be handy to define the intersection number locally at a point. Suppose $C$ and $D$ are two curves defined by $f$ and $g$, respectively, without a common component. Define the *intersection multiplicity* $(C.\,D)_p$ to be the length of $\mathscr{O}_{p,X}/(f,g)$, i.e. the dimension of the $k$-vector space $k[[x,y]]/(f,g)$.

**Example 6.4.** Let $f = y$ and $g = y - x^2$ over $\mathbb{C}$. Then $k[[x,y]]/(f,g) \cong k[[x]]/x^2$ is generated by 1 and $x$ as a $k$-vector space, hence the length is two.

**Proposition 6.5.** *If $C$ and $D$ do not have a common component, then*

$$C.\,D = \sum_{p \in C \cap D} (C.\,D)_p.$$

*Proof.* Recall the exact sequence

$$0 \to \mathscr{O}_C(-D) \to \mathscr{O}_C \to \mathscr{O}_{C \cap D} \to 0,$$

where (as a scheme) $C \cap D$ is defined by $(f,g)$ locally, namely the structure sheaf at $p$ is $\mathscr{O}_{p,X}/(f,g)$. Then we have

$$h^0(\mathscr{O}_{C\cap D}) = \chi(\mathscr{O}_C) - \chi(\mathscr{O}_C(-D)).$$

The left hand side equals $\sum_{p \in C\cap D}(C.\,D)_p$ by definition. The right hand side only depends on the linear equivalence classes of $C$ and $D$, hence we can replace them by differences of smooth curves that are transversal to each other appropriately. Therefore, it reduces to $\deg(\mathscr{O}_C(D)) = C.\,D$ as in the definition of the pairing. □

**Exercise 6.6.** Consider two plane curves $Q : YZ - X^2 = 0$ and $C : YZ^2 - X^3 = 0$. Calculate the intersection multiplicity of $Q$ and $C$ at every intersection point and confirm that they satisfy Proposition 6.5.

**Example 6.7** (Self-intersection)**.** Even if $C$ is nonsingular, we cannot directly see how to define $C^2 = C.\,C$. Instead, we use $C^2 = \deg(\mathscr{O}_C(C))$, where $\mathscr{O}_C(C)$ is the *normal bundle* of $C$ in $X$, often denoted by $N_{C/X}$ or $N_C$.

Let us treat normal bundles in a more general setting. Suppose $X$ is a smooth complex manifold and $D \subset X$ a smooth submanifold of codimension $k$. Denote by $N_{D/X}$ or simply $N_D$ the *normal bundle* of $D$ in $X$ as

$$N_D := (T_X|_D)/T_D.$$

Similarly, let $N_D^*$ be the *conormal bundle* that fits in the exact sequence

$$0 \to N_D^* \to \Omega_X|_D \to \Omega_D \to 0,$$

where $\Omega$ is the cotangent bundle, i.e. the sheaf of differentials. Both $N_D$ and $N_D^*$ are of rank $k$.

**Proposition 6.8** (Adjunction formula I)**.** *Suppose $D$ is a smooth, effective divisor in $X$. Then we have*

$$N_D \cong \mathscr{O}_X(D)|_D, \quad N_D^* \cong \mathscr{O}_X(-D)|_D.$$

*Proof.* Suppose $D$ is defined by $(U_\alpha, f_\alpha)$. Then the line bundle $\mathscr{O}_X(D)$ has transition functions $g_{\alpha\beta} = f_\alpha/f_\beta$. Since $f_\alpha \equiv 0$ on $D \cap U_\alpha$, the differential $df_\alpha$ defines a local section of $N_D^*$ on $U_\alpha$. Moreover,

$$df_\alpha = g_{\alpha\beta} \cdot df_\beta$$

on $D \cap U_{\alpha\beta}$, hence the union $(U_\alpha, df_\alpha)$ form a global section of $N_D^* \otimes \mathscr{O}_X(D)$, which is nowhere zero, for otherwise $D$ would be singular at the vanishing locus. It implies that $N_D^* \otimes \mathscr{O}_X(D)$ is a trivial line bundle. □

**Proposition 6.9** (Adjunction formula II)**.** *In the above setting, suppose $D$ is a divisor and let $K$ be the canonical line bundle of $X$. Then*

$$K_D \sim (K_X \otimes \mathscr{O}_X(D))|_D.$$

*Proof.* By the defining sequence of $N_D^*$ and Propostion 6.8, we have

$$c_1(\mathscr{O}_D(-D)) = c_1(N_D^*) = c_1(K_X|_D) - c_1(K_D),$$

hence $K_D \sim (K_X \otimes \mathscr{O}_X(D))|_D$. □

**Example 6.10** (Canonical line bundle of a hypersurface)**.** Let $X$ be a degree $k$ smooth hypersurface in $\mathbb{P}^n$. Then we have

$$K_X \sim (K_{\mathbb{P}^n} \otimes \mathscr{O}_{\mathbb{P}^n}(k))|_X = \mathscr{O}_X(k-n-1).$$

Now we specialize to the surface case.

**Proposition 6.11** (Adjunction formula for surfaces)**.** *If $C$ is a smooth curve of genus $g$ on $X$ and if $K_X$ is the canonical divisor of $X$, then*

$$2g - 2 = C.(C + K_X).$$

*Proof.* We know that

$$K_C \sim (K_X \otimes \mathscr{O}(C))|_C.$$

Take the degree of both sides. The left hand side is $2g-2$ and the right hand side is $C.(C+K_X)$. Hence the desired formula follows. □

**Example 6.12** (Plane curves)**.** Suppose $D$ and $E$ are plane curves of degree $d$ and $e$, respectively. Then $D.E = de$ as we saw in Bézout's theorem. Moreover, if $C$ is a smooth plane curve of degree $d$, then by Proposition 6.11 we recover that

$$g = \frac{(d-1)(d-2)}{2}.$$

**Example 6.13** (Quadric surfaces)**.** Let $Q$ be a smooth quadric surface in $\mathbb{P}^3$. One can show that $\mathrm{Pic}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by lines in the two families of rulings. Denote the generators by $f_1 = (1,0)$ and $f_2 = (0,1)$. We have

$$f_1^2 = f_2^2 = 0, \quad f_1.f_2 = 1,$$

because two lines in the same family are disjoint and two lines of opposite families meet transversally at a point. A hyperplane section of $Q$ has class $(1,1)$ and the canonical line bundle $K$ of $Q$ has class $(-2,-2)$ by adjunction.

Let $C$ be a smooth curve of type $(a,b)$. Then $C+K$ has class $(a-2, b-2)$. By Proposition 6.11, we have

$$2g - 2 = (a,b).(a-2,b-2) = 2ab - 2a - 2b,$$

$$g = (a-1)(b-1).$$

In particular, if $g = 0$ and if $C$ has degree $d$ in $\mathbb{P}^3$, we have $(a, b) = (1, d-1)$ or $(d-1, 1)$.

Recall the Riemann-Roch formula on curves. It says that $\chi(L) - \chi(C) = \deg(L)$ for a line bundle $L$ on a curve $C$. In other words, for fixed degree the holomorphic Euler characteristic of $L$ only depends on the topology of $C$. An analogous result holds for surfaces (as well as for higher dimensional varieties).

**Theorem 6.14** (Riemann-Roch for surfaces)**.** *Let $D$ be a divisor on a surface $X$. Then we have*

$$\chi(\mathscr{O}_X(D)) - \chi(\mathscr{O}_X) = \frac{1}{2} D.\,(D - K_X).$$

*Proof.* Since the result only depends on the linear equivalence classes, we can write $D \sim C - E$ for two nonsingular curves $C$ and $E$. By the exact sequences

$$0 \to \mathscr{O}_X(C - E) \to \mathscr{O}_X(C) \to \mathscr{O}_E(C) \to 0,$$
$$0 \to \mathscr{O}_X \to \mathscr{O}_X(C) \to \mathscr{O}_C(C) \to 0,$$

we obtain that

$$\chi(\mathscr{O}_X(C - E)) - \chi(\mathscr{O}_X) = \chi(\mathscr{O}_C(C)) - \chi(\mathscr{O}_E(C)).$$

Moreover, by Riemann-Roch on curves we have

$$\chi(\mathscr{O}_C(C)) = 1 - g_C + C^2,$$
$$\chi(\mathscr{O}_E(C)) = 1 - g_E + C.\,E.$$

Finally, by Proposition 6.11 we know

$$g_C = \frac{1}{2} C.\,(C + K_X) + 1,$$
$$g_E = \frac{1}{2} E.\,(E + K_X) + 1.$$

Combining the above, we thus obtain the desired formula. □

**Exercise 6.15** (Bertini)**.** Let $X \subset \mathbb{P}^n$ be a smooth projective variety. Prove that a general hyperplane cuts out a nonsingular subvariety of $X$. Use this to show that if $D$ is a very ample divisor on $X$, then a general element in the linear system $|D|$ is a smooth divisor.

**Exercise 6.16.** Suppose that $X$ is a smooth surface of degree $d$ in $\mathbb{P}^3$ and suppose it contains a line $L$. Prove that $L^2 = 2 - d$ on $X$.

**Exercise 6.17.** Let $X$ be a smooth surface of degree $d$ in $\mathbb{P}^3$. Calculate the self-intersection $K_X^2$.

6.2. **Blow up.** Blow up is a general procedure which applies to a subspace of a variety. To illustrate the idea, here we restrict to blowing up a point on a smooth surface $S$. Since the operation is local, we use affine coordinates $x, y$ for $S = \mathbb{A}^2$ or an open disk and suppose that $p = (0, 0)$ is the origin.

Consider all lines passing through $p$. We get a one-dimensional family of lines that are distinguished by the direction $y/x$ if $x \neq 0$ or $x/y$ if $y \neq 0$. Use $[u, v]$ to be the coordinates of $\mathbb{P}^1$. Define the *blow up* of $S$ at $p$ by $\widetilde{S}$ as follows:

$$\widetilde{S} := \{(x, y, [u, v]) \mid xv = yu\} \subset S \times \mathbb{P}^1.$$

Note that $\widetilde{S}$ admits a *contraction* $\pi$ to $S$ by forgetting the coordinates $[u, v]$.

**Proposition 6.18.** *In the above setting, $\widetilde{S}$ is a smooth surface, $\widetilde{S}\backslash\pi^{-1}(p) \to S\backslash p$ is an isomorphism and $\pi^{-1}(p) \cong \mathbb{P}^1$.*

*Proof.* For any $q \neq p$, i.e. $q$ has coordinate $(x, y)$ such that $x, y$ are not both zero, without loss of generality suppose that $x \neq 0$. Then $v/u = y/x$ is uniquely determined by $q$, hence $\pi^{-1}(q)$ is a single point. Indeed one can use $x, y$ to be the local coordinates of $\widetilde{S}$ nearby $\pi^{-1}(q)$, hence $\widetilde{S}$ is smooth away from $\pi^{-1}(p)$ and $\widetilde{S}\backslash\pi^{-1}(p) \cong S\backslash p$.

At $p$, since $x = y = 0$, there is no constraint to $[u, v]$. We thus obtain that $\pi^{-1}(p) \cong \mathbb{P}^1$.

Finally for a point $q = (0, 0, [u, v]) \in \pi^{-1}(p)$, without loss of generality suppose that $u \neq 0$ and $v/u = t$. Then a local coordinate system nearby $q$ can be given by $x, t$, namely, $(x, tx, [1, t])$. Therefore, $\widetilde{S}$ is also smooth along $\pi^{-1}(p)$. $\square$

Denote by $E = \pi^{-1}(p) \cong \mathbb{P}^1$ the *exceptional curve* of the blow up. Let $C \subset S$ be an effective curve. Denote by $\widetilde{C}$ the closure of $\pi^{-1}(C\backslash p)$ and call it the *proper transform* of $C$. If $C$ does not contain $p$, then $\widetilde{C}$ is just $\pi^{-1}(C)$ and $\pi : \widetilde{C} \to C$ is an isomorphism. The purpose of doing blow up is to separate tangent directions of curves that pass through $p$.

**Example 6.19.** Let $L$ be a line passing through $p$. Suppose $(a, b)$ is a point on $L$ not equal to $p$. Then $a, b$ cannot be both zero. Suppose $a$ is not zero and let $t = b/a$. Then $\pi^{-1}(L\backslash p)$ parameterize $(x, y, [u, v])$ such that $y/x = v/u = t$. In other words, the equations that define $\widetilde{L}$ are $y - tx = 0$ and $v - tu = 0$. Note that $\widetilde{L} \cap E$ is a single point $(0, 0, [1, t]) = (0, 0, [a, b])$. Therefore, the proper transforms of lines through $p$ form a disjoint union of lines in $\widetilde{S}$, each of which meets $E$ at a distinct point corresponding to the direction of the line.

**Remark 6.20.** The proper transform of a curve is different from the total inverse image under $\pi$, if the curve contains $p$. For instance, $\pi^{-1}(L) = \widetilde{L} \cup E$ is reducible, consisting of the proper transform as well as the exceptional curve.

**Example 6.21.** Consider a plane nodal cubic $C : y^2 - x^2(x + 1) = 0$. Blow up $p = (0, 0)$ and consider the proper transform of $C$. In $\widetilde{\mathbb{A}}^2$, let $v/u = b$, then $\pi^{-1}(C)$ is defined by $y - bx = 0$ and $x(x+1-b^2) = 0$, i.e. it consists of the exceptional curve $E$ along with the proper transform $\widetilde{C}$ defined by $uy - xv = 0$ and $(x+1)u^2 - v^2 = 0$. Note that $\widetilde{C}$ meets $E$ at two points $q_1 = (0, 0, [1, 1])$ and $q_2 = (0, 0, [1, -1])$, where $q_1, q_2$ correspond to the tangent directions of the two branches of $C$ at the node $p$. Moreover, $\widetilde{C}$ is smooth at both $q_1$ and $q_2$. This example illustrates that blow up is a useful tool for resolving singularities.

**Exercise 6.22.** Consider a plane cuspidal curve $C : y^2 - x^3 = 0$. Blow up the origin and write down explicitly the equations that define the proper transform $\widetilde{C}$ of $C$. Moreover, find the intersection of $\widetilde{C}$ and the exceptional curve $E$.

Now let us study the intersection paring associated to the blown up surface. Let $X$ be a smooth, projective surface and $p \in X$ a point. Denote by $\widetilde{X}$ the blow up of $X$ at $p$ and $E$ the exceptional curve. We have seen that $E \cong \mathbb{P}^1$ parameterizing tangent directions at $p$. Pulling back a line bundle induces a natural map $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$. On the other hand, if $D$ is an effective curve on $\widetilde{X}$, $\pi_* D$

is effective on $X$, which induces a projection $\pi_* : \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(X)$. In particular, $\pi_* E = 0$, which can also be seen from Proposition 6.23 (5) below.

**Proposition 6.23.** *In the above setting, we have the following results:*
*(1)* $E^2 = -1$*;*
*(2)* $\mathrm{Pic}(\widetilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$*;*
*(3)* $(\pi^* C).(\pi^* D) = C.D$ *for any* $C, D \in \mathrm{Pic}(X)$*;*
*(4)* $(\pi^* C).E = 0$ *for any* $C \in \mathrm{Pic}(X)$*;*
*(5)* $(\pi^* C).D = C.(\pi_* D)$ *for any* $C \in \mathrm{Pic}(X)$ *and* $D \in \mathrm{Pic}(\widetilde{X})$*;*
*(6)* $K_{\widetilde{X}} = \pi^* K_X + E$ *and* $K_{\widetilde{X}}^2 = K_X^2 - 1$.

*Proof.* For (1), since $E \cong \mathbb{P}^1$ parameterizes all linear directions at $p$, the normal bundle $N_{E/X}$ is isomorphic to the tautological bundle $\mathscr{O}_{\mathbb{P}^1}(-1)$. By the adjunction formula, we have $E^2 = \deg(N_{E/X}) = -1$.

For (2), we have the exact sequence

$$\mathbb{Z} \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(X) \to 0,$$

where the far left term is generated by $[E]$. Since $(nE)^2 = -n^2 \neq 0$, the far left map is an injection. Moreover, since a line bundle or a divisor is uniquely determined modulo any codimension two locus, we know $\mathrm{Pic}(X) \cong \mathrm{Pic}(X \backslash p) \cong \mathrm{Pic}(\widetilde{X} \backslash E)$, hence $\pi^*$ splits the exact sequence.

For (3), we can write $C \sim C_1 - C_2$ and $D \sim D_1 - D_2$ such that $C_i, D_j$ are very ample, hence can be chosen to not contain $p$. Then the claim follows since $C_i.D_j = \pi^*(C_i).\pi^*(D_j)$ in $X \backslash p = \widetilde{X} \backslash E$.

For (4), as in (3) we write $C \sim C_1 - C_2$ such that $C_i$ does not meet $p$, then $\pi^* C_i$ is disjoint with $E$.

For (5), the same idea applies.

Finally for (6), suppose $K_{\widetilde{X}} = \pi^* K_X + nE$ for certain $n \in \mathbb{Z}$. By adjunction, we have

$$\begin{aligned} 2g_E - 2 &= (K_{\widetilde{X}} + E).E \\ &= (\pi^* K_X).E - (n+1) \\ &= -n - 1. \end{aligned}$$

Since $g_E = 0$, we read off $n = 1$ and the self-intersection of $K_{\widetilde{X}}$ follows. ☐

**Remark 6.24.** We may also verify (6) via a local calculation, as we did for proving the Riemann-Hurwitz formula. Suppose in an open subset $U$ of $\widetilde{X}$ we have $u \neq 0$ and let $t = v/u$. Then $y = tx$ and $v = tu$, hence we can use $x, t$ as the local coordinates in $U$. Suppose $\omega$ is a section of $K_X$ with a local expression $\omega = f(x, y) dx \wedge dy$. Then $\pi^* \omega$ gives rise to a section of $K_{\widetilde{X}}$. Restricted to $U$ it can be written as

$$f dx \wedge (t dx + x dt) = (xf) dx \wedge dt.$$

Then the associated divisor of $\pi^* \omega$ consists of the zero of $x$, i.e. the exceptional curve $E$ as well as the associated divisor of the pullback of $f$, i.e. $\pi^* K_X$.

Next, we define the multiplicity of a curve at a point. Suppose that a curve $C$ in a smooth surface is locally defined by $f(x, y)$ such that at the origin $p$ we have $f = f_r$ modulo $(x, y)^{r+1}$, where $f_r$ is a homogenous polynomial of degree $r$ in $x, y$. In other words, $r$ is the largest integer such that $f \in (x, y)^r$. Then we say that the *multiplicity* of $C$ at $p$ is $r$ and denote it by $\mu_p(C) = r$.

**Example 6.25.** It is clear that $\mu_p(C) = 1$ if and only if $C$ is nonsingular at $p$. For a plane nodal cubic $C : y^2 - x^2(x+1) = 0$ and $p = (0,0)$, we have $\mu_p(C) = 2$. For a plane cuspidal cubic $C : y^2 - x^3 = 0$, we also have $\mu_p(C) = 2$. For three distinct lines in $\mathbb{A}^2$ meeting at the origin, say $C : y^3 - x^3 = 0$, we have $\mu_p(C) = 3$. In general, $\mu_p(C)$ measures how singular $C$ is at $p$.

**Proposition 6.26.** *Let $C$ be an effective curve in a smooth surface $X$ with multiplicity $\mu_p(C) = r$ at a point $p$. Let $\pi : \widetilde{X} \to X$ be the map associated to the blow up of $X$ at $p$, $E$ the exceptional curve and $\widetilde{C}$ the proper transform of $C$. Then*

$$\pi^* C = \widetilde{C} + rE.$$

*Proof.* Write $f = f_r + g$ in local coordinates $x, y$ in $X$ such that $g \in (x,y)^{r+1}$. Consider the open subset $U$ of $\widetilde{X}$ where $u \neq 0$ and let $t = v/u$. Then $v = ut$ and $y = xt$, hence $x, t$ give rise to a local coordinate system of $U$. Pulling back $f$, we have

$$\pi^* f = f_r(x, xt) + g(x, xt) = x^r(f_r(1,t) + xh(x,t)),$$

where $h(x,t) = g(x,xt)/x^{r+1}$ is holomorphic and $f_r(1,t) + xh(x,t)$ does not contain $E$ in its vanishing locus. From this expression, we see that the associated divisor of $\pi^* f$, i.e. $\pi^* C$ consists of two parts, one coming from the zeros of $x^r$, i.e. $r$ copies of $E$ and the other coming from the proper transform of $C$. $\square$

**Corollary 6.27.** *In the above setting, we have $\widetilde{C}.\, E = r$.*

*Proof.* By Propositions 6.23 and 6.26, we have

$$\begin{aligned} \widetilde{C}.\, E &= (\pi^* C - rE).\, E \\ &= 0 - r \cdot (-1) \\ &= r. \end{aligned}$$

$\square$

**Exercise 6.28.** Let $C, D$ be two effective curves on a surface meeting at a point $p$. Blow up $p$ and denote by $\widetilde{C}, \widetilde{D}$ the proper transforms of $C, D$, respectively. Show that $\widetilde{C}.\, \widetilde{D} = C.\, D - \mu_p(C) \cdot \mu_p(D)$.

**Exercise 6.29.** Consider the plane nodal cubic $C : Y^2 Z - X^2(X+Z) = 0$ in $\mathbb{P}^2$. Blow up the origin to obtain a surface $\widetilde{\mathbb{P}}^2$ and denote by $\widetilde{C}$ the proper transform of $C$. Calculate $\widetilde{C}^2$ and $K_{\widetilde{\mathbb{P}}^2}.\, \widetilde{C}$. Then use the adjunction formula to verify that $\widetilde{C}$ is isomorphic to $\mathbb{P}^1$, which is the normalization of $C$.

## 7. Schemes

Recall that a variety, roughly speaking, is the vanishing locus of a family of polynomials. There is certain ambiguity, say, $x^2$ and $x$ both cut out the line $x = 0$ in $\mathbb{A}^2$, but $x^2$ has vanishing order two along the line. In other words, $x$ is a nilpotent in the ring of functions $K[x,y]/(x^2)$ on the "double line" $x^2 = 0$. The language of schemes allows us to enlarge the category of varieties such that we can deal with "vanishing locus with multiplicity" in a formal way.

7.1. **The spectrum of a ring.** Let $R$ be a commutative ring. Define the *spectrum* $\operatorname{Spec} R$ as the set of all (proper) prime ideals of $R$. We sometimes use $[\mathfrak{p}]$ to denote a point in $\operatorname{Spec} R$ corresponding to a prime ideal $\mathfrak{p}$. Note that the zero ideal $(0)$ is prime if and only if $R$ is an integral domain.

**Example 7.1.** If $R$ is $\mathbb{Z}$, then $\operatorname{Spec}\mathbb{Z}$ consists of ideals $(p)$ where $p = 0$ or $p$ is a prime number. If $R = \mathbb{C}[x]/x^2$, every element of type $a + bx$ with $a \neq 0$ is a unit, hence the only prime ideal of $R$ is $(x)$.

Each element $f \in R$ gives rise to a "function" in the following sense. For a point $x = [\mathfrak{p}] \in \operatorname{Spec} R$, write $\kappa(x)$ or $\kappa(\mathfrak{p})$ to denote the quotient field of the integral domain $R/\mathfrak{p}$, i.e. the field of fractions $a/b$ for $a, b \in R/\mathfrak{p}$ and $b \neq 0$. We call $\kappa(x)$ the *residue field* at $x$. Then we can define $f(x) \in \kappa(x)$ by the image of $f$ under the canonical map $R \to R/\mathfrak{p} \to \kappa(x)$.

**Example 7.2.** Consider $\operatorname{Spec}\mathbb{Z}$. The "function" 10 at the point $(7)$ and at the point $(5)$ has value 3 and 0, by taking the remainder modulo 7 and 5, respectively.

**Example 7.3.** Consider the ring $\mathbb{C}[x]$ and let $p(x)$ be a polynomial. For $a \in \mathbb{C}$, $(x - a)$ is a prime ideal (indeed maximal). It is easy to see that $\kappa((x - a)) \cong \mathbb{C}$ and the function $p(x)$ at $(x - a) \in \operatorname{Spec}\mathbb{C}[x]$ has value equal to $p(a)$. In addition, $(0)$ is the only prime ideal not of type $(x - a)$. Hence, $\operatorname{Spec}\mathbb{C}[x]$ as a set compared to $\mathbb{A}^1$ is almost the same, except with one additional point corresponding to $(0)$.

More generally, if $R$ is the coordinate ring of regular functions on a variety $V$ defined over an algebraically closed field $K$ and if $\mathfrak{p}$ is the maximal ideal of a point $x \in V$ (i.e. all functions vanishing at $x$), then $\kappa(x) = K$ and $f(x)$ is the value of $f \in R$ at $x$.

**Example 7.4.** Consider $\operatorname{Spec}\mathbb{C}[x]/x^2$. It consists of a unique point $(x)$ with residue field $\mathbb{C}$. Then 0 and $x$ are both vanishing at $(x)$, hence as functions they are the same. In other words, $f \in R$ may not be determined by its function values on $\operatorname{Spec} R$.

7.2. **Zariski topology revisit.** We want to make $X = \operatorname{Spec} R$ a topological space. For every subset $S \subset R$, define

$$V(S) = \{[\mathfrak{p}] \in \operatorname{Spec} R \mid \mathfrak{p} \supset S\}.$$

It is easy to see that the definition is equivalent to

$$V(S) = \{x \in X \mid f(x) = 0 \text{ for all } f \in S\}.$$

Think of $f$ as a function and $x$ as a point. Then it is natural to use such $V(S)$ as closed sets for the space $\operatorname{Spec} R$ and $V(S)$ is just the vanishing locus of all "functions" in $S$. The resulting topology is called the *Zariski topology*. Since $\bigcap_\alpha V(S_\alpha) = V(\bigcup_\alpha S_\alpha)$, the Zariski topology is well-defined. Moreover, if $I$ is the ideal generated by $S$, then $V(I) = V(S)$.

The open sets of $\operatorname{Spec} R$ are given by the complements of $V(S)$. In particular, for every $f \in R$ we have a *distinguished open set* $X_f = X - V(f) = |\operatorname{Spec} R_f|$, where $R_f$ is the localization of $R$ by adjoining $1/f$ and the last equality is based on the correspondence $[\mathfrak{p}] \mapsto [\mathfrak{p}R_f]$ for a prime ideal $\mathfrak{p}$ not containing $f$. The distinguished open sets provide a base for the Zariski topology:

$$\operatorname{Spec} R - V(S) = \operatorname{Spec} R - \bigcap_{f \in S} V(f) = \bigcup_{f \in S} (\operatorname{Spec} R)_f.$$

**Remark 7.5.** The distinguished open set makes sense even if $f$ is nilpotent, i.e. the multiplicative set $\{f^n\}$ contains 0. In that case, by the definition of localization we have $R_f = 0$ and $\operatorname{Spec} R_f = \emptyset$. On the other hand, since $f$ is nilpotent, every prime ideal in $R$ contains $f$, hence $X_f = \operatorname{Spec} R - V(f) = \emptyset$.

**Proposition 7.6.** *The closure of a point* $[\mathfrak{p}]$ *consists of all* $[\mathfrak{q}]$ *such that* $\mathfrak{q} \supset \mathfrak{p}$. *In particular,* $[\mathfrak{p}]$ *is a closed point if and only if* $\mathfrak{p}$ *is maximal.*

*Proof.* The smallest closed set containing $[\mathfrak{p}]$ is $V(\mathfrak{p})$ by definition. Therefore, $V(\mathfrak{p}) = [\mathfrak{p}]$ if and only if no other prime ideal contains $\mathfrak{p}$, namely, if and only if $\mathfrak{p}$ is maximal, since every proper ideal of $R$ is contained in a maximal ideal. □

If $X$ is an affine variety defined over an algebraically closed field, then the closed points of $X$ correspond precisely to maximal ideals of its ring of functions. The closed points in the closure of the point $[\mathfrak{p}]$ are precisely the points of $X$ in the subvariety determined by $\mathfrak{p}$.

**Remark 7.7.** Usually $\operatorname{Spec} R$ is not a Hausdorff space. For instance, if $R$ is an integral domain, then $(0)$ is a prime ideal and is contained in every prime ideal, hence its closure $V(0) = \operatorname{Spec} R$ is the total space. Due to this reason $[(0)]$ is called the *generic point*. In general, every irreducible component of an affine scheme possesses a generic point corresponding to the minimal prime defining the component.

**Example 7.8.** Consider $R = \mathbb{C}[x, y]$. We want to compare $\operatorname{Spec} R$ with the affine plane $\mathbb{A}^2$. For $a, b \in \mathbb{C}$, the ideal $(x - a, y - b)$ is maximal and corresponds to the closed point $(a, b)$ in $\mathbb{A}^2$. Let $f \in R$ be an irreducible polynomial. Then $(x-a, y-b)$ contains $(f)$ if and only if $f(a, b) = 0$. Geometrically it says that the closed points in the closure of $[(f)]$ form the curve in $\mathbb{A}^2$ defined by the zero locus of $f$.

7.3. **Structure sheaves.** The last ingredient to define a scheme is an additional structure on the topological space $X = \operatorname{Spec} R$, analogous to differential structures on smooth manifolds or complex structures on complex manifolds. It is called the *structure sheaf* or *sheaf of regular functions* on $X$ and denoted by $\mathscr{O}_X$. Let $f \in R$ be a non-zero element. We first define on the distinguished open set $X_f = |\operatorname{Spec} R_f|$ that

$$\mathscr{O}_X(X_f) = R_f.$$

If $X_g \subset X_f$, then $V(f) \subset V(g)$, hence $g \in \operatorname{rad}(f)$, i.e. certain power $g^n$ is a multiple of $f$. Then the restriction map $R_f \to R_g$ can be defined via the localization map $R_f \to R_{fg} = R_g$. Modulo some detail in commutative algebra, the sections on the distinguished open sets $X_f$ over all $f \in R$ uniquely determine $\mathscr{O}_X$ as a sheaf. In particular, take $f = 1$ and then $X_f = X$, hence $\mathscr{O}_X(X) = R$, i.e. elements of $R$ are "regular functions" on $X$ as we expect. Similarly, sections of $\mathscr{O}_X(U)$ for an open (dense) subset $U$ are "rational functions" on $X$.

Now we are ready to define schemes in general. A *scheme* is simply a topological space $X$ together with a sheaf $\mathscr{O}_X$ such that $(X, \mathscr{O}_X)$ is locally affine, i.e. $X$ can be covered by open sets $U_\alpha$ such that there exists a ring $R_\alpha$ satisfying $U_\alpha \cong |\operatorname{Spec} R|$ and $\mathscr{O}_X|_{U_\alpha} \cong \mathscr{O}_{\operatorname{Spec} R_\alpha}$. If there is no confusion, we simply use $X$ to denote the scheme.

Let us mention some important notions and leave to the reader to study their properties in detail. We deal with affine schemes only, but an appropriate gluing generalizes to arbitrary schemes.

A *closed subscheme* of an affine scheme $X = \operatorname{Spec} R$ is the spectrum of a quotient ring of $R$. Therefore, closed subschemes of $X$ correspond one to one with the ideals in $R$.

Let $\operatorname{Spec} R/I$ and $\operatorname{Spec} R/J$ be two closed subschemes. Define their *union* by $\operatorname{Spec} R/(I \cap J)$ and their *intersection* by $\operatorname{Spec} R/(I + J)$.

Let $R_{\mathrm{red}}$ be the quotient of $R$ modulo the ideal of nilpotent elements. Define $X_{\mathrm{red}} = \operatorname{Spec} R_{\mathrm{red}}$ to be the *reduced scheme associated to* $X$. Note that as topological spaces $|X|$ and $|X_{\mathrm{red}}|$ are identical. If $X = X_{\mathrm{red}}$ as schemes, we say that $X$ is *reduced*.

If $|X|$ is not a union of two properly contained closed sets, then $X$ is called *irreducible*.

**Example 7.9.** The double line corresponding to the ideal $(x^2)$ is a non-reduced closed subscheme of the affine plane $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$. The cuspidal cubic $C$ corresponding to the ideal $(y^2 - x^3)$ is a reduced closed subscheme of $\mathbb{A}^2$. Moreover, $C$ is irreducible. On the other hand, consider a closed subscheme $D$ corresponding to the ideal $(y^2 - x^2)$. Then $(y^2 - x^2) = (y - x) \cap (y + x)$, hence $D$ is reducible and geometrically it is the union of two lines defined by $y - x = 0$ and $y + x = 0$.

7.4. **Local rings.** Let $x \in X$ be a point on a scheme $X$. Define the *local ring* of $X$ at $x$ by

$$\mathscr{O}_{X,x} := \varinjlim_{x \in U} \mathscr{O}_X(U),$$

i.e. the germ of the structure sheaf at $x$.

Since the definition is local, we can assume that $X = \operatorname{Spec} R$ is affine and $x = [\mathfrak{p}]$ corresponds to a prime ideal $\mathfrak{p}$ in $R$. Next we restrict to the distinguished open sets $\operatorname{Spec} R_f = X - V(f)$ containing $x$, i.e. $f(x) \neq 0$ or equivalently $f \notin \mathfrak{p}$. Thus we obtain that

$$\mathscr{O}_{X,x} := \varinjlim_{f \notin \mathfrak{p}} R_f = R_{\mathfrak{p}}$$

as well as its (unique) maximal ideal

$$\mathfrak{m}_{X,x} := \mathfrak{p} R_{\mathfrak{p}}.$$

The *dimension of* $X$ *at* $x$, denoted by $\dim(X, x)$ is the dimension of $\mathscr{O}_{X,x}$, which is defined as the supremum of lengths of chains (i.e. the number of strict inclusions) of prime ideals in $\mathscr{O}_{X,x}$. The *dimension of* $X$ is defined by the supremum of these local dimensions.

The *Zariski cotangent space* to $X$ at $x$ is $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$, regarded as a vector space over the residue field $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}$. The dual of this vector space is called the *Zariski tangent space* at $x$. Let us explain this definition when $X$ is a complex algebraic variety. In that case the tangent space at $x$ is just the vector space of derivations from the ring of germs of analytic functions at $x$ to $\mathbb{C}$, $\mathfrak{m}_{X,x}$ is the ideal of germs of regular functions vanishing at $x$ and $\mathbb{C}$ is the residue field. Therefore, the tangent space at $x$ (in the traditional sense) can be identified with $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, \mathbb{C}) = (\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^*$.

If the dimension of the Zariski tangent space is the same as $\dim(X, x)$, we say that $X$ is *nonsingular* at $x$. Otherwise we say that $X$ is *singular* at $x$.

**Example 7.10.** Consider $R = \mathbb{C}[x_1, \dots, x_n]$. At the origin $x = [(x_1, \dots, x_n)] \in \mathbb{A}^n = \operatorname{Spec} R$, we have

$$\mathscr{O}_{\mathbb{A}^n,x} = R_{(x_1,\dots,x_n)} = \{f/g \mid g(0,\dots,0) \neq 0\},$$
$$\mathfrak{m}_{\mathbb{A}^n,x} = \{f/g \mid f(0,\dots,0) = 0 \text{ and } g(0,\dots,0) \neq 0\}.$$

Then $\mathfrak{m}_{\mathbb{A}^n,x}/\mathfrak{m}^2_{\mathbb{A}^n,x}$ is a $n$-dimensional vector space over $\mathbb{C}$ with a basis given by $x_1, \dots, x_n$.

**Example 7.11.** Consider $R = \mathbb{C}[x]/x^2$ and let $X = \operatorname{Spec} R$. At $x = [(x)]$ the unique prime ideal in $R$, we have

$$\mathscr{O}_{X,x} = R_{(x)} = R,$$
$$\mathfrak{m}_{X,x} = (x),$$
$$\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} = (x).$$

Thus we have $\dim(X, x) = 0$, since there is no strict inclusion of two prime ideals in $R$. On the other hand, the Zariski tangent space is one-dimensional at $x$. Hence we conclude that $X$ is a zero-dimensional singular scheme. Indeed, $X$ can be regarded as a "double point" with support $X_{\text{red}} = \operatorname{Spec} \mathbb{C} = [(0)]$.

**Example 7.12.** Consider $R = \mathbb{C}[x, y]/(y^2 - x^3)$, $C = \operatorname{Spec} R$ and let $p = [(x, y)] \in C$. Geometrically $p$ is the cusp on the cuspidal cubic $C$ in the affine plane. We have

$$\mathscr{O}_{C,p} = \{f/g \mid g(0,0) \neq 0\},$$
$$\mathfrak{m}_{C,p} = \{f/g \mid f(0,0) = 0 \text{ and } g(0,0) \neq 0\}.$$

Then $\mathfrak{m}_{C,p}/\mathfrak{m}^2_{C,p} = (x, y)/(x, y)^2$ as a vector space over $\mathbb{C}$ is generated by $x$ and $y$, hence has dimension two. On the other hand, note that $(x)$ and $(y)$ are not prime ideals in $R$, because $y^2 = x^3$, and any other function not in $(x, y)$ is a unit in $\mathscr{O}_{C,p}$, hence $\dim(C, p) = 1$ by the inclusion $0 \subset (x, y)$. As a result $C$ is singular at $p$.

7.5. **Morphisms.** Morally speaking, a morphism from one scheme to another is locally a homomorphism of their local rings. The rigorous definition is a bit technical, so we simplify the situation by considering morphisms between affine schemes and leave to the reader to "patch them together".

Let $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$ be two affine schemes. Then a *morphism* $\psi : X \to Y$ is the same as a morphism $\phi : R \to S$ via the relation $\psi([\mathfrak{p}]) = [\phi^{-1}(\mathfrak{p})]$. Therefore, we have the following correspondence.

**Corollary 7.13.** *The category of affine schemes is equivalent to the category of commutative rings, with reversed arrows.*

**Example 7.14.** Consider the morphism $\psi : C = \operatorname{Spec} \mathbb{C}[x, y]/(x - y^2) \to D = \operatorname{Spec} \mathbb{C}[x]$ induced by $\phi : \mathbb{C}[x] \to \mathbb{C}[x, y]/(x - y^2)$ such that $\phi(x) = x$. Then $\psi([(x - a^2, y - a)]) = [\phi^{-1}(x - a^2, y - a)] = [(x - a^2)]$. But $\psi([(x - a^2, y + a)])$ has the same image. Geometrically this is the vertical projection $C \to D$ by forgetting the $y$-coordinate, which is a double cover between two affine lines. For $a \neq 0$, the closed point $(x - a^2)$ in $D$ has two inverse images $(x - a^2, y - a)$ and $(x - a^2, y + a)$ in $C$. At the origin $(x)$ in $D$, we see a branch point, since it has only one inverse image given by $(x, y)$ in $C$.